Harmonic Analysis on Number Fields

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19th December 2005

Abstract

This paper aims to give a review of John Tate's classical PhD thesis from 1950. We introduce and explain the necessary prerequisites concerning locally compact groups, harmonic analysis on these groups, and how this theory is applied to the adèle ring and the idèle group of a number field. Following Tate, we then prove the functional equation and analytic continuation of the so-called Hecke L-functions. This class of functions includes the Dedekind zeta functions attached to number fields and in particular, the classical Riemann zeta function.

Contents

1 Introduction

Number theory is different from most other areas of mathematics in that it uses techniques from almost every part of modern pure mathematics. In particular, algebraic and analytic methods are both indispensable for the working number theorist. In this paper we will have a look at John Tate's PhD thesis from 1950, in which algebra and analysis come together in a beautiful way to prove a classical result of Hecke concerning a class of so-called *L-functions*.
Although the results proved by Tate are very interesting in their own right, the real reason for studying Tate's thesis is that it is a necessary prerequisite for understanding the various generalizations discovered during the last 40 years by Langlands, Jacquet, Drinfeld, Cogdell, Shahidi, Piatetski-Shapiro, Harris, Taylor, Henniart, Lafforgue, and many others. Unfortunately, these generalizations are too intricate to be discussed in any depth in this paper, but in section 9 we shall attempt to give a tiny glimpse of this remarkably fascinating world.

1.1 Outline

The heart of this paper is section 8, where we present, with full proofs, the content of Tate's thesis. The purpose of sections 3 to 7 is to equip the reader with the tools required in section 8. In some of these chapters we are very brief, while in others we give some more details and also prove many of our assertions. For each section we give references for the full proofs and for alternative presentations of the material. Finally, we have added section 2 on L-functions and section 9 on the Langlands program with the aim of giving some background and motivation for the rest of the paper.

1.2 Preliminaries

The main obstacle to approaching Tate's thesis is that it uses a large amount of both algebra and analysis. We will assume that the reader has a thorough knowledge of algebra up to the level of typical first graduate courses, and some familiarity with analysis.

1.2.1 Algebra

We assume that the reader is familiar with standard properties of groups, rings, fields and modules. We refer the reader to the excellent book of Grillet [9] for all this. From commutative algebra we implicitly use everything that is needed for basic algebraic number theory. Some of this is included in Grillet's book, but we also refer the reader to Atiyah and Macdonald [1].

1.2.2 Topology and analysis

We will freely use definitions and facts from elementary topology. This material is covered in the first chapter of Bredon [3]. From analysis we need basic facts concerning classical Fourier analysis, topological groups, measure theory, and elementary functional analysis. In section 3 we shall give a short summary of this.

1.2.3 Number theory

We do not assume any analytic number theory. On the other hand, standard material from a first course in algebraic number theory will be ubiquitous throughout the paper. This includes definitions and elementary properties of number fields and of p-adic numbers, and we try to give an overview of this in section 5.

1.3 Notation and conventions

Our notation is fairly standard – $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all have their usual meaning; $\mathbb{N}$ is $\{0, 1, 2, \dots\}$ and $\mathbb{F}_q$ is the finite field with q elements, where q is a power of a prime. If A is any ring (most often this will be a field), then $A^\times$ denotes the group of units in A. We will write $\mathbb{R}_+^\times$ for the multiplicative group of positive real numbers, $\mathbb{R}_{\geq 0}$ for the set of non-negative reals, and S^1 for the group of complex numbers of absolute value 1. If M is any set, $\mathcal{P}(M)$ is the set of all subsets of M, and if s is a complex number, $\Re(s)$ denotes the real part of s. For our topological considerations the notion of a neighbourhood will of course play an important rôle, and for us a neighbourhood of a point x in a topological space will be any set that contains an open set containing x; thus a neighbourhood is *not* necessarily open.

2 L-functions

Much of modern research in number theory is in some way or another related to so-called *L-functions*[1]. These functions are complex-valued functions constructed from various mathematical, often algebraic, objects. Because the L-functions capture a lot of information about the objects they are constructed from, they can be used to prove otherwise inaccessible results about these objects. Usually, an L-function is defined by a series that is convergent only in some part of the complex plane. An important question is therefore often if the L-function has analytic (or meromorphic) continuation to the whole complex plane. A related question concerns whether there is a *functional equation* for the L-function. If $f(s)$ is the L-function, the

[1]Some of these functions are sometimes referred to as zeta functions, L-series, or other similar names.

functional equation could for example be of the form $f(s) = a(s)f(1-s)$ for some meromorphic function $a(s)$. Other issues of interest usually include the location of the zeros and poles of the L-function, and the values of the L-function at special points of the complex plane (for example integers).
In this paper we shall be concerned with L-functions constructed from number fields, but before going into these definitions we shall give a number of examples of L-functions from other contexts, in the hope that this will inspire the reader to further study.

2.1 Examples

2.1.1 The Riemann zeta function

Already the simplest example of an L-function takes us directly into some of the deepest waters of number theory. The *Riemann zeta function* is defined to be

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

where s is a complex variable. This series converges in the halfplane $\Re(s) > 1$ and is analytic in this domain. It has an analytic continuation to all of $\mathbb{C}$, with only a simple pole at $s = 1$. The zeta function is zero when s is a negative even integer, and it can be shown that all other zeroes lie in the so-called *critical strip* $0 < \Re(s) < 1$. One of the most famous problems in all of mathematics is the

Conjecture 1 (Riemann Hypothesis). All zeros of $\zeta(s)$ in the critical strip lie on the line $\Re(s) = \frac{1}{2}$

As the reader probably knows, there is a very profound connection between this conjecture and the distribution of prime numbers.
The Riemann zeta function satisfies a functional equation, namely

$$\Lambda(s) = \Lambda(1-s)$$

where we have defined the *completed zeta function* by

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Concerning values of the zeta function at positive integers, we have that

$$\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}$$

where $k = 1, 2, 3, \ldots$ and B_{2k} are the Bernoulli numbers. There is no such simple formula for the values of $\zeta(s)$ at odd integers. These values are mysteriously connected with algebraic K-theory, and it is possible to describe

them by the theory of so-called *Tamagawa measure* developed by Bloch and Kato.
At last we mention that in the region $\Re(s) > 1$, the Riemann zeta function can also be expressed as

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

where p runs through the prime numbers. The factor $\frac{1}{1-p^{-s}}$ is called the *Euler factor* at the prime p. If we consider the completed zeta function $\Lambda(s)$ above, it can of course be written as a product of all the Euler factors for $\zeta(s)$, plus one additional factor $\pi^{-s/2}\Gamma(s/2)$. This additional factor can be seen as coming from a "prime at infinity" – we will explain this in the coming sections, when we generalize the notion of a prime to the concept of a *place*.

2.1.2 Arithmetic schemes

As a huge generalization of the Riemann zeta function, we can construct a zeta function for any arithmetic scheme, that is, a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$. For such a scheme X, the residue field at any closed point x is finite, and the cardinality of this field is denoted by $N(x)$. We then define the zeta function of X to be

$$\zeta_X(s) = \prod \left(1 - N(x)^{-s}\right)^{-1}$$

where the product is over the closed points of X. This product is absolutely convergent for $\Re(s) > \dim X$. Of course, we get back the Riemann zeta function for $X = \mathrm{Spec}(\mathbb{Z})$. Later we will deal with the *ring of integers* $\mathfrak{o}_K$ of a *number field* K. In this context we shall define the *Dedekind zeta function* of a number field – this will be the zeta function associated to the arithmetic scheme $\mathrm{Spec}(\mathfrak{o}_K)$.

2.1.3 The Weil conjectures

Now let X be a smooth projective variety of dimension d over a finite field $\mathbb{F}_q$ – then X can be regarded as an arithmetic scheme. We define a function f by the relation $f(q^{-s}) = \zeta_X(s)$. We can develop the quotient $f'(t)/f(t)$ in a power series, and the coefficients of this power series carry some very interesting arithmetic information about X, related to Diophantine equations. The Weil conjectures, formulated by André Weil in 1949, state that[2]:
(1) f is a rational function; more precisely, we have

$$f(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)}$$

[2]Some of these terms are defined in section 5

where the P_k are polynomials with complex coefficients such that $P_k(0) = 1$.
(2) The polynomials P_k are of the form

$$P_0(t) = 1 - t \qquad P_{2d}(t) = 1 - q^d t$$

and for $1 \leq k \leq 2d - 1$,

$$P_k(t) = \prod_i (1 - \omega_{k,i} t)$$

where the $\omega_{k,i}$ are certain algebraic integers.
(3) f satisfies a certain functional equation, relating $f(t)$ to $f(\frac{1}{q^d t})$.
(4) (Analogue of the Riemann hypothesis) The absolute value of each number $\omega_{k,i}$ equals $q^{k/2}$.
(5) If X is the reduction of a smooth projective variety Y defined over a number field K embedded in $\mathbb{C}$, then the degree of the polynomial P_k is equal to the k^{th} Betti number of the complex variety $Y(\mathbb{C})$.
The Weil conjectures are of great historical interest, since they were one of the main driving forces behind the development of scheme theory in general, and étale cohomology in particular, during the 60s and early 70s. Weil proved particular cases of his conjectures very early, and Dwork proved in 1960 that f is rational, using methods from p-adic analysis (see [15] for a nice exposition of Dwork's proof). After the emergence of schemes and étale cohomology, Deligne was able to give a full proof of the Weil conjectures in 1973; we refer to [6] for a treatment of this.

2.1.4 Elliptic curves

A very exciting area within number theory is the theory of *elliptic curves*. An elliptic curve E is a smooth projective curve of genus one with a specified basepoint O. One of the main features of such a curve is that it is an abelian group in a natural way, with the specified basepoint as the identity – higher-dimensional generalizations of elliptic curves are therefore called *abelian varieties*. If an elliptic curve can be defined by a polynomial with coefficients in a field k, and if the point O has coefficients in k, we say that the curve is *defined over* k, and we write E/k. In this case we define the set $E(k)$ to be the set of points on E with coefficients in k. The *Mordell-Weil* theorem states that if K is a number field (for example $\mathbb{Q}$) then $E(K)$ is finitely generated. To any elliptic curve E defined over $\mathbb{Q}$, we associate a certain positive integer called the *conductor* of E, and an L-function defined by a product over all primes:

$$L(E, s) = \prod_p \frac{1}{L_p(p^{-s})} \tag{1}$$

where the L_p are certain polynomials of low degree.
One of the hardest and most important problems in number theory is a

conjecture formulated by Birch and Swinnerton-Dyer; we state here a weak version.

Conjecture 2 (Weak Birch and Swinnerton-Dyer conjecture). Let E be an elliptic curve defined over $\mathbb{Q}$. Then the order of vanishing of $L(E, s)$ at $s = 1$ equals the rank of the group $E(\mathbb{Q})$.

Among the most fascinating developments in the number-theoretic research of the last decade is the proof of the Shimura-Taniyama-Weil conjecture. This conjecture provides an incredibly surprising link between elliptic curves and *modular forms*. The group $SL_2(\mathbb{Z})$ of integer matrices $\gamma = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ with determinant 1 acts on $\mathbb{H} \cup \{\infty\}$ ($\mathbb{H}$ is the set of complex numbers with positive imaginary part) by the action

$$\gamma \cdot z = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot z = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}.$$

For any positive integer N, we define $\Gamma(N)$ to be the group of matrices $\gamma = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ in $SL_2(\mathbb{Z})$ such that $a_{11} \equiv a_{22} \equiv 1$ and $a_{12} \equiv a_{21} \equiv 0 \pmod{N}$, and similarly we define $\Gamma_0(N)$ to be the group of matrices with $a_{21} \equiv 0 \pmod{N}$. If Γ is a subgroup of $SL_2(\mathbb{Z})$ containing some $\Gamma(N)$, we define a *modular form of weight k for* Γ to be a complex-valued holomorphic function f on $\mathbb{H}$ that is also holomorphic at certain boundary points of $\mathbb{H}$ called *cusps*, and that satisfies the transformation law

$$f(\gamma \cdot z) = (a_{21} + a_{22}z)^k f(z)$$

for all $\gamma \in \Gamma$. Introducing the variable $q = e^{2\pi i z/h}$, where h is a certain positive integer, f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c_n q^n.$$

and we call f a *cusp form* if f vanishes at all the cusps; for $\Gamma = SL_2(\mathbb{Z})$ this amounts to the coefficient c_0 in the Fourier expansion being zero. If f is a modular form, we can associate an L-function to f by

$$L(f, s) = \sum_{n=1}^{\infty} c_n n^{-s}$$

for all s where this sum converges. The set $\mathcal{M}_k(\Gamma)$ of modular forms of weight k for Γ is a finite-dimensional complex vector space, and the cusp forms form a subspace $\mathcal{S}_k(\Gamma)$ of $\mathcal{M}_k(\Gamma)$. Certain operators called *Hecke operators* act on $\mathcal{S}_k(\Gamma)$, and cusp forms that are simultaneous eigenvectors for all the Hecke operators are called *eigenforms*. A *newform* is then a certain kind of eigenform, and we can state

Theorem 1 (Shimura-Taniyama-Weil conjecture). Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. Then for some newform $f \in \mathcal{S}_2(\Gamma_0(N))$, we have $L(f,s) = L(E,s)$.

This theorem, originally formulated as a conjecture by Shimura, was proved for so-called *semistable* elliptic curves by Wiles and Taylor in 1995, and it had then been known for some time that this would imply the famous *Fermat's Last Theorem.* The theorem was proved for all elliptic curves in 1998 by Breuil, Conrad, Taylor and Diamond. A superb introduction to all this is given in Diamond and Shurman [5]. We will see in section 9 that this surprising connection between elliptic curves and modular forms is but a small part of a much bigger picture – the Langlands correspondence for number fields.

2.2 Hecke L-functions

Now we come to the L-functions that constitute the main subject of this paper. A *Hecke L-function* of a number field is a generalization of the Dedekind zeta function from classical algebraic number theory (to be defined in section 5). We will prove that any Hecke L-function satisfies a functional equation and has meromorphic continuation to all of $\mathbb{C}$. To define Hecke L-functions, we need to develop the notion of an *idèle class character* of a number field, and this will not be done until section 7. Therefore we content ourselves at this moment with mentioning that the class of Hecke L-functions includes the Riemann zeta function and more generally the Dedekind zeta functions, so our results will be valid in these cases.

3 Prerequisites from analysis

In this section, we review a few things from topology and analysis. We assume that the reader is familiar with classical Fourier analysis, as presented for example in Vretblad [30].

3.1 Topological groups

Almost everything in this paper takes place on topological groups of various kinds. We recall here the basic definitions and a few facts about such groups. A *topological group* is a group G equipped with a topology such that the group operation is a continuous function from $G \times G$ to G, and that forming of inverses is a continuous function from G to G. In this definition, and everywhere else unless otherwise stated, a finite product of topological groups is understood to have the product topology. The product topology turns such a product of groups into a topological group. A related fact is that if H is a normal subgroup of a group G, then the quotient group G/H

is a topological group under the quotient topology. In particular, if G is abelian, any quotient of G is a topological group in the natural way.
If x is any element of a topological group G, then multiplication by x defines a homeomorphism $G \to G$. This means that the topology of a topological group is *translation-invariant* in the sense that for any $x \in G$, a set U is open if and only if xU (or Ux) is open. This allows us to define a topology on any group G simply by defining a neighbourhood base at the identity element of G.

3.2 Measure theory

We shall develop the notion of a Haar measure on a topological group, and in section 8 we shall use this to compute various integrals on local fields and on the adèle group and the idèle group of a number field. It will be very satisfying to see how abstract measure theory enables us to compute concrete integrals on spaces that are completely different from the ordinary Euclidean spaces usually seen in measure-theoretic examples, and even more so as these computations lead us to deep results in number theory.
Recall that a σ-algebra on a set X is a subset Ω of $\mathcal{P}(X)$ such that:
(S1) $X \in \Omega$.
(S2) If $A \in \Omega$, then $A^c \in \Omega$ (A^c is here the complement of A in X).
(S3) Ω is closed under countable unions – this means that if $\{A_k\}_1^\infty$ is a countable collection of sets and $A_k \in \Omega$ for all k, then $\bigcup_k A_k$ is also in Ω.
It is easily shown from these axioms that a σ-algebra must contain the empty set and be closed under countable intersections. Elements of Ω are called *measurable sets*, and a set X together with a σ-algebra Ω on it is called a *measurable space*; formally we think of this as an ordered pair (X, Ω). A *positive measure* on a measurable space (X, Ω) is a function $\mu : \Omega \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ that is *countably additive* and such that $\mu(\emptyset) = 0$. The first condition means that if $\{A_k\}_1^\infty$ is a countable collection of disjoint sets in Ω, then the equality

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

holds. If $A \in \Omega$, we usually refer to the number $\mu(A)$ as the *measure* of A.
The purpose of all these definitions is that they allow us to do integration in a rigorous way. Let (X, Ω) be a measurable space and let μ be a positive measure on this space. The *characteristic function* of a set E is the function that takes the value 1 on E and 0 outside E; we denote this function by c_E.
We say that a complex-valued function h on X is *simple* if it can be written as a finite sum

$$h(x) = \sum_{i=1}^{n} \beta_i c_{E_i}(x)$$

where the β_i are complex numbers and the E_i are measurable sets with finite measure. The *integral* of such a function is defined to be

$$\int h \mathrm{d}\mu = \sum_{i=1}^{n} \beta_i \mu(E_i).$$

Let $\{f_k\}_1^\infty$ be a sequence of simple functions. We say that such a sequence is *Cauchy in the mean* if $\int |f_n - f_m| \mathrm{d}\mu \to 0$ as $m, n \to \infty$. A complex-valued function f on X is *measurable* if the inverse image $f^{-1}(U)$ of every open set U is measurable. A measurable function f is *integrable* if there exists a sequence $\{f_k\}_1^\infty$ of simple functions that is Cauchy and converges to f almost everywhere (meaning that the set of points where this is not the case is of measure zero). For such a function f, we define the integral of f to be

$$\int f \mathrm{d}\mu = \lim_{k\to\infty} \int f_k \mathrm{d}\mu.$$

Note that this is a limit of a Cauchy sequence in the complex plane, so the limit exists in $\mathbb{C}$. If Y is a subset of X, we say that f is *integrable on* Y if the function $f \cdot c_Y$ is integrable, and we define

$$\int_Y f \mathrm{d}\mu = \int f c_Y \mathrm{d}\mu.$$

When the space X is a topological space, we are interested in σ-algebras that somehow relate to the topology on X. For any subset S of $\mathcal{P}(X)$ there is a smallest σ-algebra containing S; this is called the σ-algebra *generated* by S. For any topological space X, we define the *Borel σ-algebra* $\mathcal{B}(X)$ to be the σ-algebra generated by the open sets of X; the elements of $\mathcal{B}(X)$ are called *Borel sets*. Note that with respect to the Borel σ-algebra, every continuous complex-valued function on X is measurable. A *Borel measure* on X is a positive measure on $(X, \mathcal{B}(X))$. If E is any Borel set, we say that μ is *inner regular* on E if

$$\mu(E) = \sup \{ \ \mu(K) \mid K \subseteq E, \ K \text{ compact}\}$$

and we say that μ is *outer regular* on E if

$$\mu(E) = \inf \{ \ \mu(U) \mid U \supseteq E, \ U \text{ open}\}.$$

We define a *Radon measure* on a topological space X to be a Borel measure that is inner regular on all open sets, outer regular on all Borel sets, and such that every compact set has finite measure.

If we now let G be a topological group, with Borel measure μ, we can look at how the measure behaves under translation by elements of the group. More precisely, we say that μ is *left translation invariant* (respectively, *right translation invariant*), if for every Borel set E and every element x of G, we have

$\mu(xE) = \mu(E)$ (respectively, $\mu(Ex) = \mu(E)$). If the group is abelian, these two concepts coincide, and we simply say that such a measure is translation invariant. Now we can give the very important

Definition 1. Let G be a topological group. A *left Haar measure* on G is a nonzero Radon measure that is left translation invariant. A right Haar measure is defined similarly, and if G is abelian, a *Haar measure* on G is a nonzero Radon measure that is translation invariant.

In the next section we shall introduce a class of topological groups for which we can guarantee the existence of a Haar measure, but first we briefly look at products of measurable spaces.
Let (X, Ω_X) and (Y, Ω_Y) be measurable spaces. We define the *product* of these two spaces to be the pair $(X \times Y, \Omega_{X\times Y})$, where $\Omega_{X\times Y}$ is the σ-algebra generated by the collection of all sets of the form $A \times B$, with $A \in \Omega_X$, $B \in \Omega_Y$. If μ is a positive measure on (X, Ω_X) and ν is a positive measure on (Y, Ω_Y), then there is a unique positive measure λ on $(X \times Y, \Omega_{X\times Y})$, called the *product measure of μ and ν*, such that for every pair of measurable sets $A \subseteq X$, $B \subseteq Y$ we have $\lambda(A \times B) = \mu(A)\nu(B)$.

