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Motives

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Preface

The American Mathematical Society, the Institute of Mathematical Statistics, and the Society for Industrial and Applied Mathematics held a joint summer research conference at the University of Washington at Seattle from July 20 to August 2, 1991 on the topic of motives. The conference was organized by Alexander Beilinson (MIT and Moscow), Pierre Deligne (IAS), Uwe Jannsen (Köln), Steven Kleiman (MIT, co-chair), Robert MacPherson (MIT), Jean-Pierre Serre (Collège de France), and Kari Vilonen (Brandeis, co-chair).

The theory of motives was introduced in the middle 1960s by Alexander Grothendieck to explain the analogies among the various cohomology theories for algebraic varieties, to play the role of the missing rational cohomology, and to provide a blueprint for proving Weil's conjectures about the zeta function of a variety over a finite field. Remarkably, over the last ten years or so, researchers in various areas—Hodge theory, algebraic K-theory, polylogarithms, automorphic forms, L-functions, ℓ-adic representations, trigonometric sums, and algebraic cycles—have discovered that an enlarged (and in part conjectural) theory of "mixed" motives indicates and explains phenomena appearing in each area. Thus the theory holds the potential of enriching each area and of unifying them all.

The Seattle conference was the first symposium ever held on motives. It presented a unique opportunity to bring together researchers and students in these diverse areas to exchange ideas and discover common themes. Everyone who applied was invited to attend, and about 140 people from all over the world registered and participated. About a third of the participants were students.

The scientific program ran eleven days. Each day, there were four one-hour lectures; the number was limited to encourage informal discussion. The first lectures introduced and surveyed the entire field; subsequent lectures elaborated on the individual areas. On the last day there was a single one-hour main lecture, followed by six half-hour subsidiary lectures. The lecturers were assigned topics, and were asked to paint panoramic views from their vantage points. A copy of the program is appended.

These volumes contain the proceedings of the conference. They include the revised texts of nearly all the lectures and a number of related works, forty-seven papers in all. There are general introductions, specialized surveys, and research papers. Each paper was refereed and is in final form.

The University of Washington provided a convenient, comfortable, and attractive site, which was conducive to the success of the conference. The AMS did a superb job of administration, freeing the organizing committee to concentrate on the scientific program. In particular, Carole Kohanski, the AMS Conference Coordinator, went far beyond the call of duty. On behalf of the entire organizing committee and all of the participants, the editors wish to express their gratitude to everyone who contributed to the success of the conference and to the production of these proceedings.

Uwe Jannsen
Steven Kleiman
Jean-Pierre Serre

Program

First Week

SUNDAY (Classical motives):

1. Historical introduction (Serre)
2. Standard conjectures (Kleiman)
3. Examples (Scholl)
4. An overview (Deligne)

MONDAY (Cohomology theories):

1. Étale cohomology (Katz)
2. Hodge theory (Steenbrink)
3. Crystalline cohomology (Illusie)
4. The Tate conjectures (Tate)

TUESDAY (Tannakian categories):

1. Tannakian categories and the motivic Galois group (Breen)
2. Motives for absolute Hodge cycles (Panchishkin)
3. CM-motives and the Taniyama group (Schappacher)
4. Motives over finite fields (Milne)

WEDNESDAY (*L*-functions):

1. Motivic Galois groups (Serre)
2. *L*-functions (Deninger)
3. The conjectures of Deligne and of Birch/Swinnerton-Dyer (Gross)
4. *K*-theoretic background (Grayson)

THURSDAY (Beilinson conjectures):

1. Beilinson conjectures I (Soulé)
2. Beilinson conjectures II (Nekovář)
3. Beilinson conjectures III: Reformulation in terms of mixed motives (Scholl)
4. Mixed motives and motivic sheaves (Jannsen)

FRIDAY (Bloch-Kato conjectures, Beilinson-Lichtenbaum complexes):

1. Bloch-Kato conjectures I (Perrin-Riou)
2. Bloch-Kato conjectures II (Fontaine)
3. Beilinson-Lichtenbaum complexes (Lichtenbaum)
4. Higher Chow groups (Bloch)

Second Week

SUNDAY (Mixed Tate motives):
1. Polylogarithms and the line minus three points (Hain)
2. Mixed Tate motives I (MacPherson)
3. Mixed Tate motives II: Zagier's conjecture (Goncharov)
4. Beilinson's work on the Zagier conjecture (Deligne)

MONDAY (Automorphic forms I):
1. The local Langlands conjecture (Kudla)
2. Pure motives and automorphic forms (Ramakrishnan)
3. Shimura varieties and motives (Milne)
4. L-functions of Shimura varieties (Rogawski)

TUESDAY (Automorphic forms II):
1. Hodge-de Rham structures and periods (M. Harris)
2. Mixed motives coming from Shimura varieties (Harder)
3. Galois representations congruent to those arising from Shimura varieties (Tilouine)
4. mod-p Galois representations and Serre's conjectures (Ribet)

WEDNESDAY (p-adic theory, function fields):
1. p-adic L-functions (Coates)
2. Iwasawa theory for motives (Greenberg)
3. p-adic motives (Schneider)
4. Function fields (Goss)

THURSDAY (Miscellaneous topics):
1. Exponential sums (Katz)
2. ℓ-adic representations associated to abelian varieties (Serre)
3. p-adic properties of absolute Hodge cycles (Wintenberger)
4. The motive of an abelian variety (Künnemann)
5. Parshin-Beilinson adèles for schemes (Huber)
6. Hodge modules, questions (M. Saito)
7. F_q-points of a variety and a Hodge-theoretic analogue (Esnault)

Cohomology

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The Standard Conjectures

STEVEN L. KLEIMAN

ABSTRACT. This paper introduces the formalism of Grothendieck's two "standard conjectures". We discuss the context in which the conjectures arose, their implications for the category of numerical equivalence (neq) motives, the way they explain the Weil conjectures, the basic theory of correspondences, eight important forms of the Lefschetz standard conjecture, and finally the Hodge standard conjecture and its implications.

1. Introduction

Grothendieck published only one article [**5**] about the two conjectures on algebraic cycles, which he called the "standard conjectures". The article is short and expository; it states a number of implications and indicates their significance, but gives no proofs. The proofs, together with further development of the theory, appeared at about the same time in the author's article [**12**], which was written at Grothendieck's request and with his aid and encouragement (but without his revealing that [**5**] was in the works). In this paper, we review the old theory and the subsequent developments.

Grothendieck [**5**, p. 193] wrote that the conjectures "arose from an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties ... and they were worked out about three years ago independently by Bombieri and myself." He concluded his article with these words: "The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called 'theory of motives' which is a systematic theory of 'arithmetic properties' of algebraic varieties as embodied in their groups of classes of cycles for numerical equivalence. ... Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry."

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Grothendieck formulated the two conjectures for smooth, projective varieties X over an algebraically closed ground field k using the ℓ-adic étale cohomology groups $H^i(X)$ where ℓ is a prime different from the characteristic of k. He fixed an isomorphism of the ℓ-group of roots of unity in k^* with $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ "for simplicity" so that each algebraic cycle of codimension i has a cohomology class in $H^{2i}(X)$. This act is unusual (within parentheses, Grothendieck called it a "heresy!");[1] however, it does serve to clarify the nature of the conjectures and their consequences. For example, as we shall see, the conjectures imply all three Weil conjectures—indeed, explain them—yet there is no need to keep track of twisting by roots of unity. Furthermore, the theory looks more geometric.

The two conjectures are easy to state. The first, the *Lefschetz standard conjecture*, asserts that an abstract analogue of the Λ-operator of Hodge theory is induced by an algebraic cycle on $X \times X$. This conjecture has other forms, which will be discussed in §4; however, Grothendieck [**5**, bottom of p. 196] wrote that it "seems to be most amenable" in this form. The second conjecture, the *Hodge standard conjecture*, asserts that there is an abstract version of the Hodge index theorem for the **Q**-vector space of classes of algebraic cycles. In characteristic zero, the second conjecture holds of course, but the first is still unknown.

Given the two standard conjectures, the category of *numerical equivalence* (neq) *motives* has marvelous properties indeed. Grothendieck published nothing on motives himself, but his ideas were explained and developed twenty years ago by Demazure [**2**], Manin [**14**], Saavedra [**15**], and the author [**13**]. See also Scholl's report [**16**, §1] in these proceedings. Scholl explains, among other things, a lovely variant, due to Jannsen [**8**, pp. 447–8], of Grothendieck's construction of the category of neq motives.

Grothendieck's construction of the category of neq motives proceeds in stages. First, the *neq correspondence category* is formed: an object is a smooth, projective variety X; and a set of maps $\operatorname{Hom}(X, Y)$ is the **Q**-vector space of neq classes of algebraic cycles of codimension r on $X \times Y$ where $r := \dim X$; the composition of a class u on $X \times Y$ with a class v on $Y \times Z$ is the class $p_{13*}(p_{12}^* u \cdot p_{23}^* v)$ where the p's are the projections. The category has direct sums and tensor products; they are induced by the disjoint union and Cartesian product of varieties. Next, the category of *neq effective motives* is formed by formally adding images of all projectors (idempotent endomorphisms) π; the objects of this category are the pairs (X, π). For example, the class of $X \times \{\text{point}\}$ is a projector, and if $X = \mathbf{P}^1$, then the corresponding motive is called the *Lefschetz motive* and denoted by $\mathbf{L}$. Finally, the whole category of neq motives is obtained by formally adding the tensor product inverse $\mathbf{T}$ of $\mathbf{L}$, called the *Tate motive.*

[1] The exclamation point is Grothendieck's.

The category of neq motives is conjectured to be the subcategory of semisimple objects in a larger category of "mixed motives".[2] Indeed, there should be a more refined local and relative theory of motivic sheaves and motivic cohomology. In this conjectured theory, there is an important role to be played by a host of natural equivalence relations that filter the groups of algebraic cycles and breach the gap between numerical equivalence and rational equivalence. Work in this direction has been carried out by Beilinson, Bloch, Jannsen, Murre, and many others; see Jannsen's comprehensive report **[9]** in these proceedings.

The standard conjectures imply that each neq motive has a semisimple endomorphism ring, or equivalently, that the category is semisimple abelian. (The basic reason why was given by Weil for curves, and developed by Serre **[19]** in higher dimensions.) Grothendieck called this fact a "miracle" (the author personally heard him do so on several occasions). However, in a remarkable piece of work, Jannsen **[8]** recently proved this semisimplicity without assuming the conjectures. In fact, he proved the converse: if an equivalence relation on cycles gives rise in similar fashion to a semisimple category of motives, then the relation is necessarily numerical equivalence. Jannsen's proof is short and elementary; it could have been found in the 1960s when motives were first considered. However, unlike the standard conjectures, Jannsen's work does not yield the semisimplicity of the Frobenius endomorphism, which yields the (E. Artin–Weil) Riemann hypothesis. Nevertheless, perhaps Jannsen's success is a sign that the standard conjectures are more tractable than we have come to believe.

The two standard conjectures have another important consequence, which is unknown even in characteristic zero: the coincidence of homological equivalence and numerical equivalence of algebraic cycles. When this coincidence occurs on $X \times Y$, then every map from the neq motive of X to that of Y induces a map from the cohomology of X to that of Y. Sometimes, the coincidence of the two relations can be proved by sandwiching homological equivalence between numerical equivalence and another equivalence relation. For example, an old theorem asserts that, if a divisor is numerically equivalent to 0, then some multiple is algebraically equivalent to 0, and a fortiori homologically equivalent to 0. At first, it was hoped that a similar converse would hold for cycles of arbitrary codimension, but this hope was dashed in 1969 when Griffiths found a counterexample. Recently, Beilinson speculated (private communication, fall 1989) that algebraic equivalence might be replaced by a broader relation such as this one, which he called "Drinfeld equivalence": a cycle Z on X is Drinfeld equivalent to 0 if there exists

[2] In a letter of March 25, 1992 to the author, Serre wrote: "It is difficult for me to recall my discussions (of 1964–65) with Grothendieck precisely, but I am almost sure that: (a) he dreamed about motives which we now called mixed; I remember for instance telling him that the corresponding ℓ-adic representations are no more semi-simple (the example being an extension of an Abelian variety by a multiplicative group); (b) he had no precise definition of them (that I knew of); this is probably why he did not mention them in print".

a proper, smooth, and connected family F/T, a cycle C on F, and two points s, t of T such that $F(s) = X$ and $C(s) = Z$ and $C(t) = 0$. For example, if Z is algebraically equivalent to 0, then we may take $F = X \times T$ for a suitable T.

The Lefschetz standard conjecture has a weaker form, which asserts that the composition of the projection and the inclusion,

$$H^*(X) \twoheadrightarrow H^i(X) \rightarrowtail H^*(X),$$

is induced by an algebraic cycle π^i on $X \times X$. Consequently, the neq motive of X decomposes into a direct sum of pieces that correspond to the individual groups $H^i(X)$.

The standard conjectures imply therefore that the canonical functor

$$h\colon ((\text{smooth, projective varieties})) \longrightarrow ((\text{neq motives}))$$

is a sort of universal cohomology theory: any contravariant functor that is formally like $X \mapsto H^*(X)$ and satisfies the standard conjectures factors through h. Hence, the category of motives may be used as an abstract substitute for singular cohomology to compare "motivated" properties of the various cohomology theories. For example, every endomorphism of a variety induces an endomorphism of each of its cohomology groups. Does the characteristic polynomial have integer coefficients that are independent of the theory? The answer is yes if the standard conjectures hold; see Corollary 5-5.

As are other universal objects, the category of neq motives is constructed by making the minimal modifications necessary to the original object, the category of smooth, projective varieties. From this point of view, each induced functor

$$\rho\colon ((\text{neq motives})) \longrightarrow ((\text{graded } \mathbf{Q}\text{-vector spaces}))$$

is called a "realization" of the category of motives. Nowadays, some prefer to start from the category of vector spaces and work toward the category of motives (see [1] for example). From this point of view, the various functors ρ are called "improvements".

Section 2 of this paper introduces the three Weil conjectures and their connection with the standard conjectures. Section 3 introduces the notion of a Weil cohomology and develops the theory of cohomological correspondences, which forms the basis of the theory of the two standard conjectures. Section 4 introduces the cohomology operator Λ and three related operators. Then it states eight forms of the Lefschetz standard conjecture and investigates the relationship among them. Section 5 states the Hodge standard conjecture. Then it shows that, in the presence of this conjecture, the Lefschetz standard conjecture is equivalent to another conjecture, that homological equivalence and numerical equivalence coincide. Finally, it shows how the two standard conjectures yield the refined form of the third Weil conjecture, the Riemann hypothesis, which asserts this: the action of the Frobenius endomorphism Φ on the cohomology group $H^i(X)$ is semisimple, its characteristic polynomial has integer coefficients, which are independent of the choice of cohomology theory, and its eigenvalues are of absolute value $q^{i/2}$. The proofs in this paper on occasion refer to **[12]** for details; however, the spirit of each proof normally remains intact.

2. The Weil conjectures

Let X be an irreducible, smooth, projective variety of dimension r defined by polynomials with coefficients in the finite field $\mathbf{F}_q$ with q elements, where q is a power of the characteristic p. For each integer $n \geq 1$, let ν_n denote the number of points of X with coordinates in the extension field $\mathbf{F}_{q^n}$; in other words,

$$\nu_n := \#X(\mathbf{F}_{q^n}).$$

Form the following "logarithmic" generating function for ν_n:

$$\log Z(t) := \sum_{n\geq 1} \nu_n t^n / n.$$

The function $Z(t)$ is called the *zeta function* of X, although another function $Z(q^{-s})$ is denoted by $\zeta(s)$.

By specifying the shape of $Z(t)$, the Weil conjectures describe the growth of ν_n. The function $Z(t)$ was introduced in 1923 by E. Artin. He did so after Hecke had revived interest in Dedekind's extension of Riemann's zeta function to number fields. Artin considered hyperelliptic function fields over $\mathbf{F}_q$ and proved that $Z(t)$ is a rational function with a functional equation. He conjectured that its zeros lie on the circle $|t| = q^{1/2}$ and called this conjecture the "Riemann hypothesis". In 1931, F. K. Schmidt reformulated the theory in the language of algebraic geometry and extended the formulation to arbitrary smooth, projective curves. Weil formulated his conjectures in arbitrary dimension around 1950. Some additional history is given below; to learn more, see Dieudonné's fascinating article **[3]**, his extensive book **[4]**, and Katz's masterful introduction **[10]**.

For example, consider the projective r-space $\mathbf{P}^r$. The standard decomposition,

$$\mathbf{P}^r = \mathbf{A}^0 \amalg \mathbf{A}^1 \amalg \cdots \amalg \mathbf{A}^r,$$

of $\mathbf{P}^r$ into the disjoint union of affine spaces yields

$$\nu_n = 1 + q^n + \cdots + q^{nr}.$$

So the definition of $\log Z(t)$ yields

$$\log Z(t) = \sum t^n/n + \sum q^n t^n/n + \cdots + \sum q^{nr} t^n/n.$$

Now, use the standard power series expansion,

$$\log(1-u) = -\sum u^n/n.$$

It yields the final expression for $Z(t)$ as a rational function,

$$Z(t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^r t)}.$$

To treat a general X, we use its Frobenius endomorphism Φ, which carries a point x of X to the point x^q of X, whose coordinates are the qth powers of those of x:

$$\Phi: X \to X, \qquad \Phi(x) := x^q.$$

Obviously, ν_n is equal to the number of points x of X left fixed by the nth iterate Φ^n, at least set-theoretically. In fact, ν_n is equal to the weighted number of fixed points; for the latter is, by definition, the intersection number on $X \times X$ of the graph of Φ^n with the diagonal, and the intersection is transverse. Hasse introduced this use of Φ in 1936, and then he and Deuring pointed out the relevance of the theory of correspondences.

Finally, let $H^i(X)$ be the ith ℓ-adic cohomology group as in §1. Then the *Lefschetz fixed-point formula* (or *trace formula*) expresses ν_n as the alternating sum of the traces of the endomorphisms of the $H^i(X)$ induced by pullback under Φ^n:

$$\nu_n = \sum_{i=0}^{2r} (-1)^i \operatorname{Tr}(\Phi^n | H^i(X)).$$

The groups $H^0(X)$ and $H^{2r}(X)$ are 1-dimensional vector spaces and, on them, Φ^n induces the identity and multiplication by q^{rn} respectively, because $\Phi^n: X \to X$ is a finite surjective map of degree rn. For $0 < i < 2r$, let w_{ij} be the eigenvalues of $\Phi | H^i(X)$. Then the eigenvalues of $\Phi^n | H^i(X)$ are the nth powers w_{ij}^n. Hence,

$$\nu_n = 1 + \sum_{i=1}^{2r-1} (-1)^i \sum_j w_{ij}^n + q^{rn}.$$

As in the example of $\mathbf{P}^r$ above, using the expansion of $\log(1-u)$, we find that $Z(t)$ is a rational function

$$Z(t) = \frac{P_1(t) \cdots P_{2r-1}(t)}{P_0(t) P_2(t) \cdots P_{2r-2}(t) P_{2r}(t)},$$

where the $P_i(t)$ are not exactly the characteristic polynomials, but

$$P_i(t) := \prod_j (1 - w_{ij} t) = \det((1 - t\Phi) | H^i(X)).$$

Moreover, $P_0 = 1 - t$ and $P_{2r}(t) = 1 - q^r t$.

The rationality of $Z(t)$ is the first of the three Weil conjectures; in fact, Weil himself explained the above way of using the Lefschetz formula.[3] However, the first proof of the rationality was given in 1960 by Dwork, who, instead, made an ingenious use of p-adic analytic functions; moreover, Dwork

[3] In a letter of March 25, 1992 to the author, Serre wrote: " ... the idea of counting points over $\mathbf{F}_q$ by a Lefschetz formula is entirely *an idea of Weil.* I remember how enthusiastic I was when he explained it to me, and a few years later I managed to convey my enthusiasm to Grothendieck (whose taste was not a priori directed towards finite fields)."

proved the rationality for an *arbitrary* variety, one that need not be smooth or projective. In 1963, M. Artin and Grothendieck developed enough of the theory of étale cohomology to justify the use of Weil's Lefschetz formalism for smooth, projective varieties. Furthermore, they proved a more general result, which Weil had conjectured as well, but which Dwork's methods did not yield: the rationality of certain L-functions, generalizing those introduced by E. Artin. In addition, they proved a base change theorem and a comparison theorem, which imply another part of what Weil conjectured: if X is the reduction mod p of a complex variety X' defined by equations with coefficients in a number field, then $P_i(t)$ has degree equal to the ith Betti number of X' (no w_{ij} vanishes because $\Phi|H^i(X)$ is nonsingular).

Two years later, in 1965, M. Artin, Grothendieck, and Verdier proved the rationality of L-functions of an even more general sort on an arbitrary variety, recovering Dwork's theorem in particular. To do so, they developed a theory of cohomology with compact supports and reduced the general statement to the case in which X is a smooth, projective curve. Finally, in that case, they proved a suitable version of the Lefschetz formula. However, there was still no way to rule out cancellation among the $P_i(t)$ above; so it was still conceivable that the coefficients of the $P_i(t)$ were not ordinary integers and depended on ℓ. Cancellation was ruled out in 1973 by Deligne, for it is ruled out by the Riemann hypothesis.

Poincaré duality yields the second Weil conjecture, which asserts that $Z(t)$ satisfies the following functional equation:

$$Z(1/q^n t) = (-1)^{\chi+\mu} q^{n\chi/2} t^{\chi} Z(t),$$

where χ is the Euler characteristic, the alternating sum of the dimensions of the $H^i(X)$, and where μ is 0 if r is odd, and μ is the multiplicity of $-q^{r/2}$ as an eigenvalue w_{rj} if r is even. (This μ is, unfortunately, missing from [**12**, 4.4, p. 385] as N. Katz kindly pointed out March 4, 1969.) Indeed, under the duality, the transpose of $\Phi|H^{2r-i}(X)$ is equal to $q^r(\Phi|H^i(X))^{-1}$. Hence, up to order, the numbers $w_{(2r-i)j}$ and $q^r w_{ij}^{-1}$ are equal. The functional equation follows via a simple computation. Poincaré duality and the functional equation were also proved in 1963 by Artin and Grothendieck.

The third and last Weil conjecture, the *Riemann hypothesis*, specifies the absolute value of the eigenvalues w_{ij}:

$$|w_{ij}| = q^{i/2}.$$

The w_{ij} are algebraic integers, and each appears along with all its conjugates, because the characteristic polynomial of $\Phi|H^i(X)$ is equal to $t^{b_i}P(t^{-1})$ where $b_i := \dim H^i(X)$, the ith Betti number of X. The conjecture was proved in two different ways in 1933 and 1934 for elliptic curves by Hasse, and in two different ways over the course of the 1940s for curves of arbitrary genus by Weil. The conjecture was finally proved in 1973 for arbitrary X by Deligne.

It is also generally conjectured that the endomorphisms $\Phi|H^i(X)$ are semisimple. This conjecture was proved by Weil for curves, abelian varieties, and a few other varieties, but it is still unknown in general. It is implied by the standard conjectures; see Theorem 5-6.

3. Correspondences

The theory of the standard conjectures is purely formal, so we shall develop it using an arbitrary *Weil cohomology theory*. This is a contravariant functor $X \mapsto H^*(X)$ from the category of *irreducible*, smooth, projective varieties X over an algebraically closed field to the category of graded anticommutative algebras over a "coefficient field" K of characteristic zero, with the following properties:

(1) (finiteness) Each $H^i(X)$ has finite dimension, and vanishes unless $0 \leq i \leq 2r$ where $r = \dim X$.

(2) (Poincaré duality) For each X of dimension r, there is a functorial "orientation" isomorphism $H^{2r}(X) \xrightarrow{\sim} K$ and, preceded by the cup product (multiplication) pairing, it yields a nondegenerate bilinear pairing,

$$H^i(X) \times H^{2r-i}(X) \to K \quad \text{by } x\,,\, y \mapsto \langle x \cdot y \rangle\,,$$

where, for any u in $H^*(X)$, the symbol $\langle u \rangle$ denotes the image under the orientation map of the projection of u in $H^{2r}(X)$. For convenience, given a Y of dimension s and a map $f\colon X \to Y$, let

$$f_*\colon H^i(X) \to H^{2s-2r+i}(Y)$$

denote the transpose of $f^*\colon H^{2r-i}(Y) \to H^{2r-i}(X)$.

(3) (Künneth formula) For each X and Y, the projections induce an isomorphism

$$H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y).$$

(4) (cycle map) For each X, let $C^i(X)$ denote the group of algebraic cycles of codimension i. Then there is a group homomorphism

$$\gamma_X\colon C^i(X) \to H^{2i}(X)\,,$$

called the "cycle map", satisfying

(i) (functoriality) for each map $f\colon X \to Y$,

$$f^*\gamma_Y = \gamma_X f^* \quad \text{and} \quad f_*\gamma_X = \gamma_Y f_*\,,$$

(ii) (multiplicitivity) $\gamma_{X\times Y}(Z \times W) = \gamma_X(Z) \otimes \gamma_Y(W)$, and

(iii) (calibration) if P is a point, then $\gamma_P\colon C^0(P) \to H^0(P)$ is equal to the canonical inclusion of the integers $\mathbf{Z}$ into the coefficient field K.

(5) (weak Lefschetz theorem) Let $h\colon W \to X$ be the inclusion of a smooth hyperplane section, and set $r := \dim X$. Then the induced map $h^*\colon H^i(X) \to H^i(W)$ is an isomorphism for $i \leq r-2$ and an injection for $i = r-1$.

(6) (strong Lefschetz theorem) Let W be a smooth hyperplane section of X, set $r := \dim X$, and define the Lefschetz operator

$$L\colon H^i(X) \to H^{i+2}(X) \quad \text{by } Lx := x \cdot \gamma_X(W).$$

Then, for $i \leq r$, the $(r-i)$th iterate of L is an isomorphism

$$L^{r-i}\colon H^i(X) \xrightarrow{\sim} H^{2r-i}(X).$$

All of these properties were proved in 1963 for étale cohomology by Artin and Grothendieck, except for the last one, the strong Lefschetz theorem. It was proved in 1973 by Deligne at the same time that he proved the Riemann hypothesis. To everyone's surprise, the Lefschetz theorem turned out to be the deeper result. Immediately afterwards, Katz and Messing **[11]** proved that, because the strong Lefschetz theorem holds for étale cohomology, it holds, when the ground field is the algebraic closure of a finite field, for any cohomology theory, like crystalline cohomology, that possesses all the other properties, except possibly (4), the existence of a cycle map.

The properties above imply that the cycle map γ_X preserves product; indeed, if $\delta\colon X \to X \times X$ is the diagonal map, then

$$\begin{aligned}\gamma_X(Z \cdot W) &= \gamma_X \delta^*(Z \times W) = \delta^*(\gamma_X(Z) \otimes \gamma_X(W)) \\ &= \gamma_X(Z) \cdot \gamma_X(W)\end{aligned}$$

for any two properly intersecting algebraic cycles Z and W on X. It is also easy to prove (see **[12**, 1.2.1, p. 363**]**) that if some nonzero multiple mZ is algebraically equivalent to 0, then $\gamma_X(Z) = 0$. Denote the **Q**-vector subspace of $H^{2i}(X)$ generated by the various $\gamma_X(Z)$ by $A^i(X)$.

The Künneth formula, Poincaré duality, and some linear algebra yield the following three canonical isomorphisms:

$$\begin{aligned}H^*(X \times Y) &= H^*(X) \otimes H^*(Y) = \operatorname{Hom}(H^*(X), K) \otimes H^*(Y) \\ &= \operatorname{Hom}(H^*(X), H^*(Y)).\end{aligned}$$

Thus an element u of $H^*(X \times Y)$ may be viewed as a linear map, or operator, from $H^*(X)$ to $H^*(Y)$. Viewed this way, u is called a *correspondence.* If u is in the **Q**-vector subspace $A^*(X \times X)$, then u is called *algebraic.*

It is easy to see that, if $u = a \otimes b$ in $H^*(X \times Y)$, then $u(x) = \langle x \cdot a \rangle b$. Hence, an arbitrary u is given by the formula

$$u(x) = p_{2*}(p_1^* x \cdot u),$$

where p_1 and p_2 are the projections. Moreover, if u is in $H^{2r+d}(X \times X)$ where $r = \dim X$, then u is equal to a homogeneous linear map of degree

d. On the other hand, if v is in $H^*(Y \times Z)$, then the composition vu of linear maps is identified with a cycle in $H^*(X \times Z)$, namely,

$$vu = p_{13*}(p_{12}^* u \cdot p_{23}^* v),$$

where again the p's are the projections. In particular, this formula shows that if u and v are defined by algebraic cycles, then so is their composition vu.

Here are three basic examples of correspondences; they and others are discussed in more detail in [**12**, pp. 365–6]. First, given a map $g: Y \to X$, let u in $H^*(X \times Y)$ be the class of its graph, and ${}^t u$ in $H^*(X \times Y)$ the "transpose" of u. Then $u = g^*$ and ${}^t u = g_*$. Second, given x in $H^*(X)$, let $u: H^*(X) \to H^*(X)$ be the map of right multiplication by x. Then $u = \delta_* x$ where $\delta: X \to X \times X$ is the diagonal map. In particular, u is equal to its own transpose ${}^t u$. Third, consider the diagonal subvariety Δ of $X \times X$. For $i = 1, \dots, 2r$ where $r = \dim X$, form the Künneth components of $\gamma_{X \times X} \Delta$:

$$\pi^i \in H^{2r-i}(X) \otimes H^i(X).$$

Then π^i is equal to the composition

$$\pi^i: H^*(X) \twoheadrightarrow H^i(X) \rightarrowtail H^*(X)$$

of the canonical projection and the canonical inclusion. In particular, π^i is a projector. Moreover, obviously, $\pi^{2r-i} = {}^t \pi^i$. So π^{2r-i} is algebraic if and only if π^i is.

A correspondence u in $H^{2r}(X \times X)$ induces an endomorphism of $H^i(X)$ for each i, and the endomorphism's trace is given by the following lovely formula:

(trace formula) $$\operatorname{Tr}(u|H^i(X)) = (-1)^i \langle u \cdot \pi^{2r-i} \rangle.$$

This formula is simple to check; see [**12**, 1.3.6(ii), p. 366], where a more general version is treated as well.

The next result is particularly important because, when combined with the first example above, it implies this: given $f: X \to X$, the induced endomorphism $f|H^i(X)$ is such that its characteristic polynomial has integer coefficients.

THEOREM 3-1. *Assume π^{2r-i} is algebraic where $r = \dim X$. Let u be a correspondence defined by an algebraic cycle on $X \times X$, and let t be a variable. Then* $\det((1 - ut)|H^i(X))$ *is a polynomial with integer coefficients, and these coefficients are given by universal polynomials in the rational numbers,*

$$s_n := \langle u^n \cdot \pi^{2r-i} \rangle,$$

for $n = 1, \dots, b_i$ where $b_i := \dim H^i(X)$.

Indeed, by hypothesis, there is an integer m such that $m\pi^{2r-i}$ is defined by an algebraic cycle. Moreover, since u is defined by an algebraic cycle, so is the composition u^n for all $n \geq 0$. Hence ms_n is an integer because γ converts intersection product into "cup" product. Now, the trace formula implies that s_n is equal to the sum of the nth powers of the eigenvalues of $u|H^i(X)$. It follows that the eigenvalues are algebraic integers; see [**12**, 2.8, p. 371]. Hence, the coefficients of the characteristic polynomial are algebraic integers too. However, they are also rational numbers because, solving the Newton identities, we can express them as universal polynomials with rational coefficients in s_n for $n = 1, \ldots, b_i$. Thus the proof is complete.

4. The Lefschetz standard conjecture

Fix a Weil cohomology theory $X \mapsto H^*(X)$. The Lefschetz standard conjecture has numerous forms. The most important form involves a natural quasi-inverse (one-sided inverse) Λ to the Lefschetz operator L. We define Λ on $H^i(X)$ for $0 \leq i \leq r$ where $r := \dim X$ by the following commutative diagram:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{r-i}} & H^{2r-i}(X) \\ \Lambda\downarrow & & L\downarrow \\ H^{i-2}(X) & \xrightarrow{L^{r-i+2}} & H^{2r-i+2}(X) \end{array}$$

in which the two horizontal maps are isomorphisms by the strong Lefschetz theorem. We define Λ on $H^{2r-i+2}(X)$ by the following similar commutative diagram:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{r-i}} & H^{2r-i}(X) \\ L\uparrow & & \Lambda\uparrow \\ H^{i-2}(X) & \xrightarrow{L^{r-i+2}} & H^{2r-i+2}(X) \end{array}$$

Clearly, Λ is surjective on $H^i(X)$ and injective on $H^{2r-i+2}(X)$.

