2.9 Answers and solutions

2.9.1 Exercises

E155 False. **E156** False. **E157** True. **E158** True. **E159** False. **E160** False. **E161** True. (If n is even, then $n = 2k$ for some k, and if n is odd, then $n = 2k - 1$ for some k.) **E162** False. (It should be $n^3 \equiv 2 \pmod 5$.) **E163** True. (Since $8|16$ and $16|(a-b)$, we can conclude that $8|(a-b)$.) **E164** False. (For example, 1 is congruent to 6 mod 5, but not mod 10.) **E165** True. (Multiply both sides by (-1).) **E166** True. (Add 4 to both sides.) **E167** False. (You cannot raise one side to 2 and the other side to 3.) **E168** False. (You cannot cancel 2 from both sides, because 2 is not coprime to 4.) **E169** True. (Because 3 is coprime to 7.) **E170** 0 and 1. **E171** 0, 1 and 4. **E172** Yes, because $GCD(2,3)$ divides 7. One solution is $x = 2$, but there are many other solutions also. **E173** No, because $GCD(8,12)$ does not divide 6. **E174** No, because $GCD(9,6)$ does not divide 2. **E175** Yes, because $GCD(3,8)$ divides 5. One solution is $x = 7$, but there are many other solutions also. **E176** Yes, by the Chinese Remainder Theorem. One solution is 17. **E177** Yes, by the Chinese Remainder Theorem. One solution is 71. (The positive integers congruent to 8 mod 9 are the numbers 8, 17, 26, 35, 44 and so on. Check each number in this list until you find one that gives remainder 1 when divided by 10.) **E178** Here we cannot apply the Chinese Remainder Theorem, because 4 and 6 are not coprime. If such a number existed it would have to be both even and odd, which is of course impossible. So the answer is no.

E179 Base step: $S(1)$ is the statement $1 \equiv 8 \pmod 7$, which is true.
Induction step: We assume that

$$2^{3n} \equiv 1 \pmod 7$$

Multiply both sides by 2^3. This gives

$$2^{3(n+1)} \equiv 8 \pmod 7$$

and since $8 \equiv 1 \pmod 7$, this completes the induction step.

E180 Base step: $S(1)$ is the statement $3 \equiv 3 \pmod 6$, which is true.
Induction step: We assume that

$$3^n \equiv 3 \pmod 6$$

Multiply both sides by 3. We get

$$3^{n+1} \equiv 9 \pmod 6$$

and since $9 \equiv 3 \pmod 6$, we have proved the statement $S(n+1)$.

E181 8. **E182** 60. **E183** 4. **E184** 12. **E185** 24. **E186** 18.

E187 Euler's theorem (with $n = 77$ and $a = 2$) says that

$$2^{60} \equiv 1 \pmod{77}$$

Multiply both sides by 4.
E188 Euler's theorem (with $n = 39$ and $a = 8$) says that

$$8^{24} \equiv 1 \pmod{39}$$

Multiply both sides by 8.
E189 Euler's theorem (with $n = 19$ and $a = 5$) says that

$$5^{18} \equiv 1 \pmod{19}$$

Multiply both sides by 125. This gives

$$5^{21} \equiv 125 \pmod{19}$$

in other words, $(5^{21} - 125)$ is a multiple of 19.
E190 Fermat's little theorem says

$$a^p \equiv a \pmod{p}$$

Raising both sides to p gives

$$a^{p^2} \equiv a^p \pmod{p}$$

and combining these two congruences shows the desired result.
E191 Euler's theorem (with $n = 11$ and $a = 6$) says that

$$6^{10} \equiv 1 \pmod{11}$$

Raise both sides to 9 to get

$$6^{90} \equiv 1 \pmod{11}$$

Since $1 \equiv 12 \pmod{11}$, we also have

$$6^{90} \equiv 12 \pmod{11}$$

Since 6 is coprime to 11, we may cancel 6 on both sides. This gives

$$6^{89} \equiv 2 \pmod{11}$$

so the answer is 2.
E192 The last digit is the same thing as the remainder when divided by 10. Euler's theorem (with $n = 10$ and $a = 7$) says that

$$7^4 \equiv 1 \pmod{10}$$

Raise both sides to 100. We get

$$7^{400} \equiv 1 \pmod{10}$$

Multiply both sides by 7. We see that the answer is 7.
E193 Euler's theorem (with $n = 8$ and $a = 5$) says that

$$5^4 \equiv 1 \pmod 8$$

Raise both sides to 14. This gives

$$5^{56} \equiv 1 \pmod 8$$

Now multiply both sides by 3. We get

$$3 \cdot 5^{56} \equiv 3 \pmod 8$$

so the answer is 3.
E194 9. **E195** 0. **E196** 2. **E197** 1. **E198** 10. **E199** 2. **E200** 4. **E201** 11. **E202** 1. **E203** 1. **E204** 0. **E205** 2. **E206** 1. **E207** $x = 4$. **E208** $x = 4$. **E209** $x = 4$ and $x = 5$. **E210** $x = 0$, $x = 3$ and $x = 6$. **E211** $x = 1$ and $x = 4$. **E212** $x = 0$. **E213** No solutions. **E214** $x = 0$ and $x = 4$. **E215** $x = 3$. **E216** 6. **E217** 10. **E218** 2.