Theorem 2 (Fubini's theorem). Let $f(x, y)$ be a measurable function on $(X \times Y, \Omega_{X\times Y})$ and let μ and ν be positive measures on X and Y, respectively, with product measure λ. If either $\int_X(\int_Y |f|\mathrm{d}\nu)\mathrm{d}\mu$ or $\int_Y(\int_X |f|\mathrm{d}\mu)\mathrm{d}\nu$ is finite, then f is integrable on $X \times Y$. Moreover,

$$\int_{X\times Y} f\mathrm{d}\lambda = \int_X(\int_Y f\mathrm{d}\nu)\mathrm{d}\mu = \int_Y(\int_X f\mathrm{d}\mu)\mathrm{d}\nu.$$

4 Harmonic analysis and Pontryagin duality

Here we aim to summarize the essence of the analytical input to Tate's thesis. It is indeed surprising and beautiful that the theory of harmonic analysis, so deeply rooted in classical analysis, can play such an important rôle in algebraic number theory.

4.1 Locally compact abelian groups

Classically, harmonic analysis takes place on the real line $\mathbb{R}$. However, to really take advantage of this machinery in a number-theoretic setting, the classical Fourier theory needs to be generalized. It turns out that instead of restricting ourselves to working with functions defined on $\mathbb{R}$, we can consider functions defined on any *locally compact abelian group*. In the following we shall have a closer look at this exciting theory.

4.1.1 Definitions and elementary examples

Definition 2. A topological space X is said to be *locally compact* if every point $t \in X$ is contained in some compact neighbourhood. A *locally compact group* is a topological group that is locally compact and Hausdorff as a topological space.

Example 1. Obviously, the additive groups $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact groups.

Example 2. Let G be any group equipped with the discrete topology. Then G is a locally compact group.

Example 3. Classical Lie groups, for example $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$ $(n \geq 2)$ provide examples of non-abelian locally compact groups.

In section 7 we will see some much more interesting examples of locally compact groups, when we look at the adèle group and idèle group of a number field. These groups, as well as the additive and multiplicative groups of the local fields that we will encounter in section 6.2, are all abelian. We shall therefore restrict our attention to abelian groups, although much of what we say holds in the nonabelian case as well. We introduce the abbreviation LCA for "locally compact abelian"

4.1.2 Haar measure

One of the main features of locally compact abelian groups is the following

Theorem 3 (Existence and uniqueness of Haar measure on LCA groups). Every locally compact abelian group G admits a Haar measure. If μ and ν are two Haar measures on G then $\nu = a \cdot \mu$ for some positive real number a.

Proof. See chapter XI in Halmos [10] or §1.2 in Valenza and Ramakrishnan [23]. □

Clearly, if μ is a Haar measure, then so is $a \cdot \mu$ for any positive real a, so Theorem 3 describes the "best possible situation" concerning existence and uniqueness. The choice of a Haar measure amounts to a normalization. This will usually be done by fixing the measure of some compact set. The notion of Haar measure will be of utmost importance throughout our subsequent work. Because of this, we list here the most important properties of a Haar measure; all of these are included in or follow easily from the definition in section 3.2. Let μ be a Haar measure on the LCA group G. Then
(H1) μ is translation invariant.
(H2) $\mu(U) > 0$ for all nonempty open sets U.

(H3) $\mu(K)$ is finite for all compact sets K.
(H4) If f is real-valued, non-negative and integrable on G, then

$$\int_U f \mathrm{d}\mu = \sup_K \{ \int_K f \mathrm{d}\mu \mid K \text{ compact}, \ K \subseteq U \}$$

for every open set $U \subseteq G$.
On several occasions we will deal with products $G = G_1 \times G_2$ of locally compact groups. Such products are locally compact in the product topology, and the product of the Haar measures on the factors is a Haar measure on the product. If we are given a Haar measure λ on the product G and a Haar measure μ on one of the factors, say G_1, it is clear that there is a unique Haar measure ν on G_2 such that the product measure of μ and ν is λ. The measure ν is called the *quotient measure* of λ with respect to μ. If G is a discrete group, the *counting measure* on G is the measure that assigns to each set its cardinality. For any LCA group G with Haar measure μ, we define $L^1(G)$ to be the complex vector space of functions f on G such that $|f|$ is integrable; this definition is clearly independent of the choice of Haar measure on G. When we work with integrals on an LCA group G with some fixed Haar measure μ we adopt the habit of replacing the expression $\int_G f \mathrm{d}\mu$ by $\int_G f(x)\mathrm{d}x$ (if the group operation is written additively) or $\int_G f(x)\mathrm{d}^*x$ (if the group operation is written multiplicatively); here x may be replaced with any symbol serving as an integration variable. In this case we also use the symbol $\mathrm{d}x$ for the Haar measure in question. It also happens that we refer to the measure of a measurable set E as the *volume* of E, and we then denote this by $Vol(E, \mathrm{d}x)$.

4.1.3 Characters of locally compact abelian groups

Throughout this paper the characters and quasi-characters of various locally compact abelian groups will play a very important rôle.

Definition 3. A *quasi-character* of a topological group G is a continuous homomorphism from G to $\mathbb{C}^\times$. A quasi-character χ is a *character* if for every $x \in G$ we have $|\chi(x)| = 1$.

We emphasize that continuity is included in the very definition of characters and quasi-characters. We also issue a word of warning: some texts use the term character to refer to what we call a quasi-character. In this case, they say that a character is *unitary* if it is of absolute value one.
We now establish some simple properties of characters that will be frequently used later.

Proposition 1. A quasi-character of a compact group is a character.

Proof. Let χ be a quasi-character on a compact group G, and let x be an element of G. If x is of finite order m, we have $|\chi(x)^m| = |\chi(x^m)| = |\chi(1)| =$

$|1| = 1$, so $|\chi(x)| = 1$. If x is of infinite order, we let H be the closure of the subgroup of G generated by x. This is a closed subset of a compact set, so it is compact. Since χ is continuous, the image $\chi(H)$ must be compact, which is clearly not the case if $|\chi(x)|$ does not equal 1. □

Proposition 2. Let q be some real number greater than 1. The quasi-characters of the infinite cyclic group $q^{\mathbb{Z}}$, and the quasi-characters of the multiplicative group $\mathbb{R}_+^\times$, are all of the form $t \mapsto t^s$ for some $s \in \mathbb{C}$.

Proof. For the first assertion, let χ be a quasi-character of $q^{\mathbb{Z}}$ and take $z \in \mathbb{C}$ such that $e^z = \chi(q)$ (this is always possible since the image of the complex exponential mapping is all of $\mathbb{C}^\times$). If we now set $s = \frac{z}{\log q}$ we get

$$\chi(q^m) = \chi(q)^m = e^{zm} = e^{(\log q)\cdot ms} = (q^m)^s$$

and the assertion follows. For the second case, let χ be a quasi-character of $\mathbb{R}_+^\times$. We begin by observing that $\frac{\chi}{|\chi|}$ is then a character, which we will call μ. Also, $|\chi|$ is a continuous group homomorphism from $\mathbb{R}_+^\times$ to $\mathbb{R}_+^\times$, and we set $g(x) = |\chi(x)|$, so that $\chi = \mu \cdot g$. We recall from classical Fourier analysis (see for example section 1.2.7 in [24]) that the characters of $\mathbb{R}$ are all on the form $x \mapsto e^{itx}$ for some real number t. Since the log function is an isomorphism of topological groups from $\mathbb{R}_+^\times$ to $\mathbb{R}$, the characters of $\mathbb{R}_+^\times$ must all be of the form $x \mapsto e^{it\log x} = x^{it}$, for some $t \in \mathbb{R}$; in particular, this holds for μ. For any group homomorphism g from $\mathbb{R}_+^\times$ to $\mathbb{R}_+^\times$, and any integer m we must have $g(x^m) = g(x)^m$. If n is positive, we have $g(x) = g((x^{1/n})^n) = g(x^{1/n})^n$. Combining these results give us $g(x^{m/n}) = g(x^{1/n})^m = g(x)^{m/n}$ for any rational number $\frac{m}{n}$. But by continuity of g, we must have $g(x^\alpha) = g(x)^\alpha$ for all $\alpha \in \mathbb{R}$. Put $b = g(e)$. Then for any $x \in \mathbb{R}_+^\times$ we get

$$g(x) = g(e^{\log x}) = b^{\log x} = x^{\log b}.$$

Setting $s = it + \log b$, we conclude that $\chi(x) = x^s$. □

The idea of the above proof will be used many times in later sections. In general, if $G = G_1 \times G_2$ is a topological group, every quasi-character χ on G restricts to a quasi-character χ_1 on G_1, and to a quasi-character χ_2 on G_2. Every element x of G is represented by a pair (x_1, x_2) where x_i is the projection of x onto G_i. For any x, we have $\chi(x) = \chi((x_1, 1)\cdot(1, x_2)) = \chi_1(x_1)\chi_2(x_2)$. Thus the quasi-character χ is determined by its restrictions to the factors of G, and this can be used to find all the quasi-characters of a given group G, provided that we can decompose G into a product of factors that are easier to handle.

Proposition 3. Let G be a compact group with Haar measure d^*x, and let χ be a character on G. Then the integral $\int_G \chi(x)\mathrm{d}^*x$ equals zero if χ is nontrivial, and $Vol(G, \mathrm{d}^*x)$ otherwise.

Proof. The second assertion is obvious. If χ is nontrivial, pick a g such that $\chi(g) \neq 1$. Then since the Haar measure is translation-invariant, we have $\int \chi(x)\mathrm{d}^*x = \int \chi(gx)\mathrm{d}^*x = \chi(g) \int \chi(x)\mathrm{d}^*x$, so $\int \chi(x)\mathrm{d}^*x$ must be zero. □

In general, when χ is a quasi-character and U is a subset of the kernel of χ, we say that χ *vanishes* on U. This may cause some confusion, since in most contexts a function that vanishes on some set takes the value zero there, while for us such a function takes the value 1.

4.2 The Pontryagin dual

We define the *Pontryagin dual* $\hat{G}$ of an abelian topological group G to be the set of all characters of G. This is of course a group under multiplication, with the trivial character as identity. For any compact subset C of G, and any neighbourhood U of 1 in S^1, we define a subset of $\hat{G}$ by

$$N(C, U) = \{\chi \in \hat{G} \mid \chi(C) \subseteq U\}.$$

Clearly the trivial character is in every $N(C, U)$, and we turn $\hat{G}$ into a topological group by letting the set of all the $N(C, U)$ be a neighbourhood base at the identity; the resulting topology is called the *compact-open topology.* This topology gives rise to the following beautiful duality theorem.

Theorem 4 (Pontryagin duality). Let G be an abelian topological group. Then
(i) If G is discrete, then $\hat{G}$ is compact.
(ii) If G is compact, then $\hat{G}$ is discrete.
(iii) If G is locally compact, then $\hat{G}$ is locally compact.
(iv) If G is locally compact, then G and $\hat{G}$ are mutally dual, in the sense that the natural map from G to $\hat{\hat{G}}$, sending $y \in G$ to the character $\chi \mapsto \chi(y)$ on $\hat{G}$, is an isomorphism of topological groups.

Proof. See the first chapter in the excellent treatment of Rudin [24]. The proof uses some machinery from functional analysis; this is efficiently summarized in Rudin's appendices, and treated with full proofs in the wonderful textbook of Searcóid [25]. □

We have mentioned that the Pontryagin dual of $\mathbb{R}$ is isomorphic to $\mathbb{R}$, under the correspondence sending $y \in \mathbb{R}$ to the character $x \mapsto e^{iyx}$. In general, if G is isomorphic as a topological group to $\hat{G}$, we say that G is *self-dual.* We shall encounter many locally compact groups, and in fact a great number of them will be self-dual.

Theorem 5. Let G be LCA and let H be a closed subgroup of G. Let $H^\perp \subseteq \hat{G}$ be the set of characters χ on G that are trivial on H. Then $H^\perp$ is a closed subgroup of $\hat{G}$. Moreover, $H^\perp$ is the Pontryagin dual of G/H and H is the Pontryagin dual of $\hat{G}/H^\perp$.

Proof. See section 2.1 in [24] $\square$

4.3 The Fourier transform

In this section, G denotes a locally compact abelian group.

Definition 4. Let $f \in L^1(G)$. Then we define the *Fourier transform* of f by

$$\begin{array}{rcl} \hat{f} : \hat{G} & \to & \mathbb{C} \\ \chi & \mapsto & \int_G f(y)\overline{\chi}(y)\mathrm{d}y \end{array}$$

This is clearly well-defined since $f \in L^1(G)$ implies $f\overline{\chi} \in L^1(G)$. Just as in the classical setting, the Fourier transform satisfies a *Fourier inversion formula.* This formula does not hold for all functions in $L^1(G)$, and there are several different approaches to defining appropriate subsets of $L^1(G)$ for which it does hold. For our purposes, it will be convenient and sufficient to use

Definition 5. For an LCA group G, we define $Inv(G)$ to be the set of functions $f \in L^1(G)$ such that f is continuous and $\hat{f}$ is in $L^1(\hat{G})$.

Theorem 6 (Fourier inversion formula). There exists a Haar measure $\mathrm{d}\chi$ on $\hat{G}$ such that for all $f \in Inv(G)$

$$f(y) = \int_{\hat{G}} \hat{f}(\chi)\chi(y)\mathrm{d}\chi. \tag{2}$$

Proof. From Hewitt and Ross [12], §31.44, we have that if f is in $L^1(G)$ and $\hat{f}$ is in $L^1(\hat{G})$, then the Fourier inversion formula holds *almost everywhere*, meaning that the set of all $x \in G$ such that it does not hold is of measure zero. Since $f \in Inv(G)$, f is continuous, so if we can show that the right hand side of equation (2) is also continuous, we are done. The proof of this uses an alternative characterization of the compact-open topology on $\hat{G}$. In fact, this topology is the weakest topology such that all Fourier transforms of functions in $L^1(G)$ are continuous on $\hat{G}$. From Pontryagin duality and the fact that $\hat{f} \in L^1(\hat{G})$ by assumption, it is clear that the right hand side of the inversion formula is continuous, so the formula must hold everywhere, as claimed. $\square$

We remark that the coefficient $\frac{1}{2\pi}$ in front of the classical Fourier inversion formula stems from the convention of always using the ordinary Lebesgue measure $\mathrm{d}t$ on the real line - if we instead would use the measure $\frac{\mathrm{d}t}{2\pi}$ on the "dual" real line (or the measure $\frac{\mathrm{d}t}{\sqrt{2\pi}}$ on both the real line and the dual real line), the coefficient would disappear. In other words, $\frac{\mathrm{d}t}{2\pi}$ is the Haar measure promised to exist by Theorem 6. In section 8 we will need the following consequence of the inversion formula:

Corollary 1. For functions f on G satisfying the Fourier inversion formula we have

$$\hat{\hat{f}}(x) = f(-x)$$

and in particular,

$$\hat{\hat{f}}(0) = f(0).$$

Proof. By definition, the Fourier transform of $\hat{f}$ is

$$\hat{\hat{f}}(x) = \int_{\hat{G}} \hat{f}(\chi)\overline{\chi}(x)\mathrm{d}\chi$$

and from the inversion formula it is immediate that

$$f(-x) = \int_{\hat{G}} \hat{f}(\chi)\chi(-x)\mathrm{d}\chi.$$

But for any character χ, we always have $\chi(-x) = \overline{\chi}(x)$, so the corollary follows. $\square$

Now consider the case of a self-dual LCA group G. Let $\phi : G \to \hat{G}$ be an isomorphism of topological groups. Then if $f \in Inv(G)$, and $\mathrm{d}x$ is a Haar measure on G, the Fourier inversion formula gives us the relation

$$f(y) = c \cdot \int_{\hat{G}} \hat{f}(\phi(x))\phi(x)(y)\mathrm{d}x$$

for some positive real constant c depending on ϕ and $\mathrm{d}x$. Clearly there is a unique choice of $\mathrm{d}x$ for which the above formula holds with $c = 1$, and we call this measure the *self-dual* measure with respect to the isomorphism ϕ.

5 Prerequisites from algebraic number theory

So far, we have been focusing on analysis; now we turn to number theory. In this section, we make a rapid summary of basic definitions and facts from algebraic number theory. A very good first introduction to this subject is given in Stewart & Tall [27]; a more conceptual approach is found in Neukirch [22]. There are also many other books on algebraic number theory, for example [8], [33] and [17]. Algebraic number theory is built on commutative algebra, with emphasis on Dedekind domains, and we refer the reader to [1] and to [9] for more on this. An excellent introduction to number theory in general but with emphasis on algebraic number theory is Ireland and Rosen [13].

5.1 Number fields

The principal objects of study in algebraic number theory are *number fields*. A number field is by definition a finite field extension[3] K of the rationals $\mathbb{Q}$, and we define n to be the degree $[K : \mathbb{Q}]$. Every number field has a *primitive element*, that is, an element α such that $K = \mathbb{Q}(\alpha)$. Some of the simplest examples of number fields are the *quadratic* number fields, where $\alpha = \sqrt{d}$ for some square-free $d \in \mathbb{Z}\backslash\{0, 1\}$, and the *cyclotomic* number fields, where α is a root of unity.

The ring of integers The most important substructure of a number field is its *ring of integers* $\mathfrak{o}_K$. This ring is defined to be the integral closure of $\mathbb{Z}$ in K, that is, $x \in \mathfrak{o}_K$ if and only if x is a root of some monic polynomial with coefficients in $\mathbb{Z}$. Elements of $\mathfrak{o}_K$ are called (algebraic) integers. As an abelian group, $\mathfrak{o}_K$ is of rank n, and as a ring it turns out to be a Dedekind domain, which means that any ideal $\mathfrak{a}$ of $\mathfrak{o}_K$ can be written in a unique way as a finite product $\prod \mathfrak{p}_i^{n_i}$ $(n_i > 0)$ of positive powers of distinct prime ideals. Thus the set of all ideals of $\mathfrak{o}_K$ forms a commutative monoid in a natural way, with $\mathfrak{o}_K$ as the identity element. For many reasons, it would be nice if the ideals formed a group instead of a monoid, and therefore we try to generalize the notion of an ideal. We observe that since $\mathfrak{o}_K$ is a Dedekind domain, it is in particular noetherian, so every ideal is a finitely generated $\mathfrak{o}_K$-submodule of K. This leads us to define a *fractional ideal* of K to be a finitely generated $\mathfrak{o}_K$-submodule of K. It is possible to show that every fractional ideal can be written uniquely as a finite product $\prod \mathfrak{p}_i^{n_i}$, but now with the exponents n_i in $\mathbb{Z}$. Under multiplication, the set of all fractional ideals form a group J_K, called the *ideal group* of K; it is clear that this is the free abelian group on the set of prime ideals of $\mathfrak{o}_K$. The traditional ideals are now called *integral* ideals, to distinguish them from the more general fractional ideals; when we speak of *ideals* it will be clear from the context what we mean. Since J_K is a group, it is also clear what we mean by the *inverse* of an ideal $\mathfrak{a}$ - it is the ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathfrak{o}_K$.

The class group Every nonzero element $x \in K^\times$ generates a fractional ideal (x) called the *principal ideal generated by x*. These ideals form a subgroup P_K of the ideal group and we define the *class group* of K by

$$Cl_K = J_K/P_K.$$

A major theorem of elementary algebraic number theory says that the class group is finite, and we define the *class number* h_K of K to be the order of the class group. The class group and class number are important for

[3]To disperse all possible confusion: this does not mean that a number field is a finite field, but that it has finite dimension as a vector space over $\mathbb{Q}$.

many reasons; one is that the ring $\mathfrak{o}_K$ is a UFD if and only if $h_K = 1$. It is often very difficult to determine the class number of a given number field, and experience shows that class numbers behave almost completely unpredictably.

Embeddings of K into $\mathbb{C}$ A number field can be embedded into $\mathbb{C}$ in exactly n different ways; these embeddings play a very important rôle. If the image of such an embedding is contained in $\mathbb{R}$, the embedding is called *real*; if not, we say that the embedding is *complex*. If σ is a complex embedding, then the complex conjugate $\overline{\sigma}$ is another complex embedding, so the complex embeddings come in conjugate pairs. We denote by r_1 the number of real embeddings and by r_2 the number of conjugate pairs of complex embeddings of K into $\mathbb{C}$.

The unit group The unit group of the ring $\mathfrak{o}_K$ will be denoted by U_K. We write μ_K for the group of roots of unity contained in K; this group is finite. *Dirichlet's unit theorem* says that U_K is isomorphic to $\mu_K \times \mathbb{Z}^{r_1+r_2-1}$. Historically, ideals emerged in the 19th century as a generalization of numbers. As a reaction to the lack of unique factorization in the ring $\mathfrak{o}_K$, Kummer invented "ideal numbers" with the property that every element of $\mathfrak{o}_K$ factors uniquely as product of these. In modern language, this corresponds to the fact that the principal ideal generated by $x \in \mathfrak{o}_K$ can be written as a product of prime ideals in a unique way. As we have seen, this holds also more generally for nonzero elements of K, if we allow for negative powers of prime ideals in the factorization. There is a natural map from $K^\times$ to J_K given by $x \mapsto (x)$, and this map takes us from numbers to ideals. We have an exact sequence

$$1 \to U_K \to K^\times \to J_K \to Cl_K \to 1$$

showing that in some sense, the unit group measures the contraction taking place when we pass from numbers to ideals, and the class group measures the expansion in the same process. This gives some indication of the importance of these groups in number theory.