Alternatively, we can define Λ using *primitive elements.* These are the elements of the following vector space:

$$P^i(X) := \mathrm{Ker}(L|H^i(X)).$$

Clearly $P^i(X)$ is a direct summand of $H^i(X)$, the other summand being $LH^{i-2}(X)$. Hence, each x in $H^i(X)$ has a unique decomposition of the following form, known as its *primitive decomposition*:

$$x = \sum_{j \geq \max(i-r,0)} L^j x_j \quad \text{where } x_j \in P^{i-2j}(X).$$

We can now define Λ by the following formula:

$$\Lambda x := \sum_{j \geq \max(i-r,1)} L^{j-1} x_j.$$

Similarly, we can define three additional useful operators as follows:

$$^{c}\Lambda x := \sum_{j\geq \max(i-r,\,1)} j(n-i+j+1)L^{j-1}x_j\,;$$

$$*x := \sum_{j\geq \max(i-r,\,0)} (-1)^{(i-2j)(i-2j+1)/2}L^{r-i+j}x_j\,;$$

$$p^j x := \delta_{ij}x_m \quad \text{where } m := \max(i-r,\,0) \text{ for } j = 0,\,\dots,\,2r.$$

(The formula for p^j corrects [**12**, 1.4.2.4, p. 367].) Thus p^j is the projector onto $P^j(X)$ for $j = 0, \dots, r$ and $p^j = p^{2r-j}\Lambda^{r-j}$ for $j = r, \dots, 2r$. It is easy to check that $*^2 = 1$ and $\Lambda = *L*$.

We can now state eight forms of the Lefschetz standard conjecture; they and a few more are discussed in greater detail in [**12**]. Four of the eight simply assert that the above four operators are algebraic. Three of the remaining four sound weaker, but, in fact, six of the seven are outright equivalent, and the seventh is practically equivalent. The eighth form (stated third) is, doubtless, truly weaker. The three principal forms are the following statements:

$A(X, L)$: The restriction $L^{r-2i}\colon A^i(X) \to A^{r-i}(X)$ is an isomorphism for all i.

$B(X)$: The operator Λ is algebraic.

$C(X)$: The projector π^i is algebraic for $0 \leq i \leq 2r$.

The five additional forms of the conjecture are as follows:

$^{c}B(X)$: The operator $^{c}\Lambda$ is algebraic.

$\theta(X)$: For each $i \leq r$, there exists an algebraic correspondence θ^i inducing the isomorphism $H^{2r-i}(X) \xrightarrow{\sim} H^i(X)$ inverse to L^{r-i}.

$\nu(X)$: For each $i \leq r$, there exists an algebraic correspondence ν^i inducing an isomorphism $H^{2r-i}(X) \xrightarrow{\sim} H^i(X)$.

$^{p}C(X)$: The operator p^i is algebraic for $0 \leq i \leq 2r$.

$*(X)$: The operator $*$ is algebraic.

The following result expresses the relationship among the above eight forms of the conjecture. It also justifies omitting the "L" from $B(X)$.

THEOREM 4-1.

(1) *Conjecture* $A(X \times X, L\otimes 1 + 1 \otimes L)$ *implies* $B(X)$.

(2) *Conjecture* $B(X)$ *holds for all choices of* L *if it holds for one.*

(3) *The following conjectures are equivalent*:

$$B(X),\ ^{c}B(X),\ \theta(X),\ \nu(X),\ ^{p}C(X),\ *(X).$$

(4) *Conjecture* $B(X)$ *implies* $A(X, L)$ *and* $C(X)$.

Indeed, assume $B(X)$, and set $\theta^i := \Lambda^{r-i}$. Then θ^i is algebraic, and it induces an inverse to L^{r-i}. Thus $\theta(X)$ holds.

Assume $\theta(X)$. Clearly, any algebraic correspondence carries $A^*(X)$ into itself. Hence $\theta(X)$ implies $A(X, L)$. Now, the following formula is easy to verify:

$$\pi^i = \theta^i \left(1 - \sum_{j>2r-i} \pi^j\right) L^{r-i} \left(1 - \sum_{j<i} \pi^j\right).$$

Proceeding by induction on i, we conclude that $\theta(X)$ implies $C(X)$. Therefore (4) holds. Finally, $\theta(X)$ implies $B(X)$ because of the following formula, which is easy to verify:

$$\Lambda = \sum_{i \le r} (\pi^{i-1}\theta^{i+2}L^{r-i+1}\pi^i + \pi^{2r-i}L^{r-i+1}\theta^{i+2}\pi^{2r-i+2}).$$

Trivially, $\theta(X)$ implies $\nu(X)$. Conversely, assume $\nu(X)$ and set $u := \nu^i L^{r-i}$. Then u is algebraic. So, by Theorem 3-1, its characteristic polynomial $P(t)$ has rational coefficients. By the Cayley–Hamilton theorem, $P(u) = 0$. Hence u^{-1} is a linear combination of the powers u^j for $j \ge 0$, and the combining coefficients are rational numbers. So u^{-1} is algebraic. Set $\theta^i := u^{-1}\nu^i$. Then θ is algebraic, and it is the inverse of L^{r-i} on $H^{2r-i}(X)$; in other words, $\theta(X)$ holds.

Since $\nu(X)$ does not involve L, and since $\nu(X)$ is equivalent to $\theta(X)$ and to $B(X)$, the latter two conditions hold for all possible choices of L if either holds for one choice. Thus (2) holds.

Clearly, ${}^cB(X)$ implies $\nu(X)$. Conversely, $\theta(X)$ implies ${}^pC(X)$; in fact, the p^i are given by universal (noncommutative) polynomials with integer coefficients in L and the θ^i, see [**12**, 1.4.4, p. 368]. Clearly, ${}^pC(X)$ implies ${}^cB(X)$ and $*(X)$. Since $\Lambda = *L*$, obviously $*(X)$ implies $B(X)$. Thus (3) holds.

Finally, assume $A(X \times X, L \otimes 1 + 1 \otimes L)$. Then ${}^c\Lambda \otimes 1 + 1 \otimes {}^c\Lambda$ carries $A^*(X \times X)$ into itself by [**12**, 1.4.6(ii), p. 368] and [**12**, 2.1, p. 369]. However, ${}^c\Lambda \otimes 1 + 1 \otimes {}^c\Lambda$ carries the class of the diagonal subvariety Δ into $2\,{}^c\Lambda$ by [**12**, 1.3.4, p. 365]. Thus ${}^cB(X)$ holds. So $B(X)$ holds. Thus (1) holds. The proof is now complete.

The final part of the proof is due to Jannsen (private communication, October 24, 1991). Assertion (1) was stated without proof by Grothendieck [**5**, p. 196]. In [**12**, 2.13, p. 372], only the following weaker statement was proved: $B(X)$ is implied by $A(X \times X, L \otimes 1 + 1 \otimes L)$ and $B(W)$, where W is a smooth hyperplane section of X. However, the weaker statement is enough to yield the next result.

Corollary 4-2. *Conjecture $A(X, L)$ holds for all X and L if and only if $B(X)$ holds for all X.*

The following result gives some examples of varieties X for which the conjectures are known to hold. It also gives two ways to construct new examples from old ones. Note that, if $H^1(X)$ is the étale cohomology group, then its dimension is equal to twice the dimension of the connected component of the Picard scheme $\mathrm{Pic}^0(X)$.

PROPOSITION 4-3. (1) *Conjecture* $B(X)$ *is stable under product and under hyperplane section.*

(2) *Conjecture* $B(X)$ *holds if* X *is* (a) *a curve,* (b) *a surface such that the dimension of* $H^1(X)$ *is twice that of* $\mathrm{Pic}^0(X)$, (c) *an Abelian variety, or* (d) *a generalized flag manifold* G/P.

(3) *Conjecture* $C(X)$ *holds if* X *is defined by equations with coefficients in a finite field.*

Indeed, $B(X)$ is stable under product because by **[12**, 1.4.6(ii), p. 368**]**

$$^c\Lambda_{X\times Y} = {}^c\Lambda_X \otimes 1 + 1 \otimes {}^c\Lambda_Y.$$

If W is a smooth hyperplane section of X, then by **[12**, 1.4.7(vii), p. 369**]**

$$\Lambda_W = f^*\Lambda_X^2 f_*;$$

hence, $B(X)$ implies $B(W)$. If X is a curve, then $B(X)$ is trivial. If $X = G/P$, then the algebraic cycles generate $H^*(X)$ because the class of the diagonal is of the form $\sum x_i \otimes y_i$ where x_i and y_i are the classes of algebraic cycles (this argument is in Schubert's 1879 book **[17**, §39, §41**]** for $\mathrm{GL}(4, \mathbf{C})$; the argument has been rediscovered several times since then). Since $X \times X$ is equal to $(G\times G)/(P\times P)$, therefore $B(X)$ holds trivially. If X is a surface or an abelian variety, then a few pages of argument are needed to establish $B(X)$; see **[12**, §2 Appendix, pp. 373–378**]**. For a surface, the proof is essentially due to Grothendieck; for an abelian variety, the proof grew out of discussions between the author and Lieberman. Finally, (3) was proved by Katz and Messing **[11]** using the Riemann hypothesis, which Deligne had just established.

COROLLARY 4-4. *The following three conjectures are equivalent*:

$$A(X\times X, L\otimes 1 + 1\otimes L),\ B(X),\ B(X\times X).$$

Indeed, $A(X\times X, L\otimes 1+1\otimes L)$ implies $B(X)$ by Theorem 4-1 (1). Furthermore, $B(X)$ implies $B(X\times X)$ by Proposition 4-3 (1). Finally, $B(X\times X)$ implies $A(X\times X, L\otimes 1 + 1\otimes L)$ by Theorem 4-1 (4).

5. The Hodge standard conjecture

Fix a Weil cohomology theory $X \mapsto H^*(X)$. The Hodge standard conjecture concerns the cup product pairing on the primitive algebraic cohomology classes on a smooth, projective X of dimension r:

Hdg(X) : For all $i \le r/2$, the $\mathbf{Q}$-valued pairing on $A^i(X) \cap P^{2i}(X)$,

$$x, y \mapsto (-1)^i \langle L^{r-2i} x \cdot y\rangle,$$

is positive definite.

In characteristic zero, the conjecture is true for étale cohomology; indeed, by the Lefschetz principle, we may assume that the ground field is the field of complex numbers, and then the comparison theorem and standard Hodge theory yield $\mathrm{Hdg}(X)$. In arbitrary characteristic, $\mathrm{Hdg}(X)$ holds if X is a surface. A purely algebraic proof, which works in arbitrary characteristic, was given in 1937 by B. Segre **[18]**; independently, in 1958, Grothendieck **[6]** gave a similar proof.

There is another widely believed, long-standing conjecture:

$D(X)$: If an algebraic cycle Z on X is numerically equivalent to 0, then $\gamma_X(Z) = 0$; in other words, numerical equivalence and homological equivalence coincide on X.

Of course, if $\gamma_X(Z) = 0$, then Z is numerically equivalent to 0 because γ_X converts intersection product into cup product. Hence, $D(X)$ may be put as follows: On X, homological equivalence of algebraic cycles is the same as numerical equivalence. The relationship between this conjecture and the two standard conjectures is given by the next result.

PROPOSITION 5-1. *Conjecture $D(X)$ implies $A(X, L)$, and the converse holds—in other words, the two conjectures are equivalent—in the presence of* $\mathrm{Hdg}(X)$.

Indeed, assume $A(X, L)$. Clearly, $*$ carries $A^{2r-i}(X)$ into $A^i(X)$. Assume $\mathrm{Hdg}(X)$ too. Then, therefore, the quadratic form on $A^i(X)$,

$$x, y \mapsto \langle x \cdot *y \rangle,$$

is positive definite. Consequently, the canonical pairing

$$A^i(X) \otimes A^{r-i}(X) \to \mathbf{Q} \tag{5-1}$$

is nonsingular. Hence, $D(X)$ holds.

Assume $D(X)$. Then pairing (5-1) is nonsingular. Hence $A^i(X)$ and $A^{r-i}(X)$ have the same dimension, which is finite by the following lemma because $D(X)$ holds. Since the map

$$L^{r-2i}: A^i(X) \to A^{r-i}(X)$$

is injective because of the strong Lefschetz theorem, it is therefore bijective; in other words, $A(X, L)$ holds.

LEMMA 5-2. *Let $C^i_{\mathrm{neq}}(X)$ denote the group of cycles on X modulo numerical equivalence. Then $C^i_{\mathrm{neq}}(X)$ is a free abelian group of finite rank.*

Indeed, form the K-vector subspace of $H^{2r-2i}(X)$ generated by the image of γ_X, and choose $y_1, \dots, y_m$ in the image that form a basis. Consider the map,

$$\alpha: \gamma_X C^i(X) \to \mathbf{Z}^m \quad \text{given by } \alpha(x) := (\langle x \cdot y_1 \rangle, \dots, \langle x \cdot y_m \rangle).$$

It is easy to see that the image of α is equal to $C^i_{\mathrm{neq}}(X)$. So the latter is a free group of finite rank.

COROLLARY 5-3. *Assume that the ground field is the field of complex numbers, that the cohomology theory is the de Rham theory, and that the ordinary Hodge conjecture holds for* X. *Then* $A(X, L)$ *and* $D(X)$ *hold.*

Indeed, it is an immediate consequence of standard Hodge theory that the map

$$L^{r-2p}: H^{p,p}(X) \cap H^{2p}(X, \mathbf{Q}) \to H^{r-p,r-p}(X) \cap H^{2r-2p}(X, \mathbf{Q})$$

is bijective. The ordinary Hodge conjecture asserts that the source and target are equal to $A^i(X)$ and $A^{r-i}(X)$ respectively. Thus $A(X, L)$ holds, and Proposition 5-1 implies that $D(X)$ holds.

PROPOSITION 5-4. *If* $\mathrm{Hdg}(X \times X)$ *and* $B(X)$ *hold, then the operators* Λ, ${}^c\Lambda$, $*$, *and* p^i *and* π^i *for all* i *are defined by algebraic cycles that are independent of the cohomology theory.*

Indeed, $D(X \times X)$ holds by Corollary 4-4, and the π^i are represented by algebraic cycles on $X \times X$ by Theorem 4-1, (3) and (4). These representatives may be chosen without regard to the cohomology theory because their numerical equivalence classes are intrinsically determined as elements of the ring of algebraic correspondences by the following general fact [**12**, 3.15, p. 382]: a graded, noncommutative ring $E^* = \bigoplus_{p=-r}^{r} E^p$ with 1 has at most one complete set of orthogonal idempotents $\pi^0, \dots, \pi^{2r}$ such that (a) $E^p = \bigoplus_i \pi^{i+p} E^* \pi^i$ and (b) for $i = 0, \dots, r$ there exist elements ℓ^i in E^{2r-2i} and λ^i in $E^{-(2r-2i)}$ such that $(\lambda^i \ell^i - 1)\pi^i$ and $(\ell^i \lambda^i - 1)\pi^{2r-i}$ vanish. Now, by Theorem 4-1 (3), the operator ${}^c\Lambda$ is represented by an algebraic cycle; so, see [**12**, 1.4.6, p. 368], the cycle's numerical equivalence class is uniquely determined by the formula: $[{}^c\Lambda, L] = \sum_{i=0}^{2r}(n-i)\pi^i$. Finally, the remaining operators are given by universal (noncommutative) polynomials in L and ${}^c\Lambda$; see [**12**, 1.4.3, p. 367, and 1.4.5, p. 368].

The following result gives one reason why the two standard conjectures are important.

COROLLARY 5-5. *Assume* $B(X)$ *and* $\mathrm{Hdg}(X \times X)$.

(1) *Then the Betti numbers* $\dim H^i(X)$ *are independent of the cohomology theory.*

(2) *Let* u *be a correspondence defined by an algebraic cycle on* $X \times X$. *Then its characteristic polynomial has integer coefficients, which are independent of the cohomology theory.*

Indeed, by Proposition 5-4, all the π^i are defined by algebraic cycles that are independent of the cohomology theory. Hence (1) follows from the trace formula in §3, applied to $u := 1_X$. Moreover, (2) follows from Theorem 3-1.

The final result addresses the issue of semisimplicity and the standard conjectures. Connections among Tate's conjecture, semisimplicity, and Conjecture $D(X)$ were explored recently by Deligne, by Jannsen, and by Katz and

Messing; their work appeared in informally distributed handwritten notes of July 1991, and it was incorporated in Tate's article **[20]** in these proceedings.

THEOREM 5-6. *Assume* $B(X)$ *and* $\mathrm{Hdg}(X \times X)$.

(1) *Then, under composition of correspondences,* $A^*(X \times X)$ *is a semisimple* **Q**-*algebra.*

(2) (generalized Riemann hypothesis) *Assume* X *is defined by equations with coefficients in the finite field with* q *elements, and let* ϕ *denote its Frobenius endomorphism. Then the induced endomorphism* $\Phi|H^i(X)$ *is semisimple, its characteristic polynomial has integer coefficients, which are independent of the choice of cohomology theory, and its eigenvalues are of absolute value* $q^{i/2}$.

Indeed, given any correspondence u, set $u' := *{}^{\mathrm{t}}u*$ where ${}^{\mathrm{t}}u$ is the transpose of u. Suppose u is algebraic. Then so is u', because $*(X)$ holds by Theorem 4-1 (3). Now, $C(X)$ holds by Theorem 4-1 (3); hence, the trace formula in §3 implies that $\mathrm{Tr}(u'u)$ is in $\mathbf{Q}$. Furthermore, a calculation shows that $\mathrm{Hdg}(X \times X)$ implies that $\mathrm{Tr}(u'u) > 0$ if $u \neq 0$; see **[12**, 3.11, p. 381].

To prove (1), suppose u is a nonzero element of the radical. Then $u'u$ is nilpotent, but $u'u \neq 0$ since $\mathrm{Tr}(u'u) \neq 0$. Say $(u'u)^{2^m} = 0$, but $v := (u'u)^{2^{m-1}} \neq 0$. Then $v'v = v^2 = 0$, but $\mathrm{Tr}(v'v) \neq 0$, a contradiction. Thus (1) holds.

Consider (2). By Corollary 5-4, the characteristic polynomial of $\Phi|H^i(X)$ has integer coefficients, which are intrinsic. Finally, set $\Phi_i := \Phi|H^i(X)$ and $g := \sum_i \Phi_i/q^{i/2}$. Then g is an automorphism of the algebra $H^*(X)$ and $g|H^{2r}(X) = 1$. It follows formally **[12**, 4.2, p. 384] that $g^{-1} = {}^{\mathrm{t}}g$. Clearly, g carries the class of a hyperplane section into itself. Hence, g induces an automorphism of each primitive subspace $P^i(X)$. Therefore, ${}^{\mathrm{t}}g$ commutes with $*$. Hence $g' = *{}^{\mathrm{t}}g* = {}^{\mathrm{t}}g$. By the preceding paragraph, the pairing

$$u\,,\, v \mapsto \mathrm{Tr}(u'v)$$

is an inner product on the $\mathbf{Q}(q^{1/2})$-algebra generated by g. Since $g'g = 1$, left translation by g preserves this inner product. It follows that g is semisimple and its eigenvalues have absolute value 1. (The final argument is found in Serre's paper **[19]**.) The proof is now complete.

REFERENCES

1. A. Beilinson, R. MacPherson, and V. Schechtman, *Notes on motivic cohomology*, Duke Math. J. **54** (1987), 679–710.
2. M. Demazure, *Motifs de variétés algébriques*, Sém. Bourbaki **365** (1969).
3. J. A. Dieudonné, *On the history of the Weil conjectures*, reprinted in Etale Cohomology and the Weil Conjectures, by E. Freitag and R. Kiehl, Springer-Verlag, 1988.
4. ——, *Cours de géométrie algébrique*. I, Presses Univ. France, Paris, 1974.
5. A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic Geometry — Bombay Colloquium, 1968, Oxford, 1969, pp. 193–199.

6. ——, *Sur une note de Mattuck–Tate*, J. Reine Angew. Math. **200** (1958), 208–215.
7. U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, New York and Berlin, 1990.
8. ——, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), 447–452.
9. ——, *Motivic sheaves and filtrations on Chow groups*, these Proceedings. vol. 1, pp. 245–302.
10. N. Katz, *An overview of Deligne's proof of the Riemann hypothesis for varieties over finite fields* (*Hilbert's problem* 8), Math. Developments Arising from Hilbert problems, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, RI, 1976, pp. 275–305.
11. N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.
12. S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposeś sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 359–386.
13. ——, *Motives*, Algebraic geometry—Oslo, 1970, Sijthoff & Noordhoff, Groningen, 1970, pp. 53–82.
14. Y. I. Manin, *Correspondences, motives and monoidal transformations*, Math. USSR Sb. **6** (1968), 439–470.
15. N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, New York and Berlin, 1972.
16. A. J. Scholl, *Classical motives*, these Proceedings, vol. 1, pp. 163–187.
17. H. Schubert, *Kalkül der abzählenden Geometrie*, Teubner, Leipzig, 1879; reprint, Springer-Verlag, 1979 .
18. B. Segre, *Intorno ad teorema di Hodge sulla teoria della base per le curve di una superficie algebrica*, Ann. Mat. **16** (1937), 157–163.
19. J.-P. Serre, *Analogues kählériens de certaines conjectures de Weil*, Ann. of Math. (2) **71** (1960), 392–394; reprinted in Collected Papers, vol. 2, paper 45, Springer-Verlag, New York and Berlin, 1986.
20. J. Tate, *Conjectures on algebraic cycles in ℓ-adic cohomology*, these Proceedings, vol. 1, pp. 71–83.

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Review of ℓ-adic Cohomology

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In this lecture, we will review some of the basic facts about étale cohomology and recall some of the open questions.

Basic definitions

All rings are understood to be commutative, associative, and unitary. Given a ring A, an A-algebra B is said to be *elementary étale* over A if it has a finite presentation

$$B = A[x_1, \dots, x_N]/(f_1, \dots, f_N)$$

such that the Jacobian determinant $\det(\partial f_i/\partial x_j)$ is invertible in B.

Let X and Y be arbitrary schemes. A morphism $f: Y \to X$ is said to be étale if there exists an affine open covering of X, say $X = \bigcup_i \operatorname{Spec}(A_i)$, and for each i an affine open covering of $f^{-1}(\operatorname{Spec}(A_i))$, say $f^{-1}(\operatorname{Spec}(A_i)) = \bigcup_j \operatorname{Spec}(B_{ij})$, such that, for all i, j, B_{ij} is elementary étale over A_i. One knows that an étale morphism is open.

Given an arbitrary scheme X, the (small) étale site of X is the category $(\mathrm{Sch\,Et}/X)$ of all schemes étale over X, endowed with the following notion of covering: a family of morphisms of étale X-schemes with common target $\varphi_i: U_i \to V$ is said to be a covering of V if $V = \bigcup_i \varphi_i(U_i)$. [One knows that each φ_i is automatically étale, being an X-morphism of étale X-schemes. Hence each $\varphi_i(U_i)$ is open in V.]

An "étale sheaf" $\mathscr{F}$ of sets on X is a contravariant functor $\mathscr{F}$ from $(\mathrm{Sch\,Et}/X)$ to (Sets), such that, for any covering $\{\varphi_i: U_i \to V\}_i$ as above, the following diagram of sets is exact:

$$\mathscr{F}(V) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{ij} \mathscr{F}(U_i \times_V U_j).$$

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For example, for any X-scheme Y, the tautological functor $\mathrm{Hom}_{\mathrm{Sch}/X}(-, Y)$ is known ("étale descent") to be an étale sheaf of sets on X.

The abelian étale sheaves on X form an abelian category with enough injectives. The right derived functors of the left exact "global sections" functor $\mathscr{F} \mapsto \mathscr{F}(X)$ are by definition the étale cohomology groups $H^i(X, \mathscr{F})$. The groups are functorial in the pair $(X, \mathscr{F})$. In particular for any integer $N \geq 1$, we can form the constant sheaf $\mathbb{Z}/N\mathbb{Z}$ and speak of the cohomology groups $H^i(X, \mathbb{Z}/N\mathbb{Z})$. But in this generality we cannot say much about them.

Schemes of finite type over a ground field

Suppose henceforth that k is a field and that X is separated and of finite type over k. Although it makes sense to look at $H^i(X, \mathbb{Z}/N\mathbb{Z})$, for our present purposes these are not the "good" objects. Rather we choose an algebraic closure $\overline{k}$ of k, form the $\overline{k}$-scheme $X \otimes_k \overline{k}$, and consider its étale cohomology groups $H^i(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$. These are abelian groups killed by N, endowed with an action of $\mathrm{Gal}(\overline{k}/k)$.

In fact, we need to go one step further and define "cohomology with compact support". According to a result of Nagata **[Na]**, X admits an open k-immersion, $j: X \to \overline{X}$, into a scheme $\overline{X}$ which is proper over k. For any étale sheaf $\mathscr{F}$ on X, one can define its extension by zero $j_!\mathscr{F}$ on $\overline{X}$, pull it back to $\overline{X} \otimes_k \overline{k}$, and consider the cohomology groups $H^i(\overline{X} \otimes_k \overline{k}, j_!\mathscr{F})$. It results from the proper base change theorem that these groups are independent of the auxiliary choice of the compactification $j: X \to \overline{X}$: they are by definition the cohomology groups with compact support, denoted $H^i_c(X \otimes_k \overline{k}, \mathscr{F})$. They are abelian groups endowed with an action of $\mathrm{Gal}(\overline{k}/k)$. If U is open in X, with closed complement Z, we have an excision long exact sequence

$$\to H^i_c(U \otimes_k \overline{k}, \mathscr{F}|U) \to H^i_c(X \otimes_k \overline{k}, \mathscr{F}) \to H^i_c(Z \otimes_k \overline{k}, \mathscr{F}|Z) \to .$$

Taking $\mathscr{F}$ to be $\mathbb{Z}/N\mathbb{Z}$, we obtain the groups $H^i_c(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$. According to the proper mapping theorem and the proper base change theorem, we have

THEOREM **[SGA 4,** XIV, 1.2 and XII, 5.4 and 5.3]. *For X separated and of finite type over a field k, the cohomology groups with compact support $H^i_c(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$ are all finite and invariant under extension of algebraically closed field from $\overline{k}$ to any algebraically closed overfield. Moreover, they vanish for $i > 2\dim(X)$.*

The finiteness and the invariance, but not the vanishing, fail in general for the ordinary groups $H^i(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$: for example, they fail for $H^1(\mathbb{A}^1 \otimes_k \overline{k}, \mathbb{Z}/p\mathbb{Z})$ if k has finite characteristic p. But this is in some sense a "pathology", for we have

THEOREM **[SGA 4 1/2,** Théorèmes Fin., 1.9(ii) and 1.10; **SGA 4,** XIV, 3.2 and X, 4.2]. *For X separated and of finite type over a field k, and N an* *integer invertible in* k, *the ordinary cohomology groups* $H^i(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$ *are all finite, and invariant under extension from* $\overline{k}$ *to any algebraically closed overfield. Moreover, they vanish for* $i > 2\dim(X)$. *If* X *is affine, they vanish for* $i > \dim(X)$.

The intuition is that if N is invertible in k, then the cohomology groups $H^i(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$ and $H^i_c(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})$ behave "just like" ordinary and compactly supported topological cohomology groups of usual locally compact spaces. For example, we have the following comparison theorem:

THEOREM **[SGA 4,** XVI, 4.1]. *Suppose that* X *is a separated* $\mathbb{C}$*-scheme of finite type, and denote by* X^{an} *the complex space* $X(\mathbb{C})$ *in its classical topology. There are canonical isomorphisms*

$$H^i(X, \mathbb{Z}/N\mathbb{Z}) \approx H^i_{\text{top}}(X^{\text{an}}, \mathbb{Z}/N\mathbb{Z}),$$
$$H^i_c(X, \mathbb{Z}/N\mathbb{Z}) \approx H^i_{c\,\text{top}}(X^{\text{an}}, \mathbb{Z}/N\mathbb{Z}).$$

Also in keeping with this intuition, we have Poincaré duality. To state it, we first recall the notion of a Tate twist. For any field k in which N is invertible, we denote by $\mathbb{Z}/N\mathbb{Z}(1)$ the $\mathbb{Z}/N\mathbb{Z}[\text{Gal}(\overline{k}/k)]$-module $\mu_N(\overline{k})$, the group of Nth roots of unity in $\overline{k}$, with the natural action of $\text{Gal}(\overline{k}/k)$. Thus $\mathbb{Z}/N\mathbb{Z}(1)$ is an invertible $\mathbb{Z}/N\mathbb{Z}$-module on which $\text{Gal}(\overline{k}/k)$ acts by the mod N cyclotomic character $\chi\colon \text{Gal}(\overline{k}/k) \to (\mathbb{Z}/N\mathbb{Z})^\times$, defined by $\sigma(\zeta) = \zeta^{\chi(\sigma)}$, for σ in $\text{Gal}(\overline{k}/k)$ and ζ in $\mu_N(\overline{k})$. For any integer $i \geq 0$, we define the $\mathbb{Z}/N\mathbb{Z}[\text{Gal}(\overline{k}/k)]$-module $\mathbb{Z}/N\mathbb{Z}(i)$ to be the i-fold tensor power $(\mu_N(\overline{k}))^{\otimes i}$ (tensor product over $\mathbb{Z}/N\mathbb{Z}$). For $i \leq 0$, we define $\mathbb{Z}/N\mathbb{Z}(i)$ to be $\text{Hom}_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}(-i), \mathbb{Z}/N\mathbb{Z})$. Thus for any integer i, $\mathbb{Z}/N\mathbb{Z}(i)$ is an invertible $\mathbb{Z}/N\mathbb{Z}$-module on which Gal acts by the ith power of the mod N cyclotomic character. For any $\mathbb{Z}/N\mathbb{Z}[\text{Gal}(\overline{k}/k)]$-module V, and any i, its Tate twist $V(i)$ is defined to be the $\mathbb{Z}/N\mathbb{Z}[\text{Gal}(\overline{k}/k)]$-module $V \otimes \mathbb{Z}/N\mathbb{Z}(i)$ (tensor product over $\mathbb{Z}/N\mathbb{Z}$).

THEOREM **[SGA 4,** XVIII, 2.9, 2.14, 3.2.6.2]. *Suppose that* N *is invertible in the field* k. *Let* X/k *be a separated* k*-scheme of finite type.*

(1) *If* X *is a geometrically irreducible* k*-scheme of dimension* d, *then* $H^{2d}_c(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(d) \approx \mathbb{Z}/N\mathbb{Z}$.

(2) *If in addition* X/k *is smooth, then for any integers* a *and* b *with* $a + b = d$, *and any* i, *the cup-product pairing*

$$H^i_c(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(a) \times H^{2d-i}(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(b) \to H^{2d}_c(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(d) \approx \mathbb{Z}/N\mathbb{Z}$$

is a Gal*-equivariant pairing which identifies each of* H^i_c *and* H^{2d-i} *with the* $\mathbb{Z}/N\mathbb{Z}$ *dual of the other.*

As an application of Poincaré duality, one can define, following Tate **[Ta,** §2], the cohomology class of a subvariety. Thus let X/k be smooth and geometrically connected, say of dimension d, and let $Z \subset X$ be a geometrically irreducible subscheme of codimension i. Because the inclusion of Z in X is proper, we have a restriction map on cohomology with compact supports

$$H_c^{2d-2i}(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z}) \to H_c^{2d-2i}(Z \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z}).$$

Twisting by $(d-i)$ we obtain a map

$$H_c^{2d-2i}(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(d-i) \to H_c^{2d-2i}(Z \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(d-i) \approx \mathbb{Z}/N\mathbb{Z}.$$

By Poincaré duality, this map is the cup-product with a unique element of $H^{2i}(X \otimes_k \overline{k}, \mathbb{Z}/N\mathbb{Z})(i)$, which is by definition the cohomology class of Z.

We next pass from $\mathbb{Z}/N\mathbb{Z}$-cohomology to ℓ-adic cohomology. We continue to work with a scheme separated and of finite type over a field k. We choose a prime number ℓ which is invertible in k. We define the compact and ordinary $\mathbb{Z}_\ell$-cohomology groups as inverse limits:

$$H_c^i(X \otimes_k \overline{k}, \mathbb{Z}_\ell) := \lim \mathrm{inv}_n H_c^i(X \otimes_k \overline{k}, \mathbb{Z}/l^n\mathbb{Z}),$$
$$H^i(X \otimes_k \overline{k}, \mathbb{Z}_\ell) := \lim \mathrm{inv}_n H^i(X \otimes_k \overline{k}, \mathbb{Z}/l^n\mathbb{Z}).$$

These are finitely generated $\mathbb{Z}_\ell$-modules with a continuous action of $\mathrm{Gal}(\overline{k}/k)$, which vanish for $i > 2\dim(X)$ and which satisfy the expected universal coefficient long exact sequences.

We next pass from $\mathbb{Z}_\ell$-cohomology to $\mathbb{Q}_\ell$-cohomology, by defining

$$H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) := H_c^i(X \otimes_k \overline{k}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$
$$H^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) := H^i(X \otimes_k \overline{k}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

These are finite-dimensional $\mathbb{Q}_\ell$-spaces with a continuous action of $\mathrm{Gal}(\overline{k}/k)$, which vanish for $i > 2\dim(X)$. They also satisfy the Künneth formula (cf. [**SGA 4,** XVII, 5.4.3; **SGA 4 1/2,** Théorème Fin., 1.11]): given X/k and Y/k, both separated of finite type, we have isomorphisms of graded $\mathbb{Q}_\ell$-algebras

$$H_c^*((X \times_k Y) \otimes_k \overline{k}, \mathbb{Q}_\ell) \approx H_c^*(X \otimes_k \overline{k}, \mathbb{Q}_\ell) \otimes H_c^*(Y \otimes_k \overline{k}, \mathbb{Q}_\ell),$$
$$H^*((X \times_k Y) \otimes_k \overline{k}, \mathbb{Q}_\ell) \approx H^*(X \otimes_k \overline{k}, \mathbb{Q}_\ell) \otimes H^*(Y \otimes_k \overline{k}, \mathbb{Q}_\ell).$$

If X/k is smooth and geometrically connected, Poincaré duality holds: replacing $\mathbb{Z}/N\mathbb{Z}$ by $\mathbb{Q}_\ell$ in the statement of $\mathbb{Z}/N\mathbb{Z}$-Poincaré duality above gives the correct statement.

The case of a finite ground field

We now suppose in addition the field k is finite. We denote by p the characteristic of k, by q the cardinality of k, and by ℓ a prime number different from p. We denote by F in $\mathrm{Gal}(\overline{k}/k)$ the "geometric Frobenius", defined as the inverse of the automorphism $x \mapsto x^q$ of $\overline{k}$. For every integer

$n \geq 1$, we denote by k_n the unique extension of k of degree n (inside the fixed $\overline{k}$) and by N_n the (finite!) number of points of X with values in k_n. The most fundamental diophantine invariant of X/k is its zeta function, the formal series in $1 + T\mathbb{Q}[[T]]$ defined by

$$Z(X/k, T) := \exp\left(\sum_{n\geq 1} N_n T^n/n\right).$$

In fact, this zeta function lies in $1 + T\mathbb{Z}[[T]]$, because it is also given by a formal Euler product over all the closed points x in X. Each closed point x in X has residue field a finite extension of k, whose degree over k is called the degree of the closed point x. There are only finitely many closed points of each degree in X. If we denote by B_r the number of closed points of degree r in X, we have the identity

$$\begin{aligned} Z(X/k, T) :=& \exp\left(\sum_{n\geq 1} N_n T^n/n\right) \\ =& \prod_{\text{cl. pt.'s } x \text{ in } X} [1 - T^{\deg(x)}]^{-1} = \prod_{r\geq 1}[1 - t^r]^{-B_r}. \end{aligned}$$

[This amounts to the elementary identity $N_n = \sum_{r|n} rB_r$.]

Interlude: the geometric meaning of geometric Frobenius

Before going further, let us explain "why" it is the geometric Frobenius element F of $\mathrm{Gal}(\overline{k}/k)$ whose characteristic polynomial occurs below. For *any* $\mathbb{F}_p$-scheme Y and any power q of p, denote by $\mathrm{Fr}_{q,Y}$ the "absolute qth power endomorphism" of Y, i.e., the map that is the identity on the underlying topological space, and $f \mapsto f^q$ on $\mathcal{O}_Y$. For any *étale* Y-scheme $\varphi: Z \to Y$, the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\mathrm{Fr}_{q,Z}} & Z \\ \varphi\downarrow & & \downarrow\varphi \\ Y & \xrightarrow[\mathrm{Fr}_{q,Y}]{} & Y \end{array}$$

is *Cartesian*. This implies that for any étale sheaf $\mathcal{F}$ on Y, both the direct image sheaf $(\mathrm{Fr}_{q,Y})_*\mathcal{F}$ and the inverse image sheaf $(\mathrm{Fr}_{q,Y})^*\mathcal{F}$ are canonically equal to $\mathcal{F}$ and $\mathrm{Fr}_{q,Y}$ acts as the identity on $\mathcal{F}(Y) = H^0(Y, \mathcal{F})$, hence as the identity on $H^i(Y, \mathcal{F})$ for all i (cf. **[SGA 5,** XV, §2, Propositions 1 and 2]).

Now suppose that k is a finite field with q elements, that X/k is separated of finite type, and that $j: X \to \overline{X}$ is a compactification. Applying the above on $Y = \overline{X} \otimes_k \overline{k}$ to all the sheaves $\mathcal{F}_n = j_!(\mathbb{Z}/\ell^n\mathbb{Z})$, we find that $\mathrm{Fr}_{q,\overline{X}\otimes_k\overline{k}}$ acts as the identity on $H_c^*(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$. Denote by φ in $\mathrm{Gal}(\overline{k}/k)$

the "arithmetic Frobenius" automorphism $\alpha \mapsto \alpha^q$ of $\overline{k}$, so that $\varphi^{-1} = F$. Denote by $\mathrm{Fr}_{\overline{X}/k}$ the "relative Frobenius" $\overline{k}$-endomorphism of $\overline{X} \otimes_k \overline{k}$ defined as $(\mathrm{Fr}_{q,\overline{X}}) \otimes (\mathrm{id}_{\overline{k}})$. Clearly we have

$$\mathrm{Fr}_{q,\overline{X}\otimes_k\overline{k}} = (\mathrm{Fr}_{\overline{X}/k}) \circ (\mathrm{id} \otimes \varphi) = (\mathrm{id} \otimes \varphi) \circ (\mathrm{Fr}_{\overline{X}/k}).$$

But $\mathrm{Fr}_{q,\overline{X}\otimes_k\overline{k}}$ acts as the identity on $H_c^*(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$, so the relative Frobenius morphism $(\mathrm{Fr}_{X/k})^*$ $(:= (\mathrm{Fr}_{\overline{X}/k})^*)$ and the geometric Frobenius element F^* $(:= (\mathrm{id} \otimes F)^*)$ agree on $H_c^*(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ (cf. **[SGA 5,** XV, §2, Corollary of Proposition 3, and discussion following it]).

The idea that the action of $(\mathrm{Fr}_{X/k})^*$ on $H_c^*(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ should be related to the zeta function comes from the observation, already emphasized by Weil (cf. **[We,** pp. 555–556]) that for every $n \geq 1$, the k_n-valued points $X(k_n)$ are precisely the *fixed points* of the nth iterate of the relative Frobenius $\mathrm{Fr}_{X/k}$ on $X \otimes_k \overline{k}$.

What we know over finite fields

Dwork **[Dw]** proved (using p-adic methods) that $Z(X/k, T)$ is a rational function of T, i.e., lies in $\mathbb{Q}(T)$. The more precise relation to ℓ-adic cohomology came a few years later:

THEOREM **[Gr-FL; SGA 4 1/2,** Rapport, 3.1]. *Let k be a finite field and X/k separated of finite type. For any prime $\ell \neq p$, $Z(X/k, T)$ is the alternating product of the characteristic polynomials of geometric Frobenius on the ℓ-adic cohomology with compact support:*

$$Z(X/k, T) = \prod_{i=0,\dots,2\dim(X)} \det(1 - TF|H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell))^{(-1)^{i+1}},$$

equality in $\mathbb{Q}_\ell[[T]]$.

[This formula for zeta is equivalent (take the logarithms of both sides and equate coefficients) to the Lefschetz Trace Formula with constant coefficients $\mathbb{Q}_\ell$: For every $n \geq 1$,

$$\mathrm{Card}(X(k_n)) = \sum_{i=0,\dots,2\dim(X)} (-1)^i \,\mathrm{Trace}(F^n|H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell)),$$

or, more geometrically,

$$\begin{aligned}&\mathrm{Card}\{\text{Fixed points of } (\mathrm{Fr}_{X/k})^n \text{ on } X \otimes_k \overline{k}\}\\ &= \sum_{i=0,\dots,2\dim(X)} (-1)^i \,\mathrm{Trace}((\mathrm{Fr}_{X/k})^n|H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell)).]\end{aligned}$$

A decade later came Deligne's fundamental result:

THEOREM **[De,** 3.3.4]. *Let k be a finite field and X/k separated of finite type. Let ℓ be a prime $\neq p$ and $i \geq 0$ an integer. Then*

(1) *every eigenvalue* $\alpha_{i,j,\ell}$ *of* F *on* $H^i_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$ *is an algebraic integer* (*inside* $\overline{\mathbb{Q}}_\ell$) (*cf.* **[SGA 7, XXI,** 5.2.2]).

(2) *For each such eigenvalue* $\alpha_{i,j,\ell}$, *there exists an integer* $w(\alpha_{i,j,\ell})$ *with*

$$0 \le w(\alpha_{i,j,\ell}) \le i,$$

called the "weight" of $\alpha_{i,j,\ell}$, *such that for every embedding* ι *of* $\overline{\mathbb{Q}}_\ell$ *into* $\mathbb{C}$, *the complex absolute value of* $\iota(\alpha_{i,j,\ell})$ *is given by*

$$|\iota(\alpha_{i,j,\ell})| = (q^{1/2})^{w(\alpha_{i,j,\ell})}.$$

(3) *If* X/k *is proper and smooth, then* $w(\alpha_{i,j,\ell}) = i$ *for every eigenvalue* $\alpha_{i,j,\ell}$, *i.e.,* H^i_c *is "pure of weight* i*". If in addition* X *is purely of dimension* d, *the eigenvalues of* F *on* $H^{2d-i}_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$ *are related to those on* $H^i_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$ *by*

$$\{\alpha_{2d-i,j,\ell}\}_j = \{q^d/\alpha_{i,j,\ell}\}_j.$$

COROLLARY. *Let* X/k *be proper and smooth,* $i \ge 0$ *an integer. For each* $\ell \ne p$, *the characteristic polynomial*

$$P_{i,\ell}(T) := \det(1 - TF|H^i_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell))$$

has $\mathbb{Z}$*-coefficients independent of the auxiliary choice of* $\ell \ne p$.

Indeed, thanks to (3) above, for given $\ell \ne p$ there is *no cancellation* in the expression of zeta as the alternating product of the $P_{i,\ell}$'s. So we recover $P_{i,\ell}(T)$ intrinsically from the zeta function, in terms of those reciprocal zeros or poles of zeta which, together with all their algebraic conjugates, have absolute value $q^{i/2}$.

COROLLARY. *Let* X/k *be proper and smooth,* $i \ge 0$ *an integer. For each* $\ell \ne p$, *the* i*'th* ℓ*-adic Betti number*

$$B_{i,\ell} := \dim_{\mathbb{Q}_\ell} H^i_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$$

is independent of the auxiliary choice of $\ell \ne p$.

Indeed, since F is an automorphism of H^i_c (because the action of F on H^i_c is part of a $\mathrm{Gal}(\overline{k}/k)$ action), we have $B_{i,\ell} = \deg(P_{i,\ell})$.

What we do not know over finite fields

(1) ("semisimplicity of Frobenius") For X/k proper and smooth, it is conjectured, but not known in general, that the action of F on $H^i_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$ is diagonalizable over $\overline{\mathbb{Q}}_\ell$. It is known for abelian varieties.

(2) ("independence of ℓ of characteristic polynomials") It is conjectured that for X/k separated of finite type, for each i the characteristic polynomials

$$P_{i,\ell}(T) := \det(1 - TF|H^i_c(X\otimes_k\overline{k}, \mathbb{Q}_\ell))$$

have $\mathbb{Q}$-coefficients (or equivalently $\mathbb{Z}$-coefficients, since they are algebraic integers), independent of the auxiliary choice of $\ell \neq p$.

(2a) ("independence of ℓ of Betti numbers") If (2) holds, then by taking degrees, we would get that the ℓ-adic Betti numbers

$$B_{i,\ell} := \dim_{\mathbb{Q}_\ell} H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) = \deg(P_{i,\ell})$$

are independent of the auxiliary choice of $\ell \neq p$. But even this is not known. All we know is that the ℓ-adic Euler-Poincaré characteristic $\chi_\ell := \sum_i (-1)^i B_{i,\ell}$ is independent of $\ell \neq p$, because $-\chi_\ell$ is the degree of the zeta function as rational function.

(2b) ("independence of ℓ of weight j Betti numbers") Again if (2) holds, for each integer $i \geq 0$ and each integer $j \geq 0$ the "weight j Betti numbers"

$$B_{i,j,\ell} := \text{number of weight } j \text{ eigenvalues of } F \text{ on } H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$$

would be independent of $\ell \neq p$. Again, all we know is that the "weight j Euler-Poincaré characteristic" $\chi_{j,\ell} := \sum_i (-1)^i B_{i,j,\ell}$ (which Serre called the "jth virtual Betti number", cf. [**Gr-RS,** Deuxième Partie, pp. 185 and 191]) is independent of $\ell \neq p$, because $-\chi_{j,\ell}$ is the degree of the "weight j part" of the zeta function.

(3) ("independence of ℓ for kernels and cokernels of maps") Suppose that X/k and Y/k are both proper and smooth and that we are given a k-morphism $f: Y \to X$. For each integer $i \geq 0$ and each $\ell \neq p$, we have an induced map (because f is proper) which is $\mathrm{Gal}(\overline{k}/k)$-equivariant

$$(f^*)_{i,\ell}: H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) \to H_c^i(Y \otimes_k \overline{k}, \mathbb{Q}_\ell).$$

It is conjectured that the characteristic polynomials of F on both the kernel and cokernel of $(f^*)_{i,\ell}$ have $\mathbb{Z}$-coefficients, independent of $\ell \neq p$.

Suppose that numerical equivalence coincides with $\mathbb{Q}_\ell$-cohomological equivalence for every $\ell \neq p$. Then motives for numerical equivalence have ℓ-adic realizations for every $\ell \neq p$. According to Jannsen [**Ja**], the category of motives for numerical equivalence is abelian (and semisimple, though we will not use this fact below). For a given variety proper and smooth over a finite field, the Künneth components of the diagonal are rationally algebraic, represented by universal (i.e., independent of $\ell \neq p$) $\mathbb{Q}$-linear combinations of the graphs of iterates of Frobenius [**K-M,** Theorem 2, 1)]. So the individual cohomology groups $H_c^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ and $H_c^i(Y \otimes_k \overline{k}, \mathbb{Q}_\ell)$ are the ℓ-adic realizations of motives. By Jannsen [**Ja**], the corresponding motivic kernels and cokernels of f^* exist. Our ℓ-adic kernels and cokernels are the ℓ-adic realizations of these motivic kernels and cokernels. Then (2) follows, because by [**K-M,** Theorem 2, 2)] "a motive has an L-function".

We should also recall [**Ka,** end of 3.2] that conjecture (3) for the case when f is a closed immersion is (unconditionally) equivalent to conjecture (2) for the open variety $X - Y$. More generally, consider a proper smooth X and a union of smooth divisors D_i in X with normal crossings. We have an excision spectral sequence

$$E_1^{a,b} = \bigoplus_{a\text{-fold intersections of the } D_i} H_c^b\left(\left(\bigcap D_i\right) \otimes_k \overline{k}, \mathbb{Q}_\ell\right)$$
$$\Rightarrow H_c^{a+b}\left(\left(X - \bigcup_i D_i\right) \otimes_k \overline{k}, \mathbb{Q}_\ell\right)$$

in which d_1 is induced by the obvious restriction maps. This spectral sequence degenerates at E_2 (by weights: $E_1^{a,b}$, and hence $E_r^{a,b}$ for any $r \geq 1$, is pure of weight b, but d_r has bidegree $(r, 1-r)$; hence d_r must vanish for $r \geq 2$). Moreover, $E_2^{a,b}$ $(= E_\infty^{a,b})$ is the weight b subquotient of $H_c^{a+b}((X - \bigcup_i D_i) \otimes_k \overline{k}, \mathbb{Q}_\ell)$. Thus conjecture (2) for the open smooth variety $X - \bigcup_i D_i$ holds if and only if each $E_2^{a,b}$ has characteristic polynomial with $\mathbb{Z}$-coefficients independent of $\ell \neq p$. If numerical equivalence coincides with $\mathbb{Q}_\ell$-cohomological equivalence for every $\ell \neq p$, then by Jannsen the E_2 terms are the ℓ-adic realizations of motives and hence have characteristic polynomials independent of $\ell \neq p$.

(4) ("independence of ℓ for Leray spectral sequences") Suppose given X/k and S/k, separated schemes of finite type, and $f: X \to S$ a k-morphism. For any $\ell \neq p$, we have the Leray spectral sequence

$$E_2^{a,b} = H^a(S \otimes_k \overline{k}, R^b f_* \mathbb{Q}_\ell) \Rightarrow H^{a+b}(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$$

and its "compact support" analogue

$$E_2^{a,b} = H_c^a(S \otimes_k \overline{k}, R^b f_! \mathbb{Q}_\ell) \Rightarrow H_c^{a+b}(X \otimes_k \overline{k}, \mathbb{Q}_\ell).$$

Each is a spectral sequence of finite-dimensional $\overline{\mathbb{Q}}_\ell$-spaces with continuous actions of $\mathrm{Gal}(\overline{k}/k)$. For each, it is hoped that for each $(a, b, r \geq 2)$ the characteristic polynomial of F on $E_r^{a,b}$ has $\mathbb{Z}$-coefficients, independent of $\ell \neq p$.

References

[De] P. Deligne, *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 313–428.

[Dw] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648.

[Gr-FL] A. Grothendieck, *Formule de Lefschetz et rationalité des fonctions L*, Sém. Bourbaki 1964–65, Exposé 279; reprinted in Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968.

[Gr-RS] ———, *Récoltes et semailles: Réflexions et témoignage sur un passé de mathématicien*, Université des Sciences et Techniques du Languedoc (Montpellier) and CNRS, 1985.

[Ja] U. Jannsen, *Motives, numerical equivalence, and semisimplicity*, Invent. Math. **107** (1992), 447–452.

[K-M] N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

[Ka] N. Katz, *Sommes exponentielles*, Astérisque no. 79, Soc. Math. France, Paris, 1980.

[Na] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2-1** (1962), 1–10.

[SGA] A. Grothendieck et al., *Séminaire de géométrie algébrique du Bois-Marie*, SGA 1, SGA 4 Parts I, II, and III, SGA 4 1/2, SGA 5, SGA 7 Parts I and II, Lecture Notes in Math., vols. 224, 269–270–305, 569, 589, 288–340, Springer–Verlag, New York, 1971–1977.

[Ta] J. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetic Algebraic Geometry (O. F. G. Schilling, ed.), Harper and Row, New York, 1965, pp. 93–110.

[We] A. Weil, *Abstract versus classical algebraic geometry*, Proc. 1954 Internat. Congr. Math. (1954), vol. 3, North-Holland, Amsterdam, 1956, pp. 550–558.

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A Summary of Mixed Hodge Theory

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Introduction

In this paper we give a summary of concepts from mixed Hodge theory which are relevant to the theory of motives, for the convenience of the reader of these proceedings. We do not claim any originality. An excellent up-to-date overview can be found in **[BZ]**.

The history of Hodge theory shows a growing amount of abstraction: from periods of integrals to derived categories of mixed Hodge modules. We will start our summary somewhere in the middle of this development, with the concept of a mixed Hodge structure and some of its formal properties. We show the geometric origins of this concept and how it has been generalized recently. We also treat the contributions of Hodge structures over $\mathbb{R}$ to the L-functions of motives.

Notation and conventions

We will denote increasing filtrations by subscripts:

$$W_\cdot = (\cdots \subset W_i \subset W_{i+1} \subset \cdots)$$

and decreasing filtrations by superscripts:

$$F^\cdot = (\cdots \supset F^p \supset F^{p+1} \supset \cdots).$$

We let $\mathrm{Gr}_i^W = W_i/W_{i-1}$, $\mathrm{Gr}_F^p = F^p/F^{p+1}$. If an object carries two filtrations $F^\cdot$ and $W_\cdot$, we let $F^p\,\mathrm{Gr}_i^W$ denote the image of $F^p \cap W_i$ in Gr_i^W.

If R is a ring, V an R-module, and S an R-algebra, we let $V_S := V \otimes_R S$. If $F^\cdot$ is a filtration on V, we let $F^p V_S$ denote the image of $(F^p V)_S$ in V_S.

All filtrations $F^\cdot$ of V considered will satisfy

$$\bigcup_p F^p = V, \qquad \bigcap_p F^p = (0) \qquad (\text{resp. } \bigcup_i W_i = V,\ \bigcap_i W_i = (0)).$$

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1. Mixed Hodge structures

The main reference for this section is **[D1]**.

DEFINITION. A *mixed $\mathbb{Q}$-Hodge structure* is a triple $(V, W_., F^{\cdot})$ consisting of

- a finite-dimensional $\mathbb{Q}$-vector space V;
- an increasing filtration $W_.$ of V, called the weight filtration;
- a decreasing filtration $F^{\cdot}$ of $V_{\mathbb{C}}$, called the Hodge filtration, such that

(1) $$\mathrm{Gr}_n^W V_{\mathbb{C}} = F^p \mathrm{Gr}_n^W V_{\mathbb{C}} \oplus \overline{F^{n-p+1}} \mathrm{Gr}_n^W V_{\mathbb{C}}$$

for all $n, p \in \mathbb{Z}$. Here $\overline{F^q}$ is the complex conjugate of F^q with respect to $V_{\mathbb{R}} \subset V_{\mathbb{C}}$.

Let $(V, W_., F^{\cdot})$ be a mixed $\mathbb{Q}$-Hodge structure. Define

$$V^{p,q} = F^p \mathrm{Gr}_{p+q}^W V_{\mathbb{C}} \cap \overline{F^q} \mathrm{Gr}_{p+q}^W V_{\mathbb{C}}.$$

Then (1) is equivalent with the property

(2) $$\mathrm{Gr}_n^W V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \quad \text{for all } n \in \mathbb{Z}.$$

This is called the *Hodge decomposition* of $\mathrm{Gr}_n^W V_{\mathbb{C}}$. One calls $(V, W_., F^{\cdot})$ *pure of weight* n if $W_{n-1} = (0)$ and $W_n = V$, i.e., $\mathrm{Gr}_i^W = (0)$ for all $i \neq n$.

DEFINITION. A *morphism of mixed $\mathbb{Q}$-Hodge structures* is a $\mathbb{Q}$-linear map $\varphi: U \to V$ satisfying $\varphi(W_i U) \subset W_i V$ and $\varphi_{\mathbb{C}}(F^p U_{\mathbb{C}}) \subset F^p V_{\mathbb{C}}$.

Let $(V, W_., F^{\cdot})$ be a mixed $\mathbb{Q}$-Hodge structure. Define

$$I^{p,q} := F^p \cap W_{p+q} \cap \left[\overline{F^q} \cap W_{p+q} + \sum_{i \geq 2} \overline{F^{q-i+1}} \cap W_{p+q-i} \right];$$

then $I^{p,q} \subset W_{p+q} V_{\mathbb{C}}$ maps isomorphically to $V^{p,q}$ in $\mathrm{Gr}_{p+q}^W V_{\mathbb{C}}$. One can show that

$$F^p V_{\mathbb{C}} = \bigoplus_{p' \geq p} \bigoplus_q I^{p',q}, \qquad W_n V_{\mathbb{C}} = \bigoplus_{p+q \leq n} I^{pq}.$$

Hence $\{I^{p,q}\}$ gives a $\mathbb{C}$-splitting of our mixed Hodge structure. It can be shown that $\{I^{pq}\}$ is the unique $\mathbb{C}$-splitting that satisfies

$$I^{p,q} \equiv \overline{I^{q,p}} \quad \left(\mathrm{mod} \bigoplus_{\substack{r<p \\ s<q}} I^{r,s} \right)$$

(cf. **[D4]**). In general, $I^{pq} \neq \overline{I^{qp}}$.

If $\varphi: (U, W_\cdot, F^\cdot) \to (V, W_\cdot, F^\cdot)$ is a morphism of mixed $\mathbb{Q}$-Hodge structures then clearly $\varphi_\mathbb{C}$ maps $I^{p,q}(U)$ to $I^{p,q}(V)$ for all p, q. Hence φ is strictly compatible with the filtrations $F^\cdot$ and $W_\cdot$; i.e.,

$$\varphi_\mathbb{C}(F^p U_\mathbb{C}) = \varphi_\mathbb{C}(U_\mathbb{C}) \cap F^p V_\mathbb{C} \quad \text{for all } p;$$
$$\varphi(W_i U) = \varphi(U) \cap W_i V \quad \text{for all } i.$$

The mixed $\mathbb{Q}$-Hodge structures with their morphisms form an abelian category $\mathbb{Q}$-MHS, and the functors Gr_F^p and Gr_W^i from $\mathbb{Q}$-MHS to the category of $\mathbb{C}$- (resp. $\mathbb{Q}$-)vector spaces are exact.

The *Tate Hodge structure* $\mathbb{Q}(n)$ $(n \in \mathbb{Z})$ is the 1-dimensional Hodge structure with underlying rational vector space $(2\pi i)^n\mathbb{Q}$, which is purely of weight $-2n : \mathbb{Q}(n)_\mathbb{C} = \mathbb{C} = Q(n)_\mathbb{C}^{-n,-n}$.

Let $(U, W_\cdot, F^\cdot)$ and $(V, W_\cdot, F^\cdot)$ be mixed $\mathbb{Q}$-Hodge structures. We obtain filtrations $W_\cdot$ and $F^\cdot$ on $U \otimes_\mathbb{Q} V$ and $U_\mathbb{C} \otimes_\mathbb{C} V_\mathbb{C}$ respectively by

$$W_m(U \otimes_\mathbb{Q} V) = \sum_{i+j=m} W_i(U) \otimes_\mathbb{Q} W_j(V);$$
$$F^p(U_\mathbb{C} \otimes_\mathbb{C} V_\mathbb{C}) = \sum_{r+s=p} F^r(U_\mathbb{C}) \otimes_\mathbb{C} F^s(V_\mathbb{C}).$$

Then $(U \otimes_\mathbb{Q} V, W_\cdot, F^\cdot)$ is again a mixed $\mathbb{Q}$-Hodge structure. The *Tate twist* on the category $\mathbb{Q}$-MHS is defined as $V(n) := V \otimes \mathbb{Q}(n)$.

For mixed $\mathbb{Q}$-Hodge structures $(U, W_\cdot, F^\cdot)$ and $(V, W_\cdot, F^\cdot)$ we obtain filtrations $W_\cdot$ and $F^\cdot$ on $\mathrm{Hom}_\mathbb{Q}(U, V)$ and $\mathrm{Hom}_\mathbb{C}(U_\mathbb{C}, V_\mathbb{C})$ by

$$W_i \mathrm{Hom}_\mathbb{Q}(U, V) = \{\varphi: U \to V \mid \varphi(W_j(U)) \subset W_{i+j}(V) \text{ for all } j\};$$
$$F^p \mathrm{Hom}_\mathbb{C}(U_\mathbb{C}, V_\mathbb{C}) = \{\varphi: U_\mathbb{C} \to V_\mathbb{C} \mid \varphi(F^r(U_\mathbb{C})) \subset F^{p+r}(V_\mathbb{C}) \text{ for all } r\}.$$

This defines a mixed $\mathbb{Q}$-Hodge structure $\mathcal{H}om(U, V)$.

These constructions equip $\mathbb{Q}$-MHS with the structure of a neutral *Tannakian category,* with the fibre functor $\omega: \mathbb{Q}\text{-MHS} \to \mathbb{Q}\text{-Mod}$ given by

$$\omega(U, W_\cdot, F^\cdot) = U.$$

All of this also makes sense with $\mathbb{Q}$ replaced by $\mathbb{R}$, and one obtains then the category $\mathbb{R}$-MHS of mixed $\mathbb{R}$-Hodge structures. One can even define mixed $\mathbb{C}$-Hodge structures in the following way. A $\mathbb{C}$-Hodge structure of weight n is just a finite-dimensional bigraded $\mathbb{C}$-vector space $V = \bigoplus_{p+q=n} V^{p,q}$; equivalently, a finite-dimensional $\mathbb{C}$-vector space V with decreasing filtrations $F^\cdot$ and $\overline{F}^\cdot$ such that $V = F^p \oplus \overline{F}^q$ whenever $p+q = n-1$. A mixed $\mathbb{C}$-Hodge structure is a finite-dimensional $\mathbb{C}$-vector space V with an increasing filtration W and decreasing filtrations $F^\cdot$ and $\overline{F}^\cdot$ which induce for each $n \in \mathbb{Z}$ a $\mathbb{C}$-Hodge structure of weight n on $\mathrm{Gr}_n^W V$. It is often better to view $\overline{F}$ as a filtration on the complex conjugate vector space $\overline{V}$; this conjugate space, given with $\overline{F}$ and F, is again a mixed $\mathbb{C}$-Hodge structure.

Each Tannakian category with a given fibre functor is equivalent with the category of finite-dimensional representations of a certain group. In the case of $\mathbb{R}$-MHS this group has been described in **[D4]**.

Let us first consider the full subcategory $\mathbb{R}$-HS of $\mathbb{R}$-MHS whose objects are direct sums of pure objects, i.e., satisfy $V \cong \bigoplus_i \mathrm{Gr}_i^W(V)$. The category $\mathbb{R}$-HS is equivalent to the category of real representations of the group $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$; one has $S(\mathbb{R}) \cong \mathbb{C}^*$, $S(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$, and the complex conjugation on $S(\mathbb{C})$ is given by $(z, w) \mapsto (\overline{w}, \overline{z})$. Any finite-dimensional complex representation $\rho: S(\mathbb{C}) \to \mathrm{Aut}(V_\mathbb{C})$ induces a bigrading $V_\mathbb{C} = \bigoplus_{p,q} V_\mathbb{C}^{p,q}$ with $V_\mathbb{C}^{p,q} = \{v \in V \mid \rho(z, w)v = z^{-p}w^{-q}v\}$ and if ρ arises by complexification from a representation $S(\mathbb{R}) \to \mathrm{Aut}(V)$, we have $\overline{V_\mathbb{C}^{p,q}} = V_\mathbb{C}^{q,p}$; i.e., we have on V the structure of a mixed $\mathbb{R}$-Hodge structure.

The group associated to the category $\mathbb{R}$-MHS is an extension of S by a projective limit of unipotent algebraic groups, which takes care of the obstructions to splitting the mixed $\mathbb{R}$-Hodge structure.

Another important functor from $\mathbb{Q}$-MHS to $\mathbb{Q}$-vector spaces is the functor of *Hodge classes*

$$\Gamma_H: (V, W_., F^{\cdot}) \mapsto W_0(V) \cap F^0(V_\mathbb{C}).$$

For mixed $\mathbb{Q}$-Hodge structures $(U, W_., F^{\cdot})$ and $(V, W_., F^{\cdot})$ one has

$$\Gamma_H \mathscr{H}om(U, V) = \mathrm{Hom}_{\mathbb{Q}\text{-MHS}}(U, V).$$

In particular,

$$\Gamma_H(U) = \Gamma_H \mathscr{H}om(\mathbb{Q}, U) = \mathrm{Hom}_{\mathbb{Q}\text{-MHS}}(\mathbb{Q}, U);$$

i.e., $\Gamma_H U$ is the maximal $\mathbb{Q}$-Hodge substructure of U which is purely of type $(0, 0)$.