Valuations Let us return to the factorization of a principal (fractional) ideal (x). We can write this as

$$(x) = \prod \mathfrak{p}_i^{n_i}. \tag{3}$$

If we fix a prime ideal $\mathfrak{p}$, every element x of $K^\times$ defines an $m \in \mathbb{Z}$; namely the power of the prime ideal $\mathfrak{p}$ in the factorization (3). This can be interpreted as a map $v_\mathfrak{p}$ from $K^\times$ to $\mathbb{Z}$. It will be convenient to extend $v_\mathfrak{p}$ to all of K by defining $v_\mathfrak{p}(0) = \infty$. It is easily verified that this map has the following

properties:
V1 $v_\mathfrak{p}(x) = \infty \iff x = 0$.
V2 $\forall\, x, y \in K : v_\mathfrak{p}(xy) = v_\mathfrak{p}(x) + v_\mathfrak{p}(y)$.
V3 $\forall\, x, y \in K : v_\mathfrak{p}(x + y) \geq \min\{v_\mathfrak{p}(x), v_\mathfrak{p}(y)\}$.
and we define an *exponential valuation* to be any map satisfying (V1) - (V3); however, we exclude from the definition the trivial case where the map takes the value zero on all of $K^\times$.

Norm and trace In algebraic number theory, there are several related concepts called *norm*. First, there is the ordinary norm defined for any finite field extension. We recall that for such an extension K/L, K is a vector-space over L and any element $x \in K$ defines an L-linear transformation of K by $y \mapsto xy$. The norm $N_L^K(x)$ of an element $x \in K$ is defined to be the determinant of this transformation. A closely related concept is the trace $Tr_L^K(x)$ which simply is the trace of the same transformation. Thus, for $x \in K$, $N_L^K(x)$ and $Tr_L^K(x)$ are elements of L. In our number-theoretic setting, we take $L = \mathbb{Q}$ and K a number field, and our definitions apply to elements of K.
A second norm that will be of interest to us is the *absolute norm* $N(\mathfrak{a})$ of an integral ideal $\mathfrak{a}$. This is defined to be the cardinality of the quotient ring $\mathfrak{o}_K/\mathfrak{a}$; this number is always finite.
In section 7.3 we shall encounter a third norm, which will be defined on the *idèle group* of a number field.

The different, the discriminant and the regulator For any fractional ideal $\mathfrak{a}$ we define its *dual* to be the set

$$\{x \in K \mid Tr_\mathbb{Q}^K(x\mathfrak{a}) \subseteq \mathbb{Z}\}$$

and we denote this by $\mathfrak{a}'$. It is possible to show (see chapter III of [16]) that this is always a fractional ideal. The *different* $\mathfrak{D}_K$ of K is defined to be the inverse of the dual of $\mathfrak{o}_K$ – it can be shown that this is an integral ideal.
Let $\omega_1, \omega_2, \ldots, \omega_n$ be a $\mathbb{Z}$-basis for $\mathfrak{o}_K$, and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the embeddings of K into $\mathbb{C}$. We construct an $n \times n$ matrix by the formula $a_{ij} = \sigma_i(\omega_j)$, and define the *discriminant* d_K of K to be the square of the determinant of this matrix.
Let us define a map from the unit group U_K to $\mathbb{R}^{r_1+r_2}$ by the formula

$$l(u) = \big(\log|\sigma_1(u)|, \log|\sigma_2(u)|, \ldots, \log|\sigma_{r_1+r_2}(u)|\big)$$

where we have chosen a set of embeddings $\sigma_1, \ldots, \sigma_{r_1+r_2}$ that is maximal such that no two of the embeddings are complex conjugates. The image of this map is a $(r_1 + r_2 - 1)$-dimensional lattice in $\mathbb{R}^{r_1+r_2}$, and we define the *regulator* R_K of K to be the volume of a fundamental domain for this lattice (the volume taken with respect to Lebesgue measure on the subspace spanned by the lattice).

The Dedekind zeta function The most important L-function associated to a number field is the *Dedekind zeta function*. It is defined by

$$\zeta_K(s) = \sum \frac{1}{N(\mathfrak{a})^s} \tag{4}$$

where the sum is over all nonzero integral ideals of $\mathfrak{o}_K$. One of the features of this function is that it can be used to calculate the class number. The Dedekind zeta function has a pole at $s = 1$, and the residue of this pole appears in the following surprising equation, called the *analytic class number formula*:

$$Res_{s=1}\ \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K}{|\mu_K|\sqrt{|d_K|}} \cdot h_K. \tag{5}$$

Although this residue and the terms in the right-hand fraction are not always easy to calculate, it is often possible to do it, and in these cases the above formula gives us the class number of K.

5.2 The field $\mathbb{Q}_p$ of p-adic numbers

The p-adic numbers were introduced by Hensel in the beginning of the 20th century, in order to apply analytical methods related to power series expansions to problems in number theory. Today, these numbers play a central role in number theory; we give here only the basic definitions and refer the reader to [15] and to chapter 2 in [22] for more details and other equivalent definitions.
Let p be a prime number. As a set, we define the set $\mathbb{Z}_p$ of *p-adic integers* to be the set of all formal sums

$$\sum_{k=0}^{\infty} a_k p^k \tag{6}$$

where $a_k \in \{0, 1, \ldots, p-1\}$. We turn $\mathbb{Z}_p$ into a ring by viewing it as the projective limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$ $(n \geq 1)$ under the projection morphisms $\pi_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. More precisely, every element of $\mathbb{Z}/p^n\mathbb{Z}$ is representable in a unique way by a sum of the form

$$\sum_{k=0}^{n-1} a_k p^k$$

with $a_k \in \{0, 1, \ldots, p-1\}$ and it is clear that the image in $\mathbb{Z}/p^{n-1}\mathbb{Z}$ of this sum is the sum

$$\sum_{k=0}^{n-2} a_k p^k$$

so the infinite formal sum (6) can indeed be viewed as an element of the above-mentioned projective limit. Conversely, it is clear that every element

of this projective limit can be represented by such a sum.
We now define the field $\mathbb{Q}_p$ as the field of fractions of the integral domain $\mathbb{Z}_p$. It is possible to show that every element of $\mathbb{Q}_p$ can be represented in a unique way as a sum

$$\sum_{k=m}^{\infty} a_k p^k$$

where now m is in $\mathbb{Z}$ and the a_k still are in $\{0, 1, \ldots, p-1\}$.

6 Local and global fields

In this paper, and in many other number-theoretic situations, the notions of global and local fields play a crucial role. In this section, we aim to review some definitions and facts concerning these fields.

6.1 Global fields

The reader is probably aware of the similarities between the integers $\mathbb{Z}$ and the polynomial rings $\mathbb{F}_q[t]$, where $\mathbb{F}_q$ is the finite field with q elements. These rings are PIDs, and their prime elements give rise to finite fields as their residue class rings. $\mathbb{Z}$ and $\mathbb{F}_q[t]$ have quotient fields $\mathbb{Q}$ and $\mathbb{F}_q(t)$, respectively. In algebraic number theory one studies *number fields*, that is, finite field extensions of $\mathbb{Q}$. We define a *function field* to be a finite field extension of $\mathbb{F}_q(t)$, for some q. Because of the similarities between $\mathbb{Z}$ and $\mathbb{F}_q[t]$, it is useful to define a *global field* to be either a number field or a function field. The analogy between these two kinds of fields is a rich source of inspiration for number theorists, and since function fields are easier to work with, they often give rise to conjectures about how number fields should behave. In this paper, we focus entirely on number fields. It is very unfortunate that we do not have the space to treat both types of global fields, but we refer the reader to [23] for the function field analogue of the theory presented here for number fields. For a general introduction to the connections between function fields and number fields, we recommend Lorenzini [19].

6.1.1 Places

One of the most important tools used in the analysis of global fields is absolute values. Recall that if K is a field, then an absolute value on K is defined to be a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying
AV1 $|x| = 0 \iff x = 0$.
AV2 $\forall\, x, y \in K : |xy| = |x| \cdot |y|$.
AV3 $\forall\, x, y \in K : |x + y| \leq |x| + |y|$.
We exclude from this definition the map sending all nonzero x to 1. An absolute value is called *nonarchimedean* if $|x + y| \leq max\{|x|, |y|\}$ and

archimedean otherwise. Any absolute value on K induces a metric $d(x, y) = |x-y|$, and hence a topology, on K. Since our focus will be on this topology rather than the absolute value itself, it is natural to define two absolute values to be *equivalent* if they induce the same topology on K. We define a *place* of K to be an equivalence class of absolute values. It is quite easy to show (see proposition 9.1.4 in Grillet [9]) that if $|\cdot|_1$ and $|\cdot|_2$ are equivalent absolute values on a field K, then there exists a positive real number t such that $|\cdot|_1 = |\cdot|_2^t$. Therefore, a sequence $\{x_k\}_1^\infty$ in K is a Cauchy sequence with respect to $|\cdot|_1$ if and only if it is a Cauchy sequence with respect to $|\cdot|_2$. Thus it makes sense to speak of the completion of K at a place v, that is, the completion with respect to the metric induced by any absolute value representing v. Of course, a field K embeds naturally into any of its completions, and when K is a number field, these embeddings turn out to be extremely useful. The absolute value extends to an absolute value on the completion, and this extension is a continuous mapping of the completion into $\mathbb{R}_{\geq 0}$.

Now consider the field $\mathbb{Q}$, the simplest possible number field. For every prime number p, we define the *p-adic absolute value* $|\cdot|_p$ on $\mathbb{Q}$ as follows: For every rational number $r \in \mathbb{Q}$, there is a unique $k \in \mathbb{Z}$ such that r can be written as

$$r = p^k \frac{a}{b}$$

where a and b are integers not divisible by p. Using this k, the p-adic absolute value of r is defined to be

$$|r|_p = \frac{1}{p^k}.$$

A well-known theorem of Ostrowski says that any place of K is represented either by a p-adic absolute value or by the ordinary absolute value, and that all these absolute values represent different places. The corresponding completions are the fields $\mathbb{Q}_p$ of p-adic numbers and $\mathbb{R}$, respectively. Of course, the rational primes p are in one-to-one correspondence with the non-zero prime ideals of $\mathbb{Z}$, so these prime ideals are in one-to-one correspondence with the places of $\mathbb{Q}$, if we exclude the place corresponding to the ordinary absolute value.

For a general number field K of degree n over $\mathbb{Q}$, the situation is very similar. Any prime ideal $\mathfrak{p}$ gives rise to a place of K through the absolute value $x \mapsto e^{-v_\mathfrak{p}(x)}$ (here e can of course be replaced by any other real number greater than 1). These places are called *finite*, and K has exactly n places not arising in this way. The latter places are called infinite, and they come from the n embeddings of K into $\mathbb{C}$, each embedding inducing an absolute value on K through the ordinary absolute value on $\mathbb{C}$. We say that a place τ of K *lies over* a place v of $\mathbb{Q}$ if an absolute value representing τ restricts on $\mathbb{Q}$ to an absolute value in the equivalence class v. The infinite places of K

all lie over the unique infinite place of $\mathbb{Q}$, and for a prime ideal $\mathfrak{p}$ of K lying over the prime ideal $p\mathbb{Z}$, the place corresponding to $\mathfrak{p}$ lies over the place of $\mathbb{Q}$ corresponding to p, as it should be.

6.1.2 Completions

Given a number field K of degree n over $\mathbb{Q}$, and a place τ of K, we consider the completion K_τ of K with respect to τ. First, we let τ be an infinite place, so that it corresponds to an embedding of K into $\mathbb{C}$. If the image of this embedding is contained in $\mathbb{R}$, this image is dense in $\mathbb{R}$ and the completion of K with respect to τ is $\mathbb{R}$. Otherwise, the image of K is dense in $\mathbb{C}$, and the completion is $\mathbb{C}$. In these two cases, the absolute values are archimedean. Now let τ be a finite place, corresponding to a prime ideal $\mathfrak{p}$ lying over $p\mathbb{Z}$. Then the completion of K with respect to τ is a finite extension of $\mathbb{Q}_p$. The degree d of this extension is less than or equal to n, and the field is isomorphic, as a topological vector space, to $\mathbb{Q}_p^d$, with the product topology. In this case, the absolute value is non-archimedean.

6.2 Local fields

Often in algebra, we have a set of prime ideals (for example the set of closed points of an affine scheme), and we focus our attention on one prime ideal at a time. Structures arising as we do this are often called *local*, for obvious reasons. In view of the above discussion of completions of a number field at primes, or more general at a place, it is natural to introduce the following

Definition 6. We say that a field is a *local field* if it is $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$.

In view of the previous section, we know that any local field naturally comes equipped with a family of equivalent absolute values. When we speak of an absolute value on some local field F, it usually doesn't matter which one of these absolute values we use, but we shall shortly define a so-called *normalized* absolute value for any local field, to be used in the sequel.
In a local field F, any set of the form $B_t = \{x \in F \mid |x| \leq t\}$ is compact. From this it follows that a local field is locally compact, and in fact the above definition is equivalent to the definition of a local field as a locally compact, non-discrete field of characteristic zero (see chapter 4 of [23] for a proof of this). Some authors drop the requirement of characteristic zero from the definition, to include the completions of function fields, but as we are concerned only with number fields we will not do this. The reader will see that we very often define and prove things concerning local fields by cases, for $\mathbb{R}$, $\mathbb{C}$, and finite extensions of $\mathbb{Q}_p$.

6.2.1 Some structure theory for local fields

When proving things about local fields, the argument is often easy to follow when the local field is archimedean ($\mathbb{R}$ or $\mathbb{C}$), since these fields are very familiar. Here we will describe the structure of local fields, with emphasis on the non-archimedean case; for proofs and more details, we refer to [26], section II.5 in [22], and chapter 4 of [23]. We begin with the observation that the absolute value on a local field F is a group homomorphism from $F^\times$ to $\mathbb{R}_+^\times$. Recall that the image of $\mathbb{Q}^\times$ under the p-adic absolute value is the set $\{p^k \mid k \in \mathbb{Z}\}$, so it is a discrete set, isomorphic (as a group) to $\mathbb{Z}$. In general, we define the *value group* $\mathcal{G}_F$ of a local field F to be the image of $F^\times$ under the absolute value. We also define U_F to be the kernel of the absolute value, so that $F^\times \cong \mathcal{G}_F \times U_F$. This decomposition will be very useful later, when we look at the quasi-characters of $F^\times$. The value group is of course $\mathbb{R}_+^\times$ if the local field is archimedean, and if the local field is non-archimedean, the value group is isomorphic to $\mathbb{Z}$, just as in the special case of $\mathbb{Q}_p$.
Now let F be a non-archimedean local field. We define a *uniformizing parameter* for F to be an element of F of maximal absolute value less than 1. We also define the following subsets of F:

$$\begin{aligned} R &= \{x \in F \mid |x| \leq 1\}. \\ M &= \{x \in F \mid |x| < 1\}. \end{aligned}$$

Then R is open in F, and it is the unique maximal compact subring of F. Note that U_F, which clearly is the group of units of R, is the inverse image under the absolute value of the set $\{1\}$. This implies that U_F is closed and hence compact, since it is a subset of the compact set R. If we fix a uniformizing parameter π, every element of $F^\times$ can be written uniquely as $u\pi^m$ for some $u \in U_F$ and some $m \in \mathbb{Z}$. The elements of R are called *local integers* – for $\mathbb{Q}_p$ the local integers are $\mathbb{Z}_p$. Every element x of a number field K can be identified with an element x_v in each completion K_v. From the factorization (3) of a principal fractional ideal into a finite product of powers of prime ideals, it is immediate that x_v is a local integer for almost all non-archimedean places v.
Recall that a discrete valuation ring is a PID with a unique nonzero prime ideal. The ring R is a discrete valuation ring (and hence a local ring), in which the unique nonzero prime ideal is $M = R\pi$, where π is any uniformizing parameter of F. The residue field R/M is finite, and the cardinality of this residue field is a very important parameter, which we denote by q. We are now in a position to define the *normalized absolute value* on a local field F. The definition is by cases:
Case F = $\mathbb{R}$: $|x|$ = the ordinary absolute value.
Case F = $\mathbb{C}$: $|z| = z\overline{z}$. (This is the square of the ordinary absolute value.)
Case F = finite extension of $\mathbb{Q}_p$: $|x| = q^{-k}$, where k is the unique integer such that $x = u\pi^k$ for some unit u.

This definition is not as arbitrary as it looks. To see why, consider an subset A of F with finite volume with respect to a Haar measure on F. For any $x \in F$, the volume of the set xA will now be exactly $|x| \cdot Vol(A)$ – which would not be the case with any other choice of normalization.

6.2.2 Characters and duality

In section 8, we will study local fields, and the characters of the additive group (additive characters) and the quasi-characters of the multiplicative group (multiplicative quasi-characters) will play an important role. Here we begin by defining, for any local field, a distinguished non-trivial additive character that will be very useful later. We adopt a notational convention: unless otherwise stated, ψ will denote an additive character, χ will denote a multiplicative quasi-character, and μ will denote a multiplicative character.

Definition 7. Let F be a local field. Then we define the *standard character* ψ_F of F by cases, as follows:
Case F $= \mathbb{R}$: $\psi(x) = e^{-2\pi i x}$ (Here, and in the following cases, π is of course the real number $3.1415\ldots$ and not a uniformizing parameter.)
Case F $= \mathbb{C}$: $\psi(z) = e^{-2\pi i(z+\overline{z})}$
Case F = finite extension of $\mathbb{Q}_p$: In this case the definition is more involved - we will construct the standard character as a composition of four continuous group homomorphisms. First, we use the trace map from F to $\mathbb{Q}_p$ defined for any finite field extension in section 5.1. If the field extension is separable, which is always the case when the characteristic of the ground field is zero, then the trace map is not identically zero. Since $\mathbb{Z}_p$ is a subgroup of $\mathbb{Q}_p$, there is a natural projection $P : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$. Each coset of $\mathbb{Q}_p/\mathbb{Z}_p$ is represented by a unique p-adic number of the form $a_{-m}p^{-m} + \ldots + a_{-1}p^{-1}$, where $0 \leq a_k \leq p-1$, and we define a group homomorphism λ from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\mathbb{Q}/\mathbb{Z}$ by sending a coset to its representative $a_{-m}p^{-m} + \ldots + a_{-1}p^{-1}$, now interpreting this sum as being in $\mathbb{Q}$, where it falls in the interval $[0, 1)$. At last, we define $\mathbf{e}(x) = e^{2\pi i x}$, and the standard character as

$$\psi = \mathbf{e} \circ \lambda \circ P \circ Tr.$$

Theorem 7. Let F be a local field and let ψ be any nontrivial additive character of F (for example the standard character). For any $\alpha \in F$, we write ψ_α for the character $x \mapsto \psi(\alpha x)$. Every character of F is of this form for some α, and the mapping $\alpha \mapsto \psi_\alpha$ is an isomorphism of topological groups. Thus the additive group of a local field is self-dual.

Proof. See Lang [17], p 276. □

Now to multiplicative characters. We will try to understand the space $X(F^\times)$ of quasi-characters of $F^\times$. Any quasi-character $\chi \in X(F^\times)$ is determined by its restrictions to U_F and $\mathcal{G}_F$ (call these restrictions χ_U and

χ_G, respectively), and $\chi(x)$ factors as $\chi_U(u)\chi_G(g)$ where u and g are the projections of x on U_F and $\mathcal{G}_F$, respectively. Since U_F is a compact group, any quasi-character on U_F is in fact a character. Since $\mathcal{G}_F$ is isomorphic to $\mathbb{R}_+^\times$ in the archimedean case, and to $q^{\mathbb{Z}}$ in the non-archimedean case, the quasi-characters of $\mathcal{G}_F$ are all of the form $x \to x^s$ for some complex number s, so that any quasi-character χ of $F^\times$ can be written as a product of quasi-characters $\mu|\cdot|^s$ for some $s \in \mathbb{C}$, where $\mu(\frac{x}{|x|})$ is a character depending only on the projection $\frac{x}{|x|}$ of x onto U_F. In this decomposition, the number s is uniquely defined only if F is archimedean, but it is easily verifiable that the real part of s is always uniquely defined, and we call $\Re(s)$ the *exponent* of the quasi-character χ.

6.2.3 Conductors and the local different

Before we turn to integration on local fields, we define some key concepts for non-archimedean fields that will be used in section 8. For every positive integer r, we let U_r be the multiplicative subgroup $1 + M^r$ of $F^\times$; we also define U_0 to be $\mathfrak{o}_F^\times$. Every quasi-character χ must vanish on some U_r, and we define the *conductor* of χ to be the smallest natural number for which this happens. Similarly, every additive character ψ must vanish on some M^r and we define the conductor of ψ to be the smallest $r \in \mathbb{Z}$ with this property (we always understand M^0 to mean $\mathfrak{o}_F$). For a quasi-character χ on $F^\times$, we say that χ is *unramified* if the conductor of χ is zero, or equivalently, if χ is trivial on U_F. Recall from section 5.1 that the different $\mathfrak{D}_K$ of a number field K is an integral ideal of $\mathfrak{o}_K$. For a completion F of K at a prime ideal $\mathfrak{p}$ lying above $p\mathbb{Z}$, we define $d = d_F = v_{\mathfrak{p}}(\mathfrak{D}_K)$, so d is a natural number that is zero for all but finitely many prime ideals. Note that M^{-d} is the dual of R with respect to the trace map from F to $\mathbb{Q}_p$. We define the *local different* $\mathfrak{D}_F$ of F to be M^d.