Let $(V, F^{\cdot})$ be a pure $\mathbb{Q}$-Hodge structure of weight n. A *polarization* of V is a morphism $V \otimes V \to \mathbb{Q}(-n)$ inducing a $(-1)^n$-symmetric nondegenerate bilinear form Q on V, such that the form

$$(x, y) \mapsto i^{p-q} Q(x, \overline{y})$$

is Hermitian positive definite on $V^{p,q}$ for all p, q. A polarization of a mixed $\mathbb{Q}$-Hodge structure V will mean: a collection of polarizations $\{Q_i\}$, one for each graded piece $\mathrm{Gr}_i^W(V)$. One has similar definitions with $\mathbb{R}$ instead of $\mathbb{Q}$. For $\mathbb{C}$, a polarization form is sesquilinear: it is a morphism $V \otimes \overline{V} \to \mathbb{C}(-n)$.

2. The geometric case

The concept of Hodge structure has its origin in the structure of the cohomology of compact Kähler manifolds. If X is a compact Kähler manifold and $H^{p,q}(X)$ denotes the subspace of $H^{p+q}(X, \mathbb{C})$ consisting of classes represented by a closed differential form of type (p, q), then

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X), \qquad H^{q,p}(X) = \overline{H^{p,q}}(X),$$

so $H^n(X)$ is a pure Hodge structure of weight n. The class of the Kähler form Ω is of type $(1, 1)$, and cup product with Ω gives morphisms of $\mathbb{R}$-Hodge structures $L\colon H^n(X, \mathbb{R}) \to H^{n+2}(X, \mathbb{R})(1)$. If $d = \dim X$, then by the hard Lefschetz theorem

$$L^k\colon H^{d-k}(X, \mathbb{R}) \xrightarrow{\sim} H^{d+k}(X, \mathbb{R})(k).$$

Let

$$P^{d-k}(X, \mathbb{R}) = \ker(L^{k+1}\colon H^{d-k}(X, \mathbb{R}) \to H^{d+k+2}(X, \mathbb{R})(k+1));$$

then $P^q(X, \mathbb{R})$ is polarized by the form $(\omega, \eta) \mapsto (-1)^{q(q+1)/2} \int L^{d-q}\omega \wedge \eta$. Moreover,

$$H^n(X, \mathbb{R}) = \bigoplus_{k\geq 0} L^k P^{n-2k}(X, \mathbb{R})(-k) \quad \text{(primitive decomposition)}.$$

By a theorem of Kodaira, Ω represents the class of an ample line bundle on X if and only if $[\Omega] \in H^2(X, \mathbb{Q})$, i.e., $[\Omega]$ is a Hodge class. In that case, the primitive cohomology groups are even polarized $\mathbb{Q}$-Hodge structures, and hence $H^n(X, \mathbb{Q})$ is a polarizable $\mathbb{Q}$-Hodge structure.

P. Deligne **[D2, D3]** has extended all this to the case of algebraic varieties over $\mathbb{C}$ that are possibly singular or noncompact. If X is a smooth complete complex variety, there exist a smooth projective variety Y and a birational morphism $Y \xrightarrow{\pi} X$. Via π^*, $H^n(X)$ can be considered as a subspace of $H^n(Y)$ which is a $\mathbb{Q}$-Hodge substructure. In this way $H^n(X)$ gets a $\mathbb{Q}$-Hodge structure of weight n. If X is smooth but not necessarily complete, there exists a smooth compactification $X \overset{i}{\hookrightarrow} \overline{X}$ such that $\overline{X} \setminus X = D$ is a divisor with normal crossings on $\overline{X}$. The logarithmic de Rham complex $\Omega^{\cdot}_{\overline{X}}(\log D)$ is quasi-isomorphic to $i_*\Omega^{\cdot}_X$; hence

$$\mathbb{H}^n(\overline{X}, \Omega^{\cdot}_{\overline{X}}(\log D)) \cong \mathbb{H}^n(\overline{X}, i_*\Omega^{\cdot}_X) \cong \mathbb{H}^n(X, \Omega^{\cdot}_X) \cong H^n(X, \mathbb{C}).$$

Here i_* is the direct image in the strong topology, and we work with the holomorphic de Rham complex. Note that i is a Stein map; hence, $R^q i_* F = 0$ for $q > 0$ and F a coherent analytic sheaf on X. Hence $i_*\Omega^{\cdot}_X$ and $Ri_*\Omega^{\cdot}_X \simeq Ri_*\mathbb{C}_X$ are quasi-isomorphic. The sheaf complex $\Omega^{\cdot}_{\overline{X}}(\log D)$ carries a Hodge filtration $F^{\cdot}$ and a weight filtration $W_{\cdot}$ given by

$$\begin{cases} F^p\Omega^r_{\overline{X}}(\log D) = \Omega^r_{\overline{X}}(\log D) \quad \text{if } r \geq p, \text{ and } 0 \text{ otherwise;} \\ W_k\Omega^r_{\overline{X}}(\log D) = \text{image of } \Omega^r_{\overline{X}}(\log D) \otimes \Omega^{r-k}_{\overline{X}} \\ \text{under cup product in } \Omega^r_{\overline{X}}(\log D) \text{ if } k \leq r, \text{ and } \Omega^r_{\overline{X}}(\log D) \text{ otherwise.} \end{cases}$$

These filtrations induce the Hodge and weight filtrations on $H^n(X, \mathbb{C})$, and one can show that $W_{\cdot}$ is in fact already defined over $\mathbb{Q}$. Independence of the choice of a smooth compactification follows from the fact that any two such are dominated by a common one.

In the singular case one can construct a smooth complete simplicial scheme $\overline{X}_.$ together with a divisor $D_. \subset \overline{X}_.$ with normal crossings which gives a simplicial resolution of X. We refer to **[D3]** for details.

Let X be a smooth complete variety $/\mathbb{C}$, and let $Z \subset X$ be a subvariety of codimension p. Choose a resolution of singularities $\widetilde{Z} \to Z$ of Z, and let $\rho: \widetilde{Z} \to X$. The Poincaré dual of $\rho^*: H^i(X) \to H^i(\widetilde{Z})$ gives a morphism

$$\rho_*: H^{n-2p}(\widetilde{Z})(p) \to H^n(X)$$

of $\mathbb{Q}$-Hodge structures, whose image is a $\mathbb{Q}$-Hodge substructure of $H^n(X)$ of *Hodge level* $\leq n - 2p + 1$, i.e., for which $V^{p,q} = 0$ as soon as $|p - q| > n - 2p + 1$.

Moreover, the sequence

$$H^{n-2p}(\widetilde{Z})(p) \xrightarrow{\rho_*} H^n(X) \to H^n(X \backslash Z)$$

is exact. The *general Hodge conjecture* as corrected by Grothendieck **[G]** states that for every $\mathbb{Q}$-Hodge substructure V of $H^n(X)$ of Hodge level $\leq n+1-2p$ there should exist a subscheme $Z \subset X$ of codimension p such that $V \subset \ker(H^n(X) \to H^n(X \backslash Z))$. The particular case of $n = 2p$ states that each class in $\Gamma_H H^{2p}(X)(-p)$ should be carried by an algebraic cycle of codimension p in X.

There also exists a version of the Hodge conjecture in the singular case, cf. **[J]**.

3. Hodge structures over $\mathbb{R}$ and Γ-factors

Let X be a smooth projective variety over $\mathbb{R}$. Then complex conjugation acts on $X(\mathbb{C})$, and hence it leads to an involution F_∞ of $H^*(X(\mathbb{C}))$ which maps $H^{p,q}$ to $H^{q,p}$. This additional structure is important for the contribution of the infinite places to the L-function of a motive over a number field.

Recall that an $\mathbb{R}$-Hodge structure, more precisely a split mixed $\mathbb{R}$-Hodge structure, is nothing but a representation of the group S into $\mathrm{GL}(V)$ for V a real vector space. Observe that a representation of $S(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ into $\mathrm{GL}(U)$ for U a $\mathbb{C}$-vector space is equivalent to a bigrading $U = \bigoplus_{p,q} U^{p,q}$, i.e., a $\mathbb{C}$-Hodge structure on U (which is a direct sum of pure ones). We define a $\mathbb{Q}$- (resp. $\mathbb{C}$-)Hodge structure over $\mathbb{R}$ to be a $\mathbb{Q}$- (resp. $\mathbb{C}$-)Hodge structure U together with an involution F_∞ which maps $U^{p,q}$ to $U^{q,p}$.

Example. Let H be a $\mathbb{Q}$-Hodge structure (resp. a $\mathbb{Q}$-Hodge structure over $\mathbb{R}$), F a finite field extension of $\mathbb{Q}$, $j: F \hookrightarrow \mathrm{End}_{\mathbb{Q}} H$ an F-module structure on H, and $\sigma: F \hookrightarrow \mathbb{C}$ a complex embedding. Then $H \otimes_{F,\sigma} \mathbb{C}$ is in a natural way a $\mathbb{C}$-Hodge structure (resp. a $\mathbb{C}$-Hodge structure over $\mathbb{R}$).

The Weil group $\mathscr{W} = \mathscr{W}(\mathbb{C}/\mathbb{R})$ of $\mathbb{C}$ over $\mathbb{R}$ is an extension

$$1 \to \mathbb{C}^* \to \mathscr{W} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

equipped with an element $w \in \mathscr{W}$ mapping onto complex conjugation in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and such that $w^2 = -1$ and $wzw^{-1} = \overline{z}$ for $z \in \mathbb{C}^*$.

The group $\mathscr{W}$ can be embedded in $\mathbb{H}^*$ (the multiplicative group of the quaternions) as the normalizer of $\mathbb{C}^* \subset \mathbb{H}^*$. In the representation of $\mathbb{H}$ as $\{\left(\begin{smallmatrix} z & u \\ -\bar{u} & \bar{z} \end{smallmatrix}\right) | z, u \in \mathbb{C}\} \subset \operatorname{End}(\mathbb{C}^2)$, one can take $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Given a $\mathbb{C}$-Hodge structure over $\mathbb{R}$,

$$V = \bigoplus_{p,q} V^{p,q}, \quad F_\infty \colon V^{p,q} \xrightarrow{\sim} V^{q,p}, \quad F_\infty^2 = \mathrm{Id},$$

we have the representation $\rho \colon \mathbb{C}^* \to \mathrm{GL}(V)$ with $\rho(z)(v) = z^{-p}\bar{z}^{-q} v$ for $z \in \mathbb{C}^*$, $v \in V^{p,q}$. We extend ρ to a representation of $\mathscr{W}$ into $\mathrm{GL}(V)$ by

$$\rho(w)(v) = i^{p+q} F_\infty(v) \quad \text{for } v \in V^{p,q}.$$

This establishes an equivalence of categories between $\mathbb{C}$-Hodge structures over $\mathbb{R}$ and finite-dimensional complex representations of $\mathscr{W}$, which are algebraic on $\mathbb{C}^* \subset \mathscr{W}$ viewed as a real algebraic group.

EXAMPLE. The norm map on $\mathbb{H}^*$ restricts to a 1-dimensional real representation of $\mathscr{W}$ which corresponds to $\mathbb{R}(1)$ with the involution $F_\infty = -1$.

Let $(H = \bigoplus_{p,q} H^{p,q}, F_\infty)$ be a $\mathbb{C}$-Hodge structure over $\mathbb{R}$. Then its Γ-factors are given by

$$L_\infty(s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \prod_p \Gamma_{\mathbb{R}}(s-p)^{h^{p+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p-}}$$

where

$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$,

$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$,

$h^{pq} = \dim H^{pq}$,

$h^{p\pm} =$ the dimension of the $\pm(-1)^p$-eigenspace of F_∞ acting on $H^{p,p}$.

4. Variations of mixed Hodge structure and generalizations

Let S be a complex manifold. A *variation of mixed $\mathbb{Q}$-Hodge structure* over S is a triple $(\mathbb{V}, W_., F^{\cdot})$ where

$\mathbb{V}$ is a local system of $\mathbb{Q}$-vector spaces on S;

$W_.$ is an increasing filtration of $\mathbb{V}$ by local subsystems;

$F^{\cdot}$ is a decreasing filtration of the holomorphic vector bundle $\mathbb{V} \otimes_{\mathbb{Q}_S} O_S$ by holomorphic subbundles.

These data must satisfy the following properties:

for all $s \in S$, the triple $(\mathbb{V}_s, (W_.)_s, F^{\cdot}(s))$ is a mixed Hodge structure;

$\nabla F^p \subset \Omega^1_S \otimes F^{p-1}$ for all p (*Griffiths's transversality*).

Here $\nabla \colon \mathbb{V} \otimes_{\mathbb{Q}_S} O_S \to \mathbb{V} \otimes_{\mathbb{Q}_S} \Omega^1_S$ is the integrable connection given by $\nabla(v \otimes f) = v \otimes df$ for local sections v, f of $\mathbb{V}$ and O_S respectively.

A *polarization* of $(\mathbb{V}, W_., F^{\cdot})$ is a collection of bilinear forms $\mathrm{Gr}_i^W \mathbb{V} \otimes \mathrm{Gr}_i^W \mathbb{V} \to \mathbb{Q}(-i)$ $(i \in \mathbb{N})$ which induce a polarization of the pure Hodge structure $\mathrm{Gr}_i^W \mathbb{V}_s$ for each $i \in \mathbb{N}$, $s \in S$.

The notion of *variation of mixed* $\mathbb{R}$*-Hodge structure* over S is defined analogously. For a *variation of mixed* $\mathbb{C}$*-Hodge structure* over S one has a local system $\mathbb{V}$ of $\mathbb{C}$-vector spaces on S with filtrations $W_\cdot$, $F^\cdot$, and $\overline{F}^\cdot$; here $F^\cdot$ is again a filtration by holomorphic subbundles of $\mathbb{V} \otimes_{\mathbb{C}_S} \mathcal{O}_S$, but $\overline{F}^\cdot$ consists of holomorphic subbundles of $\overline{V} \otimes_{\mathbb{C}_S} \mathcal{O}_S$, where $\overline{\mathbb{V}}$ has the complex conjugate $\mathbb{C}_S$-module structure. Both $F^\cdot$ and $\overline{F}^\cdot$ must satisfy the Griffiths's transversality condition, and for all $s \in S$, the quadruple $(\mathbb{V}_s, (W_\cdot)_s, F^\cdot(s), \overline{F}^\cdot(s))$ is required to be a mixed $\mathbb{C}$-Hodge structure.

Consider a polarized variation of a mixed $\mathbb{Q}$-Hodge structure $(\mathbb{V}, W_\cdot, F^\cdot)$, on the punctured unit disc Δ^* in $\mathbb{C}$. In the pure case, on the punctured tangent space to Δ at 0 there exists a variation of mixed Hodge structure of a very special kind (a *nilpotent orbit*) which near 0 is asymptotic to the given one. For this to be true, polarizability is essential **[Sch]**. In the mixed case, the presence of a polarization does not suffice to guarantee reasonable asymptotic properties near 0, and admissibility conditions have to be satisfied for this to be the case **[SZ]**. For arbitrary S, a variation of mixed $\mathbb{Q}$-Hodge structure over S is called admissible if its restriction to any curve on S is admissible **[K1]**. Part of the admissibility conditions is the polarizability of the pure variations $\mathrm{Gr}_i^W \mathbb{V}$.

If $\mathbb{V}$ is an admissible variation of mixed $\mathbb{Q}$-Hodge structure on a quasi-projective complex manifold X, then $\mathbb{H}^*(X, \mathbb{V})$ carries a functorial mixed Hodge structure **[Sa6]**.

The standard example of a polarized variation of $\mathbb{Q}$-Hodge structure arises from a smooth projective morphism $f: X \to S$ between complex manifolds X and S. The underlying local system is $R^m f_* \mathbb{Q}_X$ for some $m \in \mathbb{N}$. One has

$$R^m f_* \mathbb{Q}_X \otimes_{\mathbb{Q}_S} O_S \cong R^m f_* f^{-1} O_S \cong \mathbb{R}^m f_*(\Omega^\cdot_{X/S}),$$

where $\Omega^\cdot_{X/S}$ is the *relative de Rham complex* of f; the Hodge filtration is given by

$$F^p \mathbb{R}^m f_*(\Omega^\cdot_{X/S}) = \mathbb{R}^m f_*(\sigma_p \Omega^\cdot_{X/S})$$

with

$$\sigma_p \Omega^q_{X/S} = \begin{cases} 0 & \text{for } q < p, \\ \Omega^q_{X/S} & \text{otherwise}. \end{cases}$$

If f is no longer smooth or projective, there always exists a dense open subset U of S such that $R^m f_* \mathbb{Q}_{X|U}$ underlies an admissible variation of mixed $\mathbb{Q}$-Hodge structure. Some weaknesses of the concept of variation of mixed $\mathbb{Q}$-Hodge structure are clear: one always needs a local system as underlying $\mathbb{Q}$-structure; one misses analogous objects for constructible sheaves or even constructible sheaf complexes, which arise naturally in geometry. Another aspect is the following. The purity of the cohomology groups of smooth projective varieties is closely related to the validity of Poincaré duality. For singular projective varieties, Poincaré duality is valid for intersection homology groups (over $\mathbb{Q}$, with middle perversity), but classical Hodge theory applies to these only in special cases (using L_2-cohomology). These problems have been solved by the introduction of the notion of *mixed Hodge modules* **[Sa2]**, **[Sa3]**. The starting point of this theory was Saito's description of the mixed Hodge structure on the vanishing cycles of isolated singularities by putting a filtration on the associated D-module (Gauss-Manin system) **[Sa1]**.

Let us give an outline of the nature of the pure objects in this theory: the *polarizable Hodge modules*. For X a complex manifold of dimension n, let D_X denote the sheaf of germs of holomorphic differential operators on X. For any right D_X-module M we define its de Rham complex $DR(M)$ by $DR(M)^p = M \otimes_{O_X} \Lambda^{-p} T_X$ with differential given by

$$d(m \otimes \partial_{i_1} \wedge \cdots \wedge \partial_{i_p}) = \sum_{j=1}^{n} (-1)^j\, m\partial_{i_j} \otimes \partial_{i_1} \wedge \cdots \hat{\partial}_{i_j} \cdots \wedge \partial_{i_p}$$

in terms of local coordinates on X. If M is holonomic, $DR(M)$ is a constructible complex of $\mathbb{C}_X$-modules **[K2]**. The de Rham functor induces equivalences of categories (the *Riemann-Hilbert correspondence*, cf. **[Me1]**):

$$D^b_{rh}(D_X) \to D^b_c(\mathbb{C}_X)$$

and

$$\mathrm{Mod}_{rh}(D_X) \to \mathrm{Perv}(\mathbb{C}_X).$$

Here b stands for bounded, r for regular, h for holonomic, c for constructible; $\mathrm{Perv}(\mathbb{C}_X)$ is the category of perverse sheaves of $\mathbb{C}_X$-modules.

The category $MF_h(D_X, \mathbb{Q})$ of filtered holonomic D_X-modules with $\mathbb{Q}$-structure has as objects quadruples (M, F, K, α) where M is a holonomic right D_X-module, F a good filtration on M, K a perverse $\mathbb{Q}_X$-module, and α an isomorphism of $DR(M)$ with $K \otimes \mathbb{C}$ in $\mathrm{Perv}(\mathbb{C}_X)$. The morphisms in $MF_h(D_X, \mathbb{Q})$ are the obvious ones.

If X is a point, then $MF_h(D_X, \mathbb{Q})$ is the category of finite-dimensional filtered $\mathbb{C}$-vector spaces with $\mathbb{Q}$-structure, which contains as full subcategories the (polarizable) $\mathbb{Q}$-Hodge structures. Likewise, for X arbitrary, the category $MH(X, w)^p$ of polarizable Hodge modules on X of weight w is a certain full subcategory of $MF_h(D_X, \mathbb{Q})$. The axioms defining this subcategory are formulated in terms of the formalism of vanishing cycle functors.

Let X be compact, U a Zariski-dense open subset of X. Let $(\mathbb{V}, F^{\cdot})$ be a polarizable variation of $\mathbb{Q}$-Hodge structure of weight w on U, and let $V = \mathbb{V} \otimes_{\mathbb{Q}_U} O_U$. Then V has a natural structure of a left D_U-module, and with $F_p = F^{-p}$ we obtain a good filtration on it (by Griffiths's transversality). Hence $V \otimes \Omega^n_U$ is a right D_U-module, filtered by $F_p(V \otimes \Omega^n_U) = F_{p+n}(V) \otimes \Omega^n_U$, whose de Rham complex is just $V \otimes \Omega^{\cdot}_U[n]$, which is quasi-isomorphic to $\mathbb{V}[n] \otimes \mathbb{C}$. Furthermore $\mathbb{V}[n]$ can be extended in a unique minimal way to an object $IC(\mathbb{V})$ of $\mathrm{Perv}(\mathbb{Q}_X)$, and $IC(\mathbb{V}) \otimes \mathbb{C} \cong DR(M)$ where M is the minimal extension of $V \otimes \Omega_U^n$ to a regular holonomic right D_X-module. The filtration F extends to M in a suitable way, and $(M, F, IC(\mathbb{V}))$ is a polarizable Hodge module of weight $w + n$ with *strict support* X. A similar construction can be done for U a Zariski-dense open subset of an irreducible closed subset of X, giving rise to polarizable Hodge modules with smaller strict support. Every polarizable Hodge module on X is a direct sum of polarizable Hodge modules with strict support in irreducible closed analytic subspaces of X. So the category $MH(X, w)^p$ is a semisimple abelian category, and there is a one-to-one correspondence between its indecomposable objects and indecomposable polarizable variations of Hodge structure on Zariski locally closed smooth subsets of X **[Sa6,** Theorem 2.5].

If $f: X \to Y$ is a projective morphism and (M, F, K) is a polarizable Hodge module on X of weight w, then $\mathbb{H}^i f_*(M, F, K)$ (an extension of the perverse higher direct image ${}^p\mathbb{H} f_* K)$ is a polarizable Hodge module on Y of weight $w + i$. Let us take the situation that X is a projective smooth variety, Z an irreducible subvariety of X, and U the regular locus of Z; we take for f the constant map to a point. Applying the above to the polarizable Hodge module with $K = IC(\mathbb{Q}_U)$ we obtain a pure Hodge structure on the intersection cohomology groups of Z.

The mixed Hodge modules are certain extensions of polarizable Hodge modules. The conditions that the extension data must fulfill are again expressed in terms of vanishing cycles. All six operations of Grothendieck: $\otimes^L$, $R\underline{\mathrm{Hom}}$, f_*, f^*, $f_!$, and $f^!$ (see **[Me2]**) can be extended to the derived category of mixed Hodge modules, as well as the vanishing cycle functors ψ and φ. Every admissible variation of mixed Hodge structure is a mixed Hodge module **[Sa6]**.

Saito has also developed certain conjectures concerning mixed Hodge modules of geometric origin, analogous to the Hodge conjecture **[Sa4, Sa5]**.

For an elementary introduction to mixed Hodge modules, the reader is recommended to consult **[Sa7]**.

REFERENCES

[BZ] J.-L. Brylinski and S. Zucker, *An overview of recent advances in Hodge theory*, Encyclopaedia of Mathematical Sciences, vol. 69, Several Complex Variables VI, Complex Manifolds, Springer-Verlag, Berlin and Heidelberg, 1990, pp. 39–142.

[D1] P. Deligne, *Théorie de Hodge*. I, Proc. Internat. Congr. Math. (Paris, 1970), vol. I, Gauthier-Villars, Paris, 1971, pp. 425–430.

[D2] ——, *Théorie de Hodge*. II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–58.

[D3] ——, *Théorie de Hodge*, III, Inst. Hautes Études Sci. Publ. Math. **44** (1975), 5–77.

[D4] ——, *Structures de Hodge mixtes réelles*, these Proceedings, vol. 1, pp. 509–514.

[G] A. Grothendieck, *Hodge's general conjecture is false for trivial reasons*, Topology **8** (1969), 299–303.

[J] U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, Berlin, 1990.

[K1] M. Kashiwara, *A study of a variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22** (1986), 991–1024.

[K2] ——, *On the maximally overdetermined system of linear differential equations.* I, Publ. Res. Inst. Math. Sci. **10** (1974/75), 563–579.

[Me1] Z. Mebkhout, *Une equivalence de catégories*, Compositio Math. **51** (1984), 51–62; *Une autre équivalence de catégories*, Compositio Math. **51** (1984), 63–88.

[Me2] ——, *Systèmes differentiels—Le formalisme des six opérations de Grothendieck pour les D_X-modules coherents*, Hermann, Paris, 1989 (Travaux en cours 35).

[Sa1] M. Saito, *Gauss-Manin systems and mixed Hodge structure*, Proc. Japan Acad. Sci. Ser. A Math. Sci. **58** (1982), 29–32.

[Sa2] ——, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), 849–995.

[Sa3] ——, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.

[Sa4] ——, *Hodge conjecture and mixed motives.* I, Complex Geometry and Lie Theory, Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, pp. 283–303.

[Sa5] ——, *Hodge conjecture and mixed motives.* II, Preprint Res. Inst. Math. Sci., no. 692, Kyoto University, 1990.

[Sa6] ——, *Mixed Hodge modules and admissible variations*, C. R. Acad. Sci. Paris Sér I Math. **309** (1989), 351–356.

[Sa7] ——, *Introduction to mixed Hodge modules*, Théorie de Hodge, Luminy, Juin 1987, Astérisque, no. 179–180, Soc. Math. France, Paris, 1989, pp. 145–162.

[Sch] W. Schmid, *Variation of Hodge structure: The singularities of the period mapping*, Invent. Math. **22** (1973), 211–320.

[SZ] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure.* I, Invent. Math. **80** (1985), 489–542.

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Crystalline Cohomology

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Crystalline cohomology was invented by Grothendieck in 1966 to fill in the gap at p in the family of ℓ-adic cohomologies. The word *crystal* appears for the first time in his letter to J. Tate **[47]**. In the fall of 1966, he gave a series of lectures at the IHES **[48]**, in which he outlined a program that was to be carried out by Berthelot in his thesis **[5]**. Since then, much progress has been made, especially on Grothendieck's problem of the *mysterious functor* **[47, 50]**, about the comparison between de Rham cohomology and p-adic étale cohomology in mixed characteristic, with crystalline cohomology as go-between. After the pioneering work of Fontaine in the seventies, setting up fundamental constructions and conjectures **[38, 39, 40]**, extensive results were obtained in the eighties, first under some restrictions by Bloch-Kato **[19]** and Fontaine-Messing **[44]**, and then in general by Faltings **[35, 36]**. In the case of semi-stable reduction, new structures on de Rham cohomology inspired by the theory of degeneration of Hodge structures have lately been discovered by Fontaine, Jannsen, Hyodo-Kato, and others, raising new questions (see **[41]**). I will give an overview of this and of some further developments, after first recalling some basic definitions and properties of crystalline cohomology. This report considerably overlaps with other surveys **[55, 56, 42, 62]**, in which the reader will find more details on some of the topics discussed here. Sections 2 and 3, especially, are nothing but a digest of parts of **[42]** and **[62]**.

Contents

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1. Crystalline cohomology of proper and smooth varieties over perfect fields of characteristic $p > 0$

1.1. Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ the ring of Witt vectors on k, $W_n = W/p^n W$ the ring of Witt vectors of length n, σ the Frobenius automorphism of W, K the field of fractions of W.

Let X be a scheme over k and n an integer ≥ 1. Crystalline cohomology of X over W_n, $H^*(X/W_n)$, is defined as the cohomology of a site $(X/W_n)_{\text{crys}}$, the *crystalline site* of X over W_n, with value in a sheaf of rings $\mathscr{O} = \mathscr{O}_{X/W_n}$:

$$H^*(X/W_n) := H^*((X/W_n)_{\text{crys}}, \mathscr{O}). \tag{1.1.1}$$

Objects of $(X/W_n)_{\text{crys}}$ are pairs $(i\colon U \to T, \gamma)$, where U is an open subscheme of X, i a closed immersion of W_n-schemes, and γ a PD-structure on i, i.e., a structure of divided powers on the ideal of i, compatible with the standard divided powers on the ideal (p) (see **[5]** or **[16]** for the definition of divided powers); note that such an immersion is automatically a nil-immersion, so that U and T have the same underlying topological space. Maps in $(X/W_n)_{\text{crys}}$ are defined in the obvious way. A family $(U_\alpha \to T_\alpha) \to (U \to T)$ is said to be a covering if $(T_\alpha \to T)$ is an open Zariski cover of T. Associating to $(U \to T)$ the sheaf of rings $\mathscr{O}_T$ defines the sheaf of rings $\mathscr{O}_{X/W_n}$ mentioned above. The topos of sheaves on $(X/W_n)_{\text{crys}}$, hence the groups (1.1.1), depend functorially on X/k. For variable n, the W_n-modules (1.1.1) form an inverse system, whose limit is by definition the crystalline cohomology of X over W:

$$H^*(X/W) := \varprojlim_n H^*(X/W_n). \tag{1.1.2}$$

This is a graded W-module depending functorially on X/k.

1.2. It is a miracle of differential calculus—and a consequence of Grothendieck's fabulous insight into it—that such bizarre-looking objects are amenable to calculation. The main tool for computing them is the following simple result, which is one of the key points of the whole theory. Assume that we have a closed immersion $i\colon X \to Z$ of X into a smooth W_n-scheme Z. Then one can "add" divided powers to the ideal of i in a universal way, obtaining a factorization of i into $X \to D \to Z$ where D is the so-called *divided-power envelope* of i (for example, if X is smooth and Z is a smooth lifting of X, D is just Z itself). The scheme D is affine over Z, and $\mathcal{O}_D$, viewed as an $\mathcal{O}_Z$-module, has a natural integrable connection ∇ with respect to W_n, satisfying $\nabla x^{[m]} = x^{[m-1]} \otimes dx$, for x in the ideal of X in D, $x^{[m]}$ denoting the mth divided power of x. One can therefore consider the de Rham complex of Z/W_n with coefficients in $\mathcal{O}_D$,

$$\mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n} = (\mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_Z} \Omega^1_{Z/W_n} \to \cdots),$$

a complex of $\mathcal{O}_D$-modules with differential given by differential operators of order 1 (which can also be viewed as the quotient of the de Rham complex of D/W_n by the differential graded ideal generated by the relations $dx^{[m]} = x^{[m-1]}dx$). The hypercohomology of D (or, equivalently, of X, since X and D have the same underlying space) with values in $\mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}$ calculates $H^*(X/W_n)$: there is a canonical isomorphism

$$H^*(X/W_n) \xrightarrow{\sim} H^*(X, \mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}), \tag{1.2.1}$$

compatible in a natural way with maps of embeddings $(X' \to Z') \to (X \to Z)$. In particular, if $X = \operatorname{Spec} k$ is embedded in the affine line $Z = \operatorname{Spec} W_n[t]$ by the inclusion of $\operatorname{Spec} k$ into $\operatorname{Spec} W_n$ followed by the zero section, then $D = \operatorname{Spec} W_n\langle t\rangle$, where $W_n\langle t\rangle$ is the divided-power polynomial algebra, and a consequence of (1.2.1) is that the natural augmentation

$$W_n \to W_n\langle t\rangle \otimes_{W_n[t]} \Omega^{\cdot}_{W_n[t]/W_n}$$

is a quasi-isomorphism: this is of course obvious since $dt^{[m]} = t^{[m-1]} \otimes dt$, but this observation is in fact the key ingredient in the construction of (1.2.1) ("*divided-power Poincaré lemma*").

Crystalline cohomology can be defined in a much more general framework than that considered above (see **[5]**, or **[16]**, which can be used as an introduction to **[5]**). However, even in the restricted context we have chosen, the crystalline cohomology groups (1.1.2) are well behaved only for schemes that are proper and smooth over k. We shall now recall the main results in this situation.

1.3. Properties of $H^*(X/W)$ for X/k proper and smooth.[1]

(a) *Weil cohomology.* Let $\mathcal{C}$ be the category of proper and smooth schemes over k. For X in $\mathcal{C}$, $H^i(X/W)$ is of finite type over W, and $H^i(X/W) = 0$ for $i > 2\dim(X)$. If X/k is projective, or if X is liftable to a proper and smooth scheme over W, then

$$\operatorname{rk}_W H^i(X/W) = \dim H^i(X \otimes \bar{k}, \mathbb{Q}_\ell), \tag{1.3.1}$$

where $\bar{k}$ is an algebraic closure of k and ℓ a prime $\neq p$. In the latter case, this is a consequence of the basic comparison result recalled below (1.3.8); in

[1]The results below are due to Berthelot [5], except when other specific indications are given.

the former one, this is a much deeper fact, due to Katz-Messing **[79]**, which follows from the Weil conjectures **[27, 28]**.

On $\mathscr{C}$, $X \mapsto K \otimes_W H^*(X/W)$ is a Weil cohomology theory in the sense of Kleiman **[80]**. Namely, it satisfies the Künneth formula and Poincaré duality, weak and hard Lefschetz theorems when X is projective **[6, 79]**, and there is a cycle class map

$$cl: CH^i(X) \to H^{2i}(X/W) \otimes K, \tag{1.3.2}$$

where $CH^i(X)$ is the Chow group of codimension i cycles on X, satisfying

$$cl(Y \cdot Z) = cl(Y)cl(Z) \tag{1.3.3}$$

for cycles Y, Z intersecting properly. In Berthelot's thesis **[5]**, the cycle class map was defined only on smooth cycles, and (1.3.3) was proven only in the case of transversally intersecting smooth cycles. The general case is due to Gillet-Messing **[45]**. This was also obtained independently by Gros **[46]**, who constructs a finer class map with value in $H^{2i}(X/W)$, satisfying again (1.3.3). As for the Künneth isomorphism and Poincaré duality, there are statements not neglecting torsion, involving a perfect complex $R\Gamma(X/W)$ such that $H^iR\Gamma(X/W) = H^i(X/W)$, and yielding the usual short exact sequence of "universal coefficients" (see **[5]**, or **[55]** for a summary).

As a consequence of this formalism, when $k = \mathbb{F}_q$, $q = p^a$, one has a cohomological expression for the zeta function of X/k, namely

$$Z(X/k, t) = \prod_{0 \leq i \leq 2d} \det(1 - F_q^* t; H^i(X/W)) \otimes K)^{(-1)^{i+1}}, \tag{1.3.4}$$

for X in $\mathscr{C}$, purely of dimension d, where $F_q: X \to X$ is the k-linear Frobenius (ath power of the absolute one). When X/k is projective, it was again derived by Katz-Messing **[79]** from the Weil conjectures that the crystalline characteristic polynomials coincide with the ℓ-adic ones (which are known to have integral coefficients and be independent of ℓ):

(1.3.5)
$$\det(1 - F_q^* t; H^i(X/W) \otimes K) = \det(1 - F_q^* t; H^i(X \otimes \bar{k}, \mathbb{Q}_\ell)) \qquad (\ell \neq p)$$

((1.3.1) is in fact a by-product of this).

(b) *Comparison with de Rham cohomology.* There are two main comparison theorems: with the de Rham cohomology of X/k on the one hand, and with the de Rham cohomology of a proper and smooth lifting over W, on the other hand. Both follow from (1.2.1). Let X be in $\mathscr{C}$.

(i) By (1.2.1) for $n = 1$ and $Z = X$, there is a canonical isomorphism

$$H^*(X/k) \xrightarrow{\sim} H^*_{\mathrm{DR}}(X/k) \quad (\underset{\mathrm{dfn}}{=} H^*(X, \Omega^\cdot_{X/k})) \tag{1.3.6}$$

(N.B. : $W_1 = k$). Moreover, there is a canonical isomorphism

$$H^*(X/W) \otimes_W k \xrightarrow{\sim} H^*(X/k)$$

when $H^*(X/W)$ is torsionfree, and in general a short exact sequence

$$(1.3.7) \quad 0 \to H^*(X/W) \otimes k \to H^*_{\mathrm{DR}}(X/k) \to \mathrm{Tor}_1^W(H^{*+1}(X/W), k) \to 0.$$

(ii) Suppose Z/W is a proper and smooth lifting of X/k. By (1.2.1) applied to $Z_n = Z \otimes W_n$ and a passage to the limit, one obtains a canonical isomorphism

$$(1.3.8) \qquad H^*(X/W) \xrightarrow{\sim} H^*_{\mathrm{DR}}(Z/W) \quad (\underset{\mathrm{dfn}}{=} H^*(Z, \Omega^{\cdot}_{Z/W})).$$

The fact that $H^*_{\mathrm{DR}}(Z/W)$ depends only on $Z \otimes k$ is quite a miracle. Conjectured by Grothendieck in **[48]**, by analogy with Monsky-Washnitzer's invariance result in the affine case **[92]**, it was a main motivation for the development of crystalline cohomology. We shall discuss in §2 several generalizations of (1.3.8).

(c) *Frobenius.* Let X be in $\mathscr{C}$. By functoriality, the absolute Frobenius endomorphism F_{abs} of X induces a σ-linear map

$$\phi = F^*_{\mathrm{abs}}: H^*(X/W) \to H^*(X/W).$$

This map is an isogeny, i.e., $\phi \otimes \mathbb{Q}$ is bijective. Thus, if T is the torsion submodule of $H^i(X/W)$, $(H^i(X/W)/T, \phi)$ is an *F-crystal* on k (i.e., a free finitely generated W-module together with a σ-linear endomorphism F such that $F \otimes \mathbb{Q}$ is bijective; for a thorough discussion of F-crystals, see Katz's excellent exposition **[78]**). For example, if $i = 1$, then $T = 0$, and $(H^1(X/W), \phi)$ is the Dieudonné module of the p-divisible group associated to the Albanese variety of X (see **[86]** in the case X is an abelian variety and **[58,** II 3.11] in general).

This σ-linear action of Frobenius is what gives crystalline cohomology its charm—and its distinctive feature among other cohomology theories. It was the subject of extensive study in the seventies and early eighties. One central theorem is the so-called *Katz conjecture* on Newton and Hodge polygons. Before stating it, let me briefly recall what the Newton polygon of an F-crystal is. Let $\bar{k}$ be an algebraic closure of k. If H is an F-crystal on k, then, by a theorem of Dieudonné–Manin, the σ-linear extension $H \otimes W(\bar{k})$ of H is isogenous (i.e., becomes isomorphic over the field of fractions of $W(\bar{k})$) to a direct sum of F-crystals of the form $P(m, n) = W(\bar{k})[T]/(T^m - p^n)$, F acting σ-linearly by multiplication by T, with m, $n \in \mathbb{Z}$, $n/m \geq 0$, $(m, n) = 1$; the rational numbers n/m appearing in this decomposition are called the *slopes* of H; if λ is a slope, the multiplicity of λ, denoted $\mathrm{mult}(\lambda)$, is m times the number of factors $P(m, n)$ such that $\lambda = n/m$; slopes and their multiplicities depend only on H. The *Newton polygon* of H is the increasing polygonal line starting at $(0, 0)$, having slope λ_i with multiplicity $\mathrm{mult}(\lambda_i)$, where $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r$ is the sequence of slopes of H. If $k = \mathbb{F}_q$, $q = p^a$, the Newton polygon of H coincides with the usual Newton polygon of the polynomial $\det(1 - F^a t, H) \in W[t]$,

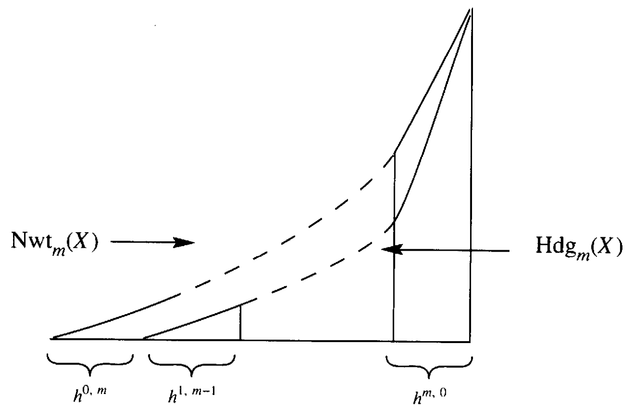

Figure 1

i.e., the slopes are the valuations (normalized by $v(q) = 1$) of the eigenvalues of the W-linear operator F^a, and their multiplicities are the multiplicities of the corresponding eigenvalues. The Katz conjecture is part (i) of the following result:

THEOREM 1.3.9 (Mazur-Ogus). *Let X be in $\mathscr{C}$. Denote by $\mathrm{Nwt}_m(X)$ the Newton polygon of $(H^m(X/W)/\text{torsion}, \phi)$. Define the Hodge polygon of X in degree m denoted $\mathrm{Hdg}_m(X)$, as the increasing polygonal line starting at $(0, 0)$, having slope i with multiplicity the ith Hodge number $h^{i,m-i} = \dim_k H^{m-i}(X, \Omega^i_{X/K})$. Then*

(i) *$\mathrm{Nwt}_m(X)$ lies on or above $\mathrm{Hdg}_m(X)$;*

(ii) *Assume that $H^*(X/W)$ is torsion free and that the Hodge to de Rham spectral sequence $E_1^{ij} = H^j(X, \Omega^i_{X/K}) \Rightarrow H^*_{\mathrm{DR}}(X/k)$ degenerates at E_1. Then $h^{i,m-i}$ is the multiplicity of the elementary divisor p^i of the W-linear map $\phi^\sim : \sigma^* H^m(X/W) \to H^m(X/W)$ defined by ϕ; moreover $\mathrm{Nwt}_m(X)$ and $\mathrm{Hdg}_m(X)$ have the same endpoint (b_m, c_m), where $b_m = \mathrm{rk}\, H^m(X/W)$, $c_m = \sum i h^{m-i} = \mathrm{length}\, H^m(X/W)/\mathrm{Im}\, \phi^\sim$.*[2]

In case (ii), the picture is as shown in Figure 1.

As a corollary of (i) and of the cohomological expression (1.3.4) for the zeta function, one finds that, if $k = \mathbb{F}_q$ and X/k is a smooth, complete intersection of dimension d and multidegree $\mathbf{a} = (a_1, \dots, a_r)$ in $\mathbb{P}_k^{d+r}$, then one has

$$Z(X/k, t)/Z(\mathbb{P}_k^d/k, t) \in \mathbb{Z}[[q^c t]], \tag{1.3.10}$$

where c is the smallest nonnegative integer such that $h^{c,d-c} \neq 0$ (see [26] for a formula giving c in terms of $\mathbf{a}$), or equivalently

$$N_s(X/k) = N_s(\mathbb{P}_k^d/k) \mod q^{cs}$$

[2]When X is projective, the strong Lefschetz theorem implies that $c_m = mb_m/2$.

for any integer $s \geq 1$, where N_s denotes a number of points with values in $\mathbb{F}_{q^s}$. Katz proved the above congruence in **[75]** as a consequence of a more general one, valid for affine, possibly singular varieties, generalizing previous results of Chevalley–Warning **[115]** and Ax **[4]**, and he raised the question of deriving it from an inequality of type (i). This inequality was first proven, as well as (ii), by Mazur **[85]** for varieties X admitting a projective and smooth lifting Z over W whose Hodge cohomology $H^*(Z, \Omega^*_{Z/W})$ has no torsion. Then 1.3.9 was proven in general by Ogus **[16**, §8**]**. A different proof, using the theory of the de Rham-Witt complex (see (d)), was given by Nygaard **[96]**. Quite recently, far-reaching generalizations of 1.3.9 for crystalline cohomology with coefficients in certain F-crystals have been obtained by Ogus **[104]**.

Coming back to 1.3.9, we see that the lowest possible position for $\mathrm{Nwt}_m(X)$ is $\mathrm{Hdg}_m(X)$. When $H^*(X/W)$ has no torsion and $\mathrm{Nwt}_m(X) = \mathrm{Hdg}_m(X)$ for all m, X is said to be *ordinary.*[3] For example, if X is an abelian variety of dimension g over k, then X is ordinary if and only if X is ordinary in the usual sense, i.e., ${}_pX(\overline{k}) = (\mathbb{Z}/p\mathbb{Z})^g$. It is known that X is ordinary if X is a "sufficiently general" curve of genus g **[89]**, or polarized abelian variety **[93, 95]**, or polarized $K3$ surface **[100]**, or smooth complete intersection (of fixed dimension and multi-degree) **[61]**.

The fact that "general" varieties tend to be ordinary is related to another basic result on Newton polygons, namely Grothendieck's specialization theorem, which can be stated as follows:

THEOREM 1.3.11. *Let S be a k-scheme of finite type and $X \to S$ be a proper and smooth morphism. Then there is a finite partition of S into locally closed subsets S_i $(i \in I)$ such that for any $i \in I$ and $m \in \mathbb{Z}$, the Newton polygon* $\mathrm{Nwt}_m(X_{\overline{s}})$ *of $H^m(X_{\overline{s}}/W(k(\overline{s})))/torsion$ is independent of the geometric point $\overline{s}$ of S_i (where "geometric" means "with value in the spectrum of an algebraically closed extension of k"). Furthermore, if $\overline{s}$ (resp. $\overline{t}$) is a geometric point of S localized at s (resp. t), and if s is a specialization of t, i.e. $s \in \overline{\{t\}}$, then* $\mathrm{Nwt}_m(X_{\overline{s}})$ *is above* $\mathrm{Nwt}_m(X_{\overline{t}})$, *and both polygons have the same endpoint.*

This result, with a sketch of proof, is mentioned by Grothendieck in a letter to Barsotti **[51]**. An abstract variant of 1.3.11 for F-crystals is proven in **[31]**, **[51]**, and **[78]**. This variant is unfortunately not sufficient for 1.3.11, whose proof in full generality was given by Crew **[23]**, using the theory of convergent crystals of Berthelot-Ogus **[101]**.

For curves, abelian varieties, and $K3$ surfaces, there are much more precise results available than those of "ordinariness" quoted above. Indeed, for the corresponding modular varieties, stratifications by the Newton polygon,

[3] A generalization of this notion, taking torsion into account, is discussed in **[19]** and **[65]**.

whose existence is ensured by 1.3.11,[4] have been the focus of considerable study for the past twenty years. Let me just mention Koblitz's thesis **[81]**, which is the first systematic work on this topic, the works of Artin **[1]** and Ogus **[100]** on $K3$ surfaces, and the report of Oort **[106]**, giving a survey of the very complete results obtained by Oort and others, in the case of abelian varieties.

(d) *De Rham-Witt theory.* For X in $\mathscr{C}$, $H^*(X/W)$ can be calculated as the hypercohomology of a very nice complex on X, called the de Rham-Witt complex of X, and denoted by $W\Omega_X^{\cdot}$. This complex was first constructed for $p > 2$ and $\dim(X) < p$ by Bloch in his seminal paper **[18]**. A general construction, following an idea of Deligne (inspired by Lubkin's work **[84]**), was given in **[58]**. The de Rham-Witt complex

$$W\Omega_X^{\cdot} = (W\mathscr{O}_X \to W\Omega_X^1 \to \cdots \to W\Omega_X^i \to \cdots),$$

which is a certain quotient of the (suitably completed) de Rham complex of the sheaf of Witt vectors $W\mathscr{O}_X$ over W, comes equipped with operators F, V in each degree, extending the classical ones on $W\mathscr{O}_X$, and satisfying $FV = VF = p$, $FdV = d$; the endomorphism of $W\Omega_X^{\cdot}$ given by p^iF in degree i yields the Frobenius ϕ of (c) by passing to the hypercohomology; $W\Omega_X^i = 0$ for $i > \dim X$. The resulting spectral sequence

$$E_1^{ij} = H^j(X, W\Omega_X^i) \Rightarrow H^*(X/W), \tag{1.3.12}$$

called the *slope spectral sequence*, degenerates at E_1 modulo torsion, and expresses the part of $H^m(X/W)\otimes K$ of slopes $\in [i, i+1[$ as $H^{m-i}(X, W\Omega_X^i)\otimes K$, equipped with p^iF. This sequence, whose construction was the initial goal of the theory, is studied in **[58]**, and in more detail in **[65]**, where the torsion is closely analysed. Together with other remarkable properties of the de Rham-Witt complex, it proved to be a powerful tool in the study of the relations of $H^*(X/W)$ with other cohomology groups attached to X, such as Serre's groups $H^*(X, W\mathscr{O}_X)$ **[110]**, the de Rham and Hodge cohomology groups of X/k, the flat coholomology groups $H^*(X, \mu_{p^n})$ of Milne **[90]**, and the higher formal Picard groups $R^if_*(\mathbb{G}_m)^\wedge$ of Artin-Mazur **[2]**. The theory provides, in particular, a natural framework for "explaining" the origin of the torsion in $H^2(X/W)$, or the pathology in the Hodge groups of X (such as nondegeneration of the Hodge to de Rham spectral sequence) discovered much earlier by Serre, Mumford, and others (cf., e.g., **[58, 113]**). It also gives a new, direct way to construct Poincaré duality and the theory of Chern and cycle classes in crystalline cohomology. See **[58, 65, 96, 32, 33, 34, 46]** for the main results, and **[57, 59]** for surveys. Finally, deep relations between the de Rham-Witt complex and the Milnor K-groups, that were in germ in Bloch's paper **[18]**, were later found by Bloch-Gabber-Kato **[19]**, who used them as

[4] In this case, the "abstract variant" is sufficient.

a key technical tool in the comparison between crystalline cohomology and p-adic étale cohomology (see 3.3.6). These relations have also played an important role in subsequent work of Kato and others on higher-class field theory and the structure of the p-torsion part of the Chow groups (see, e.g., **[21, 22, 67, 68, 71, 73, 74]**).

2. De Rham cohomology in mixed characteristic

2.1. In this section, A is a complete discrete valuation ring of perfect residue field k of characteristic $p > 0$ and field of fractions K of characteristic 0, K_0 is the field of fractions of $W = W(k)$, and $e = [K : K_0]$ the absolute ramification index of A. We denote by $\mathscr{C}_K$ (resp. $\mathscr{C}_k$) the category of proper and smooth schemes over K.

For X_K in $\mathscr{C}_K$, the de Rham cohomology

$$H^*_{\mathrm{DR}}(X_K/K) = H^*(X_K, \Omega^{\cdot}_{X_K/K})$$

has a rather intricate structure, which we shall now describe.

2.2. Hodge filtration. The first element of structure is the Hodge filtration, which is the decreasing filtration

$$\begin{aligned}\mathrm{Fil}^0 H^m_{\mathrm{DR}}(X_K/K) = H^m_{\mathrm{DR}}(X_K/K) \supset \cdots \supset \mathrm{Fil}^i H^m_{\mathrm{DR}}(X_K/K) \\ \supset \cdots \supset \mathrm{Fil}^m H^m_{\mathrm{DR}}(X_K/K) \supset 0\end{aligned}$$

given by the (degenerate at E_1) Hodge to de Rham spectral sequence

$$E_1^{ij} = H^j(X_K, \Omega^i_{X_K/K}) \Rightarrow H^*_{\mathrm{DR}}(X_K/K).$$

Thus

$$\mathrm{Fil}^i H^m_{\mathrm{DR}}(X_K/K) = H^m(X_K, \Omega^{\geq i}_{X_K/K}),$$

and

$$\mathrm{gr}^i H^m_{\mathrm{DR}} = H^{m-i}(X_K, \Omega^i_{X_K/K}).$$

To describe the other elements of structure, we shall use the "crutch" of a model of X_K over A. As we shall see in the next section, it is hoped that one can get rid of it.

2.3. The case of good reduction. Assume that there is a proper and smooth scheme X over A such that $X \otimes K = X_K$. Let $Y = X \otimes_A k$ be the special fiber of X. Thus Y is in $\mathscr{C}_k$, and we can consider its crystalline cohomology $H^*(Y/W)$, a finitely generated graded W-module, endowed with the σ-linear Frobenius operator ϕ, such that $\phi \otimes \mathbb{Q}$ is bijective. If $e \leq p - 1$, so that the maximal ideal of A has divided powers, there is defined in Berthelot's thesis **[5]** a canonical isomorphism

$$(2.3.1) \qquad H^*(Y, W) \otimes_W A \xrightarrow{\sim} H^*_{\mathrm{DR}}(X/A) \quad (= H^*(X, \Omega^{\cdot}_{X/A}))$$

(a slight generalization of (1.3.8)). When $e > p-1$, it is no longer true in general that $H^*_{\mathrm{DR}}(X/A)$ is isomorphic to $H^*(Y/W) \otimes A$, but Berthelot-Ogus **[15]** construct a canonical isomorphism

$$H^*(Y/W) \otimes_W K \xrightarrow{\sim} H^*_{\mathrm{DR}}(X/A) \otimes K = H^*_{\mathrm{DR}}(X_K/K). \tag{2.3.2}$$

By (2.3.2),

$$H^*_0(X_K) := H^*(Y/W) \otimes_W K_0$$

is a K_0-structure of $H^*_{\mathrm{DR}}(X_K/K)$, endowed with the σ-linear automorphism ϕ defined by Frobenius. The structure thus obtained on $H^*_{\mathrm{DR}}(X_K/K)$,

$$(\mathrm{Fil}^{\bullet}, H^*_0(X_K), \phi), \tag{2.3.3}$$

makes $H^*_{\mathrm{DR}}(X_K/K)$ into an object of the category $MF_K(\phi)$ of Fontaine, i.e., a *filtered ϕ-module over K* (see **[41]** or **[42]**). It follows from Falting's results (see 3.2.3), that (2.3.3) depends only on X_K and not on the model X. That had been directly proven much earlier by Messing **[45**, Appendix B]. This verification played in fact a key role in the formulation of Fontaine's crystalline conjecture (see 3.2).

2.4. The case of semi-stable reduction. Assume that there is a proper and flat scheme X over A such that $X \otimes K = X_K$ and such that X has semi-stable reduction, which means that, locally for the étale topology (both on the source and on the target), X is isomorphic to the subscheme of $\mathrm{Spec}\, A[t_1, \ldots, t_m]$ defined by the equation $t_1 \cdots t_r = t$, where t is a uniformizing parameter of A. Thus X is regular, and the special fiber $Y = X \otimes_A k$ is a divisor with normal crossings in X. When Y is singular, the crystalline cohomology groups $H^*(Y/W)$ defined by (1.1.2) are bad (e.g., $H^*(Y/W) \otimes K_0$ may be infinite dimensional), but Y inherits from X an additional structure, called a *log structure* **[67]** (depending in fact only on the reduction $X \bmod t^2$), by means of which it is possible to define new crystalline cohomology groups

$$H^*(\underline{Y}/\underline{W}), \tag{2.4.1}$$

having good properties (the underlined letter $\underline{Y}$ (resp. $\underline{W}$) means that Y (resp. W) is equipped with a certain log structure). The basic facts concerning the theory of log structures are given in **[67]**. The groups (2.4.1) were first defined by Hyodo **[53]** using a variant of the de Rham-Witt complex for log schemes of the form $\underline{Y}$. His constructions are extended to more general log schemes in **[54]**. See also **[42, 62]** for an introduction and **[64]** for a survey. I will limit myself to briefly explaining the structure and properties of these groups.

First of all, $H^*(\underline{Y}/\underline{W})$ is finitely generated over W, and endowed with a σ-linear Frobenius operator ϕ such that $\phi \otimes \mathbb{Q}$ is bijective (so $H^*(\underline{Y}/\underline{W})$ / torsion is an F-crystal). When X_A is smooth, $H^*(\underline{Y}/\underline{W}) = H^*(Y/W)$, and ϕ is the operator considered previously. Next—this is the new thing— there is defined a W-linear map, called the *monodromy operator*, $N\colon H^*(\underline{Y}/\underline{W}) \to H^*(\underline{Y}, \underline{W})$ satisfying

$$N\phi = p\phi N \tag{2.4.2}$$

(which implies in particular that $N \otimes \mathbb{Q}$ is nilpotent). This operator N is the analogue of the residue at 0 of the Gauss-Manin connection in the case of a proper and flat family over the complex disc having semi-stable reduction at the origin, or of the logarithm of the t_ℓ-part of the monodromy on $H^*(X_{\overline{K}}, \mathbb{Q}_\ell)$ $(\ell \neq p)$, where $t_\ell\colon I \to \mathbb{Z}_\ell(1)$ is the ℓ-tame quotient of the inertia group (cf. (2.6.2) below); see **[42]** for a discussion along these lines. Finally Hyodo-Kato **[54]** construct a canonical isomorphism (depending on the choice of a uniformizing parameter t)

$$\rho_t\colon H^*(\underline{Y}/\underline{W}) \otimes_W K \xrightarrow{\sim} H^*_{\mathrm{DR}}(X_K/K), \tag{2.4.3}$$

generalizing (2.3.2) (if u is a unit, $\rho_t = \rho_{ut}\exp(\log(u)N)$, where log is the usual p-adic logarithm, vanishing on k^*). (An independent construction of (2.4.3), at least in the case $e = 1$, has recently been given by Ogus **[105]**.) In particular, the rank of $H^m(\underline{Y}, \underline{W})$ is the mth Betti number of X_K. By (2.4.3),

$$H^*_0(X) := H^*(\underline{Y}/\underline{W}) \otimes_W K_0 \tag{2.4.4}$$

is a K_0-structure of $H^*_{\mathrm{DR}}(X_K/K)$, endowed with the σ-linear automorphism ϕ and the nilpotent endomorphism N deduced by extension of scalars, related by (2.4.2). The resulting structure on $H^*_{\mathrm{DR}}(X_K/K)$,

$$(\mathrm{Fil}^{\cdot}, H^*_0(X), \phi, N),$$

which is an analogue of the *limit Hodge structure* studied in **[112]**, makes $H^*_{\mathrm{DR}}(X_K/K)$ into an object of the category $MF_K(\phi, N)$ of Fontaine, i.e., a *filtered* (ϕ, N)*-module over* K **[41, 42]**. As we shall see in the next section, it follows from the conjecture C_{st} of Fontaine that this structure should depend only on X_K and not on the semi-stable model X. This should also follow from a suitable log variant of the arguments of Gillet-Messing **[45]**.

2.5. The general case. Let $\overline{K}$ be an algebraic closure of K, $G = \mathrm{Gal}(\overline{K}/K)$, K_0^{ur} the maximal unramified extension of K_0 contained in $\overline{K}$. One can hope that there exists a finite Galois extension K' of K contained in $\overline{K}$ over which X_K acquires semi-stable reduction (this is the case for curves by the semi-stable reduction theorem of Grothendieck et al. **[49, 30, 3]** and in general, thanks to Hironaka's resolution of singularities and Mumford's theorem **[94]**, in the analogous situation in equal characteristic zero). Let X be a semi-stable model of $X_{K'} = X_K \otimes K'$ over the ring of integers of K'. Then $H^*_0(X)$ is a K'_0-structure of $H^*_{\mathrm{DR}}(X_{K'}/K')$ (where K'_0 is the maximal unramified extension of K_0 contained in K'), equipped with the pair of operators (ϕ, N) described above. Denote again by (ϕ, N) the operators obtained on $K_0^{\text{ur}} \otimes_{K_0'} H_0^*(X)$ by extension of scalars. On this module, we have in addition a discrete action of G (i.e., such that the stabilizer of every element is open). Again a general conjecture of Fontaine (see §3.3) predicts that the structure

$$(\text{Fil}^{\cdot}, K_0^{\text{ur}} \otimes_{K_0'} H_0^*(X), \phi, N, \text{ action of } G)$$

on $H^*_{\text{DR}}(X_K/K)$ should depend only on X_K.

2.6. Conjectures of Weil type. Let us come back to the hypotheses of 2.4, with $k = \mathbb{F}_q$, $q = p^a$. The conjectures concern the comparison of the characteristic polynomial

$$P_W^m = \det(1 - \phi^a t, H_0^m(X)) \in W[t] \tag{2.6.1}$$

with the analogous ℓ-adic ones. Let $\overline{K}$ and G be as in §2.5, $\overline{k}$ be the corresponding algebraic closure of k, and $I = \operatorname{Ker} G \to \operatorname{Gal}(\overline{k}/k)$, be the inertia subgroup. Let $F_q \in \operatorname{Gal}(\overline{k}/k)$ be the geometric Frobenius $(x \mapsto x^{1/q})$ and let ℓ be a prime $\neq p$. Since X has semi-stable reduction, I acts unipotently on $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$ through its ℓ-tame quotient $t_\ell : I \to \mathbb{Z}_\ell(1)$ by the formula

$$g \cdot x = \exp(Nt_\ell(g))x, \qquad g \in I, \ x \in H^m(X_{\overline{K}}, \mathbb{Q}_\ell), \tag{2.6.2}$$

where

$$N : H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(1) \to H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is a nilpotent map (the logarithm of the monodromy) uniquely determined by (2.6.2) (see **[SGA 7, I]**, **[28, 107, 63]** for a survey). By the uniqueness, N is G-equivariant; so if $F \in G$ lifts F_q, one has the formula

$$NF = qFN, \tag{2.6.3}$$

which is similar to $N\phi^a = q\phi^a N$ deduced from (2.4.2). It follows from (2.6.2) and (2.6.3) that the characteristic polynomial

$$P_\ell^m = \det(1 - Ft, H^m(X_{\overline{K}}, \mathbb{Q}_\ell)) \in \mathbb{Z}_\ell[t] \tag{2.6.4}$$

depends only on F_q. The first form of the conjecture is the following:

CONJECTURE 2.6.5. *The polynomials* (2.6.1) *and* (2.6.4) *coincide and have integer coefficients. Moreover, if* α *is a reciprocal root of* $P_W^m = P_\ell^m$, α *is a Weil number of weight* $\in [0, 2m]$.

When X/A is smooth, $N = 0$, F acts through F_q on $H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \simeq H^m(X_{\overline{k}}, \mathbb{Q}_\ell)$, and (at least for X projective) 2.6.5 boils down to Katz-Messing's result (1.3.5). The case of curves should be accessible, but, to our knowledge, has not yet been treated. A stronger form of 2.6.5 involves the monodromy filtrations. Let (M_r) be the increasing filtration on $H_0^m(X)$ (resp. $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$) determined by the nilpotent operator N, i.e., characterized by $NM_i \subset M_{i-2}$ and $N^r : \operatorname{gr}_r^M \xrightarrow{\sim} \operatorname{gr}_{-r}^M$. Because of (2.4.2) (resp.

(2.6.3)), this filtration is stable under ϕ^a (resp. F). Moreover I acts trivially on gr_i^M, so the action of F on gr_i^M depends only on F_q. The polynomials P_W^m and P_ℓ^m decompose as

$$P_W^m = \prod P_{W,r}^m, \qquad P_\ell^m = \prod P_{\ell,r}^m,$$

where

$$P_{W,r}^m = \det(1-\phi^a t, \mathrm{gr}_r^M H_0^m(X)), \qquad P_{\ell,r}^m = \det(1-F_q t, \mathrm{gr}_r^M H^m(X_{\overline{K}}, \mathbb{Q}_\ell)).$$

Then one can make the following more precise conjecture:

CONJECTURE 2.6.6. (a) *If α is a reciprocal root of $P_{W,r}^m$ (resp. $P_{\ell,r}^m$), α is a Weil number of weight $m+r$.*

(b) *One has $P_{W,r}^m = P_{\ell,r}^m$ and $P_{W,r}^m$ has integer coefficients.*

Part (a) is usually called the *conjecture of purity of the weight filtration.* It is known for $\dim X_K \leq 2$, X projective, and Y with simple normal crossing in the ℓ-adic case **[107]**, and for $\dim X_K \leq 2$ in the crystalline case **[91]**. It seems that the only known thing on (b) is that, if $\dim X_K \leq 1$, the polynomials $P_{\ell,r}^m$ are independent of ℓ and have integer coefficients by Grothendieck's theorem (SGA 7, IX, 4.3) (modulo some identification essentially contained in the "Picard Lefschetz formula" (SGA 7, IX, 12.5)).

3. De Rham cohomology and p-adic étale cohomology

We keep the notations of 2.1 and denote again by $\overline{K}$ an algebraic closure of K and $G = \mathrm{Gal}(\overline{K}/K)$. For X_K in $\mathscr{C}_K$, G acts on the p-adic étale cohomology $H^*(X_{\overline{K}}, \mathbb{Q}_p)$. The so-called theory of p-adic periods compares this representation with the de Rham cohomology $H^*_{\mathrm{DR}}(X_K/K)$, endowed with the structure described in §2. According to the type of assumption made on X_K, there are several period isomorphisms, involving the various Barsotti-Tate rings constructed by Fontaine

$$B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{DR}}, B_{\mathrm{HT}}.$$

The definitions and main properties of these rings are summarized in the surveys **[42]** and **[62]**. Proofs are given in **[41]**. The period isomorphisms are extensively discussed in **[42]** and **[62]**, to which we could just refer the reader. For his convenience we shall nevertheless briefly recall them below.