6.2.4 Haar measure and integration

Since a local field F is locally compact, the additive and multiplicative groups of F are both equipped with a Haar measure. In fact, we have the following connection between the two Haar measures:

Proposition 4. Let F be a local field, let $\mathrm{d}x$ be a Haar measure on the additive group of F, and let c be any positive real constant. Then $c \cdot \frac{\mathrm{d}x}{|x|}$ is a Haar measure on $F^\times$.

Proof. See Lang [16], p. 95. □

There is a useful geometric picture of a non-archimedean local field that we shall now try to describe. Of course, we can think of $\mathbb{C}^\times$ as the union of all circles $C_r = \{z \in \mathbb{C} \mid |z| = r\}$, where r ranges over all positive real numbers.

In a similar way, we can think of the nonzero elements of a non-archimedean local field as the union of all "circles" $\{x \in F \mid |x| = r\}$, but in this case r ranges only over the integer powers of q, so that F can be thought of as the union of countably many concentric circles. The differences between the "complex circles" and the "nonarchimedean circles" include that the natural group structure on such a circle is much more complicated in the latter case, and that the volume of a nonarchimedean circle is positive (therefore we sometimes speak of such a circle as an "annulus"). In this geometric picture, the local integers R constitute the "closed unit disc", and the maximal ideal M is the disc $\{x \in F \mid |x| \leq \frac{1}{q}\}$ - this can also be thought of as the "open unit disc". Any ideal of R is of the form M^k, and as k grows, this ideal correspond to smaller and smaller discs. If we let k assume negative values, the sets M^k will no longer be contained in R, but they will still be discs, that grow larger and larger as $|k|$ grows. We will later normalize the Haar measure on non-archimedean local fields by fixing the volume of R. Then the volume of the maximal ideal M will equal $Vol(R)/q$, and for any $k \in \mathbb{Z}$ we will have

$$Vol(M^k) = \frac{Vol(R)}{q^k}.$$

It will be useful to introduce the notation A_k for the annulus $M^k \backslash M^{k+1}$. From the above formula it is immediate that

$$Vol(A_k) = Vol(R) \cdot \frac{q-1}{q^{k+1}}. \tag{7}$$

The functions we shall need to integrate will be very nice; they will vanish outside some disc $\{x \mid |x| \leq r\}$, and they will be constant on any annulus.

7 Adèles and idèles

Eventually, we want to apply harmonic analysis to number fields. The locally compact groups of interest in this situation are not the additive or multiplicative groups of the number field itself, but two more sophisticated objects, called the *adèle group* and the *idèle group*. These objects are essential to a proper understanding of class field theory, and they are also crucial in any effort to understand the Langlands program. Their usefulness can be explained by the fact that they keep track of all the completions of a number field at the same time. In order to construct these groups, we shall first describe a construction that can be seen as a generalization of the classical direct sum from elementary group theory.

7.1 The restricted direct product

Let I be a set of indices. Suppose that we are given a family of locally compact abelian groups $\{G_v \mid v \in I\}$ and for almost all $v \in I$ (meaning all but a finite number of v), an open compact subgroup H_v of G_v. We denote by I_∞ the finite subset of I for which H_v is not defined. As an abstract group, we define the *restricted direct product of the G_v with respect to the H_v* to be the subgroup G of the direct product $\prod G_v = \{(\ldots, g_v, \ldots)\}$ consisting of those elements for which $g_v \in H_v$ for all but a finite number of v. This subgroup of course has a topology that is induced by the product topology on $\prod G_v$, but this is *not* the topology we shall be interested in. Instead, we define a topology on G by specifying a neighbourhood base at the identity element, namely all sets of the form $\prod N_v$, where the product is over all of I and the N_v are open neighbourhoods of 1 in G_v such that $N_v = H_v$ for all but finitely many v. In order to understand this topology on G, we introduce some useful subgroups of G.

Definition 8. Let the G_v and the H_v be as above, and let S be a finite subset of the index set I containing I_∞. We define G_S to be the subgroup of G consisting of all elements g such that $g_v \in H_v$ for all $v \notin S$.

For any such S, the group G_S is naturally isomorphic to

$$\prod_{v \in S} G_v \times \prod_{v \notin S} H_v$$

which is a product of a finite number of locally compact groups with a compact group, so G_S is locally compact in the product topology given by the above product. It is fairly easy to see that this topology on the groups G_S is the same as the one induced by the topology on G, and that the natural projections $G \to G_v$ are continuous. But the G_S are clearly open, and every element of G is contained in some G_S, so G is locally compact.
The restricted direct product is often denoted by $\prod' G_v$. In this notation the subgroups H_v are suppressed, but it shall always be clear from the context what they are.
It will be useful to have the following

Proposition 5. A subset Y has compact closure if and only if it is contained in a set of the form $\prod_v K_v$, where K_v is a compact subset of G_v for all v and $K_v = H_v$ for almost all v. In particular, every compact subset of G is contained in some G_S.

Proof. Let K be a compact subset of G. Since the sets G_S cover G, there is a finite collection of sets $\{G_{S_k}\}$ that cover K. In fact, K is contained in one single set G_S, since we can set $S = \bigcup S_k$. Let K_v be the image of K in G_v under the natural projection map. Since this map is continuous, every K_v is compact. Clearly K is contained in $\prod K_v$ and $K_v \subseteq H_v$ for all $v \notin S$. The converse is obvious since a closed subset of a compact space is compact. □

7.1.1 Quasi-characters

In section 8 the characters of the adèle group and the quasi-characters of the idèle group will play a central rôle. Here we shall describe the quasi-characters of an arbitrary restricted direct product. We keep the notation of the previous section, and we also write y_v for the projection of an element $y \in G$ onto the factor G_v.
Let χ be a quasi-character of G. Then the restriction of χ to G_v is a quasi-character on G_v, which we shall denote by χ_v. We have the following

Theorem 8. The quasi-characters of G are exactly the functions of the form

$$\chi(y) = \prod_v \chi_v(y_v)$$

where χ_v is a quasi-character of G_v for all v, and the restriction of χ_v to H_v is trivial for all but finitely many v. Moreover, a quasi-character given on this form is a character if and only of all the χ_v are characters.

Proof. Suppose first that χ is a function of this form. It is clear that χ is then well-defined, and also that χ is a group homomorphism from G to $\mathbb{C}^\times$. We must prove that χ is continuous, and for this it suffices to show that given a neighbourhood U of $1 \in \mathbb{C}^\times$, there is a a member N of the neighbourhood base at the identity of G such that $\chi(N) \subseteq U$. So let U be such a neighbourhood. Let S be the finite set $\{v \mid \chi_v|_{H_v} \text{ nontrivial}\}$. Let m be a rational integer larger than the cardinality of S. Take a neighbourhood V of $1 \in \mathbb{C}^\times$ such that the product of m complex numbers in V always lies in U. For every v, the continuity of the quasi-character χ_v implies that there is a neighbourhood N_v of $1 \in G_v$ such that $\chi_v(N_v) \subseteq V$. Also, when $v \notin S$, we can take N_v to be H_v. But from the choice of V, it now follows that $N = \prod N_v$ is a member of the neighbourhood basis such that $\chi(N) \subseteq U$.
For the other direction, let χ be an arbitrary quasi-character of G. Clearly χ restricts to a quasi-character χ_v on every factor G_v of the restricted direct product. To see that these restrictions are trivial on H_v for almost all v, choose a neighbourhood of $1 \in \mathbb{C}^\times$ containing no nontrivial subgroups of $\mathbb{C}^\times$. By continuity of χ, there is a open neighbourhood $N = \prod N_v$ of the identity in G such that $N_v = H_v$ for all v outside some finite set S and such that $\chi(N) \subseteq U$. For any v, the image of H_v under χ is a subgroup of $\mathbb{C}^\times$ contained in U. By the choice of U, χ must be trivial on H_v for all $v \notin S$. Now it is clear that for any $y \in G$, the product $\prod \chi_v(y_v)$ is finite and equals χ_v, and the last part of the lemma also follows immediately. □

Given this theorem, it is not difficult to give an explicit description of the dual group $\hat{G}$ of characters on G. Recall that $H_v^\perp$ is the subgroup of $\hat{G}_v$ of characters that are trivial on H_v.

Theorem 9. The subgroups $H_v^\perp$ are compact and open subgroups of $\hat{G}_v$ for all v such that H_v is defined. The dual group of $\hat{G}$ is $\prod' \hat{G}_v$, the restricted product of the $\hat{G}_v$ with respect to the $H_v^\perp$.

Proof. From the previous theorem, it is clear that the dual group $\hat{G}$ is isomorphic as an abstract group to the restricted product above. We must prove that this isomorphism is also topological. Let $\beta : \prod' \hat{G}_v \to \hat{G}$ be the isomorphism, sending (χ_v) to $\prod \chi_v$. Recall from section 4.2 that the topology on $\hat{G}$ is defined by a neighbourhood base at the identity given by sets of the form

$$N(C, U) = \{\chi \in \hat{G} \mid \chi(C) \subseteq U\}.$$

where C is compact in G and U is open in $\mathbb{C}$. Take an arbitrary set N of this kind. To show that β is continuous, it suffices to construct a neighbourhood of the identity of $\prod' \hat{G}_v$ that is sent into N by β.
We know from Proposition 5 that C is of the form $\prod K_v$, where K_v is a compact neighbourhood of 1 in G_v and $K_v = H_v$ for almost all v. Let S be the set of v such that $K_v \neq H_v$, and let m be the cardinality of S. As before, we can find a neighbourhood V of $1 \in \mathbb{C}$ such that the product of m elements in V always lies in U. For such a V, we consider the set

$$\prod_{v \in S} N(K_v, V) \times \prod_{v \notin S} {H_v}^\perp.$$

But by the choice of V, this is a neighbourhood of the identity in $\prod \hat{G}_v$ with image contained in N, so β is continuous.
It remains to prove that β^{-1} is also continuous. But if we take an element L of the neighborhood basis of $\hat{G}$, it is a set of the form $N(K, U)$, for some compact $K \subseteq G$ and some open $U \subseteq \mathbb{C}$. By Proposition 5, K can be written as $\prod K_v$ with $K_v = H_v$ for almost all v. But it is clear that $\prod N(K_v, U)$ is a neighbourhood of the identity in $\prod \hat{G}_v$ that contains $\beta^{-1}(L)$, so β^{-1} is also continuous. This completes the proof. □

7.1.2 Integration

Several of the crucial steps towards our results on Hecke L-functions will use integration on the adèle and idèle groups. Since these groups are quite abstract already before you do analysis on them, it can be hard to grasp the integration arguments. We hope that the results in this section, together with the fact that the functions we integrate will be fairly simple, will ease the reader's burden.
First we discuss measures on restricted direct products of locally compact groups.

Theorem 10. Let G_v, H_v, and G be as before. Let $\mathrm{d}g_v$ denote Haar measure on G_v, and suppose that these Haar measures are normalized so that the volume of H_v equals 1 for almost all v. Then there is a unique Haar measure $\mathrm{d}g$ on G such that for each finite index set S containing I_∞, the restriction $\mathrm{d}g_S$ of $\mathrm{d}g$ to G_S is the product measure on G_S.

Proof. Since G is locally compact, it admits a Haar measure $\mathrm{d}g$, unique up to multiplication by a constant. For any finite index set $S \supseteq I_\infty$, this Haar measure restricts to a Haar measure on the subgroup G_S. On the other hand, the product measure on G_S is clearly a Haar measure. If we fix a set S, we can normalize $\mathrm{d}g$ by requiring that the restriction to G_S shall equal the product measure on G_S. Thus the theorem is proven, if we can show that this normalization is independent of the set S. Let T be another finite index set containing I_∞. We consider the subgroup $G_{S\cup T}$ of G. The measure $\mathrm{d}g$ restricts to a Haar measure on $G_{S\cup T}$, which must be a constant times the product measure. Since $G_S \subseteq G_{S\cup T}$ this constant must be 1, because the restriction of $\mathrm{d}g$ to G_S is the product measure. But since also $G_T \subseteq G_{S\cup T}$, the restriction of $\mathrm{d}g$ to G_T is the product measure, so the normalization of $\mathrm{d}g$ is independent of the chosen index set. □

We sometimes use the symbolic notation $\prod_v \mathrm{d}g_v$ for the measure $\mathrm{d}g$ in the theorem, and we call it the measure *induced* by the measures $\mathrm{d}g_v$. We now come to a theorem that will allow us to integrate so-called *adelic Schwartz-Bruhat functions* in the next section.

Theorem 11 (Integration on restricted direct products). With G_v, H_v, and G as above, we have:
(i) If f is an integrable function on G, then

$$\int_G f(g)\mathrm{d}g = \lim_S \int_{G_S} f(g_S)\mathrm{d}g_S$$

where g_S denotes an integration variable, and the limit is taken over larger and larger S.
(ii) Let T be a finite index set containing I_∞ and all indices for which $Vol(H_v, \mathrm{d}g_v) \neq 1$. Let f_v be a continuous function on G_v for each v, and suppose that $f_v|_{H_v} = 1$ for all $v \notin T$. For $g \in G$, define

$$f(g) = \prod_v f_v(g_v).$$

Then f is a well-defined and continuous function on G. If the f_v are also integrable, and if $S \supseteq T$ is a finite index set, we have

$$\int_{G_S} f(g_S)\mathrm{d}g_S = \prod_{v\in S} \Big(\int_{G_v} f_v(g_v)\mathrm{d}g_v\Big). \tag{8}$$

Also, if the product

$$\prod_v \Big(\int_{G_v} f_v(g_v)\mathrm{d}g_v\Big) \tag{9}$$

is finite, then f is integrable and

$$\int_G f(g)\mathrm{d}g = \prod_v \Big(\int_{G_v} f_v(g_v)\mathrm{d}g_v \Big). \tag{10}$$

Proof. **(i)** Since $\mathrm{d}g$ is a Haar measure, $\int_G f(g)\mathrm{d}g$ can be computed as

$$\sup\{\int_C f(g)\mathrm{d}g\}$$

where the supremum is taken over all compact subsets C of G. But by Proposition 5 every compact subset of G is contained in some G_S, so the limit formula follows.

(ii) From the hypotheses on f_v it is clear that $f(g)$ is well-defined, since the product in the definition is a finite product. Also, any $g \in G$ has a neighbourhood that is contained in some G_S, and we can assume that the set S contains all v such that $f_v|_{H_v}$ is not 1. But in this neighbourhood, which carries the product topology of G_S, f is a finite product of continuous functions, so it is continuous at g. By the hypotheses on S, we see that for any element g_S of G_S, we have $f(g_S) = \prod_{v\in S} f_v(g_v)$. But since the measure $\mathrm{d}g_s$ is the product measure of G_S, equation (8) follows. Now suppose that the product in equation (9) is finite. Then clearly

$$\prod_v \Big(\int_{G_v} f_v(g_v)\mathrm{d}g_v \Big) = \lim_S \int_{G_S} f(g_S)\mathrm{d}g_S = \int_G f(g)\mathrm{d}g$$

by equation (8) and part (i). □

7.2 The adèle group

Now let K be a number field, and let K_v be the completion of K at a place v. If v is a finite place, we denote the local ring of integers by $\mathfrak{o}_v$; we have seen that it is an open compact additive subgroup of K_v. We define the *adèle group* $\mathbb{A}_K$ of K to be the restricted direct product of the additive groups K_v (v ranging over all places) with respect to the groups $\mathfrak{o}_v$ (at the finite places), and we define an *adèle* to be an element of the adèle group. Since every element of K is a local integer for all but finitely many places, we see that for any $x \in K$, the sequence $(x, x, x, \ldots)$ is an adèle, so that K can be regarded as a subgroup of $\mathbb{A}_K$. We have the following

Theorem 12. K is discrete in $\mathbb{A}_K$, and $\mathbb{A}_K/K$ is compact.

Proof. See chapter IV, § 2, in Weil [32]. □

In other contexts, it is of great importance that $\mathbb{A}_K$ carries a natural ring structure, in which multiplication is carried out component-wise, and therefore $\mathbb{A}_K$ is usually called the *adèle ring*. On a few occasions, we will have to multiply adèles, but otherwise we will only be concerned with the additive structure of $\mathbb{A}_K$.

7.2.1 Characters and duality

Let ψ_v be the standard character of the local field K_v, and define for any adèle x

$$\Psi(x) = \prod \psi_v(x_v).$$

Then Ψ is a non-trivial character on $\mathbb{A}_K$, and we call Ψ the *standard character* of $\mathbb{A}_K$. If α is any adèle, we write Ψ_α for the character $x \mapsto \Psi(\alpha x)$. The adèle group is built from the self-dual local fields associated with a number field, and the self-duality of these fields is reflected in the following

Theorem 13. Any character of $\mathbb{A}_K$ is of the form Ψ_α for some α, and the mapping

$$\begin{array}{ccc} \mathbb{A}_K & \to & \hat{\mathbb{A}}_K \\ \alpha & \mapsto & \Psi_\alpha \end{array}$$

is an isomorphism of topological groups, so $\mathbb{A}_K$ is self-dual.

Proof. In view of the selfduality of local fields (Theorem 7), it follows from Theorem 9 that the Pontryagin dual of $\mathbb{A}_K$ is isomorphic to $\prod' K_v$, where the restricted direct product is taken with respect to the subgroups

$$\mathfrak{o}_v^\perp = \{y \in K_v \mid x \mapsto \psi_v(yx) \text{ is trivial on } \mathfrak{o}_v\}.$$

But such a subgroup is of the form M^k for some $k \in \mathbb{Z}$, where M is the maximal ideal of $\mathfrak{o}_v$. It is easy to verify that if π is a uniformizing parameter of K_v, then the map $x \mapsto \pi x$ is a automorphism of K_v, regarded as a topological additive group. But some power of this automorphism identifies $\mathfrak{o}_v$ with M^k, so the theorem follows. □

We shall also be interested in characters of $\mathbb{A}_K/K$. Recall that $\mathbb{Z}$, which of course is a discrete subgroup of $\mathbb{R}$, is the dual group of $\mathbb{R}/\mathbb{Z}$ (which we usually think of as S^1). For the adèle group, we have a similar situation, and this will later lead us to an analogue of the classical Poisson summation formula.

Theorem 14. (i) The standard character Ψ is trivial on K, and therefore induces a character on $\mathbb{A}_K/K$.
(ii) The Pontryagin dual of K (with its induced discrete topology) is isomorphic to $\mathbb{A}_K/K$ (and, of course, vice versa).

Proof. **(i)** Recall that we defined the standard character as $\Psi = \prod \psi_v$, where ψ_v is the standard character associated with the place v. We divide the proof into two steps: First we prove that the statement for a general number field K follows from the statement for $\mathbb{Q}$, then we prove the statement for $\mathbb{Q}$ by some calculations. For the first step we need the following fact: If p is a place of $\mathbb{Q}$ and x an element of K, then we have

$$Tr_{\mathbb{Q}}^{K}(x) = \sum_{v|p} Tr_{\mathbb{Q}_p}^{K_v}(x) \tag{11}$$

(see Serre [26], p. 41 for a proof). Let us define a map $T : \mathbb{A}_K \to \mathbb{A}_{\mathbb{Q}}$ sending an adèle $(y_v)_v$ of K to the adèle $(\sum_{v|p} Tr_{\mathbb{Q}_p}^{K_v}(y_v))_p$ of $\mathbb{Q}$. From the relation (11) it is clear that the image of $K \subseteq \mathbb{A}_K$ under T is contained in $\mathbb{Q}$. But for any adele $x = (x_v)_v \in \mathbb{A}_K$, we have

$$\begin{aligned}
\Psi_K(x) &= \prod_v \psi_v(x_v) = \prod_p \prod_{v|p} \psi_v(x_v) \\
&= \prod_p \prod_{v|p} \psi_p \circ Tr_{\mathbb{Q}_p}^{K_v}(x_v) = \prod_p \psi_p(\sum_{v|p} Tr_{\mathbb{Q}_p}^{K_v}(x_v)) \\
&= \Psi_{\mathbb{Q}} \circ T(x).
\end{aligned}$$

In other words, Ψ_K factors through $\Psi_{\mathbb{Q}}$. Since T sends K into $\mathbb{Q}$, it is enough to show that $\Psi_{\mathbb{Q}}$ is trivial on $\mathbb{Q}$. This will be easy to prove for rational numbers of the form $\frac{1}{p^m}$ where p is a prime and m is a positive integer. We need the following

Lemma 1. Every positive rational number can be written as a finite sum of rationals of the form $\pm\frac{1}{p^m}$.

Proof. Let $r = \frac{s}{t}$ for some integers s and t. Then r is a finite sum of s copies of $\text{sign}(r) \cdot \frac{1}{t}$. Let $p_1^{m_1} \cdots p_g^{m_g}$ be the prime factorization of t. By Bézout's lemma from elementary number theory, there are integers a and b such that $1 = bp_1^{m_1} + ap_2^{m_2} \cdots p_g^{m_g}$, that is

$$\frac{1}{t} = \frac{a}{p_1^{m_1}} + \frac{b}{p_2^{m_2} \cdots p_g^{m_g}}.$$

A trivial induction argument shows that there are integers $a_1, \ldots, a_l$ such that

$$\frac{1}{t} = \frac{a_1}{p_1^{m_1}} + \frac{a_2}{p_2^{m_2}} + \cdots + \frac{a_l}{p_g^{m_g}}$$

and the lemma follows. □

For a rational number of the form $\frac{1}{p^m}$ we consider the value $\psi_l(1/p^m)$ for l a place of $\mathbb{Q}$. It is clear that if $l = p$, then this value is $e^{2\pi i p^{-m}}$. Also, if l is a finite prime distinct from p, this value is 1, and if l is the place at infinity, then $\psi_l(1/p^m) = e^{-2\pi i p^{-m}}$. Therefore, the product $\Psi_{\mathbb{Q}}(\frac{1}{p^m}) = \prod_l \psi_l(\frac{1}{p^m})$ equals 1. But by the lemma we can write any rational number as a finite sum of rationals of this form, so $\Psi_{\mathbb{Q}}$ is trivial on $\mathbb{Q}$.

(ii) Since K is a closed subgroup of $\mathbb{A}_K$, this follows immediately from Theorem 5, if we can show that $K = K^\perp$, that is, the character Ψ_α is trivial on K if and only if $\alpha \in K$. From part (i) it is clear that if $\alpha \in K$, then Ψ_α is trivial on K, so that $K \subseteq K^\perp$. Since $K^\perp$ is the Pontryagin dual of the compact group $\mathbb{A}_K/K$, it must be discrete as a topological group. Consider the quotient group $K^\perp/K$. This group is finite, since it is a discrete subgroup of the compact group $\mathbb{A}_K/K$. But $K^\perp$ also carries the structure of a vector space over K, and since K is not finite it follows that the index $(K^\perp : K)$ must be 1, so $K^\perp = K$, as required. □

7.3 The idèle group

Let again K_v be the completion of a number field K at a place v. If v is $\mathfrak{p}$-adic, the multiplicative group $K_v^\times$ of K_v contains the local units, here denoted by $\mathfrak{o}_v^\times$, as an open compact subgroup, and we define the *idèle group* $\mathbb{I}_K$ of K to be the restricted direct product of the $K_v^\times$ with respect to these subgroups. In the idèle group, it is the multiplicative group $K^\times$ that can be regarded as an embedded subgroup, and we define the *idèle class group* C_K to be the quotient $\mathbb{I}_K/K^\times$. In contrast with the adèlic situation, the idèle class group is not compact. It should be noted that the idèle group is naturally isomorphic to the multiplicative group of the adèle ring, but the topology on the idèle group is *not* the one induced by the topology of the adèle ring.

We define an "absolute value" on the idèle group by assigning to any idèle ξ the positive real number $\|\xi\| = \prod |\xi_v|_v$ where the product is over all places of K and $|\cdot|_v$ denotes the normalized absolute value on K_v. Of course, this is not an absolute value in the sense of section 6.1.1; $\mathbb{I}_K$ is not a field, so the conditions AV1 and AV3 make no sense. However, the mapping satisfies AV2, in other words, it is a group homomorphism from $\mathbb{I}_K$ to $\mathbb{R}_+^\times$, and we call it the *norm* of the idèle group. Before we use the norm to define some very important groups related to $\mathbb{I}_K$, we note a few properties. First, the norm is surjective. To see this, take an arbitrary idèle ξ, and choose an infinite place τ. We can multiply the component of ξ at τ with any real number t without changing the fact that ξ is an idèle, and given a positive real number r, we can clearly choose t so that the norm of ξ is r. Another property of the norm is that $\|x\| = 1$ for any $x \in K^\times$ - this is the statement of the classical *Artin product formula* (for a proof, see for example page 185 in Neukirch [22]).

Now we define $\mathbb{I}_K^1$ to be the kernel of the norm, that is, the group of idèles of norm one. Since $K^\times \subseteq \mathbb{I}_K^1$, we can consider the quotient group $\mathbb{I}_K^1/K^\times$; this group is the *norm-one idèle class group* C_K^1 of K.

Theorem 15. $K^\times$ is discrete in $\mathbb{I}_K^1$, and C_K^1 is compact.

Proof. See chapter IV, § 4, in Weil [32]. □

Since C_K^1 is compact, it has finite volume, and this volume will play an important rôle in section 8.2, where it appears in some expressions for residues of global zeta functions.

7.3.1 Integration

The global zeta functions used in section 8 are defined as certain integrals over the idèle group. To manipulate these integrals, we need some prerequisites in addition to the integration techniques in section 7.1.2. Let us fix an infinite place τ of K. For any $t \in \mathbb{R}_+^\times$ and $\xi \in \mathbb{I}_K$, we define $t * \xi$ to be the idèle that has local component ξ_v at all places except at τ, where it has component $t\xi_\tau$ if τ is real and $\sqrt{t}\xi_\tau$ if τ is complex. From this definition it is immediate that $\|t * \xi\| = |t| \cdot \|\xi\|$, and it is clear that every idèle can be expressed uniquely as $t * \xi$ for some $t \in \mathbb{R}_+^\times$ and $\xi \in \mathbb{I}_K^1$. Thus we can write the idèle group $\mathbb{I}$ as a direct product $\mathbb{R}_+^\times \times \mathbb{I}_K^1$. If we fix the Haar measure on $\mathbb{R}_+^\times$ to be $\frac{\mathrm{d}x}{x}$, where $\mathrm{d}x$ is the ordinary Lebesgue measure on the real line, there is a unique Haar measure d^*b (the quotient measure) on $\mathbb{I}_K^1$ such that for any integrable function f on $\mathbb{I}$ we have

$$\int_{\mathbb{I}_K} f(x)\mathrm{d}^*x = \int_{\mathbb{R}_+^\times}\int_{\mathbb{I}_K^1} f(t * b)\mathrm{d}^*b \, \frac{\mathrm{d}t}{t}. \tag{12}$$

Through this we can reduce integration on $\mathbb{I}_K$ to integration on $\mathbb{I}_K^1$. But if we now fix the measure on $K^\times$ to be the counting measure, there is a unique measure d^*y on C_K^1 such that for any integrable function g on $\mathbb{I}_K^1$ we can write

$$\int_{\mathbb{I}_K^1} g(b)\mathrm{d}^*b = \int_{C_K^1}\Big(\sum_{a\in K^\times} g(ay)\Big)\mathrm{d}^*y. \tag{13}$$

Here the sum is of course just the integral over $K^\times$ with respect to the counting measure. If this sum is well-behaved, which is the case since g was assumed integrable, we have reduced our integration to an integration over C_K^1, and this will be easy to handle because C_K^1 is compact.

7.4 Idèle class characters

As we mentioned in section 2.2 the definition of Hecke L-functions will make use of *idèle class characters.* By definition, an idèle class character is a quasi-character on the idèle class group, or equivalently, a quasi-character on the idèle group that is trivial on K. It is reasonable to argue that we should call this an idèle class *quasi*-character, but the terminology is traditional, so we stick to it and ask the reader to be careful.
If we take any idèle class character χ, it decomposes into a product of a quasi-character χ_1 on $\mathbb{R}_+^\times$ and a quasi-character χ_2 on $\mathbb{I}_K^1$. But since χ is trivial on K we see that χ_2 induces a quasi-character on C_K^1, and because

C_K^1 is compact, this must be a character. By Proposition 2, χ_1 is of the form $|\cdot|^s$ for some complex s. It follows that any idèle class character can be written as $x \mapsto \mu(x)\|\cdot\|^s$, where μ is a character depending only on the projection of x onto C_K^1. We also note that since χ is trivial on K, we have $\chi(-\xi) = \chi(-1)\cdot\chi(\xi) = \chi(\xi)$ for any idèle ξ, that is, idèle class characters are *indifferent to sign.*

8 Tate's thesis

Tate's thesis has had a great impact on algebraic number theory, because of its new approach to L-functions and its connections with the Langlands program. Although it was completed in 1950 at Princeton, it was not officially published until 1967, when it appeared in [4]. The arguments in Tate's thesis are not very easy to follow, but we hope that the structure theory for local fields in section 6.2.1, and the properties of adèles and idèles in section 7 will be of some help.

For the material in this section there are two main references, in addition to Tate's thesis itself. The first is chapter 14 in the classical text book of Lang [17] and the other is chapter 7 in the recent book of Valenza and Ramakrishnan [23] which we have already referred to several times. Tate's own thesis, as well as the text of Lang, are very compact with many steps left to the reader as exercises or simply omitted as "obvious", while Valenza and Ramakrishnan give the arguments in some more detail, although still leaving some steps to the reader. We have drawn on all three texts, but on the points where they differ, we have usually followed Valenza and Ramakrishnan. The material in Tate's thesis is also presented in the first chapter of Moreno [21], but from a slightly different point of view, using distribution theory.

8.1 Local zeta functions

Let us fix a number field K. We will begin this section with discussing the local fields that arise as completions of this number field at archimedean and non-archimedean places. For these fields, we will define so-called *local zeta functions* and establish a functional equation for these functions. This functional equation will then be used in section 2.2 together with a functional equation for so-called *global zeta functions*, to prove the functional equation and analytic continuation of the Hecke L-functions.

8.1.1 Choice of Haar measure

In this and the coming sections, we let F be a local field with normalized absolute value $|\cdot|$ and Haar measure $\mathrm{d}x$. It will be convenient for us to identify F with its Pontryagin dual through the isomorphism induced by

the standard character (see Theorem 7). The Haar measure $\mathrm{d}x$ will be normalized so that it is self-dual with respect to this isomorphism. In concrete terms, this means that we choose:
Case F = $\mathbb{R}$: $\mathrm{d}x$ = the ordinary Lebesgue measure.
Case F = $\mathbb{C}$: $\mathrm{d}x$ = twice the ordinary Lebesgue measure on the complex plane.
Case F = a $\mathfrak{p}$-adic field: $\mathrm{d}x$ = the measure that makes the formula $Vol(\mathfrak{o}_F) = q^{-d/2}$ hold.
We observe that for the non-archimedean local completions, the ring of integers $\mathfrak{o}_v$ now has volume 1 for almost all places, so that we can apply Theorem 11 in our later calculations on the adèle group. To work with the idèle group, we also have to make a choice of the factor c for each completion (recall the relation $\mathrm{d}^*x = c\frac{\mathrm{d}x}{|x|}$ from Proposition 4). Since we want to apply Theorem 11 also to the idèle group we must choose this factor so that also $Vol(\mathfrak{o}_v^\times, \mathrm{d}^*x)$ equals 1 for almost all nonarchimedean completions. In view of equation 7, we choose $c = \frac{q}{q-1}$ – this implies $Vol(\mathfrak{o}_v^\times, \mathrm{d}^*x) = Vol(\mathfrak{o}_v, \mathrm{d}x)$ for all nonarchimedean places v. For archimedean completions, we always choose $c = 1$.

8.1.2 Local L-factors

We now define, for each completion F of K, a *local L-factor*, which is a complex-valued function defined on the space $X(F^\times)$ of quasi-characters of $F^\times$. As usual, we define the local L-factor by cases, for $F = \mathbb{R}$, $F = \mathbb{C}$ and F a $\mathfrak{p}$-adic field. The motivation behind these definitions, which at first seem to be drawn out of a hat, is that they are exactly what is needed to prove Theorem 17 concerning local zeta functions (to be defined in the next section).
Recall that the Γ-function is analytic in the whole complex plane, except for simple poles at $0, -1, -2, \ldots$, and that $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\mathrm{d}t$ for any s with positive real part.

Case $F = \mathbb{R}$**:** In this case $U_F = \{\pm 1\}$ and $\mathcal{G}_F = \mathbb{R}_+^\times$ and any quasi-character χ decomposes as $\mu|\cdot|^s$ with μ and s uniquely defined. We define the local L-factor to be

$$L(\chi) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } \mu = 1. \\ \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2}) & \text{if } \mu = \mathrm{sgn}. \end{cases} \tag{14}$$

where sgn denotes the sign character $x \mapsto \frac{x}{|x|}$ on U_F.

Case $F = \mathbb{C}$**:** For $F = \mathbb{C}$, we have $U_F = S^1$ and $\mathcal{G}_F = \mathbb{R}_+^\times$. The continuous characters of S^1 are all of the form $e^{i\theta} \mapsto e^{in\theta}$, with $n \in \mathbb{Z}$, so any quasi-character of $\mathbb{C}$ can be written as

$$\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$$

with s and n uniquely defined. We define the local L-factor as

$$L(\chi_{s,n}) = (2\pi)^{-(s+|n|)/2}\Gamma((s+|n|)/2). \tag{15}$$

Case $F =$ a $\mathfrak{p}$-adic field: Let π_F be a uniformizing parameter for F. We define

$$L(\chi) = \begin{cases} \left(1-\chi(\pi_F)\right)^{-1} & \text{if } \chi \text{ is unramified.} \\ 1 & \text{otherwise.} \end{cases} \tag{16}$$

Note that the fact that χ is unramified implies that the L-factor is independent of the choice of π_F.

8.1.3 Definition of the local zeta functions

For an archimedean local field, we define a Schwarz-Bruhat function to be a complex-valued, smooth ($\mathcal{C}^\infty$) function f satisfying $\lim_{|x|\to\infty} p(x)f(x) = 0$ for every polynomial $p(x)$. For a non-archimedean field we define a Schwartz-Bruhat function to be a finite complex-linear combination of characteristic functions of closed discs. Such a function is thus continuous, constant on any set of the form $\pi_F^m\mathfrak{o}$, and there is an $m_0 \in \mathbb{Z}$ such that the function is zero on $\pi_F^m\mathfrak{o}$ for all $m > m_0$. Remembering these properties is one of the keys to understanding the calculations on the adèle group in section 8.2. The complex vector space of Schwartz-Bruhat functions on the local field F will be denoted by $\mathcal{S}(F)$.
For any $f \in \mathcal{S}(F)$ and quasi-character $\chi \in X(F)$ we define the *local zeta function* by

$$\zeta(f,\chi) = \int_{F^\times} f(x)\chi(x)\mathrm{d}^*x.$$

Before we formulate and prove the functional equation for the local zeta functions, we give two very important definitions.

Definition 9. If χ is a quasi-character on $F^\times$ and s is a complex number, than of course $\chi|\cdot|^s$ also is a quasi-character, and we define $L(s,\chi)$ to be $L(\chi|\cdot|^s)$.

Definition 10. For any quasi-character χ we define its *shifted dual* $\chi^\vee$ to be the quasi-character $x \mapsto \frac{|x|}{\chi(x)}$.

From these definitions it is immediate that $L((\chi|\cdot|^s)^\vee) = L(1-s, \frac{1}{\chi})$.

8.1.4 Local functional equation

Now let ψ be the standard character on F. By Proposition 7, the map $a \mapsto \psi(ax)$ identifies F with $\hat{F}$. In this section, we will define the Fourier transform $\hat{f}$ of a Schwartz-Bruhat function to be

$$\hat{f}(y) = \int_F f(x)\psi(yx)\mathrm{d}x$$

thus dropping the usual conjugation of the second factor in the integrand. The only difference this makes is that the conjugation will reappear in the Fourier inversion formula. In order to work with the Fourier transform, we will need

Theorem 16. Let F be a local field and let $f \in \mathcal{S}(F)$. Then $\hat{f} \in \mathcal{S}(F)$ and the Fourier inversion formula holds for f.

Proof. For the real and complex case this is well-known from classical Fourier analysis. For the non-archimedean case, it is clear that any $f \in \mathcal{S}(F)$ is continuous and lies in $L^1(F)$, so if we can show that $f \in \mathcal{S}(F)$ implies $\hat{f} \in \mathcal{S}(F)$ it follows that $\mathcal{S}(F) \subseteq Inv(F)$ and we are done. It suffices to consider functions that are characteristic functions of closed discs. Let f be the characteristic function of $\pi^k \mathfrak{o}_F$, for some $k \in \mathbb{Z}$. Then

$$\hat{f}(y) = \int_{\pi^k \mathfrak{o}_F} \psi(yx)\mathrm{d}x. \tag{17}$$

For a fixed y, the function $\psi(yx)$ is a character on F, and it restricts to a character on the compact additive group $\pi^k \mathfrak{o}_F$. Hence the integral equals $Vol(\pi^k \mathfrak{o}_F, \mathrm{d}x)$ if this character is trivial, and zero otherwise. We observe that for any $u \in \mathfrak{o}_F^\times$, the character $\psi(yx)$ is trivial if and only $\psi(uyx)$ is trivial, so the integral (17) depends only on the absolute value of y. Let m be the conductor of ψ, regarded as a character on F. Then the character $\psi_y(x) = \psi(yx)$ has conductor $m-k$, where k is the integer such that $y = u\pi^k$ for some unit u. But this implies that the integral (17) is zero for $|y| > q^{k-m}$ and equals $Vol(\pi^k \mathfrak{o}_F, \mathrm{d}x)$ otherwise. The theorem follows. □

Already local zeta functions are quite abstract, and global zeta functions will be even worse. Therefore, we give here two simple examples of local zeta functions, before we proceed.

Example 4. Let us start with a non-archimedean local zeta-function. Take $F = \mathbb{Q}_p$, choose f to be the characteristic function of $\pi\mathfrak{o}_F = p\mathbb{Z}_p$ and let

$\chi(x) = |x|^s$. Then, with $A_k = p^k\mathbb{Z}_p \backslash p^{k+1}\mathbb{Z}_p$, we compute

$$\begin{aligned}
\zeta(f,\chi) &= \int_{\mathbb{Q}_p^\times} f(x)\chi(x)\mathrm{d}^*x = \int_{p\mathbb{Z}_p-\{0\}} |x|^s \mathrm{d}^*x \\
&= \int_{p\mathbb{Z}_p} |x|^{s-1}\mathrm{d}x = \sum_{k=1}^{\infty} \int_{A_k} |x|^{s-1}\mathrm{d}x \\
&= \sum_{k=1}^{\infty} Vol(A_k, \mathrm{d}x)\left(\frac{1}{p^k}\right)^{s-1} = \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}}\left(\frac{1}{p^k}\right)^{s-1} \\
&= \frac{p-1}{p(p^s-1)}
\end{aligned}$$

where the last geometric sum is convergent if $s > 0$.

Example 5. For an archimedean example, take $F = \mathbb{R}$, $f(x) = e^{-x^2}$ and $\chi(x) = |x|^s$. Then we have, for $s > 0$,

$$\begin{aligned}
\zeta(f,\chi) &= \int_{\mathbb{R}^\times} f(x)\chi(x)\mathrm{d}^*x = \int_{\mathbb{R}} e^{-x^2}|x|^{s-1}\mathrm{d}x \\
&= 2\int_0^\infty e^{-x^2}x^{s-1}\mathrm{d}x = \Gamma(s/2)
\end{aligned}$$

where we have cheated using a table of definite integrals for the last step.

Note that in the first (p-adic) example, the zeta function is a rational function in p^{-s}, just as the Euler factors that appeared in the Euler products in the examples of section 2. Also, in the archimedean example the zeta function is expressed by the Γ-function – we will see that this zeta function corresponds to the factor we put in front of for example the Riemann zeta function to get a nice-looking functional equation.
We now come to the main theorem for local zeta functions.

Theorem 17 (Local functional equation). Let $f \in \mathcal{S}(F)$ and $\chi \in X(F)$ with $\chi = \mu|\cdot|^s$ as above, and let $\sigma = \Re(s)$ be the exponent of χ. Then:
(i) $\zeta(f,\chi)$ is absolutely convergent if σ is positive.
(ii) If $\sigma \in (0,1)$ there is a functional equation

$$\zeta(\hat{f},\chi^\vee) = \gamma(\chi)\zeta(f,\chi)$$

for some $\gamma(\chi)$ which is independent of f.
(iii) $\gamma(\chi)$ can be expressed as

$$\gamma(\chi) = \varepsilon(\chi)\frac{L(\chi^\vee)}{L(\chi)} \tag{18}$$

where $\varepsilon \in \mathbb{C}^\times$ for all χ. Moreover, if we fix a character μ and put $\chi = \mu|\cdot|^s$, then γ is meromorphic as a function of s.
(iv) $\zeta(f,\chi)$ has meromorphic continuation to the whole complex plane.