3.1. Hodge-Tate decompositions and the C_{DR} conjecture. Let C be the completion of $\overline{K}$. C is again algebraically closed, and G acts continuously on it. By Tate **[114]**, one has

$$H^0(G, C(i)) = \begin{cases} K & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \tag{3.1.1}$$

where (i) denotes the usual Tate twist by the ith-power of the cyclotomic character.

THEOREM 3.1.2 (Faltings [35]). *For X_K in $\mathscr{C}_K$, there exists a natural G-equivariant isomorphism*

$$\bigoplus(C \otimes_K H^{m-i}(X_K, \Omega^i)(-i)) \xrightarrow{\sim} C \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p)$$

(when $\Omega^i = \Omega^i_{X_K/K}$ and G acts diagonally on the right-hand side).

See [62] for the history of this isomorphism, called the *Hodge-Tate decomposition*, initially conjectured by Tate [111].

From 3.1.2 one deduces, using (3.1.1), that the p-adic representation $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ determines the Hodge numbers $h^{i,m-i} = \dim H^{m-i}(X_K, \Omega^i)$:

$$h^{i,m-i} = \dim_K(C \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p)(i))^G.$$

By introducing, with Fontaine, the graded ring

$$B_{\mathrm{HT}} := \bigoplus_{i \in \mathbb{Z}} C(i)$$

(with its natural action of G), one can rewrite the Hodge-Tate decomposition in the form of a G-equivariant isomorphism

$$B_{\mathrm{HT}} \otimes_K \left(\bigoplus H^{m-i}(X_K, \Omega^i)\right) \xrightarrow{\sim} B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p), \tag{3.1.3}$$

compatible with the natural gradings on both sides ($C(j) \otimes H^{m-i}(X_K, \Omega^i)$ being of degree $i + j$). Then the Hodge cohomology $H^{m-*}(X_K, \Omega^*) = \bigoplus H^{m-i}(X_K, \Omega^i)$ is recovered from the p-adic étale cohomology by

$$H^{m-*}(X_K, \Omega^*) \xrightarrow{\sim} (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p))^G.$$

There is a similar formula for recovering the de Rham cohomology, which involves the ring B_{DR}, a K-algebra that is a complete discrete valuation field, endowed with an action of G, and whose associated graded algebra for the filtration $\mathrm{Fil}^{\cdot}$ defined by the valuation is just B_{HT} (with its G-action). The following result was conjectured by Fontaine [39] (C_{DR} *conjecture*):

THEOREM 3.1.4 (Faltings [36]). *For X_K in $\mathscr{C}_K$, there exists a natural isomorphism*

$$B_{\mathrm{DR}} \otimes_K H^m_{\mathrm{DR}}(X_K/K) \xrightarrow{\sim} B_{\mathrm{DR}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p),$$

compatible with the filtrations and actions of G on both sides.

By (3.1.1), $(B_{\mathrm{DR}})^G = K$, and therefore $H^m_{\mathrm{DR}}(X_K/K)$, a filtered K-module, is recovered from $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ by

$$H^m_{\mathrm{DR}}(X_K/K) \xrightarrow{\sim} (B_{\mathrm{DR}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p))^G.$$

The period isomorphism 3.1.4 does not permit, of course, to recover the p-adic étale cohomology, as Galois representation, from the de Rham cohomology, as filtered K-vector space. However, finer period isomorphisms, taking into account the extra structure on the de Rham cohomology discussed in §2, and involving the other rings B_{cris}, B_{st} of Fontaine, enable one to do it in some cases (and, conjecturally, in general).

3.2. Good reduction and the C_{cris} conjecture. The ring B_{cris} is a sub G-K_0-algebra of B_{DR}, such that $K \otimes_{K_0} B_{\text{cris}} \to B_{\text{DR}}$ is injective and induces an isomorphism on the associated graded objects, $K \otimes_{K_0} B_{\text{cris}}$ being endowed with the induced filtration. In addition, B_{cris} is equipped with a σ-K_0-linear endomorphism ϕ, and

$$\mathbb{Q}_p = \{x \in B_{\text{cris}} | 1 \otimes x \in \text{Fil}^0 B_{\text{DR}} \text{ and } \phi x = x\}. \tag{3.2.1}$$

Moreover

$$B_{\text{cris}}^G = K_0. \tag{3.2.2}$$

The following result, conjectured by Fontaine **[40]** (C_{cris} *conjecture*), was first proven by Fontaine-Messing **[44]** in the case $e = 1$ and relative dimension $< p$ (or $m < p$):

THEOREM 3.2.3 (Faltings **[36]**). *For X/A proper and smooth, there exists a natural isomorphism*

$$B_{\text{cris}} \otimes_{K_0} H_0^m(X_K) \xrightarrow{\sim} B_{\text{cris}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p)$$

(*with $H_0^m(X_K)$ as in* (2.3.3)), *compatible with the actions of ϕ and G on both sides, as well as with the filtrations after extending the scalars to K.*

As above, thanks to (3.2.2), $H_0^m(X_K)$, with the action of ϕ, is recovered from $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ by

$$H_0^m(X_K) \xrightarrow{\sim} (B_{\text{cris}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p))^G,$$

and the Hodge filtration on $K \otimes_{K_0} H_0^m(X_K) \xrightarrow{\sim} H_{\text{DR}}^m(X_K/K)$ is obtained by a similar formula: $H_{\text{DR}}^m(X_K/K)$ as an object of $MF_K(\phi)$ is determined by the Galois representation $H^m(X_{\overline{K}}, \mathbb{Q}_p)$. But conversely this representation is determined by $H_{\text{DR}}^m(X_K/K)$ as object of $MF_K(\phi)$: thanks to (3.2.1), the period isomorphism 3.2.3 gives

$$H^m(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\sim} \{x \in B_{\text{cris}} \otimes H_0^m(X_K) \mid \phi x = x, \ 1 \otimes x \in \text{Fil}^0\}$$

as representations of G.

When $\dim X_K < p$ and $e = 1$ (i.e. $A = W$), more precise results due independently to Fontaine-Messing (unpublished) and Faltings **[36]** show that the Galois representation $H^m(X_{\overline{K}}, \mathbb{Z}_p)$ and the de Rham cohomology $H_{\text{DR}}^m(X/A)$ with its W-structure $H^m(Y/W)$ endowed with the Frobenius ϕ, where $Y = X \otimes_A K$ (cf. (1.3.8)), mutually determine each other. In particular, if $(-)_{\text{tor}}$ denotes the torsion part,

$$\text{lgth}\, H_{\text{DR}}^m(X/A)_{\text{tor}} = \text{lgth}\, H^m(X_{\overline{K}}, \mathbb{Z}_p)_{\text{tor}}. \tag{3.2.4}$$

There is even a stronger result, stated at the very end of **[44]**, namely that the invariant factors for $H^m(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$ coincide with those for $H_{\text{DR}}^m(X \otimes W_n/W_n)$.

3.3. Semi-stable reduction: the C_{st} and C_{pst} conjectures. Associated to a uniformizing parameter π of A there is defined a sub-G-K_0-algebra B_{st} of B_{DR}, containing B_{cris}, such that $K \otimes_{K_0} B_{st} \to B_{DR}$ is injective, and which is a polynomial algebra in one generator as a B_{cris}-algebra. This algebra is endowed with a σ-K_0-linear operator ϕ extending the operator ϕ on B_{cris} and a B_{cris}-derivation N, called the *monodromy operator*, satisfying

(3.3.1) $$N\phi = p\phi N.$$

One has

(3.3.2) $$B_{cris} = \operatorname{Ker} N \colon B_{st} \to B_{st};$$

hence, by (3.2.1),

(3.3.3) $$\mathbb{Q}_p = \{x \in B_{st} \mid 1 \otimes x \in \operatorname{Fil}^0 B_{DR},\ \phi x = x,\ Nx = 0\}.$$

Furthermore,

(3.3.4) $$B_{st}^G = K_0.$$

The following conjecture, made by Fontaine [**41**], was inspired by ideas and work of Jannsen (cf. [**66**]):

CONJECTURE 3.3.5 (C_{st} conjecture). *For X/A proper and with semi-stable reduction, there exists a natural isomorphism*

$$B_{st} \otimes_{K_0} H_0^m(X) \xrightarrow{\sim} B_{st} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p)$$

(with $H_0^m(X)$ as in (2.4.4))[5] *compatible with the actions of ϕ, N, G on both sides as well as with the filtrations after extending the scalars to K (on the left-hand side N acts by $N \otimes 1 + 1 \otimes N$).*

THEOREM 3.3.6 (Kato [**70**]). *The C_{st} conjecture is true for* $\dim X_k < (p-1)/2$.

The proof uses a variant of the method of Fontaine–Messing [**44**], relying on a calculation of p-adic vanishing cycles due to Hyodo [**52**] and Kurihara [**82**], generalizing to the semi-stable case the one made by Bloch-Kato [**19**] in the good reduction case. It has not been possible so far to apply Falting's method in the proofs of 3.1.4 and 3.2.3 to remove the restriction of dimension. Even the case $m = 1$ (but $\dim X_K$ arbitrary) is unknown. See [**37**], however, for variants "with coefficients" of 3.3.5 in the case $\dim X_K = 1$.

The C_{st} conjecture implies that the Galois representation $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ and the object of $MF_K(\phi, N)$ defined by $H^m_{DR}(X_K/K)$ with its additional structures (2.4) mutually determine each other. Indeed, by (3.3.4), $H_0^m(X)$ with the action of ϕ and N is recovered from $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ by

(3.3.6.1) $$H_0^m(X) \xrightarrow{\sim} (B_{st} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p))^G$$

[5] and considered as a K_0-structure of $H^m_{DR}(X_K/K)$ by means of the isomorphism (2.4.3) associated to the same uniformizing parameter π.

(and similarly for the Hodge filtration on $K \otimes_{K_0} H_0^m(X) \xrightarrow{\sim} H_{\mathrm{DR}}^m(X_K/K)$), and conversely, by (3.3.3), 3.3.5 gives

(3.3.6.2)

$$H^m(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\sim} \{x \in B_{\mathrm{st}} \otimes H_0^m(X) \mid \phi x = x,\ Nx = 0,\ 1 \otimes x \in \mathrm{Fil}^0\}.$$

as representations of G.

However, for $e = 1$ and $\dim X_K < (p-1)/2$, a formula of the form (3.2.4) (with $H_{\mathrm{DR}}^m(X/W)$ defined as $H^m(X, \omega^{\cdot}_{X/W})$ where $\omega^{\cdot}_{X/W}$ is the relative de Rham complex with log poles along $X \otimes k$ (cf. [**64**])) is not known to hold—or even to be plausible!

Finally, consider the general case of an arbitrary proper and smooth K-scheme X_K. Assume first that the *geometric semi-stable reduction conjecture* is true, namely that there exists a finite Galois extension K' of K contained in $\overline{K}$ such that X_K acquires semi-stable reduction over K'. Let

$$H_0^m(X_K) := K_0^{\mathrm{ur}} \otimes_{K_0'} H_0^m(X')$$

be the K_0^{ur}-structure of $\overline{K} \otimes_K H_{\mathrm{DR}}^m(X_K/K)$ considered in 2.5, with its actions of ϕ, N, and G. The C_{st} conjecture would then imply the existence of a natural isomorphism

$$B_{\mathrm{st}} \otimes_{K_0^{\mathrm{ur}}} H_0^m(X_K) \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p) \qquad (3.3.7)$$

compatible with the actions of ϕ, N, G on both sides and the filtrations after extending the scalars to K (it is known that if ℓ is a finite extension of K contained in $\overline{K}$, then the natural map from the ring B_{st} relative to K to that relative to L is an isomorphism, so in particular B_{st} contains K_0^{ur}). By (3.3.7), the Galois representation $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ and the de Rham cohomology $H_{\mathrm{DR}}^m(X_K/K)$ with its additional structures described above would mutually determine each other through formulas analogous to (3.3.6.1) and (3.3.6.2): to get $H_0^m(X_K)$, apply $\varinjlim K_0^{\mathrm{ur}} \otimes_L (-)^{G_L}$ to $(-) = B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p)$ where L runs through the finite Galois extensions L of K contained in $\overline{K}$ and $G_L = \mathrm{Gal}(\overline{K}/L)$ (this $\varinjlim$ stabilizes); conversely, to get $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ as representation of G, take an L such that G_L acts trivially on $H_0^m(X_K)$, so that $H_0^m(X_K)$ comes from a G-L_0-vector space V_0 and take $\{x \in B_{\mathrm{st}} \otimes_{L_0} V_0 \mid \phi x = x,\ Nx = 0,\ 1 \otimes x \in \mathrm{Fil}^0\}$ (this gives a p-adic representation of G_L *together with* an action of $\mathrm{Gal}(L/K)$, which is the desired representation of G). If one does not believe in the geometric semi-stable reduction conjecture, one can still hope that $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ is associated to an abstract object of the form $H_0^m(X_K)$ by a formula of type (3.3.7): this is the C_{pst} *conjecture* of Fontaine ([**41, 42**]), or *p-adic monodromy conjecture*, which is in some sense an analogue of Grothendieck's local ℓ-adic monodromy theorem (SGA 7 I).

4. Towards a theory of crystalline coefficients

Elaborate as it is, crystalline cohomology is still far from having reached the degree of achievement of the other existing cohomology theories, i.e., Hodge theory and ℓ-adic cohomology (see Steenbrink's and Katz's exposés in this volume). What is lacking is a theory of *coefficients* in the sense of Grothendieck, i.e., the construction of a good derived category of crystalline sheaves satisfying a formalism of *six operations* ($\overset{L}{\otimes}$, $R\operatorname{Hom}$, Lf^*, Rf_*, $Rf_!$, $Rf^!$). That does not mean that crystalline cohomology has always been confined to the rather narrow framework discussed in §1. From the very start, some nice coefficients, called *F-crystals*, have been considered and used successfully in a number of questions, and in the past few years, there have been several interesting developments, about which I will briefly report below.

4.1. ***F*-crystals, periods and Dieudonné theory.** The basic notion of coefficients is that of crystal, which, as was said at the beginning, was introduced by Grothendieck in his letter to Tate **[47]**:

> "un cristal possède deux propriétés caractéristiques: la rigidité et la faculté de croître, dans un voisinage approprié. Il y a des cristaux de toute espèce de substance: des cristaux de soude, de soufre, de modules, d'anneaux, de schémas relatifs, etc. $(\cdots)$".

In order to recall the definitions, let us consider, to fix ideas, a k-scheme of finite type S as in 1.1. A sheaf of $\mathcal{O}$-modules E on $(S/W_n)_{\text{crys}}$ amounts to the data, for each object $(U \to T)$ of $(S/W_n)_{\text{crys}}$, of a sheaf of $\mathcal{O}_T$-modules E_T on (the Zariski site of) T, and for each map $g\colon (U' \to T') \to (U \to T)$, of a map $E_g\colon g^*E_T \to E_{T'}$, satisfying a transitivity condition for a composition, and such that E_g is an isomorphism when g is an open immersion. One says that E is a *crystal* when, for *any* g, E_g is an isomorphism; E is said to be quasi-coherent (resp. coherent, etc.) if each E_T is. A crystal (in $\mathcal{O}$-modules) on S/W is an inverse system of crystals (in $\mathcal{O}$-modules) E_n on S/W_n, inducing each other in the obvious way. In what follows, "crystal" will mean "crystal in $\mathcal{O}$-modules" unless otherwise stated. If S is embedded in a smooth W-scheme Z, and if D denotes the p-adically completed divided power envelope of S in Z, the data of a coherent crystal E on S/W is equivalent to the data of a coherent $\mathcal{O}_D$-module M, equipped with an integrable connection ∇ relative to W, satisfying a topological nilpotence condition, and the crystalline cohomology of S/W with coefficients in E can be calculated as de Rham cohomology of Z/W with coefficients in M (see **[16,** §7] for a more precise statement). An *F-crystal* on S/W is a crystal E together with a map $F\colon F_{\text{abs}}^*E \to E$, where F_{abs} is the absolute Frobenius endomorphism of the crystalline topos S/W_n. It is usually assumed in addition that E is locally free of finite type, and nondegenerate; i.e., there exists a map $V: E \to F_{\text{abs}}^* E$ such that $FV = p^d$, $VF = p^d$ for some integer $d \geq 0$, called the level (when $d = 0$, i.e., when F is an isomorphism, one says that E is a *unit-root* F-crystal). Over a point, i.e., for $S = \operatorname{Spec} k$, one recovers the notion of 1.3(c). The category of unit-root F-crystals of rank r on S/W is naturally equivalent to that of lisse p-adic sheaves of rank r over S (see [**76,** 3.5; **77,** 4.1] for particular cases and [**15,** 2.4.10] for the general one). For a general discussion of general properties of F-crystals, see [**76**] and [**78**].

The notion of F-crystal can be thought of as a coarse analogue of that of lisse ℓ-adic sheaf, or—as Ogus likes to advocate—of variation of Hodge structure. F-crystals usually arise as relative crystalline cohomology of good proper and smooth S-schemes. One basic example is that of an abelian scheme $f: X \to S$ of relative dimension g. Then $R^1 f_* \mathcal{O}_{X/W}$ is an F-crystal on S/W, locally free of rank $2g$, and of level 1 (V being given by the Verschiebung) [**14,** 2.5.5], called the *Dieudonné crystal of* X. Grothendieck introduced this object in [**47**] (with a slightly different definition), and made several conjectures about it, which were the starting point of a vast program giving a common generalization of the Serre-Tate theory of deformation of abelian schemes (see, e.g., [**87**] or [**60,** Appendix 1] and classical Dieudonné theory for p-divisible groups over k [**31**]. He reported on the first main theorems in [**50**] and [**51**], but it is only recently that his program has been completed, as a result of the efforts of several other people, especially joint work of Berthelot, Breen, and Messing on the one hand, and Kato on the other hand [**86, 87, 60, 13, 14, 15,** 72, **88**]) Let me just mention one central result in the theory. Define a *Dieudonné crystal* on S/W as a crystal E on S/W, locally free of finite type, equipped with maps $F: F_{\text{abs}}^* E \to E$ and $V: E \to F_{\text{abs}}^* E$ satisfying $FV = p$, $VF = p$. Let BT(S) (resp. Dieud(S/W)) be the category of (p-) Barsotti-Tate groups on S (resp. Dieudonné crystals on S/W). Then there is defined a contravariant functor

$$\mathbf{D}: BT(S) \to \operatorname{Dieud}(S/W),$$

compatible with any base change, and such that for $S = \operatorname{Spec} k$, $\mathbf{D}$ is the usual Dieudonné functor (see, e.g., [**87**] or [**51**]). Moreover, if G is the Barsotti-Tate group associated to the abelian scheme X/S, then $\mathbf{D}(G)$ is the Dieudonné crystal of X considered above. (For all this, see [**14**].) The main result is:

THEOREM 4.1.2. *Assume* S/k *smooth. Then the functor* $\mathbf{D}$ (4.1.1) *is an equivalence of categories.*

The full faithfulness is proven in [**15**], the essential surjectivity in [**72**] for $p > 2$ and in [**88**] in the general case.

In the same vein, since $K3$ surfaces are motivically close to abelian varieties, it is natural to look at the F-crystals arising from them: if $f: X \to S$ is a proper and smooth map whose geometric fibers are $K3$ surfaces, $R^2 f_* \mathcal{O}_{X/W}$ is an F-crystal on S, locally free of rank 22. Ogus looked more closely at the case where all fibers are supersingular. Inspired by the work of Rudakov–Shafarevitch **[109]** on the one hand, and that of Artin **[1]** on the other hand, he established a global Torelli theorem in this case; i.e., he constructed a period map from a fine moduli space of supersingular $K3$ surfaces to a certain classifying space of F-crystals and proved it to be an isomorphism **[100, 102]**. See also related work of Nygaard **[97]** and Nygaard-Ogus **[98]** on the Tate conjecture for nonsupersingular $K3$ surfaces.

4.2. Fontaine-Laffaille sheaves. Let us come back to the fundamental isomorphism (1.3.8). While $H^*_{\mathrm{DR}}(Z/W)$ depends only on $X = A \otimes k$, the Hodge filtration $\mathrm{Fil}^{\cdot} H^*_{\mathrm{DR}}(Z/W)$, defined as in 2.2 as the abutment filtration of the Hodge to de Rham spectral sequence $E_1^{ij} = H^j(Z, \Omega^i) \Rightarrow H^*_{\mathrm{DR}}(Z/W)$, does depend on Z. For example, when X is an abelian variety, the submodule $\mathrm{Fil}^1 = H^0(Z, \Omega^1) \subset H^1(X/W)$ determines the lifting Z up to isomorphism, at least for $p > 2$ **[87, 50]**. A close study of the dependence of $\mathrm{Fil}^{\cdot}$ on Z in general was made by Ogus **[103]**. We have also seen that the triple $(H^*(X/W), \mathrm{Fil}^{\cdot}, \phi)$, tensored with $\mathbb{Q}_p$, determines the p-adic representation $H^*(Z \otimes \overline{k}, \mathbb{Q}_p)$ (3.2.3). But it is interesting to consider the triple itself, i.e., not tensored with $\mathbb{Q}_p$. The effect of ϕ on $\mathrm{Fil}^{\cdot}$, which is quite subtle, was first investigated by Mazur **[85]**, who obtained what seems to be the best possible estimates on the p-adic divisibility of ϕ on the steps of the filtration. A simple case of these estimates, which played a key role in subsequent work of Fontaine and others, is the following one (see **[40]**, where it is deduced from **[85]**):

THEOREM 4.2.1. *Assume that $H^j(Z, \Omega^i)$ is torsion free for all i, j. Then for $m < p$ the σ-linear endomorphism ϕ of $H^m(X/W)$ is divisible by p^i on Fil^i and*

$$H^m(X/W) = \sum p^{-i} \phi(\mathrm{Fil}^i).$$

(Ogus **[103]** has given examples where the conclusion is false if $m \geq p$.) Elaborations on this result as well as on part (ii) of Mazur-Ogus's theorem 1.3.1 led to the discovery of strong degeneration results, such as the following one, for which an entirely elementary proof is given in **[29]**:

THEOREM 4.2.2. *Let X/k be proper and smooth of dimension $< p$, and let $F: X \to X'$ be the relative Frobenius. Then any (proper and smooth) lifting of X over W_2 canonically decomposes $F_* \Omega^{\cdot}_{X/k}$, in $D(X', \mathcal{O}_{X'})$, into $\bigoplus \Omega^i_{X'/k}[-i]$ (hence forces the Hodge to de Rham spectral sequence of X/k to degenerate at E_1).*

The degeneration of the Hodge to de Rham spectral sequence in characteristic zero as well as the vanishing theorem of Kodaira-Akizuki-Nakano are simple consequences of 4.2.2 (see **[29]**).

F-crystals (M, ϕ) endowed with a (decreasing) filtration by sub-W-modules Fil^i such that ϕ is divisible by p^i on Fil^i and $\sum p^{-i}\phi(\mathrm{Fil}^i) = M$ (and variants of this notion) have been systematically studied by Fontaine-Laffaille **[43]** in connection with the construction of p-adic representations. Their theory was then "sheafified" by Faltings **[36]**, yielding in particular "relative" variants and generalizations of 4.2.1 and 4.2.2. There remained, however, the problem of finding a common generalization of this and of the finer aspects of the Mazur-Ogus Theorem 1.3.1. This has just been achieved by Ogus **[104]**. The key ingredient in his work is the notion of *T-crystal*, i.e., a crystal together with a filtration by submodules (not subcrystals) satisfying an analogue of the Griffiths transversality property (T is for "transversal").

4.3. Rigid cohomology, overconvergent crystals, and $D^{\dagger}_{\mathscr{X}\mathbb{Q}}$-modules. On open or singular varieties, crystals (and their more sophisticated variants of Fontaine-Laffaille, Faltings, and Ogus alluded to above) do not provide a good cohomology theory. Their crystalline cohomology is not finitely generated in general (see, e.g., [**5**, p. 19; **48**, p. 318]). But there is an even more serious limitation. As we have said above, crystals are just more or less precise approximations of what *lisse* crystalline coefficients should be. They can form at most a small part of the desired derived category stable under Grothendieck's *six operations*. Inspired by the *dagger* theory of Monsky-Washnitzer and Dwork on the one hand and the theory of $\mathscr{D}$-modules of Kashiwara-Mebkhout on the other hand,[6] Berthelot has started to build a new theory [**7, 10, 11**] (see [**9**] for an overview), which looks quite promising. His theory has developed into three steps, of increasing generality.

(a) *Rigid cohomology.* Let k be a perfect field of characteristic $p > 0$ and let K_0 be the fraction field of $W = W(k)$. To any scheme X/k, separated and of finite type, Berthelot [**7**] associates K_0-vector spaces $H^i(X/K_0)_{\mathrm{rig}}$, called *rigid cohomology groups* of X, and similar groups with compact supports $H^i_c(X/K_0)_{\mathrm{rig}}$ or with support in a closed subscheme Y, $H^i_Y(X/k_0)_{\mathrm{rig}}$. Their construction uses techniques of rigid analytic geometry (for a general reference on this topic, see, e.g., [**20**]). They enjoy the following properties.

(1) $H^*(X/K_0)_{\mathrm{rig}}$ is contravariant in X; $H^*_c(X/K_0)_{\mathrm{rig}}$ is contravariant in X for proper morphisms $X \to Y$.

(2) If Y is a closed subscheme of X and $U = X - Y$, there are long exact sequences

$$\cdots \to H^*_Y(X/K_0)_{\mathrm{rig}} \to H^*(X/K_0)_{\mathrm{rig}} \to H^*(Y/K_0)_{\mathrm{rig}} \to \cdots,$$
$$\cdots \to H^*_c(U/K_0)_{\mathrm{rig}} \to H^*_c(X/K_0)_{\mathrm{rig}} \to H^*_c(Y/K_0)_{\mathrm{rig}} \to \cdots,$$

[6] These two theories have had huge developments; the reader may consult [**108**] for a survey of the first one and [**83, 99**] for surveys of the second one.

(3) If X/k is proper and smooth, there is a natural isomorphism

$$H^*(X/W) \otimes_W K_0 \xrightarrow{\sim} H^*(X/K_0)_{\text{rig}},$$

where $H^*(X/W)$ is the crystalline cohomology defined in (1.1.2).

(4) If A is a complete discrete valuation ring of residue field k, and if X/k is proper and smooth and admits a proper and smooth lifting Z over $S = \operatorname{Spec} A$, then there is a *built-in* natural isomorphism

$$H^*(X/K_0)_{\text{rig}} \otimes_{K_0} K \xrightarrow{\sim} H^*_{\text{DR}}(Z/S) \otimes_A K,$$

which, combined with (3), provides another construction of the Berthelot-Ogus isomorphism (2.3.2).

(5) If X/k is smooth and affine, then there is a natural isomorphism

$$H^*(X/K_0)_{\text{rig}} \simeq H^*_{\text{MW}}(X),$$

where the right-hand side is the *dagger cohomology* of Monsky–Washnitzer **[92]**.

A central open problem in this theory is the finiteness of $H^*(X/K_0)_{\text{rig}}$ and $H^*_c(X/K_0)_{\text{rig}}$. The only known general result is Monsky's theorem [P. Monsky, "One dimensional formal cohomology", Proc. Internat. Congr. Math. (Nice, 1970), vol. 1, Gauthier-Villars, 1971, pp. 451–456] to the effect that for X/k smooth, $H^1(X/K_0)_{\text{rig}}$ is finite-dimensional. If we assume the resolution of singularities holds over k, then more can be said.

(i) It is a straightforward consequence of (1), (2), (3) that $H^*_c(X/K_0)_{\text{rig}}$ is finite dimensional for any X/k.

(ii) If X/k is smooth of dimension d, $H^*(X/K_0)_{\text{rig}}$ is also finite dimensional, and there is a Poincaré duality

$$H^i(X/K_0)_{\text{rig}} \otimes H^{2d-i}_c(X/K_0)_{\text{rig}} \to K_0$$

between finite-dimensional K_0-vector spaces.

(The proof of (ii) relies on the construction of a trace map and a certain topological pairing between $R\Gamma(X/K_0)_{\text{rig}}$ and $R\Gamma_c(X/K_0)_{\text{rig}}$.)

But even assuming resolution, the finite dimensionality of $H^*(X/K_0)_{\text{rig}}$ is still open for X/K nonsmooth.

(b) *Overconvergent isocrystals and Dwork theory.* It is an observation which goes back to Dwork that formal horizontal sections of (usual) F-crystals as in 4.1 converge in bigger domains than what would be expected (see **[76,** 3.1] for a precise statement). This led to the notion of convergent crystal, developed by Berthelot and Ogus in **[17]** and **[101]**. However, on nonproper schemes convergent crystals do not behave so well, and it is convenient to impose an extra convergence condition at infinity, called overconvergence (see **[7, 9, 10]**). To any X/k as in (a) is functorially associated a category $F\text{-isocrys}^\dagger(X)$, whose objects are called overconvergent F-isocrystals on X.

The definition of rigid cohomology of (a) can be generalized to coefficients in this category **[10]**. Berthelot **[9]** has used this formalism to give a cohomological interpretation of results of Dwork and Robba on exponential sums (see **[8]** for references). It is not unreasonable to believe that overconvergent F-isocrystals are a correct analogue of lisse $\mathbb{Q}_\ell$-sheaves. As an illustration of this, Crew **[24, 25]** has defined the monodromy group of an overconvergent F-isocrystal, proved an analogue of Grothendieck's global monodromy theorem, and shown in some cases that these p-adic monodromy groups turn out to be "the same" as the ℓ-adic ones (the overconvergence hypothesis is here essential). One hopes, of course, that for any X/k (separated and of finite type) and any object of M of $F\text{-isocrys}^\dagger(X)$, the corresponding rigid cohomology groups $H^i(X, M)_{\mathrm{rig}}$ and $H^i_c(X, M)_{\mathrm{rig}}$ are finite-dimensional K_0-vector spaces. Very little is known on this at the moment.

(c) $\mathcal{D}^\dagger$*-modules.* Let A be as in (1), (4) and let $\mathcal{X}$ be a smooth formal A-scheme. Let $\mathcal{D}_{\mathcal{X}}$ be the ring of (usual) differential operators on X/A. For $m \in \mathbb{N}$, let $\mathcal{D}^{(m)}_{\mathcal{X}}$ be the subring generated locally by the operators $\partial^{p^j}_{x_i}/(p^j)!$ for $j \leq m$ where (x_i) is a system of coordinates on $\mathcal{X}$. Let $\mathcal{D}^{(m)\wedge}_{\mathcal{X}}$ be the p-adic completion of $\mathcal{D}^{(m)}_{\mathcal{X}}$ and

$$\mathcal{D}^\dagger_{\mathcal{X}} := \bigcup_{m \in \mathbb{N}} \mathcal{D}^{(m)\wedge}_{\mathcal{X}},$$

which is a strict subring of $\mathcal{D}^\wedge_{\mathcal{X}}$. Berthelot **[11]** shows that

$$\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}} := \mathcal{D}^\dagger_{\mathcal{X}} \otimes \mathbb{Q}$$

is a coherent sheaf of rings and that the category of coherent $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-modules *depends only* on the special fiber $X = \mathcal{X} \otimes k$. Moreover, Berthelot can define the characteristic cycle of a coherent $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-module endowed with a Frobenius automorphism, and the corresponding notion of *holonomic* F-$\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$*-module.* When X is proper, the de Rham cohomology of such a module is finite dimensional, and, in the projective case, an analogue of the index theorem of Dubson-Kashiwara holds **[12]**. Still assuming that X is proper, let $U \subset X$ be the complement of a divisor; if M is an overconvergent F-isocrystal on U as in (b), one can canonically associate to M an F-$\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-module $\mathcal{M}$, which is conjectured to be holonomic. Moreover, $H^i(U, M)_{\mathrm{rig}}$ can be calculated as $H^i_{\mathrm{DR}}(\mathcal{X}, \mathcal{M}) \simeq \mathrm{Ext}^i_{\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}}(\mathcal{O}_{\mathcal{X}\mathbb{Q}}, \mathcal{M})$ **[9]**, so that, in this case, a proof of this holonomicity conjecture would answer the finiteness question raised in (b). Berthelot hopes that the subcategory $D^b_h(F\text{-}\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}})$ of the derived category of F-$\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-modules with bounded, holonomic cohomology—and suitable variants defined intrinsically in terms of an arbitrary (separated and of finite type) scheme X/k will yield a category of coefficients stable under the six operations. Berthelot's theory also includes formalisms of $\bmod p^n$ coefficients generalizing the notions of crystals recalled in 4.1. However, the problem of lifting these various coefficients to W_n or W or A as above has not yet been considered. A common generalization of this, of the works of Faltings and Ogus alluded to at the end of 4.2 as well as of Hyodo-Kato's log crystalline cohomology and the theory of p-adic periods of §3 remains to be found.