Proof. **(i)** The measure d^*x is a constant times $\frac{\mathrm{d}x}{|x|}$ so we have to show that the integral

$$I(f,\sigma) := \int_{F-\{0\}} |f(x)| \cdot |x|^{\sigma-1}\mathrm{d}x$$

is finite. First, suppose F is archimedean. Let us fix some positive R such that for all x of absolute value greater than R, the product $|f(x)| \cdot |x|^{\sigma+1}$ is smaller than 1. We can write the above integral as a sum of one integral for $|x| \leq R$ and one for $|x| > R$. In the first integral, f is continuous on a compact set, so f is bounded, and the integral is finite since $\int |x|^{\sigma-1}\mathrm{d}x$ is finite. In the second integral, the integrand is bounded by $|x|^{-2}$ because of our choice of R, so the integral is finite. Now suppose F is non-archimedean. Then clearly there is a constant b such that $|f| \leq b \cdot h_k$, and therefore, $I(f,\sigma) \leq b \cdot I(h_k,\sigma)$, where h_k is the characteristic function of $\pi_F^k\mathfrak{o}_F$. But $\pi_F^k\mathfrak{o}_F - \{0\}$ is the disjoint union of all the $\pi_F^j\mathfrak{o}_F$ with $j \geq k$ so we have

$$\begin{aligned} I(h_k,\sigma) &= \int_{F-\{0\}} |h_k(x)| \cdot |x|^{\sigma-1}\mathrm{d}x \\ &= \int_{F^\times} |h_k(x)| \cdot |x|^{\sigma}\mathrm{d}^*x \\ &= Vol\big(\mathfrak{o}_F^\times, \mathrm{d}^*x\big) \sum_{j\geq k} q^{-j\sigma} \end{aligned}$$

and since $\sigma > 0$ the sum is finite (and equals $\frac{q^{-k\sigma}}{1-q^{-\sigma}}$). Hence $I(f,\sigma)$ is finite in both the archimedean and non-archimedean case.
(ii) We follow Tate in proving the following lemma:

Lemma 2. For all χ with exponent $\sigma \in (0,1)$, and for all $h \in \mathcal{S}(F)$ we have

$$\zeta(f,\chi)\zeta(\hat{h},\chi^\vee) = \zeta(\hat{f},\chi^\vee)\zeta(h,\chi).$$

Proof. Note first that $\chi^\vee$ is of exponent $(1-\sigma)$, so by part (i) and Theorem 16, all these zeta functions are well-defined for $\sigma \in (0,1)$. To prove the lemma, we will use Fubini's theorem. For this purpose, we observe that since functions in $\mathcal{S}(F)$ are integrable, continuous and bounded, the product of two such functions is in particular a product of an integrable function with a continuous and bounded function, so it is integrable. Also, multiplying a function $f(y) \in \mathcal{S}(F)$ by $|y|$ clearly gives an integrable function.
From the definitions, it is immediate that

$$\zeta(f,\chi)\zeta(\hat{h},\chi^\vee) = \iint_{F^\times \times F^\times} f(x)\hat{h}(y)\chi(xy^{-1})|y|\mathrm{d}^*x\mathrm{d}^*y.$$

Since d^*y is a Haar measure, it is translation-invariant, we can replace y by xy everywhere in the above expression without changing the value of the integral[4]. This gives us

$$\zeta(f,\chi)\zeta(\hat{h},\chi^\vee) = \iint_{F^\times \times F^\times} f(x)\hat{h}(xy)\chi(y^{-1})|xy|\mathrm{d}^*x\ \mathrm{d}^*y$$

which equals

$$\int_{F^\times} \{f,h\}(y)\chi(y^{-1})|y|\mathrm{d}^*y$$

where we have defined

$$\{f,h\}(y) = \int_{F^\times} f(x)\hat{h}(xy)|x|\mathrm{d}^*x.$$

To prove the lemma, it clearly suffices to prove that $\{f,h\} = \{h,f\}$. But from the definition of the Fourier transform and the relation $\mathrm{d}^*x = c\frac{\mathrm{d}x}{|x|}$ we get

$$\{f,h\}(y) = c\iint_{F\times F} f(x)h(z)\psi(xyz)\mathrm{d}z\mathrm{d}x$$

and by Fubini we can interchange the order of integration, so the integral is symmetric in f and h, and the lemma follows. □

Since the function h in the lemma was arbitrary, the quotient $\frac{\zeta(\hat{h},\chi^\vee)}{\zeta(h,\chi)}$ must be independent of h. We denote this quotient by $\gamma(\chi)$, and the proof of (ii) is complete.

(iii) We shall prove the existence of the ε-factor by cases, according to whether F is real, complex or $\mathfrak{p}$-adic. From part (ii), it follows that $\gamma(\chi)$ can be computed as $\frac{\zeta(\hat{f},\chi^\vee)}{\zeta(f,\chi)}$ for any $f \in \mathcal{S}(F)$. We will, for each of the three cases, choose a function f and actually compute $\zeta(f,\chi)$ and $\zeta(\hat{f},\chi^\vee)$. From this it will be clear that the ε-factor exists and lies in $\mathbb{C}^\times$, and that $\gamma(\chi)$ is meromorphic as a function of s. The calculations will be long and not particularly enlightening, so we defer the details to a separate section (see below).

(iv) If χ is of exponent σ, then $\chi^\vee$ is of exponent $1-\sigma$. Therefore $\zeta(\hat{f},\chi^\vee)$ is absolutely convergent in the region $\sigma < 1$. But since the two functions $\zeta(f,\chi)$ and $\frac{\zeta(\hat{f},\chi^\vee)}{\gamma(\chi)}$ agree on the strip $0 < \sigma < 1$, we can use the latter to define a meromorphic continuation of the former. □

8.1.5 Some local computations

Here we shall present, for each type of local field F, the function f promised to exist in the previous section, and verify that it indeed has the required properties, so that part (iii) of Theorem 17 is true. All calculations take place under the assumption that $0 < \sigma < 1$.

[4]If this seems strange, recall that we write the group operation multiplicatively - if the integrals were over $\mathbb{R}$ we would normally write $x+y$ instead of xy.

Case $F = \mathbb{R}$: We have seen that there are two possibilities for quasi-characters χ on $\mathbb{R}^\times$ – they are either of the form $x \mapsto |x|^s$ or of the form $x \mapsto \text{sgn(x)} \cdot |x|^s$. In the first case, we can take the Schwartz-Bruhat function

$$f(x) = e^{-\pi x^2}$$

and compute

$$\zeta(f, \chi) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \mathrm{d}^* x = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} \mathrm{d}x$$

which by the change of variables $u = \pi x^2$ equals

$$\pi^{-s/2} \int_0^\infty e^{-u} u^{s/2-1} \mathrm{d}u = \pi^{-s/2} \Gamma(s/2)$$

which equals $L(\chi)$. From a table of Fourier transforms we get that the function f is its own transform, so that the same calculation gives us

$$\zeta(\hat{f}, \chi^\vee) = L(\chi^\vee) \tag{19}$$

and hence

$$\gamma(\chi) = \frac{\zeta(\hat{f}, \chi^\vee)}{\zeta(f, \chi)} = \frac{L(\chi^\vee)}{L(\chi)}$$

so that we can take $\varepsilon(\chi) = 1$ in equation (18). In the second case, where $\chi(x) = \text{sgn}(x)|x|^s$, we take

$$f(x) = xe^{-\pi x^2}.$$

The local zeta function of f and χ is now

$$\zeta(f, \chi) = \int_{\mathbb{R}^\times} xe^{-\pi x^2} \cdot \frac{x}{|x|} \cdot |x|^s \mathrm{d}^* x = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^{s+1} \mathrm{d}^* x$$

and by our first calculation this equals

$$\pi^{-(\frac{s+1}{2})} \Gamma(\frac{s+1}{2}) = L(\chi).$$

Again, we use a table of Fourier transforms (or alternatively, contour integration) to find

$$\hat{f}(y) = iye^{-\pi y^2}$$

and the calculation that gave us $\zeta(f, \chi)$ also gives us

$$\zeta(\hat{f}, \chi^\vee) = iL(\chi^\vee)$$

which implies that

$$\gamma(\chi) = \frac{\zeta(\hat{f}, \chi^\vee)}{\zeta(f, \chi)} = i \cdot \frac{L(\chi^\vee)}{L(\chi)}$$

so equation (18) holds with $\varepsilon = i$. Thus we can take $\varepsilon(\chi) = i$. Let us now turn to the complex case.

Case $F = \mathbb{C}$: As discussed earlier, every quasi-character on $\mathbb{C}^\times$ is of the form

$$\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$$

for some (unique) $s \in \mathbb{C}$ and $n \in \mathbb{Z}$. For the computations, we shall choose different functions f for different values of n. Let us define

$$f_n(z) = \begin{cases} (2\pi)^{-1}\overline{z}^n e^{-2\pi z\overline{z}} & \text{if } n \geq 0. \\ (2\pi)^{-1} z^{-n} e^{-2\pi z\overline{z}} & \text{if } n < 0. \end{cases} \tag{20}$$

Then it is possible to show (see Lang [17], p. 283, for a proof by induction) that

$$\hat{f}_n(z) = (2\pi)^{-1} i^{|n|} f_{-n}(z)$$

for all n. We shall compute $\zeta(f_n, \chi_{s,n})$, and to do this we note that $\mathrm{d}^*x = (2/r)\mathrm{d}r\mathrm{d}\theta$ because of our normalizations of the Haar measure and the absolute value. Hence, for $n \geq 0$:

$$\begin{aligned} \zeta(f_n, \chi_{s,n}) &= \int_{\mathbb{C}^\times} f_n(z)\chi_{s,n}(z)\mathrm{d}^*z \\ &= \frac{1}{2\pi}\int_{\mathbb{C}^\times} \overline{z}^n e^{-2\pi z\overline{z}}(z\overline{z})^{s/2} e^{in\theta(z)}\mathrm{d}^*z \\ &= \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty r^n e^{-2\pi r^2} r^s \frac{1}{r}\mathrm{d}r\mathrm{d}\theta \\ &= 2\int_0^\infty r^{n+s-1} e^{-2\pi r^2}\mathrm{d}r \end{aligned}$$

and by the substitution $t = 2\pi r^2$ we get

$$\begin{aligned} \zeta(f_n, \chi_{s,n}) &= \Big(\frac{1}{2\pi}\Big)^{\frac{n+s}{2}} \int_0^\infty t^{\frac{n+s}{2}-1} e^{-t}\mathrm{d}t \\ &= \Big(\frac{1}{2\pi}\Big)^{\frac{n+s}{2}} \Gamma\Big(\frac{n+s}{2}\Big) \\ &= L(\chi_{s,n}). \end{aligned}$$

The same calculation for negative n gives us

$$\zeta(f_n, \chi_{s,n}) = \Big(\frac{1}{2\pi}\Big)^{\frac{-n+s}{2}} \Gamma\Big(\frac{-n+s}{2}\Big)$$

so that in all cases,

$$\zeta(f_n, \chi_{s,n}) = \Big(\frac{1}{2\pi}\Big)^{\frac{|n|+s}{2}} \Gamma\Big(\frac{|n|+s}{2}\Big) = L(\chi_{s,n}). \tag{21}$$

For computing $\zeta(\hat{f}, \chi_{s,n}^\vee)$, we observe that $\chi_{s,n}^\vee = \chi_{1-s,-n}$, and given the above formula for $\hat{f}$, we can use exactly the same calculation as for $\zeta(f, \chi_{s,n})$ to find

$$\zeta(\hat{f}, \chi_{s,n}^\vee) = \frac{1}{2\pi} i^{|n|} \Big(\frac{1}{2\pi}\Big)^{\frac{|n|+1-s}{2}} \Gamma\Big(\frac{|n|+1-s}{2}\Big) = \frac{i^{|n|}}{2\pi} L(\chi_{s,n}^\vee).$$

This shows that we can take $\varepsilon(\chi_{s,n}) = \frac{i^{|n|}}{2\pi}$, and Theorem 17 is true also in the complex case.

Case $F =$ a $\mathfrak{p}$-adic field: Let us recycle the symbol $\chi_{s,n}$ from the complex case; this will now signify the quasi-character $x \mapsto \mu(x/|x|) \cdot |x|^s$, where μ is a character of conductor n. From the considerations in section 8.1.2 it is clear that every quasi-character is of this form, for some $s \in \mathbb{C}$ and $n \in \mathbb{N}$. Let m be the conductor of the standard character on F; by definition of the parameter d in section 6.2.3 we have the relation $m = -d$. Therefore, m is always less than or equal to zero, and $m = 0$ for almost all non-archimedean completions.
We define

$$f(x) = \begin{cases} \psi_F(x) & \text{if } x \in M^{m-n}. \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

With this choice of f, we shall now calculate $\zeta(f, \chi_{s,n})$ and $\zeta(\hat{f}, \chi_{s,n})$, and this will prove the existence of the ε-factor just as in the real and complex cases. We begin by computing $\zeta(f, \chi_{s,n})$, and first we take the case $n = 0$. That $n = 0$ means that μ is the trivial character, and by definition of m, ψ_F is trivial on M^m. Therefore,

$$\begin{aligned} \zeta(f, \chi_{s,n}) &= \int_{F^\times} f(x)\chi_{s,n}(x)\mathrm{d}^*x \\ &= \int_{M^m - \{0\}} |x|^s \mathrm{d}^*x \\ &= Vol(\mathfrak{o}_F^\times, \mathrm{d}^*x) \sum_{j \geq m} q^{-js} \\ &= Vol(\mathfrak{o}_F^\times, \mathrm{d}^*x) \frac{q^{-ms}}{1 - q^{-s}} \\ &= q^{-ms} Vol(\mathfrak{o}_F^\times, \mathrm{d}^*x) L(\chi_{s,0}). \end{aligned} \tag{23}$$

Now to the case $n \geq 1$. This will be more complicated, and we need to introduce a new concept. For any pair (ω, λ) of a multiplicative character ω on $\mathfrak{o}_F^\times$ and an additive character λ on $\mathfrak{o}_F$, we define the *Gauss sum* $g(\omega, \lambda)$ by

$$g(\omega, \lambda) = \int_{\mathfrak{o}_F^\times} \omega(u)\lambda(u)\mathrm{d}^*u.$$

Lemma 3. Let ω be of conductor n and λ of conductor r.
(i) If $r < n$, then $g(\omega, \lambda) = 0$.
(ii) If $r = n$, then $|g(\omega, \lambda)|^2 = cVol(\mathfrak{o}_F, \mathrm{d}x)Vol(U_n, \mathrm{d}^*x)$.
(iii) If $r > n$, then $|g(\omega, \lambda)|^2 = cVol(\mathfrak{o}_F, \mathrm{d}x) \cdot \left[Vol(U_n, \mathrm{d}^*x) - \frac{1}{q}Vol(U_{r-1}, \mathrm{d}^*x)\right]$.

Proof. (i) To compute the value of the Gauss sum, we decompose $U = \mathfrak{o}_F^\times$ into the cosets of the subgroup $U_r = 1 + M^r$. A typical element of the coset

aU_r is of the form $a(1+u\pi^r)$ for some unit u. We observe that for such an element x we have $\lambda(a(1+u\pi^r)) = \lambda(a)\lambda(\pi^r au) = \lambda(a)$. This gives us an expression for $g(\omega,\lambda)$ as a sum over cosets:

$$g(\omega,\lambda) = \sum_{aU_r} \int_{aU_r} \omega(x)\lambda(x)\mathrm{d}^*x = \sum_{aU_r} \lambda(a)\omega(a) \int_{U_r} \omega(x)\mathrm{d}^*x.$$

But since $r < n$, the above integral is zero since ω is nontrivial on U_r.
(ii) Let $r = n$. We compute, introducing the variable $z = xy^{-1}$ in the third line:

$$\begin{aligned} |g(\omega,\lambda)|^2 &= \int_U \omega(x)\lambda(x)\mathrm{d}^*x \overline{\int_U \omega(y)\lambda(y)\mathrm{d}^*y} \\ &= \int_U\int_U \omega(xy^{-1})\lambda(x-y)\mathrm{d}^*x\mathrm{d}^*y \\ &= \int_U\int_U \omega(z)\lambda(zy-y)\mathrm{d}^*z\mathrm{d}^*y \\ &= \int_U \omega(z)\Big[\int_U \lambda(y(z-1))\mathrm{d}^*y\Big]\mathrm{d}^*z. \end{aligned} \tag{24}$$

Let us write $h(z)$ for the expression between brackets. Since $U = \mathfrak{o}_F\backslash M$, we have

$$h(z) = c\int_{\mathfrak{o}_F} \lambda(y(z-1))\mathrm{d}y - c\int_M \lambda(y(z-1))\mathrm{d}y.$$

The integrands in these integrals are characters on compact groups, so the integrals equal zero or the volume of the group. From the definition of r it follows that the first integral is zero (that is, the character $y \mapsto \lambda(y(z-1))$ is nontrivial) if and only if $(z-1) \notin M^r$. In other words, the integral equals $Vol(\mathfrak{o}_F, \mathrm{d}x)$ if and only if $z \in U_r$. Similarly, the second integral equals $Vol(M, \mathrm{d}x)$ if $z \in U_{r-1}$, and zero otherwise. Since $Vol(M) = q^{-1}Vol(\mathfrak{o}_F)$, we can write

$$h(z) = cVol(\mathfrak{o}_F)C_r(z) - cq^{-1}Vol(\mathfrak{o}_F)C_{r-1}(z)$$

where we have introduced the notation C_r for the characteristic function of U_r. Inserting this expression for $h(z)$ into our original expression for $|g(\omega,\lambda)|^2$ yields

$$\begin{aligned} |g(\omega,\lambda)|^2 &= cVol(\mathfrak{o}_F)\int_U \omega(z)C_r(z)\mathrm{d}^*z - cq^{-1}Vol(\mathfrak{o}_F)\int_U \omega(z)C_{r-1}(z)\mathrm{d}^*z \\ &= cVol(\mathfrak{o}_F)\int_{U_r} \omega(z)\mathrm{d}^*z - cq^{-1}Vol(\mathfrak{o}_F)\int_{U_{r-1}} \omega(z)\mathrm{d}^*z. \end{aligned}$$

Since $r = n$, the second integrand is a nontrivial character while the first integrand is trivial. Hence the second integral vanishes, and part (ii) of the lemma follows.
We do not need part (iii), so we omit the proof, which is very similar to the proof of part (ii). □

We now compute, for $n \geq 1$ (writing π for π_F)

$$\begin{aligned}
\zeta(f,\chi_{s,n}) &= \int_{F^\times} f(x)\chi_{s,n}(x)\mathrm{d}^*x \\
&= \int_{M^{m-n}-\{0\}} \psi(x)|x|^s\mu(x/|x|)\mathrm{d}^*x \\
&= \sum_{j=m-n}^{\infty} \int_{A^j} \psi(x)q^{-js}\mu(x/|x|)\mathrm{d}^*x \\
&= \sum_{j=m-n}^{\infty} q^{-js} \int_{\mathfrak{o}_F^\times} \psi(\pi^j u)\mu(u)\mathrm{d}^*u \\
&= \sum_{j=m-n}^{\infty} q^{-js} g(\mu,\psi_{\pi^j}). \qquad (25)
\end{aligned}$$

Here $\psi_t(x)$ stands for $\psi(tx)$, and the fourth equality sign comes from the change of variables $u = \pi^{-j}x$. We observe that the conductors of μ and $\psi_{\pi^{m-n}}$ are both n, and by part (i) of the lemma all the terms in the sum except the first one are zero. Therefore, by part (ii) of the lemma, we have

$$\zeta(f,\chi_{s,n}) = q^{-(m-n)s}g(\mu,\psi_{\pi^{m-n}}) = q^{-(m-n)s}g(\mu,\psi_{\pi^{m-n}})L(\chi_{s,n}). \qquad (26)$$

So much for $\zeta(f,\chi)$; let us now turn to $\zeta(\hat{f},\chi^\vee)$. Our first step is to determine $\hat{f}$. By definition, we have

$$\hat{f}(y) = \int_F f(x)\psi(xy)\mathrm{d}x = \int_{M^{m-n}} \psi(x(y+1))\mathrm{d}x. \qquad (27)$$

The character ψ_{y+1} restricts to a character on M^{m-n} and the above integral is either 0 or $Vol(M^{m-n},\mathrm{d}x)$ depending on whether this character is trivial or not. As in the proof of Theorem 16, this depends only on $|y+1|$. In the case $n=0$, the conductor of ψ is m, and ψ_{y+1} is trivial on M^m if and only if $y+1 \in \mathfrak{o}_F$, which is equivalent with $y \in \mathfrak{o}_F$. Hence the Fourier transform of f equals $Vol(M^m,\mathrm{d}x)$ times the characteristic function of $\mathfrak{o}_F$. Now suppose that $n > 0$. By the same argument again, ψ_{y+1} is trivial on M^{m-n} if and only if $|y+1| \leq q^{-n}$. But this happens if and only if $y \in M^n - 1$, so the Fourier transform of f equals $Vol(M^{m-n})$ times the characteristic function of $M^n - 1$.