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References

1. M. Artin, *Supersingular $K3$ surfaces*, Ann. Sci. École Norm. Sup. (4) 7 (1974), 543–568.
2. M. Artin and B. Mazur, *Formal groups arising from algebraic varieties*, Ann. Sci. École Norm. Sup. (4) **10** (1977), 87–132.
3. M. Artin and G. Winters, *Degenerate fibers and reduction of curves*, Topology **10** (1971), 373–383.
4. J. Ax, *Zeroes of polynomials over finite fields*, Amer. J. Math. **86** (1964), 255–264.
5. P. Berthelot, *Cohomologie cristalline des schémas de caractéristique $p > 0$*, Lecture Notes in Math., vol. 407, Springer-Verlag, New York and Berlin, 1974.
6. ——, *Sur le "théorème de Lefschetz faible" en cohomologie cristalline*, C. R. Acad. Sci. Paris. Sér. I Math. **277** (1973), 955–958.
7. ——, *Géométrie rigide et cohomologie des variétés algébriques de caractéristique p*, Journées d'analyse p-adique (1982), Introduction aux Cohomologies p-adiques, Bull. Soc. Math. France **23** (1986), 7–32.
8. ——, *Cohomologie rigide et théorie de Dwork: Le cas des sommes exponentielles*, Astérisque, no. 119–120, Soc. Math. France, Paris, 1984, pp. 17–49.
9. ——, *Cohomologie rigide et théorie des $\mathscr{D}$-modules*, p-adic Analysis (F. Baldassari, S. Bosch, and B. Dwork, eds.), Lecture Notes in Math., vol. 1454, Springer-Verlag, New York and Berlin, 1990, pp. 80–124.
10. ——, *Cohomologie rigide et cohomologie rigide à support propre*, in preparation for Astérisque.
11. ——, *$\mathscr{D}^{\dagger}_{\mathbf{Q}}$-modules cohérents*, I: *Opérateurs différentiels de niveau fini*, Institut de recherche mathématique de Rennes, prépublication 93-17, juillet 1993; *$\mathscr{D}^{\dagger}_{\mathbf{Q}}$-modules cohérents* II: *images directes et réciproques*, en préparation.
12. ——, *The index theorem for holonomic modules on characteristic p varieties*, Barsotti Memorial Symposium on Algebraic Geometry, Padova, 1991 (The results of this talk will be included in **[11]**).
13. P. Berthelot and W. Messing, *Théorie de Dieudonné cristalline*. I, Journées de Géométrie Algébrique de Rennes I, Astérisque, no. 63, Soc. Math. France, Paris, 1979, pp. 17–37.
14. P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné cristalline*. II, Lecture Notes in Math., vol. 930, Springer-Verlag, New York and Berlin, 1982.
15. P. Berthelot and W. Messing, *Théorie de Dieudonné cristalline*. III: *Théorèmes d'équivalence et de pleine fidélité*, The Grothendieck Festschrift Volume I, Progr. Math., vol. 86, Birkhäuser, Boston, 1990, pp. 173–247.
16. P. Berthelot and A. Ogus. *Notes on crystalline cohomology*, Math. Notes, vol. 21, Princeton Univ. Press, Princeton, NJ, 1978.
17. ——, *F-isocrystals and de Rham cohomology*. I, Invent. Math. **72** (1983), 159–199.
18. S. Bloch, *Algebraic K-theory and crystalline cohomology*, Inst. Hautes Études Sci. Publ. Math. **47** (1978), 187–268.
19. S. Bloch and K. Kato, *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. **63** (1986), 107–152.
20. S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis*, Grundlehren Math. Wiss, vol. 261, Springer-Verlag, New York and Berlin, 1984.

21. J.-L. Colliot-Thélène and W. Raskind, *On the reciprocity law for surfaces over finite fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 283–294.
22. J.-L. Colliot-Thélène, J.-J. Sansuc, and C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. **50** (1983), 763–801.
23. R. Crew, *Specialization of crystalline cohomology*, Duke Math. J. **53** (1986), 749–757.
24. ——, *F-isocrystals and their monodromy groups*, preprint.
25. ——, *Kloosterman sums and the monodromy of a p-adic hypergeometric equation*, preprint.
26. P. Deligne, *Cohomologie des intersections complètes*, Exp. XI in SGA 7, 39–61.
27. ——, *La conjecture de Weil*. I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–308.
28. ——, *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252.
29. P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.
30. P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.
31. M. Demazure, *Lecture on p-divisible groups*, Lecture Notes in Math., vol. 302, Springer-Verlag, New York and Berlin, 1972.
32. T. Ekedahl, *On the multiplicative properties of the de Rham-Witt complex*. I, Ark. Mat. **22** (1984), 185–239.
33. ——, *On the multiplicative properties of the de Rham-Witt complex*. II. Ark. Mat. **23** (1985).
34. ——, *Diagonal complexes and F-gauge structures*, Travaux en Cours, Hermann, Paris, 1986.
35. G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), 255–299.
36. ——, *Crystalline cohomology and p-adic Galois representations*, Algebraic Analysis, Geometry and Number Theory, The Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80.
37. ——, *Crystalline cohomology of semi-stable curves and p-adic Galois representations*, J. Algebraic Geometry **1** (1992), 61–82.
38. J.-M. Fontaine, *Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate*, Journées de Géométrie Algébrique de Rennes III, Astérisque, No. 65, Soc. Math. France, Paris, 1979, pp. 3–80.
39. ——, *Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate*, Ann. of Math. (2) **115** (1982), 529–577.
40. ——, *Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques*, Algebraic Geometry Tokyo-Kyoto, Lecture Notes in Math., vol. 1016, Springer-Verlag, New York and Berlin, 1983, pp. 86–108.
41. ——, *Séminaire sur les périodes p-adiques*, Hautes Études Sci., 1988, to appear in Astérisque.
42. J.-M. Fontaine and L. Illusie, *p-adic periods: a survey*, Bombay (1989), Prépublication Univ. Paris-Sud 90–53.
43. J.-M. Fontaine and G. Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), 547–608.
44. J.-M. Fontaine and W. Messing, *p-adic periods and p-adic étale cohomology*, Contemp. Math., no. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 179–207.
45. H. Gillet and W. Messing, *Cycle classes and Riemann-Roch for crystalline cohomology*, Duke Math. J. **55** (1987), 501–538.
46. M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Bull. Soc. Math. France, Mémoire 21, 1985.
47. A. Grothendieck, Letter to J. Tate, May 1966.
48. ——, *Crystals and the de Rham cohomology of schemes* (*notes by J. Coates and O. Jussila*), Dix Exposés sur la Cohomologie des Schémas, Masson, Paris and North-Holland, Amsterdam, 1968, pp. 306–358.
49. ——, *Modèles de Néron et monodromie*, Exp. IX in SGA 7, pp. 313–523.
50. ——, *Groupes de Barsotti-Tate et cristaux*, Proc. Internat. Congr. Math. (Nice, 1970), vol. I, Gauthiers-Villars, Paris, 1971, pp. 431–436.

51. ——, *Groupes de Barsotti et cristaux de Dieudonné*, Presses de l'Université de Montréal, Montréal, 1974.
52. O. Hyodo, *A note on p-adic étale cohomology in the semi-stable reduction case*, Invent. Math. **91** (1988), 543–557.
53. ——, *On the de Rham-Witt complex attached to a semi-stable family*, Compositio Math. **78** (1991), 241–260.
54. O. Hyodo and K. Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, preprint, to appear in **[41]**.
55. L. Illusie, *Report on crystalline cohomology*, Proc. Sympos. Pure Math., vol. 24, Amer. Math. Soc., Providence, RI, 1975, pp. 459–478.
56. ——, *Cohomologie cristalline, d'après P. Berthelot*, Séminaire Bourbaki, exp. 456, Lecture Notes in Math., vol. 514, Springer-Verlag, New York and Berlin, 1978, pp. 53–60.
57. ——, *Complexe de de Rham-Witt*, Journées de Géométrie Algébrique de Rennes I, Astérisque, vol. 63, Soc. Math. France, Paris, 1979, pp. 83–112.
58. ——, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. (4) **12** (1979), 501–661.
59. ——, *Finiteness, duality and Künneth theorems in the cohomology of the de Rham-Witt complex*, Algebraic Geometry Tokyo-Kyoto, Lecture Notes in Math., vol. 1016, Springer-Verlag, New York and Berlin, 1983, pp. 20–72.
60. ——, *Déformations de groupes de Barsotti-Tate*, Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell (L. Szpiro, ed.), Astérisque, no. 127, Soc. Math. France, Paris, 1985, pp. 151–198.
61. ——, *Ordinarité des intersections complètes générales*, The Grothendieck Festschrift Volume, II, Progr. Math., vol. 87, Birkhäuser, Boston, 1990, pp. 375–405.
62. ——, *Cohomologie de de Rham et cohomologie étale p-adique* (*d'après G. Faltings, J.-M. Fontaine et al.*), Séminaire Bourbaki 1989–90, exp. 726, Astérisque, nos. 189–190, Soc. Math. France, Paris, 1990, pp. 325–374.
63. ——, *Autour du théorème de monodromie locale*, preprint, to appear in **[41]**.
64. ——, *Logarithmic spaces, according to K. Kato*, Barsotti Memorial Symposium on Algebraic Geometry, Padova, 1991.
65. L. Illusie and M. Raynaud, *Les suites spectrales associées au complexe de de Rham-Witt*, Inst. Hautes Études Sci. Publ. Math. **57** (1983), 73–212.
66. U. Jannsen, *On the l-adic cohomology of varieties over number fields and its Galois cohomology*, Galois groups over $\mathbb{Q}$, Springer-Verlag, New York and Berlin, 1989, pp. 315–360.
67. K. Kato, *Class field theory and algebraic K-theory*, Algebraic Geometry, Tokyo/Kyoto 82, Lecture Notes in Math., vol. 1016, Springer-Verlag, New York and Berlin, 1983, pp. 109–126.
68. ——, *A Hasse principle for two dimensional global fields*, J. Reine Angew. Math. **366** (1986), 142–161.
69. K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic Analysis, Geometry and Number Theory, The Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224.
70. ——, *Semi-stable reduction and p-adic étale cohomology*, preprint, to appear in **[41]**.
71. ——, *The explicit reciprocity law and the cohomology of Fontaine-Messing*, Bull. Soc. Math. France **119** (1991), 397–441.
72. ——, *p-divisible groups and Dieudonné theory*, work in preparation.
73. K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), 241–275.
74. ——, *Global class field theory of arithmetic schemes*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory. I, Contemp. Math., no. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 255–331.
75. N. Katz, *On a theorem of Ax*, Amer. J. Math. **93** (1971), 485–499.
76. ——, *Travaux de Dwork*, Séminaire Bourbaki 1972, exp. 409, Lecture Notes in Math., vol. 317, Springer-Verlag, New York and Berlin, 1973, pp. 167–200.
77. ——, *p-adic properties of modular schemes and modular forms*, Modular Functions of One Variable III, Lecture Notes in Math., vol. 350, Springer-Verlag, New York and Berlin, 1973, pp. 69–190.

78. ——, *Slope filtration of F-crystals*, Journées de Géométrie Algébrique de Rennes I, Astérisque, no. 63, Soc. Math. France, Paris, 1979, pp. 113–163.
79. N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.
80. S. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Cohomologie des Schémas, Masson, Paris and North-Holland, Amsterdam, 1968, pp. 359–386.
81. N. Koblitz, *p-adic variation of the zeta-function over families of varieties defined over finite fields*, Compositio Math. **31** (1975), 119–218.
82. M. Kurihara, *A note on p-adic étale cohomology*, Proc. Japan Acad. Ser. A Math. Sci. **63** (1987), 275–278.
83. D. T. Lé and Z. Mebkhout, *Introduction to linear differential systems*, Proc. Sympos. Pure Math., vol. 40, part 2, Amer. Math. Soc., Providence, RI, 1983, pp. 31–63.
84. S. Lubkin, *Generalization of p-adic cohomology, bounded Witt vectors*, Compositio Math. **34** (1977), 225–277.
85. B. Mazur, *Frobenius and the Hodge filtration*, Bull. Amer. Math. Soc. **78** (1972), 653–667; *Frobenius and the Hodge filtration, estimates*, Ann. of Math. (2) **98** (1973), 58–95.
86. B. Mazur and W. Messing, *Universal extensions and one dimensional crystalline cohomology*, Lecture Notes in Math., vol. 370, Springer-Verlag, New York and Berlin, 1974.
87. W. Messing, *The crystals associated to Barsotti-Tate groups: With applications to abelian schemes*, Lecture Notes in Math., vol. 264, Springer-Verlag, New York and Berlin, 1972.
88. ——, *p-divisible groups and Dieudonné theory*, Barsotti Memorial Symposium on Algebraic Geometry, Padova, 1991.
89. L. Miller, *Curves over a finite field with invertible Hasse-Witt matrix*, Math. Ann. **197** (1972), 123–127.
90. J. Milne, *Duality in the flat cohomology of a surface*, Ann. Sci. École Norm. Sup. (4) **9** (1979), 171–202.
91. A. Mokrane, *Sur la structure mixte de la cohomologie de de Rham des variétés semi-stables*, Thèse, Orsay, 1992, to appear in Duke Math. J. **72** (1993).
92. P. Monsky and G. Washnitzer, *Formal cohomology*. I, Ann. of Math. (2) **88** (1968), 181–217.
93. D. Mumford, *Bi-extensions of formal groups*, Algebraic Geometry, Bombay Colloquium 1968, Tata Inst. Fund. Res., Bombay, and Oxford Univ. Press, New York, 1969, pp. 307–322.
94. ——, *Semi-stable reduction*, Toroidal Embeddings I (G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, eds.), Lecture Notes in Math., vol. 339, Springer-Verlag, New York and Berlin, 1973, pp. 53–108.
95. P. Norman and F. Oort, *Moduli of abelian varieties*, Ann. of Math. (2) **112** (1980), 413–439.
96. N. Nygaard, *Slopes of powers of Frobenius on crystalline cohomology*, Ann. Sci. École Norm. Sup. (4) **14** (1981), 369–401.
97. ——, *The Tate conjecture for ordinary K3 surfaces over finite fields*, Invent. Math. **74** (1983), 213–237.
98. N. Nygaard and A. Ogus, *Tate's conjecture for K3 surfaces of finite height*, Ann. of Math. (2) **122** (1985), 461–507.
99. T. Oda, *Introduction to algebraic analysis on complex manifolds*, Adv. Stud. Pure Math. vol. 1, North-Holland, Amsterdam, 1983, pp. 29–48.
100. A. Ogus, *Supersingular K3 crystals*, Journées de Géométrie Algébrique de Rennes II, Astérisque, no. 64, Soc. Math. France, Paris, 1979, pp. 3–86.
101. ——, *F-isocrystals and de Rham cohomology*. II-*convergent isocrystals*, Duke Math. J. **51** (1984), 765–850.
102. ——, *A crystalline Torelli theorem for supersingular K3 surfaces*, Arithmetic and Geometry II (M. Artin and J. Tate, eds.), Birkhäuser, Boston, 1983, pp. 361–394.
103. ——, *Griffiths transversality in crystalline cohomology*, Ann. of Math. (2) **108** (1978), 395–419.
104. ——, *F-crystals, Griffiths transversality, and the Hodge decomposition*, University of California, Berkeley, PAM-586, July 1993, to appear in Astérisque.
105. ——, *F-crystals on the logarithmic punctured disc*, manuscript notes, 1991.

106. F. Oort, *Moduli of abelian varieties in positive characteristic*, Barsotti Memorial Symposium on Algebraic Geometry, Padova, 1991.
107. M. Rapoport and T. Zink, *Ueber die lokale Zetafunktion von Shimuravarietäten, Monodromie filtration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), 21–101.
108. P. Robba, *Une introduction naïve aux cohomologies de Dwork*, Bull. Soc. Math. France, Mémoire 23 (1986), 61–105.
109. A. N. Rudakov and I. Shafarevich, *Inseparable morphisms of algebraic surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 1269–1307, 1439.
110. J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p*, Symposium Internacional de Topologia Algebraica, México, 1958, pp. 24–53.
111. ——, *Résumé des cours* 1965–66, Annuaire du Collège de France, Paris, 1967, pp. 49–58.
112. J. Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1976), 229–257.
113. N. Suwa, *De Rham cohomology of algebraic surfaces with $q = -p_a$ in characteristic p*, Algebraic Geometry, Tokyo/Kyoto 82, Lecture Notes in Math., vol. 1016, Springer-Verlag, New York and Berlin, 1983, pp. 73–85.
114. J. Tate, *p-divisible groups*, Proc. Conf. Local Fields, NUFFIC Summer School, Drieberger, Springer-Verlag, New York and Berlin, 1967, pp. 158–183.
115. E. Warning, *Bemerkung zur Vorstehenden Arbeit von Herr Chevalley*, Abh. Mat. Sem. Univ. Hamburg **11** (1936), 76–83.

SGA7. *Groupes de monodromie en géométrie algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1967–69, dirigé par A. Grothendieck, I. Lecture Notes in Math., vol. 288, Springer-Verlag, New York and Berlin, 1972; II, Lecture Notes in Math., vol. 340, Springer-Verlag, New York and Berlin, 1973.

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Conjectures on Algebraic Cycles in ℓ-adic Cohomology

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1. Introduction and notation

Throughout, k is a field finitely generated over its prime field, $\overline{k}$ is a separable algebraic closure of k, the Galois group is $G = \mathrm{Gal}(\overline{k}/k)$, and ℓ is a prime different from the characteristic of k. We consider smooth projective equidimensional k-schemes $X, Y, \ldots$ and put $\overline{X} = X \times_k \overline{k}$. The geometric ℓ-adic cohomology of X will be denoted simply by $H^i(X) := H^i(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell)$. For $0 \le j \le \dim X$, we put $V^j(X) := H^{2j}(X)(j)$, where (j) means the standard jth twist, and let $A^j(X) \subset V^j(X)$ denote the $\mathbb{Q}$-span of the image of the cycle class map $\mathscr{Z}^j(X) \to V^j(X)$. Thus $A^j(X)$ is (the isomorphic image of) the group of classes of algebraic cycles of codimension j on X, with coefficients in $\mathbb{Q}$, for ℓ-adic homological equivalence. Let $N^j(X) \subset A^j(X)$ denote the group of classes of cycles that are numerically equivalent to zero. In other words, put, for $j' = \dim X - j$,

$$N^j(X) := A^j(X) \cap A^{j'}(X)^\perp = \{a \in A^j(X) \mid \langle a, a'\rangle = 0 \quad \text{for all } a' \in A^{j'}(X)\} ,$$

where $\langle\ ,\ \rangle$, denotes the pairing via cup product

$$V^j(X) \times V^{j'}(X) \to V^{\dim X}(X) \xrightarrow{\mathrm{Tr}} \mathbb{Q}_\ell .$$

This pairing gives the Poincaré duality between V^j and $V^{j'}$ and has the property that $\langle a, a'\rangle \in \mathbb{Q}$ for $a \in A^j$ and $a' \in A^{j'}$, because $\langle a, a'\rangle = a \cdot a'$ is the total intersection multiplicity of the two cycles.

The group G acts continuously on $V(X)$, compatibly with the pairing, and fixes the elements of $A(X)$. We have a map

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} A^j(X) \to V^j(X) \tag{1.1}$$

induced by inclusion.

Here are our notations for some optimistic conjectural statements:

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$T^j(X)$: The map (1.1) is surjective, i.e., $\mathbb{Q}_\ell A^j(X) = V^j(X)^G$.

$I^j(X)$: The map (1.1) is injective.

$E^j(X)$: $N^j(X) = 0$, i.e., numerical equivalence is equal to ℓ-adic homological equivalence for algebraic cycles of codimension j on X with rational coefficients.

$SS^i(X)$: G acts semisimply on $H^i(X)$.

$S^j(X)$: The map $V^j(X)^G \to V^j(X)_G$ induced by identity is bijective.

Another statement, closely related to T^1, is now a theorem, proved by Zarhin in characteristic $p > 0$ and by Faltings in characteristic 0 [**34, 35, 8; 10,** Chapter VI]:

$H(A, B)$: The map $\mathbb{Q}_\ell \otimes \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_G(H^1(B), H^1(A))$ is bijective for the abelian varieties A and B.

(Memory aids: T is for Tate, I for injective, E for equality of equivalence relations, SS for semisimple, S for partially semisimple, and H for Homs.)

In the early 1960s I got the idea that T^j, especially T^1, might be true [**30**]. The origin of the idea for T^1 was my conviction [**29**] that the Shafarevitch group Ш of an abelian variety over a global field should be finite. If it, or at least its ℓ-primary part, were not finite, then the Galois cohomology of the abelian variety would be a mess, and the determination of the group of rational points by "descent" would be ineffective. Thinking about these things in the light of the new étale cohomology, with much help from M. Artin and D. Mumford, I was led to T^1, then to T^j. Especially helpful for T^1 was the consideration of the function field analog [**31**] of the conjecture of Birch and Swinnerton-Dyer and Artin's interpretation of the Shafarevitch group as the Brauer group (cf. [**21**] or [**12**]). Other clues to the reasonableness of T^1 were the logical equivalence

$$T^1(X \times Y) \Leftrightarrow T^1(X) + T^1(Y) + H(A, B) , \tag{1.2}$$

where A (resp. B) is the Albanese (or Picard) variety of X (resp. Y), and the implication

$$H(A, A) \Rightarrow T^1(A) \tag{1.3}$$

for an abelian variety A over k. Indeed $H(A, B)$ seemed extremely plausible to me thirty years ago, especially after Mumford explained how it followed for elliptic curves over a finite field from results of Deuring, and Serre proved it for elliptic curves over a number field if the j invariant of one of them was not an integer. Later I was very happy to prove $H(A, B)$ in general for k finite (in [**32**], where the implications (1.2) and (1.3) are explained). A key idea for that proof was suggested by Lichtenbaum, and several conversations about it with Serre were of great help.

My idea that T^j might be true for $j > 1$ was based on the analogy with $i = 1$ and some very meager evidence from the case of Fermat surfaces [**30**].

The plan of this paper is as follows. In §2 we discuss various logical interrelationships between the statements T, E, I, and S which can be deduced from Poincaré duality and the hard Lefschetz theorem. In §3 we give a proof by Deligne that T implies that the Künneth components of an algebraic cycle on a product $X \times Y$ are algebraic. In §4 we discuss relations with arithmetic cohomology and recall the equivalence, for k finite, between $T^1(X)$ and the finiteness of the ℓ-primary part of $\mathrm{Br}(X)$. In §5 we discuss, with no claim to completeness, some cases in which the conjectures have been proven. Most of these cases concern divisors, and we include a proof of the birational invariance of T^1.

Two topics we have not discussed are the related conjectures about orders of poles of L-functions for infinite k **[30]**, and the relations between the ℓ-adic conjectures and the Hodge conjectures in case k is of characteristic 0.

I would like to thank P. Deligne for several helpful discussions during the preparation of this paper, and D. Ramakrishnan for his help with §5.

2. Some folklore

This section is a distillation of ideas more or less known to experts which were informally explained at the conference by N. Katz, W. Messing, and U. Jannsen. I thank them for their help, and also J. Milne from whom I have copied (2.6) below. For a striking result involving related ideas, see **[14]**.

In this section we fix an X and j and drop them from the notation, writing simply V instead of $V^j(X)$, T for $T^j(X)$, etc. We also denote objects in the complementary dimension by a "prime", writing V' for $V^{d-j}(X)$, T' for $T^{d-j}(X)$, etc., where $d = \dim X$. The representations V and V' are noncanonically isomorphic by the hard Lefschetz theorem **[6]** and are canonically dual by Poincaré duality. Thus the space V^G is of the same dimension as $(V')^G$, and V^G is dual to $(V')_G$. Moreover, the canonical maps

$$V^G \to V_G \quad \text{and} \quad (V')^G \to (V')_G \tag{2.1}$$

are dual to each other. Hence

(2.2) Lemma.

(a) *The four spaces* V^G, V_G, $(V')^G$, *and* $(V')_G$ *have the same dimension.*

(b) *If either one of the maps* (2.1) *is either injective or surjective, then both maps are bijective, i.e.,* S *and* S' *hold. In particular,* $S \Leftrightarrow S'$.

Now consider the following diagram in which A means $A(X)$, not an abelian variety, and $*$ means dual:

$$\begin{array}{ccccc} \mathbb{Q}_\ell \otimes A & \xrightarrow{b} & V^G = (V'_G)^* & \xrightarrow{c} & ((V')^G)^* \\ \Big\downarrow a & & & & \Big\downarrow d \\ \mathbb{Q}_\ell \otimes (A/N) & \overset{f}{\hookrightarrow} & Q_\ell \otimes \mathrm{Hom}_{\mathbb{Q}}(A'_j, \mathbb{Q}) & \overset{e}{\hookrightarrow} & (\mathbb{Q}_\ell \otimes A')^* . \end{array} \tag{2.3}$$

It is commutative, and the arrows e and f are injective. From the definitions and (2.2) we have

$$
(2.4)\qquad
\begin{aligned}
&E \Leftrightarrow a \text{ is injective},\\
&T \Leftrightarrow b \text{ is surjective},\\
&I \Leftrightarrow b \text{ is injective},\\
S \Leftrightarrow\ &S' \Leftrightarrow c \text{ is bijective}, \ \Leftrightarrow c \text{ is surjective}, \ \Leftrightarrow c \text{ is injective},\\
&T' \Leftrightarrow d \text{ is injective},\\
&I' \Leftrightarrow d \text{ is surjective}.
\end{aligned}
$$

(2.5) Lemma. $\operatorname{Ker} b \subset \operatorname{Ker} a = \mathbb{Q}_\ell \otimes N$. *In particular,* $E \Rightarrow I$. *Moreover,* $\operatorname{Ker} b = \operatorname{Ker} a \Leftrightarrow E$.

Proof. We have $\operatorname{Ker} b \subset \operatorname{Ker} a$ because ef is injective. Moreover, $b(\operatorname{Ker} a) = \mathbb{Q}_\ell N$, so if $\operatorname{Ker} a = \operatorname{Ker} b$, then $\mathbb{Q}_\ell N = 0$, i.e., $N = 0$.

(2.6) Proposition [**22**, Proposition 8.4]. *The following implications hold*:

$$T + E \Rightarrow T' + S \Rightarrow E.$$

Proof. Assume $T + E$. Then a is injective and b is bijective; hence dc is injective. The injectivity of c implies both S and the bijectivity of c, which in turn gives the injectivity of d, i.e., T'. Assume $T' + S$. Then c and d are injective, so $\operatorname{Ker} b = \operatorname{Ker} a$ which implies E by (2.5).

(2.7) Corollary. *If T holds, then*

$$T' + S \Leftrightarrow E \quad \text{and} \quad S \Rightarrow E'.$$

This is an immediate consequence of (2.6).

(2.8) Proposition.
(i) $\dim_{\mathbb{Q}}(A/N) \le \dim \mathbb{Q}_\ell A$, *with equality if and only if* E.
(ii) $\dim \mathbb{Q}_\ell A \le \dim V^G$, *with equality if and only if* T.
(iii) $\dim_{\mathbb{Q}}(A/N) \le \dim V^G$, *with equality if and only if* $T + E$.

Proof. (i) follows from (2.5) and the surjection

$$\mathbb{Q}_\ell A = (\mathbb{Q}_\ell \otimes A)/\operatorname{Ker} b \to (\mathbb{Q}_\ell \otimes A)/\operatorname{Ker} a = \mathbb{Q}_\ell \otimes A/N.$$

(ii) Obvious.
(iii) Follows from (i) and (ii).

(2.9) Theorem. *The following are equivalent*:
(a) $\dim_{\mathbb{Q}}(A/N) = \dim_{\mathbb{Q}_\ell} V^G$,
(b) $T + E$,
(c) $T + T' + S$,
(d) $T + T' + E + E' + I + I' + S + S'$.
Moreover, if k is finite then these statements are independent of the prime ℓ and are equivalent to

(e) *The order of the pole of the zeta function* $Z(X, t)$ *at* $t = q^{-j}$ *is equal to* $\dim_{\mathbb{Q}}(A/N)$, *the rank of the group of numerical equivalence classes of cycles of codimension* j *on* X.

PROOF. The equivalence of (a), (b), (c), and (d) follows immediately from the preceding discussion. Suppose k is finite. By **[5]**, the order, m, of the pole in question is equal to the multiplicity of $t = 1$ as a root of $\det(1 - \varphi t, V)$, where $\varphi \in G$ is the Frobenius topological generator of G. Thus m is equal to the dimension of the kernel of the operator $(1 - \varphi)^N$ on V for large N. Hence $m \geq \dim \operatorname{Ker}(1-\varphi) = \dim V^G$, and equality holds if and only if $\operatorname{Ker}(1-\varphi) = \operatorname{Ker}(1-\varphi)^2$, i.e., $\operatorname{Ker}(1-\varphi)\cap(1-\varphi)V = 0$, i.e., S. Combining this with (2.6) (iii) we find that $m = \dim_{\mathbb{Q}}(A/N) \Leftrightarrow T + E + S$. This shows the equivalence of (e) with the other four statements, and since (e) is independent of ℓ they all are.

For use in §5 we add here one more statement in the spirit of this section:

(2.10) PROPOSITION. *Suppose* $B \subset A$ *is a subspace such that* $B \cap N = 0$. *Then* $\mathbb{Q}_\ell \otimes B \approx \mathbb{Q}_\ell B$ *is a direct summand of the G-module* V.

PROOF. Since $B \cap N = 0$, there is a subspace $B' \subset A'$ such that B and B' are put in perfect duality by the intersection pairing. Then the orthogonal space to $\mathbb{Q}_\ell B'$ is a complementary submodule to $\mathbb{Q}_\ell B$ in V.

3. Künneth components

Almost 30 years ago, Grothendieck remarked that the conjecture T should imply that the Künneth components of an algebraic cycle on a product $X \times Y$ are algebraic. Not recalling how, I asked Deligne at the conference. This section gives his answer.

(3.1) THEOREM (Deligne). *Let* $d = \dim X$. *If* $T^d(X \times X)$ *is true, then the Künneth components of the (class of) the diagonal,* $\Delta \in A^d(X \times X) \subset V^d(X \times X)$, *are algebraic.*

PROOF. We identify $V^d(X \times X)$ with the space of degree 0 endomorphisms of $H^*(X)$. Then Δ is the identity map, $A^d(X\times X)$ is the $\mathbb{Q}$-algebra of algebraic endomorphisms, the Künneth decomposition is

$$\operatorname{End}(H^*(X)) = \bigoplus_{i=0}^{2d} \operatorname{End}(H^i(X)) , \tag{3.2}$$

and for each i, the ith Künneth component of Δ is the idempotent projection

$$p_i \colon H^*(X) \longrightarrow H^i(X) \hookrightarrow H^*(X) . \tag{3.3}$$

For $u \in \operatorname{End}(H^*(X))$, let $u_i \colon H^i(X) \to H^i(X)$ denote the effect of u on H^i.

(3.4) LEMMA. *For* $u \in A^d(X \times X)$ *the characteristic polynomial* $\det(1 - u_i t, H^i(X))$ *has coefficients in* $\mathbb{Q}$.