Let us now compute $\zeta(\hat{f}, \chi_{s,n}^\vee)$. When $n = 0$, $\chi_{s,n}$ equals $|x|^s$ and we get

$$\begin{aligned}
\zeta(\hat{f}, \chi_{s,0}^\vee) &= \int_{F^\times} \hat{f}(x)\chi_{s,0}^\vee(x)\mathrm{d}^*x \\
&= Vol(M^m, \mathrm{d}x) \int_{\mathfrak{o}_F - \{0\}} |x|^{1-s}\mathrm{d}^*x \\
&= Vol(M^m, \mathrm{d}x) \sum_{j=0}^{\infty} \int_{A_j} |x|^{1-s}\mathrm{d}^*x \\
&= Vol(M^m, \mathrm{d}x) \frac{q}{q-1} \sum_{j=0}^{\infty} \int_{A_j} |x|^{-s}\mathrm{d}x \\
&= \frac{Vol(\mathfrak{o}_F, \mathrm{d}x)}{q^m} \frac{q}{q-1} \sum_{j=0}^{\infty} Vol(\mathfrak{o}_F, \mathrm{d}x) \frac{q-1}{q^{j+1}} q^{js}\mathrm{d}x \\
&= \frac{Vol(\mathfrak{o}_F, \mathrm{d}x)^2}{q^m} \frac{1}{1-q^{s-1}} = \frac{Vol(\mathfrak{o}_F, \mathrm{d}x)^2}{q^m} L(\chi_{s,0}^\vee). \qquad (28)
\end{aligned}$$

Combining this with the expression (23) we get

$$\gamma(\chi_{s,0}) = q^{m(s-1)} Vol(\mathfrak{o}_F, \mathrm{d}x) \frac{L(\chi_{s,0}^\vee)}{L(\chi_{s,0})} \qquad (29)$$

and hence we can take

$$\varepsilon(\chi_{s,0}) = q^{m(s-1)} Vol(\mathfrak{o}_F, \mathrm{d}x). \qquad (30)$$

Now only one calculation remains, and that is $\zeta(\hat{f}_n, \chi_{s,n}^\vee)$ for $n \geq 1$. We have $\chi_{s,n}^\vee(x) = \overline{\mu}(x)|x|^{1-s}$, and since μ is a trivial character on $1 + M^n$, we get

$$\begin{aligned}
\zeta(\hat{f}, \chi_{s,n}^\vee) &= \int_{F^\times} \hat{f}(x)\chi_{s,n}^\vee(x)\mathrm{d}^*x \\
&= Vol(M^{m-n}, \mathrm{d}x) \int_{M^n - 1} \overline{\mu}(x)|x|^{1-s}\mathrm{d}^*x \\
&= Vol(M^{m-n}, \mathrm{d}x) \int_{M^n - 1} \overline{\mu}(x)\mathrm{d}^*x \\
&= Vol(M^{m-n}, \mathrm{d}x) \int_{1+M^n} \overline{\mu}(-x)\mathrm{d}^*x \\
&= Vol(M^{m-n}, \mathrm{d}x)\overline{\mu}(-1) \int_{1+M^n} \overline{\mu}(x)\mathrm{d}^*x \\
&= Vol(M^{m-n}, \mathrm{d}x) Vol(1 + M^n, \mathrm{d}^*x)\overline{\mu}(-1) \\
&= Vol(M^{m-n}, \mathrm{d}x) Vol(1 + M^n, \mathrm{d}^*x)\overline{\mu}(-1) L(\chi_{s,n}^\vee) \qquad (31)
\end{aligned}$$

since $L(\chi_{s,n}) = 1$. In this calculation, the step where $|x|^{1-s}$ disappears needs to be explained. We are integrating over $M^n - 1$, so the integration

variable x is of the form $u\pi^n - 1$ where u is a unit. Consider the triangle with corners 0, 1, and $u\pi^n$. Since "every triangle in an ultrametric space is isosceles", the distance $|x| = |u\pi^n - 1|$ must equal either $|u\pi^n| = q^{-n}$ or 1. But the first case contradicts the triangle inequality, so $|x| = 1$, which justifies our calculation.
We conclude that the ε-factor exists and equals

$$\frac{Vol(M^{m-n}, \mathrm{d}x)Vol(1 + M^n, \mathrm{d}^*x)\overline{\mu}(-1)}{q^{-(m-n)s}g(\mu, \psi_{\pi^{m-n}})}.$$

Although it is not important for us, we mention that this can be simplified to yield

$$\varepsilon(\chi_{s,n}) = (q-1)q^{(m-n)(s-1)-1}g(\overline{\mu}, \psi_{\pi^{m-n}}). \tag{32}$$

All our computed ε-factors are nonzero and since the local L-factors are meromorphic, so are the γ-factors. Therefore, we can conclude that Theorem 17 holds for all local fields.
For later use, we also single out some consequences of our calculations. Suppose that χ is an idèle class character. Then by Theorem 8, the conductor of the local component χ_v is zero for almost all non-archimedean places of K. Therefore both n and m are zero for almost all such places.

Proposition 6. Let χ be a unitary idele class character (that is, a character, not a quasi-character). For every local factor χ_v of χ, we can consider the local epsilon-factor $\varepsilon(\chi_v)$ defined for the completion K_v. This local epsilon-factor equals 1 for almost all non-archimedean completions of K.

Proof. Whenever m is zero, d is also zero, so the Haar measure of $\mathfrak{o}_v$ is 1. The conductor n of χ_v is zero for almost all v by Theorem 8. But since n and m are then both zero for almost all places of K, equation (30) gives us that the epsilon-factor is 1 for almost all non-archimedean completions. □

Proposition 7. For every place v, there exists a function $f_v \in \mathcal{S}(K_v)$ such that f_v is the characteristic function of $\mathfrak{o}_v$ for almost all v, and a function $h_v(s, \chi_v)$ that is entire, everywhere nonzero, and equal to 1 for almost all v, with the property that

$$\zeta(f_v, \chi_v|\cdot|_v^s) = h_v(s, \chi_v)L(s, \chi_v).$$

where the last L-factor is the one in Definition 9.

Proof. The function f_v is the function f (or f_n in the complex case) that we have used in our calculations; from the relation (22) it is clear that f_v has the required property. From equations (19) and (21) it is immediate that h exists and equals 1 in the archimedean cases and equations (23) and (26) show that h exists in the non-archimedean case, and since h is given by (23) except for a finite number of places v it is clear that $h = 1$ for almost all places. □

8.2 Global zeta functions

Now let K be a number field with adèle group $\mathbb{A}_K$ and idèle group $\mathbb{I}_K$. As promised, we shall do analysis on these groups, and to start with, we define the function space that we shall be working with. We say that a complex-valued function on $\mathbb{A}_K$ is a *factorizable adelic Schwartz-Bruhat function* if it can be written as a product over all places of K

$$f(x) = \prod_v f_v(x_v)$$

where $f_v \in \mathcal{S}(K_v)$ for all places v, and f_v is the characteristic function of $\mathfrak{o}_v$ for almost all v. Note that these conditions imply that the product is well-defined for any adèle x. We define an *adelic Schwartz-Bruhat function* to be a finite complex-linear combination of factorizable Schwarz-Bruhat functions. The set of adelic Schwartz-Bruhat functions will be denoted by $\mathcal{S}(\mathbb{A}_K)$, and for any such function f we define the *adelic Fourier transform* to be

$$\hat{f}(y) = \int_{\mathbb{A}_K} f(x)\Psi(xy)\mathrm{d}x.$$

Here Ψ is the standard character on $\mathbb{A}_K$ and $\mathrm{d}x$ is the self-dual measure with respect to Ψ. Clearly, $\mathcal{S}(\mathbb{A}_K)$ is a complex vector space.

Theorem 18. For any $f \in \mathcal{S}(\mathbb{A}_K)$, the adelic Fourier transform is well-defined, $\hat{f}$ lies in $\mathcal{S}(\mathbb{A}_K)$, and the Fourier inversion formula holds for f.

Proof. It suffices to consider factorizable Schwartz-Bruhat functions, so let f be such a function. Recall that we write ψ_v for the standard character on the completion K_v. By Theorem 11, we have for a fixed adèle y:

$$\begin{aligned}\hat{f}(y) &= \int_{\mathbb{A}_K} f(x)\Psi(xy)\mathrm{d}x \\ &= \int_{\mathbb{A}_K} \prod_v f_v(x) \prod_v \psi_v(x_v y_v)\mathrm{d}x_v \\ &= \lim_S \prod_{v\in S} \int_{K_v} f_v(x_v)\psi_v(x_v y_v)\mathrm{d}x_v.\end{aligned}$$

For almost all v, f_v is the characteristic function of $\mathfrak{o}_v$, the volume of $\mathfrak{o}_v$ is 1, x_v and y_v lie in $\mathfrak{o}_v$ and ψ_v is trivial on $\mathfrak{o}_v$. Therefore the indicated integrals equal 1 for almost all places, so that

$$\begin{aligned}\hat{f}(y) &= \prod_v \int_{K_v} f_v(x_v)\psi_v(x_v y_v)\mathrm{d}x_v \\ &= \prod_v \hat{f}_v(y_v).\end{aligned}$$

This product is finite, so the adelic Fourier transform is indeed well-defined. To see that $\hat{f} \in \mathcal{S}(\mathbb{A}_K)$, we consider the functions $\hat{f}_v$ in the above product. By Theorem 16, $f_v \in \mathcal{S}(K_v)$ for all v. Also whenever f_v is the characteristic function of $\mathfrak{o}_v$ we have

$$\hat{f}_v(y_v) = \int_{K_v} f_v(x_v)\psi_v(x_v y_v)\mathrm{d}x_v = \int_{\mathfrak{o}_v} \psi_v(x_v y_v)\mathrm{d}x_v$$

and this equals one or zero depending on whether $y_v \in \mathfrak{o}_v^{\perp}$ or not. But when the conductor of ψ_v is zero, $\mathfrak{o}_v^{\perp}$ can be identified with $\mathfrak{o}_v$, so $\hat{f}_v$ is the characteristic function of $\mathfrak{o}_v$ for almost all places and $\hat{f} \in \mathcal{S}(\mathbb{A}_K)$.
If we can show that any factorizable $f \in \mathcal{S}(\mathbb{A}_K)$ is continuous and in $L^1(\mathbb{A}_K)$, then obviously $\mathcal{S}(\mathbb{A}_K) \subseteq Inv(\mathbb{A}_K)$ so the Fourier inversion formula holds for adelic Schwarts-Bruhat functions. By part (ii) of Theorem 11, any such f is continuous. To see that f is in $L^1(\mathbb{A}_K)$, we observe that the local integral

$$\int_{K_v} |f_v(x_v)|\mathrm{d}x_v$$

equals 1 for almost all v, and hence

$$\int_{\mathbb{A}_K} |f(x)|\mathrm{d}x = \prod_v \int_{K_v} |f_v(x_v)|\mathrm{d}x_v$$

is a finite product and $f \in L^1(\mathbb{A}_K)$. □

We now come to the global zeta functions, that are the analogue for $\mathbb{A}_K$ of the local zeta functions for local fields in section 8.1. Let χ be an idèle class character. For any $f \in \mathcal{S}(\mathbb{A}_K)$ we define the global zeta function

$$Z(f,\chi) = \int_{\mathbb{I}_K} f(x)\chi(x)\mathrm{d}^*x.$$

For the rest of this section we will be occupied with the proof of a functional equation for global zeta functions. When this functional equation is in place, it will be an easy task to establish the results about Hecke L-functions that are our main goal.

8.2.1 The Riemann-Roch theorem

Our first main ingredient in the proof of the global functional equation is the so-called Riemann-Roch theorem. The reason for naming the theorem Riemann-Roch is that if the number fields we are working with are replaced by function fields, the theorem is equivalent to the familiar geometric Riemann-Roch theorem for algebraic curves over $\mathbb{F}_q$.

To prove the Riemann-Roch theorem, we will use an analogue of the classical Poisson summation formula. If f is some function on $\mathbb{A}_K$, we define

$$\tilde{f}(x) = \sum_{\gamma \in K} f(x+\gamma)$$

for all x such that the sum is absolutely convergent. If this sum is absolutely convergent for all x, it is clear that $\tilde{f}$ is *K-periodic*, that is $\tilde{f}(x+\eta) = \tilde{f}(x)$ for all $\eta \in K$.

Proposition 8. If $f \in \mathcal{S}(\mathbb{A}_K)$, then $\tilde{f}$ is absolutely and uniformly convergent on any compact subset of $\mathbb{A}_K$.

Proof. See Valenza and Ramakrishnan [23], p. 261. □

Theorem 19 (Poisson summation formula). For all $f \in \mathcal{S}(\mathbb{A}_K)$ we have

$$\sum_{\gamma \in K} f(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma).$$

Proof. The function $\tilde{f}$ is K-periodic, so f induces a function on $\mathbb{A}_K/K$, which we will call ϕ, in an attempt to avoid confusion. We write $\overline{t}$ for the image in $\mathbb{A}_K/K$ of an element $t \in \mathbb{A}_K$. We also write $\overline{\mathrm{d}t}$ for the quotient measure on $\mathbb{A}_K/K$ induced by the counting measure on K, and $\overline{\Psi}$ for the character on $\mathbb{A}_K/K$ induced by the standard character Ψ on $\mathbb{A}_K$ (recall that this character is trivial on K, so it is K-periodic). By Theorem 14, the Pontryagin dual of $\mathbb{A}_K/K$ can be identified with K, so the Fourier transform of ϕ can be regarded as a function on K. More precisely, we can write, for any $x \in K$:

$$\hat{\phi}(x) = \int_{\mathbb{A}_K/K} \phi(\overline{t})\overline{\Psi}(\overline{t}x)\overline{\mathrm{d}t}. \tag{33}$$

We need the following

Lemma 4. For every $x \in K$, we have

$$\hat{f}(x) = \hat{\phi}(x). \tag{34}$$

Proof. By definition, we have

$$\begin{aligned}
\hat{\phi}(x) &= \int_{\mathbb{A}_K/K} \phi(\overline{t})\overline{\Psi}(\overline{t}x)\overline{\mathrm{d}t} \\
&= \int_{\mathbb{A}_K/K} \sum_{\gamma \in K} f(\gamma + t)\Psi(tx)\overline{\mathrm{d}t} \\
&= \int_{\mathbb{A}_K/K} \Big(\sum_{\gamma \in K} f(\gamma + t)\Psi((\gamma + t)x) \Big)\overline{\mathrm{d}t} \\
&= \int_{\mathbb{A}_K} f(t)\Psi(tx)\mathrm{d}t \\
&= \hat{f}(x).
\end{aligned}$$

□

We want to apply the Fourier inversion formula to the function ϕ, and in order to do that we must verify that $\phi \in Inv(\mathbb{A}_K/K)$. It is clear that ϕ is in $L^1(\mathbb{A}_K/K)$, because $f \in L^1(\mathbb{A}_K)$. It follows from Proposition 8 that ϕ is continuous. From the lemma, we see that

$$\sum_{\gamma \in K} |\hat{\phi}(\gamma)| = \sum_{\gamma \in K} |\hat{f}(\gamma)|$$

and since $\hat{f} \in \mathcal{S}(\mathbb{A}_K)$, this sum is finite by Proposition 8, so $\hat{\phi}$ is in $L^1(K)$. Thus $\phi \in Inv(\mathbb{A}_K/K)$ and the Fourier inversion formula gives us

$$\phi(x) = \sum_{\gamma \in K} \hat{\phi}(\gamma)\overline{\Psi}(\gamma x)$$

and the lemma gives us in particular

$$\phi(0) = \sum_{\gamma \in K} \hat{\phi}(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma).$$

But by definition,

$$\phi(0) = \sum_{\gamma \in K} f(\gamma)$$

and this completes the proof. □

Theorem 20 (Riemann-Roch theorem). For all $f \in \mathcal{S}(\mathbb{A}_K)$ and for all $\xi \in \mathbb{I}_K$ we have

$$\sum_{\gamma \in K} f(\gamma\xi) = \frac{1}{\|\xi\|} \sum_{\gamma \in K} \hat{f}(\gamma\xi^{-1}).$$

Proof. Take an idèle ξ and a function $f \in \mathcal{S}(\mathbb{A}_K)$. Define a function on $\mathbb{A}_K$ by $h(x) = f(\xi x)$. Then h is in $\mathcal{S}(\mathbb{A}_K)$, because the local components of an idèle are local units for almost all places, and a local ring of integers is fixed under multiplication by a local unit. This implies that if ξ_v is a local unit and $f_v(x_v)$ is the characteristic function of $\mathfrak{o}_v$, then $h_v(x_v) = f_v(\xi_v x_v)$ is also the characteristic function of $\mathfrak{o}_v$. We apply the Poisson summation formula to h, and get

$$\sum_{\gamma \in K} h(\gamma) = \sum_{\gamma \in K} \hat{h}(\gamma).$$

But we can also compute

$$\begin{aligned}\hat{h}(\gamma) &= \int_{\mathbb{A}_K} f(y\xi)\Psi(\gamma y)\mathrm{d}y \\ &= \frac{1}{\|\xi\|}\int_{\mathbb{A}_K} f(y)\Psi(\gamma y\xi^{-1})\mathrm{d}y \\ &= \frac{1}{\|\xi\|}\hat{f}(\gamma\xi^{-1})\end{aligned}$$

and the Riemann-Roch theorem follows. □

8.2.2 Global functional equation

We saw in section 7.4 that any idèle-class character χ can be written as $\mu\|\cdot\|^s$ with μ a character, and as for local characters, the number $\sigma = \Re(s)$, which is uniquely defined, is called the *exponent* of χ. Again, having fixed a character μ, we can regard $Z(f,\chi)$ as a function of s. Exactly as in the local case, we define the shifted dual of χ to be $\frac{\|\cdot\|}{\chi}$. We also introduce the notation V for the volume of C_K^1.

Theorem 21 (Global functional equation). For $f \in \mathcal{S}(\mathbb{A}_K)$ and $\chi = \mu\|\cdot\|^s$ an idèle class character, the global zeta function $Z(f,\chi)$ is absolutely convergent in the region $\sigma > 1$. Moreover, it extends to a meromorphic function in the whole complex plane, and satisfies the functional equation

$$Z(f,\chi) = Z(\hat{f},\chi^\vee).$$

The extended function $Z(f,\chi)$ is holomorphic everywhere, except when $\mu = \|\cdot\|^{-i\tau}$, $\tau \in \mathbb{R}$. In this case, it has a simple pole at $s = i\tau$ with residue $-Vf(0)$ and a simple pole at $s = 1+i\tau$ with residue $V\hat{f}(0)$.

Proof. As usual, it suffices to consider factorizable Schwartz-Bruhat functions, so let f be such a function and let $\chi = \mu\|\cdot\|^s$ be an idèle class character. First we consider the convergence in the region $\sigma > 1$. We must prove that the integral

$$\int_{\mathbb{I}_K} |f(x)\chi(x)|\mathrm{d}x = \int_{\mathbb{I}_K} |f(x)|\cdot\|x\|^\sigma = \lim_S \prod_{v\in S}\int_{K_v^\times} |f_v(x_v)|\cdot|x|^\sigma \mathrm{d}^*x_v$$

is finite. Let T be the finite set of places such that v is archimedean or f_v is not the characteristic function of $\mathfrak{o}_v$ or the volume of $\mathfrak{o}_v$ is not 1. We can split the above expression into a product of two factors:

$$\prod_{v\in T}\int_{K_v^\times} |f_v(x_v)|\cdot|x|^\sigma \mathrm{d}^*x_v \cdot \lim_S \prod_{v\in S}\int_{K_v^\times} |f_v(x_v)|\cdot|x|^\sigma \mathrm{d}^*x_v \tag{35}$$

where now S ranges over finite sets of places not meeting T. The first of these factors equals

$$\prod_{v\in T}\int_{K_v^\times} |f_v(x_v)|\cdot|x|^\sigma \mathrm{d}^*x_v = \prod_{v\in T}\zeta(|f_v|,|x|^\sigma) \tag{36}$$

and since $|f_v|$ is a local Schwartz-Bruhat function, this is a finite product of local zeta functions. Such functions are convergent for $\sigma > 0$, and since we assume $\sigma > 1$, the product (36) is finite. Now consider the second factor. For every finite set S not meeting T we have

$$\prod_{v\in S}\int_{K_v^\times} |f_v(x_v)|\cdot|x|^\sigma \mathrm{d}^*x_v = \prod_{v\in S}\int_{\mathfrak{o}_v^\times} |x|^\sigma \mathrm{d}^*x_v = \prod_{v\in S}\int_{\mathfrak{o}_v} |x|^{\sigma-1}\mathrm{d}x_v$$

and since $\sigma - 1 > 0$ we have for all $v \in S$ that

$$x_v \in \mathfrak{o}_v \implies |x_v| \le 1 \implies |x_v|^{\sigma-1} \le 1$$

and hence

$$0 \le \int_{\mathfrak{o}_v} |x|^{\sigma-1} \mathrm{d}x_v \le 1.$$

This implies that the limit

$$\lim_S \prod_{v \in S} \int_{\mathfrak{o}_v} |x|^{\sigma-1} \mathrm{d}x_v$$

taken over larger and larger S, is a limit of a monotonely decreasing sequence of non-negative numbers, and hence the limit exists and is finite. This shows that $Z(f,\chi)$ is indeed absolutely convergent in the region $\Re(s) > 1$.
The global zeta function can be written

$$Z(f,\chi) = \int_0^\infty \Big[\int_{\mathbb{I}_K^1} f(t*x)\chi(t*x)\mathrm{d}^*x \Big] \frac{\mathrm{d}t}{t}.$$

The inner integral, which we will denote by $I(f,\chi,t)$, equals

$$\int_{C_K^1} \big(\sum_{a \in K^\times} f(at*x)\chi(t*x) \big) \mathrm{d}^*x$$

where we used that $\chi(a) = 1$ for $a \in K$, since χ is an idèle class character. The sum over a invites us to look for a possibility to use the Riemann-Roch theorem, but Riemann-Roch applies to sums over K, not over $K^\times$. Therefore we consider the sum

$$I(f,\chi,t) + \int_{C_K^1} f(0)\chi(t*x)\mathrm{d}^*x = \int_{C_K^1} \big(\sum_{a \in K} f(at*x)\chi(t*x) \big) \mathrm{d}^*x$$

which by Riemann-Roch is equal to

$$\int_{C_K^1} \big(\frac{1}{\|t*x\|} \sum_{a \in K} \hat{f}(at^{-1}*x^{-1}) \big) \chi(t*x)\mathrm{d}^*x. \tag{37}$$

Note that for any t, the integral $I(f,\chi,t)$ is convergent without any assumption on the exponent of χ (this follows from the compactness of C_K^1 and the absolute convergence of $\sum f(at*x)$). We now compute $I(\hat{f},\chi^\vee,t^{-1})$ and get

$$\int_{C_K^1} \big(\sum_{a \in K^\times} \hat{f}(at^{-1}*x) \frac{\|t^{-1}*x\|}{\chi(t^{-1}*x)} \big) \mathrm{d}^*x.$$

Now, if we replace x by x^{-1} in the above expression, and add the term $\int_{C_K^1} \hat{f}(0)\chi^\vee(t^{-1}*x)\mathrm{d}^*x$, we get exactly the expression (37), so we arrive at the relation

$$I(f,\chi,t) = I(\hat{f},\chi^\vee,t^{-1}) + D(f,\chi) \tag{38}$$

where we have introduced

$$D(f,\chi) = \int_{C_K^1} \hat{f}(0)\chi^\vee(t^{-1} * x)\mathrm{d}^*x - \int_{C_K^1} f(0)\chi(t * x)\mathrm{d}^*x.$$

Let us return to the global zeta function and observe that we can write $Z(f,\chi)$ as a sum of the two integrals

$$Z_0 = \int_{x\in\mathbb{I}_K:||x||<1} f(x)\chi(x)\mathrm{d}^*x = \int_0^1 I(f,\chi,t)\frac{\mathrm{d}t}{t}$$

and

$$Z_1 = \int_{x\in\mathbb{I}_K:||x||\geq 1} f(x)\chi(x)\mathrm{d}^*x = \int_1^\infty I(f,\chi,t)\frac{\mathrm{d}t}{t}.$$

From the convergence of the global zeta function, it follows that these integrals must converge in the region $\sigma > 1$. For Z_1, we observe that if it converges for some σ_0, it converges for all $\sigma < \sigma_0$, so it must converge everywhere. But for Z_0, we can use equation 38 to write

$$Z_0 = \int_0^1 I\big(\hat{f},\chi^\vee,t^{-1}\big)\frac{\mathrm{d}t}{t} + E$$

where we have introduced an "error term"

$$E(f,\chi) = \int_0^1 D(f,\chi)\frac{\mathrm{d}t}{t}.$$

We may substitute t^{-1} for t in this expression for Z_0, and this gives us

$$Z_0 = \int_1^\infty I\big(\hat{f},\chi^\vee,t\big)\frac{\mathrm{d}t}{t} + E. \tag{39}$$

The integral here is convergent everywhere by the same argument as for Z_1. We will see shortly that E is a rational function in s, and thus we have our meromorphic continuation of $Z(f,\chi)$, with poles only at the poles of E. To see that E is indeed a rational function, we consider two cases. First, if χ is nontrivial on $\mathbb{I}_K^1$ it is nontrivial on C_K^1. Then

$$D = \hat{f}(0)\chi^\vee(t^{-1})\int_{C_K^1}\chi^\vee(x)\mathrm{d}^*x - f(0)\chi(t)\int_{C_K^1}\chi(x)\mathrm{d}^*x$$

and since C_K^1 is a compact group, the integrals are zero, and hence E must be zero as well. Second, suppose that χ is trivial on $\mathbb{I}_K^1$. Then the integrals that previously were zero now equal the volume V of C_K^1. Also χ is of the form $\|\cdot\|^{s'}$ for some $s' \in \mathbb{C}$, and we must have $\Re(s') = \sigma$, so that $s' = s - i\tau$ for some $\tau \in \mathbb{R}$. We now compute

$$\begin{aligned} E(f,\chi) &= \int_0^1 \Big(\hat{f}(0)t^{s'-1}V - f(0)t^{s'}V\Big)\frac{\mathrm{d}t}{t} \\ &= V\Big(\frac{\hat{f}(0)}{s-(1+i\tau)} - \frac{f(0)}{s-i\tau}\Big) \end{aligned}$$

so E is indeed rational.
The only thing that remains for Theorem 21 is now the functional equation itself. To start with, we prove a functional equation for $E(f,\chi)$. If χ is nontrivial on $\mathbb{I}_K^1$, then so is $\chi^\vee$, and hence $E(f,\chi) = E(\hat{f},\chi^\vee) = 0$. On the other hand, if χ is trivial on $\mathbb{I}_K^1$, we have $\chi^\vee = \|\cdot\|^{1-s'}$ where s' is as before, and using $\hat{\hat{f}}(0) = f(0)$ we can compute

$$\begin{aligned} E(\hat{f},\chi^\vee) &= \int_0^1 \Big(f(0)t^{-s'}V - \hat{f}(0)t^{1-s'}V\Big)\frac{\mathrm{d}t}{t} \\ &= V\Big(-\frac{f(0)}{s-i\tau} + \frac{\hat{f}(0)}{s-(1+i\tau)}\Big) \end{aligned}$$

which equals $E(f,\chi)$. Now to the zeta function. Recall that we could write $Z(f,\chi)$ as a sum of the two integrals Z_0 and Z_1, and that we have the expression (39) for Z_0. This gives us

$$\begin{aligned} Z(f,\chi) &= \int_1^\infty I(f,\chi,t)\frac{\mathrm{d}t}{t} + \int_1^\infty I(\hat{f},\chi^\vee,t)\frac{\mathrm{d}t}{t} + E(f,\chi) \\ &= \int_1^\infty\int_{\mathbb{I}_K^1} f(t*x)\chi(t*x)\mathrm{d}^*x\frac{\mathrm{d}t}{t} \\ &\quad + \int_1^\infty\int_{\mathbb{I}_K^1} \hat{f}(t*x)\chi^\vee(t*x)\mathrm{d}^*x\frac{\mathrm{d}t}{t} + E(f,\chi). \end{aligned}$$

Using the general relations $\hat{\hat{f}}(x) = f(-x)$ and $(\chi^\vee)^\vee = \chi$ we use the same equations to express

$$\begin{aligned} Z(\hat{f},\chi^\vee) &= \int_1^\infty I(\hat{f},\chi^\vee,t)\frac{\mathrm{d}t}{t} + \int_1^\infty I(\hat{\hat{f}},\chi,t)\frac{\mathrm{d}t}{t} + E(\hat{f},\chi^\vee) \\ &= \int_1^\infty\int_{\mathbb{I}_K^1} \hat{f}(t*x)\chi^\vee(t*x)\mathrm{d}^*x\frac{\mathrm{d}t}{t} \\ &\quad + \int_1^\infty\int_{\mathbb{I}_K^1} f(-t*x)\chi(t*x)\mathrm{d}^*x\frac{\mathrm{d}t}{t} + E(\hat{f},\chi^\vee). \end{aligned}$$

But now, since $\chi(-t*x) = \chi(t*x)$ and $E(f,\chi) = E(\hat{f},\chi^\vee)$, it is immediate that the awkward expressions for $Z(f,\chi)$ and $Z(\hat{f},\chi^\vee)$ are identical (we can of course substitute $-x$ for x in an integral over $\mathbb{I}_K^1$). This gives us our functional equation and the theorem is proved! □

8.3 Hecke L-functions

Let χ be an idèle class character of exponent σ. We recall (Theorem 8) that χ can be written as a product $\prod \chi_v$ over all places v, where the χ_v are local characters such that $\chi_v|_{\mathfrak{o}_v} = 1$ for almost all v, that is, χ_v is unramified for almost all v. We now define the *global L-function* of an idèle class character χ in terms of the local L-factors from section 8.1.2 to be

$$L(\chi) = \prod_v L(\chi_v)$$

whenever this is convergent. As in the local case we define $L(s, \chi)$ to be $L(\chi \| \cdot \|^s)$ for any complex s.

Definition 11. We call the function $L(s, \chi)$ the *Hecke L-function* associated to the idèle class character χ. For such a character, we also define the *finite Hecke L-function*

$$L_f(s, \chi) = \prod_{v \text{ finite}} L(s, \chi_v)$$

and the *infinite Hecke L-function*

$$L_\infty(s, \chi) = \prod_{v \text{ infinite}} L(s, \chi_v).$$

Of course, we have $L(s, \chi) = L_f(s, \chi) L_\infty(s, \chi)$. These definitions are not without meaning, because we have

Proposition 9. $L(\chi)$ is absolutely convergent whenever $\sigma > 1$.

Proof. We can delete a finite number of factors from the product defining $L(\chi)$ without changing the convergence properties. Let us remove the archimedean factors and all factors where χ_v is ramified. Writing χ as $\mu \| \cdot \|^s$ we must prove that the logarithm of the product

$$\prod_v |L(\chi_v)| = \prod_v \frac{1}{|1 - \mu_v(\pi_v)|q_v^{-s}|}$$

converges for $\sigma > 1$. Because of the identity

$$\log \prod_v \frac{1}{|1 - \mu_v(\pi_v)|q_v^{-s}|} = \Re(\sum_v \sum_{k=1}^{\infty} \frac{\mu_v(\pi_v)^k q_v^{-ks}}{k})$$

it is sufficient to show that $\sum_v \sum_1^\infty \frac{q_v^{-k\sigma}}{k}$ converges. If the place v lies over the prime $p \in \mathbb{Z}$, then $q_v \geq p$, so

$$\sum_v \sum_{k=1}^{\infty} \frac{q_v^{-k\sigma}}{k} \leq \sum_p \sum_{v|p} \sum_{k=1}^{\infty} \frac{p^{-k\sigma}}{k} \leq n \log(\prod_p \frac{1}{1 - p^{-\sigma}}).$$

The product in the last expression is the classical Euler product, which is absolutely convergent for $\sigma > 1$, so the proposition follows. □

We are now ready for our main theorem - the functional equation and analytic continuation of the Hecke L-functions. Recall the definition of the local epsilon factor from section 8.1.4.

Theorem 22 (Main Theorem for Hecke L-functions). Let χ be a unitary idèle class character (that is, a character, not a quasi-character). Then $L(s,\chi)$, which is *a priori* defined and holomorphic in the region $\Re(s) > 1$, admits a meromorphic continuation to all of $\mathbb{C}$, and satisfies a functional equation

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\chi^\vee)$$

where

$$\varepsilon(s,\chi) = \prod_v \varepsilon(\chi_v|\cdot|^s) \in \mathbb{C}^\times.$$

This meromorphic continuation is holomorphic everywhere, except in case $\chi = \|\cdot\|^{-i\tau}$, $\tau \in \mathbb{R}$, when it has simple poles at $s = i\tau$ and $s = 1 + i\tau$.

Proof. We begin by proving that $L(s,\chi)$ has meromorphic continuation to all of $\mathbb{C}$. By Proposition 7 there exist, for every place v, functions f_v and h_v such that $f_v \in \mathcal{S}(K_v)$ is the characteristic function of $\mathfrak{o}_v$ for almost all v, and h_v is entire as a function of s, everywhere nonzero, and equal to 1 for almost all v, and such that we have the relation

$$\zeta(f_v, \chi_v|\cdot|_v^s) = h_v(s,\chi_v)L(s,\chi_v). \tag{40}$$

The global zeta function can be written

$$Z(f,\chi) = \int_{\mathbb{I}_K} f(x)\chi(x)\mathrm{d}x = \prod_v \int_{K_v^\times} f_v(x)\chi_v(x)\mathrm{d}x = \prod_v \zeta(f_v,\chi_v)$$

and taking the product of equation (40) over all places and introducing $h = \prod_v h_v$ we get

$$Z(f,\chi\|\cdot\|^s) = h(s,\chi)L(s,\chi). \tag{41}$$

This product over all places is convergent when $\Re(s)$ is greater than 1, but since we have proved that $Z(f,\chi\|\cdot\|^s)$ has meromorphic continuation to the whole complex plane, equation (41) defines a meromorphic continuation of $L(s,\chi)$ to all of $\mathbb{C}$! Also, since h is entire and everywhere non-zero, the poles of $L(s,\chi)$ are the same as those of $Z(f,\chi\|\cdot\|^s)$.
Now to the functional equation. Let f be any factorizable adelic Schwartz-Bruhat function, and let χ be any idèle class character. From the local functional equation (Theorem 17) we have, for every place v of the number field K:

$$\frac{\zeta(\hat{f}_v,\chi_v^\vee)}{L(\chi_v^\vee)} = \frac{\varepsilon(\chi_v)\zeta(f_v,\chi_v)}{L(\chi_v)} \tag{42}$$

where the L-factors are the local ones from section 8.1. But we just saw that the global zeta function can be written

$$Z(f,\chi) = \prod_v \zeta(f_v, \chi_v).$$

in the region $\Re(s) > 1$, and similarly, for $\Re(s) < 0$:

$$Z(\hat{f},\chi^\vee) = \prod_v \zeta(\hat{f}_v, \chi_v^\vee).$$

We rewrite the last equation as

$$Z(\hat{f},\chi^\vee) = \prod_v L(\chi_v^\vee)\frac{\zeta(\hat{f}_v,\chi_v^\vee)}{L(\chi_v^\vee)} = L(\chi^\vee)\prod_v \frac{\zeta(\hat{f}_v,\chi_v^\vee)}{L(\chi_v^\vee)} \tag{43}$$

where the last equality sign is valid in the region of convergence for $L(\chi^\vee)$, that is $\sigma < 0$. By the local functional equation we have

$$\prod_v \frac{\zeta(\hat{f}_v,\chi_v^\vee)}{L(\chi_v^\vee)} = \prod_v \varepsilon(\chi_v)\frac{\zeta(f_v,\chi_v)}{L(\chi_v)}$$

in the whole complex plane, so by extracting the epsilon-factor and inserting into (43) we get

$$Z(\hat{f},\chi^\vee) = L(\chi^\vee)\varepsilon(\chi)\prod_v \frac{\zeta(f_v,\chi_v)}{L(\chi_v)}. \tag{44}$$

Since the Hecke L-function has meromorphic continuation to all of $\mathbb{C}$, everything in this expression can be meromorphically continued to the whole complex plane. Therefore, we can move to the region $\sigma > 1$ and replace the product by the quotient $\frac{Z(f,\chi)}{L(\chi)}$, so we arrive at

$$Z(\hat{f},\chi^\vee) = \varepsilon(\chi)\frac{L(\chi^\vee)}{L(\chi)}Z(f,\chi)$$

But in view of the global functional equation $Z(f,\chi) = Z(\hat{f},\chi^\vee)$, this implies

$$L(\chi) = \varepsilon(\chi)L(\chi^\vee)$$

and replacing χ by $\chi\|\cdot\|^s$ gives us

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\chi^\vee).$$

Thus we have our functional equation and analytic continuation! $\square$

It is instructive to compare Tate's approach to Hecke L-functions with Hecke's original proof, which is presented in chapter VII of Neukirch [22]. Hecke's proof involves long and complicated calculations, but the methods used are essentially only elementary complex analysis, together with a few simple concepts such as theta series and the Mellin transform. Therefore, if our only goal had been to prove the Main Theorem on Hecke L-functions, it would indeed have been unnecessary to develop all the adelic and $\mathfrak{p}$-adic techniques in this paper. But as we mentioned in the introduction, these techniques have far-reaching generalizations, which are among the deepest in all of mathematics. In the last section we will briefly indicate some of these developments, and give references for further study.
To conclude this section, we give two examples of how specific choices of the character χ give us L-functions that are familiar from classical number theory.

Example 6 (The Riemann zeta function). The Riemann zeta function is obtained as the finite Hecke L-function associated to the trivial idèle class character of the number field $\mathbb{Q}$. Including also the archimedean factor gives us what we called the completed zeta function in section 2.1.1. In fact, Tate's thesis gives the explanation as to why we have to add this factor in order to get a functional equation.

Example 7 (The Dedekind zeta function of a number field). If we instead of $\mathbb{Q}$ consider an arbitrary number field K, the finite Hecke L-function associated to the trivial idèle class character turns out to be the classical Dedekind zeta function from section 5.

9 The Langlands program

The mathematics in Tate's thesis opens a door to the very deep and incredibly fascinating ideas relating to the *Langlands program.* The Langlands program is a huge web of conjectures linking number theory to algebraic geometry, representation theory and complex analysis, predicting that "every L-function coming from geometry (in general: motives), also arises from some automorphic form". This program gives a framework for many seemingly unrelated number-theoretic questions, and the solved and unsolved problems within this framework are among the most profound in all of mathematics.

9.1 From quadratic reciprocity to class field theory

The reader is probably familiar with the *quadratic reciprocity law* from elementary number theory. This law gives an answer to the following question: Let a be an integer. For which primes p is the congruence $x^2 \equiv a \pmod{p}$ solvable? Although the quadratic reciprocity law was formulated already by Euler and Legendre, Gauss was the first to prove it (in 1796), and because of the importance he attached to it, he found no less than six different proofs during his lifetime. He called it *Theorema Aureum* – the Golden Theorem. Although quadratic reciprocity is easy to state and fairly easy to prove, it captures some quite profound symmetries inherent in the integers. Gauss, Eisenstein and others soon began the search for higher analogues of quadratic reciprocity – laws giving conditions for when $x^n \equiv a \pmod{p}$ is solvable. To prove such laws turned out to be much more difficult, and this was one of the reasons why modern algebraic number theory was developed. This development gradually shifted the focus to extensions of number fields and the associated Galois groups, and how prime ideals behave under such extensions. After ground-breaking work by Kronecker, Hilbert, Dedekind, Hasse and others, Artin and Takagi developed *class field theory* in the 1920s, with the *Artin reciprocity law* as one of the central theorems. This reciprocity law, encompassing all the classical reciprocity laws, has been described as "perhaps the most impressive theorem in all of number theory". Class field theory can be developed very abstractly (see chapter IV of [22]) and then applied to local as well as global fields, giving rise to so-called local and global class-field theory. Among other things, class field theory for number fields describes the finite extensions of a given number field, as long as we restrict attention to the extensions with abelian Galois groups (abelian extensions), and it also relates the characters of Galois groups of such extensions to characters of idèle groups. There are many books on class field theory, giving different points of view, see for example [4], [20], and the already mentioned [22]. For an introduction to quadratic reciprocity and other classical reciprocity laws, we refer the reader to Ireland and Rosen [13] and Lemmermeyer [18].

9.2 The Langlands conjectures

In a way, the Artin reciprocity law ends the quest for higher reciprocity laws initiated by Gauss. But the questions answered by class field theory lead to new questions, that can be viewed as even more far-reaching generalizations of reciprocity. This is the Langlands program, the ideas of which were first suggested by Langlands in the late 60s through the so-called Langlands conjectures. This program can be expressed in many ways, and on several different levels of generality. One way is through L-functions, as indicated in the introductory paragraph; another way is through representation theory. In the Langlands program, the concepts encountered in class field theory and in Tate's thesis are taken to a new level of abstraction and generality. The abelian extensions described by class field theory are replaced by arbitrary Galois extensions. For number fields, class field theory can be expressed using the adèle ring, and we have seen that it is possible to do harmonic analysis on this ring. In the Langlands philosophy, $\mathbb{A}_K$ is replaced by groups of matrices with coefficients in $\mathbb{A}_K$, and quotients of such groups – these objects are again locally compact topological spaces. The characters that are so important in Tate's thesis are replaced by group representations; since a quasi-character is nothing but a one-dimensional complex representation, this is indeed a huge generalization.

Just as for class field theory, the Langlands conjectures come in local and global versions. A local version was proved around 2000 by Harris and Taylor, see [11] for a proof. For function fields, Lafforgue gave a proof in 1999 building on earlier ideas by Drinfeld – both of them have been awarded the Fields medal for their work. For number fields, it seems to be far more difficult to find a general proof. The correspondence between elliptic curves and modular forms leading to the proof of Fermat's Last Theorem is only a special case of the general Langlands correspondence for number fields. The latter can be expressed as a correspondence between so-called algebraic automorphic representations of the group $GL_n(\mathbb{A}_K)$ and n-dimensional ℓ-adic Galois representations of $Gal(\overline{\mathbb{Q}}/K)$ that are "de Rham" at ℓ and almost everywhere unramified (here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$). For more on this, we recommend the survey paper [29] by Taylor and the book [2], edited by Bernstein and Gelbart.

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