PROOF. View X as the generic fiber of a smooth projective S-scheme $\mathscr{X}$, where $S =$ Spec R with R a regular integral domain of finite type over $\mathbb{Z}$ with fraction field k. It suffices to prove the statement for the special fiber $\mathscr{X}_s$ at a suitable closed point $s \in S$. We may therefore suppose k is finite. In that case the statement is Theorem 2.2 of **[15]**, in which it is shown that the projection p_i is algebraic, given by a rational linear combination of powers of the Frobenius morphism. Hence up_i is algebraic and by an argument in **[16]**, it follows that $\operatorname{Tr}(u_i)$ is rational by the Lefschetz formula namely

$$(-1)^i \operatorname{Tr}(u_i, H^i(X)) = \sum_{\nu=0}^{2d} (-1)^\nu \operatorname{Tr}(up_i, H^\nu(X)) = (up_i) \cdot \Delta \in \mathbb{Q}.$$

For the same reason, $\operatorname{Tr}(u_i^n) \in \mathbb{Q}$ for every n and the result follows.

To prove the theorem, note that p_i is fixed by G. Thus, if $T^d(X \times X)$ is true, then there exists $u \in A^d(X \times X)$ approximating p_i ℓ-adically arbitrarily closely. We can therefore choose u such that u_i and u_j for $j \neq i$ have no common eigenvalues, those of u_i being near 1 and the others near 0. By the lemma, there is a polynomial $P \in \mathbb{Q}[X]$ such that P takes the value 1 (with multiplicity if necessary) at the eigenvalues of u_i, and takes the value 0 at the others (with multiplicity if necessary.) Then $p_i = P(u) \in A^d(X \times X)$.

If we define motives in terms of algebraic cycles mod homological equivalence, then $\operatorname{Hom}(h(X), h(Y)) = A^d(X \times Y)$, where $d = \dim X$, and the statements $T^d(X \times Y)$ and $I^d(X \times Y)$ concern the surjectivity and injectivity of the map

$$\mathbb{Q}_\ell \otimes \operatorname{Hom}(h(X), h(Y)) \to \operatorname{Hom}_G(H(X), H(Y)).$$

In the spirit of this conference, we note that the general truth of T and/or I would imply the surjectivity and/or injectivity of the map

$$\mathbb{Q}_\ell \otimes \operatorname{Hom}(M, N) \to \operatorname{Hom}_G(M_\ell, N_\ell)$$

for all motives M, N. Here M_ℓ denotes the ℓ-adic realization of M.

The proof of Lemma (3.4) above shows that it applies not only to $u \in A^d(X \times X)$ that are algebraic, but also to $u \in V^d(X \times X)^G$ that are *almost algebraic*, in the sense that, with S as in the proof of (3.4) above, the reduction of u at s is algebraic for all closed points s in some open dense subset of S. This notion of almost algebraic class seems to be part of the folklore. It is mentioned explicitly in **[27, 5.2]** that the Künneth components of Δ are almost algebraic (by **[15]**). When Deligne told me the proof of (3.1) he remarked that if one defines motives in terms of almost algebraic classes then a motive M would have a grading (M^i) by weight and the traces would still be rational. Thus one might consider a weaker conjecture than T, namely, that V^G is spanned by almost algebraic classes.

In fact all three statements T, I, and E have "almost algebraic" analogues, in which A is replaced by the $\mathbb{Q}$-span of the almost algebraic classes and N by the subspace of A orthogonal to all almost algebraic classes in the complementary dimension. All of the results of §2 hold for the almost algebraic versions of T, I, and E, because the arguments there are based entirely on the following abstract situation: A group G, a field K (namely $\mathbb{Q}_\ell$), two isomorphic finite-dimensional K-representations V and V' of G which are canonically dual by a pairing $\langle\ ,\ \rangle$, a subfield $F \subset K$ (namely $F = \mathbb{Q}$), and F-subspaces $A \subset V$ and $A' \subset V'$ such that $\langle A, A'\rangle \subset F$.

4. Relation with arithmetic cohomology and the Brauer group

The cycle class map

$$\mathscr{Z}_i(X) \to V^i(X)^G = H^{2i}(\overline{X}_{\text{ét}}, \mathbb{Q}_\ell(i))^G$$

factors through the arithmetic cohomology group

$$V^i_{\text{arithm}}(X) := H^{2i}(X_{\text{ét}}, \mathbb{Q}_\ell(i)) ,$$

and the map $V^i_{\text{arithm}} \to (V^i)^G$ is an edge homomorphism at position $(0, 2i)$ in a spectral sequence

$$H(G, H_{\text{geom}}(X)) \Rightarrow H_{\text{arithm}}(X) .$$

Thus the conjecture $T^i(X)$ implies that the differentials

$$d_r^{0,2i} : E_r^{0,2i} \to E_r^{4,2i+1-r}$$

in that sequence are 0 for $r \geq 2$. For k finite this is easily seen to be true by consideration of Frobenius eigenvalues. In fact, it seems that the spectral sequence degenerates completely in all cases [3]. Thus, some obvious obstructions to the truth of $T^i(X)$ do vanish. On the other hand, for infinite k we do not have even a conjectural characterization of the $\mathbb{Q}_\ell$-span of the algebraic cycles in $V^i_{\text{arithm}}(X)$ (cf. [7]).

From now on in this section we assume k is finite. Then $G = \widehat{\mathbb{Z}}$ and for a finite G-module Λ,

$$H^0(G, \Lambda) = \Lambda^G , \quad H^1(G, \Lambda) = \Lambda_G , \qquad H^p(G, \Lambda) = 0 \quad \text{for } p \neq 0, 1 .$$

It follows that for coefficients in $(\mathbb{Z}/\ell^n\mathbb{Z})(i)$ the spectral sequence above becomes a collection of short exact sequences, one of which, on passage to the inverse limit over n, becomes

$$(4.1)\quad 0 \to H^{2i-1}(\overline{X}_{\text{ét}}, \mathbb{Z}_\ell(i))_G \to H^{2i}(X_{\text{ét}}, \mathbb{Z}_\ell(i)) \to H^{2i}(\overline{X}_{\text{ét}}, \mathbb{Z}_\ell(i))^G \to 0 .$$

By [5], the eigenvalues of the Frobenius acting on $H^i(j)$ are of absolute value $q^{j-\frac{i}{2}}$. Hence the left-hand group in (4.1) is finite, and tensoring with $\mathbb{Q}_\ell$ we find that $V^i_{\text{arithm}} = (V^i)^G$. Thus, for k finite, the conjecture $T^i(X)$ is simply that the classes of algebraic cycles of codimension i span the arithmetic cohomology $H^{2i}(X_{\text{ét}}, \mathbb{Q}_\ell(i))$ or equivalently, that they generate a $\mathbb{Z}_\ell$-submodule of finite index in $H^{2i}(X_{\text{ét}}, \mathbb{Z}_\ell(i))$.

For $i = 1$ this has a nice interpretation in terms of the Brauer group of X. The exact sequence

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 0$$

of sheaves on $X_{\text{ét}}$ gives, on taking cohomology, an exact sequence of finite groups

$$0 \to (\mathbb{Z}/\ell^n\mathbb{Z}) \otimes \text{ Pic } X \to H^2(X_{\text{ét}}, \mu_{\ell^n}) \to \text{ Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \text{Br}(X)) \to 0$$

where $\text{Br}(X)$ is the Brauer group of the scheme X (cf. **[20, 12]**.) Passing to the inverse limit we get

$$(4.2) \quad 0 \to \mathbb{Z}_\ell \otimes \text{ Pic } X \to H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1)) \to \text{ Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \text{ Br}(X)) \to 0 .$$

The ℓ-primary part of $\text{Br}(X)$ has the form $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \times$ finite, because $\text{Hom}(\mathbb{Z}/\ell\mathbb{Z}, \text{ Br}(X))$ is finite, and the right-hand group in (4.2) is then isomorphic to $\mathbb{Z}_\ell^r$. Putting these considerations together with (2.9) we obtain

(4.3) PROPOSITION. *For k finite, the following statements are equivalent:*
(i) $T^1(X)$.
(ii) *The ℓ-primary part of* $\text{Br}(X)$ *is finite.*
(ii) $\mathbb{Z}_\ell \otimes \text{ Pic } X \to H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ *is bijective.*
(iv) *The order of pole of $Z(X, t)$ at $t = q^{-1}$ is equal to the rank of* Pic X.

It is known that not only are these statements independent of $\ell \neq p =$ char k (as is clear from (iv)), but they imply the finiteness of the whole group $\text{Br}(X)$ and this finiteness is also implied by the finiteness of the p-primary part of $\text{Br}(X)$ **[31]**, **[19]**, **[18]**, **[22,** 0.4].

5. Known cases

We now discuss some cases in which the conjecture T has been proven. Most of the results concern divisors and, accordingly, most of this section deals with T^1. At the end there is a brief discussion of T^i for $i > 1$.

We begin with some general properties of T^1. It is well known (cf., e.g., SGA 6 XIII, Theorem 4.6) that numerical equivalence implies τ-equivalence for divisors. For cycles with coefficients in $\mathbb{Q}$, τ-equivalence is the same as algebraic equivalence, and implies homological equivalence. Hence $E^1(X)$ holds for all X. From (2.7) we conclude

(5.1) PROPOSITION. *Let $d = \dim X$. Then* $T^1(X) \Rightarrow T^{d-1}(X) + E^{d-1}(X) + S^1(X)$.

If we know T^1 for one type of variety X, it follows for many others.

(5.2) THEOREM. (a) $T^1(X) + T^1(Y) \Leftrightarrow T^1(X \times Y)$.

(b) *The conjecture* $T^1(X)$ *is birationally invariant. More generally, if* $X \to Y$ *is a dominant rational map between varieties, then* $T^1(X)$ *implies* $T^1(Y)$.

PROOF.

(a) As already noted (1.2) this follows from the theorem $H(A, B)$ of Faltings and Zarhin.

(b) For k finite this follows easily from (4.3) and the birational invariance of the ℓ-primary part of the Brauer group of X. When I asked Deligne what could be said for arbitrary k, he pointed out that one could get around the use of Brauer group by considering $T^1(U)$ for an arbitrary open dense subscheme $U \subset X$ and proving $T^1(X) \Leftrightarrow T^1(U)$. To get from X to U one first removes a subscheme Z of codimension ≥ 2, and then removes smooth prime divisors D_i, their singularities having been removed by the first step. The first operation has no effect on H^2, and the effect of the second is calculated by the Gysin exact sequence

$$\bigoplus_{i=1}^{N} H^0(D_i)(-1) \to H^2(X) \to H^2(U) \to \bigoplus_{i=1}^{N} H^1(D_i)(-1) .$$

Twisting once and taking the part of weight ≤ 0 gives

$$\mathbb{Z}_\ell^N \to H^2(\overline{X}_{\text{ét}}, \mathbb{Z}_\ell)(1) \to H^2(\overline{U}_{\text{ét}}, \mathbb{Z}_\ell)(1) \to 0 ,$$

because the groups $H^1(D_i)$ are of weight ≥ 1 (cf. [**6**]). Tensoring with $\mathbb{Q}$ yields a short exact sequence

$$(5.3) \qquad 0 \to \mathbb{Q}_\ell B \to V^1(X) \to V^1(U) \to 0$$

where $B \subset A^1(X) \subset V^1(X)$ is the $\mathbb{Q}$-span of the classes of the divisors D_i. By (2.10) the sequence (5.3) of G-modules splits because, as noted above, $E^1(X)$ is true, i.e., $N^1(X) = 0$. Hence

$$0 \to \mathbb{Q}_\ell B \to V^1(X)^G \to V^1(U)^G \to 0$$

is exact, and $T^1(X) \Leftrightarrow T^1(U)$ follows. From this, the birational invariance is clear.

Let $f: X \to Y$ be a dominant rational map. Let $i: X' \hookrightarrow X$ be a linear section of the same dimension as Y with $X' \to Y$ dominant. Replacing X and Y by suitable open dense subvarieties of themselves we can assume that f is a morphism, X' is smooth, and $fi: X' \to Y$ is finite, say of degree n. Then the equation $n = (fi)_*(fi)^* = (f_* i_* i^*)f^*$ shows that f^* maps the situation for Y isomorphically onto a direct summand of the situation for X. Thus $T^1(X)$ implies $T^1(Y)$.

REMARK. The proof shows that, whether or not $T^1(X)$ holds, the quotient $V^1(U)^G/\mathbb{Q}_\ell A(U)$ is independent of the open dense U in X, so is birationally invariant. Thus $T^1(X)$ is equivalent to the vanishing of $V^1(U)^G$

for one, hence all, U's such that $A(U) = 0$, i.e., the U's obtained by removing from X divisors that generate the Néron-Severi group of X mod torsion.

A nice application of (5.2) is the following theorem, noted in **[28]**.

(5.5) THEOREM. *The statement* T^1 *is true for every Fermat surface in* $\mathbb{P}^3$.

Indeed, the surface

$$a_0x_0^n + a_1x_1^n + a_2x_2^n + a_3x_3^n = 0$$

is dominated by the product of the two curves

$$a_0x_0^n + a_1x_1^n = y^n \quad \text{and} \quad a_2x_2^n + a_3x_3^n = z^n ,$$

and T^1 is trivially true for curves.

Of course much more generally, T^1 holds for every variety that is dominated by a product of curves, or a product of curves and an abelian variety, since (1.3) T^1 is true for abelian varieties.

Close to abelian varieties, in some sense, are $K3$ surfaces.

(5.6) THEOREM.

(a) T^1 *holds for all* $K3$ *surfaces* X *in characteristic* 0.

(b) *Over a finite field* k *of characteristic* $p \geq 5$, T^1 *holds for all nonsupersingular and all elliptic* $K3$ *surfaces.*

Statement (a) and its proof were told to me by D. Ramakrishnan. The proof is an easy exercise, given (1) the existence **[4]** of an abelian variety A and an absolute Hodge cycle on $X \times A$ inducing an injection $H^2(X) \hookrightarrow H^2(A)$; (2) the theorem of Faltings that T^1 is true for A; and (3) the theorem of Lefschetz that rational classes of type $(1, 1)$ are algebraic.

(b) See **[25]** for the nonsupersingular case, and **[1]** for the elliptic case.

(5.7) QUESTION. What about $K3$ surfaces over infinite fields of characteristic p?

In recent years there has been work on conjecture T^1 for various types of modular surfaces defined over number fields.

(5.8) THEOREM. T^1 *holds for*

(a) *Hilbert modular surfaces,*

(b) *many quaternionic Shimura surfaces,*

(c) *Picard modular surfaces, i.e., compactifications of congruence arithmeticquotients of the unit ball in* $\mathbb{C}^2$,

(d) *Siegel modular threefolds.*

(a) The references are **[13, 17, 23]**. In the first of these it is proved T^1 holds for $k/\mathbb{Q}$ abelian by showing that the Hirzebruch-Zagier cycles give enough in that case to fill out V^G, thereby proving T^1. Soon after, Klingenberg and, independently by quite different methods, Murty and Ramakrishnan were able to treat the case of arbitrary k, in which the existence of some more exotic cycle classes, not defined over $\mathbb{Q}^{ab}$, must be proved. Both teams show their existence only indirectly, via the Lefschetz $(1, 1)$-theorem. It is an open problem to find divisors representing these classes concretely. The intersection of such divisors with the "diagonal" modular curve might give interesting points on the modular curve. (The Heegner points occur in the intersection of Hirzebruch-Zagier cycles.)

(b) Such a surface X is constructed by means of a totally real number field F and a quaternion division algebra B over F which is split at exactly two real places $\sigma, \tau: F \to \mathbb{R}$, and is defined over an abelian extension k_0 of the field $F^\sigma F^\tau$. In a work **[24]**, $T^1(X)$ is proved for arbitrary $k \supset k_0$ in case $F^\sigma F^\tau / F^\sigma$ is solvable—in particular, in case $F/\mathbb{Q}$ is Galois. The method here consists in establishing period relations between X and certain modular surfaces where it can be shown directly that V^G is spanned by algebraic classes.

(c) See **[2]**. An important ingredient of the proof of Blasius-Rogawski is the proof of irreducibility of certain ℓ-adic representations occurring in $V(X)$. This generalizes the theorem of Ribet for elliptic modular forms. Their proof also uses p-adic Hodge theory.

(d) In **[33]** it is shown that for Siegel modular threefolds the whole of H^2 is algebraic, so T^1 holds.

What about T^i for $i > 1$? In the few cases I know in which it has been proved it is true because there are as many algebraic cycles as there are room for.

One such case is the old one **[30]** of the $2r$-dimensional Fermat hypersurface $\sum_{\nu=0}^{2r+1} x_\nu^{q+1} = 0$. Over the field with q^2 elements this equation can be viewed as $\sum \overline{x}_\nu x_\nu = 0$. This shows that there is a large group U (unitary over the finite field) of automorphisms. It turns out that the representation of U on V^r is the direct sum of the trivial representation and the irreducible representation of lowest degree > 1, forcing $\mathbb{Q}_\ell A^r = V^r$. For a more geometric proof, see **[28]**.

In characteristic 0 one does not need the dimension of $\mathbb{Q}_\ell A^r$ to be the whole $2r$th Betti number to conclude that T^r holds; it is enough to have $\dim \mathbb{Q}_\ell A^r = h^{r,r}$, because the p-adic Hodge theorem of Faltings gives for $\ell = p$ the inequality $\dim (V^r)^G \le h^{r,r}$ as a corollary. Although this seems to be well known and indeed is used implicitly in (5.8)(b) and (c) above, I first learned it from Brent Gordon who attributes the argument to Faltings in a remark in **[11]**. Indeed, if we imbed $\overline{k}$ in the completion C_p of an algebraic closure of $\mathbb{Q}_p$, let k_p be the closure of k in C_p, put $X_p = X \times_k k_p$ and $G_p = \operatorname{Gal}(\overline{k}/\overline{k} \cap k_p)$, then the p-adic Hodge decomposition **[9]** is

$$V^r(X) \underset{\mathbb{Q}_p}{\otimes} C_p = \bigoplus_\nu H^{r-\nu}(X_p, \Omega^{r+\nu}_{X_p/k_p}) \otimes_{k_p} C_p(-\nu).$$

Hence

$$(V_r(X) \underset{\mathbb{Q}_p}{\otimes} C_p)^{G_p} = H^r(X_p, \Omega^r_{X_p/k_p})$$

and this gives the required inequality because $(V^r)^G \otimes_{\mathbb{Q}_p} k_p$ is a subspace of the left-hand side.

Gordon [11] shows, for all r, that the equality $\dim \mathbb{Q}_\ell A^r = h^{r,r}$ holds, and hence T^r is true, for X a smooth compactification of the k-fold fiber product of the universal family $A \to M$ of elliptic curves with level N-structure.

We conclude with a challenge. In [26] some exotic Hodge classes on abelian fourfolds with complex multiplication by cube roots of unity are shown to be algebraic and the Hodge conjecture thereby proved in a non-trivial case. D. Ramakrishnan asks whether it is possible to prove T^2 for these fourfolds.

REFERENCES

1. M. Artin and H. P. F. Swinnerton-Dyer, *The Shafarevich-Tate conjecture for pencils of elliptic curves on K3 surfaces*, Invent. Math. **20** (1973), 249–266.
2. D. Blasius and J. Rogawski, *Tate classes and arithmetic quotients of the two ball*, in Zeta Functions of Picard Modular Surfaces, Les publications CRM (R. P. Langlands and D. Ramakrishnan, eds.), Montreal, 1992, pp. 421–444.
3. P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 107–126.
4. ——, *La conjecture de Weil pour les surfaces K3*, Invent. Math. **15** (1972), 206–226.
5. ——, *La conjecture de Weil.* I, Inst. Hautes Études Sci. Publ. Math. **43** (1974) 273–308.
6. ——, *La conjecture de Weil.* II, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 137–251.
7. ——, *A quoi servent les motifs?*, these Proceedings, vol. 1, pp. 143–161.
8. G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366.
9. ——, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), 255–299.
10. G. Faltings and G. Wüstholz, *Rational Points*, Braunschweig, Viehweg, 1984.
11. B. Gordon, *Algebraic cycles and the Hodge structure of a Kuga fiber variety*, Trans. Amer. Math. Soc. **336**, no. 2 (1993), 933–947.
12. A. Grothendieck, *Le groupe de Brauer*, I. *Algèbres d'Azumaya et interprétations diverses*, II. *Théorie cohomologique*, III. *Exemples et compléments*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 46–188.
13. G. Harder, R. P. Langlands, and M. Rapoport, *Algebraische Zyklen auf Hilbert-Blumenthal-Flächen*, J. Reine Angew. Math. **366** (1986), 53–120.
14. U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), 447–452.
15. N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.
16. S. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 359–386.
17. C. Klingenberg, *Die Tate-Vermutungen für Hilbert-Blumenthal-Flächen*, Invent. Math. **89** (1987), 291–317.
18. S. Lichtenbaum, *Zeta-functions of varieties over finite fields at $s = 1$*, Arithmetic and Geometry, Progr. Math., vol. 35, Birkhäuser, Boston, 1983, 173–194.
19. J. Milne, *On a conjecture of Artin and Tate*, Ann. of Math. (2) **102** (1975), 517–533.
20. ——, *Étale cohomology*, Princeton Math. Ser., vol. 33, Princeton Univ. Press, Princeton, NJ, 1980.
21. ——, *Comparison of the Brauer group with the Tate-Shafarevitch group*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1982), 735–743.

22. ——, *Values of zeta functions of varieties over finite fields*, Amer. J. Math. **108** (1986), 297–360.
23. V. K. Murty and D. Ramakrishnan, *Period relations and the Tate conjecture for Hilbert modular surfaces*, Invent. Math. **89** (1987), 319–345.
24. ——, *Cycles on quaternionic Shimura surfaces* (in preparation).
25. N. Nygaard and A. Ogus, *Tate's conjecture for $K3$ surfaces of finite height*, Ann. of Math. (2) **122** (1985), 461–507.
26. C. Schoen, *Hodge classes on self products of a variety with an automorphism*, Compositio Math. **65** (1988), 3–32.
27. J.-P. Serre, *Valeurs propres des endomorphismes de Frobenius*, Sém. Bourbaki, no. 446, 1974, Lecture Notes in Math., vol. 431, Springer-Verlag, 1974, pp. 190–204.
28. T. Shioda and T. Katsura, *On Fermat varieties*, Tôhoku J. Math. **31** (1979), 97–115.
29. J. Tate, *Duality theorems in Galois cohomology over number fields*, Proc. Internat. Congr. Math. (Amsterdam 1962), pp. 288–295.
30. ——, *Algebraic cycles and poles of zeta functions*, Arithmetic Algebraic Geometry, Harper and Row, New York, 1965, pp. 93–110.
31. ——, *On the conjecture of Birch and Swinnerton-Dyer and a geometric analog*, Sém. Bourbaki, no. 306, 1966, pp. 1–26.
32. ——, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.
33. R. Weissauer, *Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades*, J. Reine Angew. Math. **391** (1988), 100–156.
34. J. G. Zarhin, *Isogenies of abelian varieties over fields of finite characteristics*, Math USSR-Sb. **24** (1974), 451–461.
35. ——, *A remark on endomorphisms of abelian varieties over function fields of finite characteristics*, Math. USSR-Izv. **8** (1974), 477–480.

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Some Remarks on the Hodge Type Conjecture

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In the theory of mixed motives, the following question (which will be called the *Hodge type conjecture*) seems interesting:

(0.1) Are absolute Hodge cycles **[8, 12]** algebraic?

(See also **[16]**, for example.) In fact, it is conjectured (or hoped) by Beilinson **[1]** and others that there would exist an abelian category of *mixed motivic sheaves* $\mathcal{MM}(X)$ with a constant object $\mathbb{Q}_X^M$ in its bounded derived category $D^b\mathcal{MM}(X)$ for a smooth projective variety X over a number field k, such that the cycle map

$$(0.2) \qquad CH^p(X)_{\mathbb{Q}} \to \operatorname{Ext}^{2p}_{D^b\mathcal{MM}(X)}(\mathbb{Q}_X^M, \mathbb{Q}_X^M(p))$$

is bijective, where $CH^p(X)_{\mathbb{Q}}$ is the Chow group of algebraic cycles on X modulo rational equivalence with rational coefficients. (See also **[30]**.) This would imply a positive answer to (0.1) using the adjunction isomorphism for $a_X : X \to \operatorname{Spec} k$ (see **[28]**), if $\mathcal{MM}(\operatorname{Spec} k)$ (or its pure part) is a full subcategory of the category consisting of the families of realizations as in **[8, 9, 12, 16]**.

Conversely, we can show that the Hodge type conjecture would imply a partial positive answer to the conjecture (0.2). In fact, as the first approximation to the category of mixed motivic sheaves $\mathcal{MM}(X)$, we can consider the full subcategory $\mathcal{M}(X)^{\mathrm{go}}$ of the objects of *geometric origin* (defined as in **[2]**) in the category $\mathcal{M}(X)$ consisting of families of realizations (see **[28]** for details). Then we have a cycle map (0.2) with $\mathcal{MM}(X)$ replaced by $\mathcal{M}(X)^{\mathrm{go}}$, and its surjectivity is reduced to the Hodge type conjecture for any smooth projective varieties.

So let us consider the proof of the Hodge conjecture. It is known (cf., for example, **[16]**) that the Hodge type conjecture can be reduced to the Hodge

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conjecture for complex varieties (see also (1.10) below). For a smooth (but not necessarily proper) variety, the group of codimension p Hodge cycles is defined by

$$\text{(0.3)} \qquad \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^{2p}(X, \mathbb{Q})(p))$$

(where MHS denotes the abelian category of mixed Hodge structures **[7]**), and the Hodge conjecture is meaningful also for such varieties (cf. also **[16]**).

The idea of Griffiths **[14]** (inspired by a work of Lefschetz **[20]**) is to use a Lefschetz pencil, which reduces the problem to the study of the Abel-Jacobi map. But the image of the Abel-Jacobi map is difficult to determine in general, and this attempt succeeded so far only in some special cases (see **[33]**). To solve this difficulty, a natural idea is to use the Lefschetz pencil iteratively, or more generally, to consider a smooth projective morphism $f : X \to S$ of smooth varieties such that X is birational to the original variety (here the singular fibers are deleted). The birational replacement of the variety is allowed as long as we can solve the problem by induction on the codimension of cycles (see (1.4) below), and it is especially useful in the codimension two case, because the Hodge conjecture for divisors is true by Lefschetz-Hodge.

For $f : X \to S$ as above, we have a Leray spectral sequence in the category of mixed Hodge structures:

$$\text{(0.4)} \qquad E_2^{i,j} = H^i(S, R^jf_*\mathbb{Q}_X)(p) \Rightarrow H^{i+j}(X, \mathbb{Q})(p)$$

(see (2.1) below), and it degenerates at E_2 by the decomposition theorem (see **[10]**). Forgetting the mixed Hodge structure, (0.4) can also be expressed as

$$\text{(0.5)} \qquad E_2^{i,j} = \mathrm{Ext}^i(\mathbb{Q}_S, R^jf_*\mathbb{Q}_X(p)) \Rightarrow \mathrm{Ext}^{i+j}(\mathbb{Q}_X, \mathbb{Q}_X(p)),$$

where the Ext's are taken in the derived categories of sheaves of $\mathbb{Q}$-modules $D(\mathbb{Q}_S)$ and $D(\mathbb{Q}_X)$. In fact, we have a natural isomorphism $\mathrm{Ext}^i(\mathbb{Q}_X, \mathscr{F}) = \mathbf{H}^i(X, \mathscr{F})$ for $\mathscr{F} \in D(\mathbb{Q}_X)$ by the adjunction for $a_X : X \to \mathrm{pt}$ (same for S). Note that Ext in (0.5) can be taken in other categories; for example, in the category of mixed Hodge modules, the right-hand side is isomorphic to the $\mathbb{Q}$-Deligne cohomology by the adjunction for $a_X : X \to \mathrm{pt}$ (see **[26, I]**) using a result of Carlson-Morgan. Here the higher extension Ext^p in the abelian category of mixed Hodge structures vanishes for $p > 1$ as a corollary of the result of Carlson-Morgan (see also **[16]**), and it was first observed by Beilinson.

We now restrict (0.5) to a fiber X_s of f, and get

$$\text{(0.6)} \qquad E_2^{i,j} = \mathrm{Ext}^i(\mathbb{Q}, H^j(X_s, \mathbb{Q})(p)) \Rightarrow \mathrm{Ext}^{i+j}(\mathbb{Q}_{X_s}, \mathbb{Q}_{X_s}(p)),$$

which also degenerates at E_2. Here Ext should not be taken in the category of sheaves of $\mathbb{Q}$-modules, because the E_2-term becomes zero for $i > 0$, and we lose too much information. In the category of mixed Hodge modules, the E_2-term vanishes for $i > 1$ by the vanishing of higher extensions explained above, and we do not get enough information either, except for the case of a Lefschetz pencil.

If the mixed motivic sheaves exist so that (0.2) holds, the information lost by the restriction is minimum. In fact, for a cycle with rational coefficients on X, its restriction to the pull-back of a dense Zariski-open subset of S is zero, if its restriction to each fiber is zero (see (2.10) below). But we do not have the mixed motivic sheaves yet, and, in order that (0.2) be surjective with $\mathcal{MM}(X)$ replaced by $\mathcal{M}(X)^{go}$, we need the Hodge conjecture for any varieties by definition of geometric origin. So we need a completely new idea to solve the problem. It should be noted that **[26, I]** is obtained as a byproduct of the above attempt.

However, in some simple cases, it is possible to use (0.4) effectively. Let us denote by $\mathrm{HC}(X, p)$ the Hodge conjecture for codimension p cycles on X, and consider the following conditions:

(i) There exists a surjective smooth projective morphism $f : X \to S$ of smooth varieties such that $r := \dim X - \dim S \leq 2$ and the geometric genus p_g $(:= \dim H^r(X_s, \mathcal{O}_{X_s}))$ of the fibers X_s of f is zero.

(ii) If $H^1(X_s, \mathbb{Q}) = 0$ and X_s is connected, then $\mathrm{HC}(S, 2)$ is true. Otherwise, there exists a smooth relative hyperplane section Y of f defined over a dense Zariski-open subset U of S (see Remark after (2.1)) such that $\mathrm{HC}(Y, 2)$ is true.

In the condition (ii), we may always assume fibers are connected, replacing S by a finite covering if necessary. Note that the condition (ii) is satisfied if $\dim X = 4$. The condition (i) for $r = 1$ implies that X is uniruled.

(0.7) THEOREM. *With the above notation, if the conditions* (i), (ii) *are satisfied, then* $\mathrm{HC}(X, 2)$ *is true.*

(See (2.4) for a more general statement.) A typical example is a product of two surfaces such that the geometric genus of one surface is zero $(r = 2)$ or a ruled fourfold $(r = 1)$. Since $\mathrm{HC}(X, 2)$ is reduced to $\mathrm{HC}(X', 2)$ for a dominant rational map $X' \to X$ (see (1.7)), (0.7) gives a generalization of Murre's argument for unirational (or uniruled) varieties **[6, 23]**. I am informed by the referee that a result similar to (0.7) with $r = 2$ has been obtained by S. Zucker **[36]** in some case (which includes the case where f and S are smooth projective and $\dim X = 4$). Note also that Grothendieck **[15]** gave an argument for the proof of his generalized Hodge conjecture using a Lefschetz pencil, if the cohomology of a generic hyperplane section satisfies some condition on the coniveau filtration (see also **[32]**).

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In this note, variety means a separated complex algebraic variety unless otherwise stated.

1. Hodge cycles and cycle maps

1.1. Hodge cycles. Let X be a smooth variety over $\mathbb{C}$. The group of codimension p Hodge cycles $H^{p,p}(X, \mathbb{Q})$ is defined by

$$H^{p,p}(X, \mathbb{Q}) = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^{2p}(X, \mathbb{Q})(p)), \tag{1.1.1}$$

where MHS denotes the category of mixed Hodge structures, and $\mathbb{Q}$ the Hodge structure of type $(0, 0)$ (see [7]). By construction of mixed Hodge structure in [7], $H^{2p}(X, \mathbb{Q})$ has weights $\geq 2p$, and

$$\mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}) = \mathrm{Im}(H^{2p}(\overline{X}, \mathbb{Q}) \to H^{2p}(X, \mathbb{Q})) \tag{1.1.2}$$

for any smooth compactification $\overline{X}$ of X. This implies the surjectivity of the restriction morphism

$$H^{p,p}(X, \mathbb{Q}) \to H^{p,p}(U, \mathbb{Q}) \tag{1.1.3}$$

for a dense Zariski-open subset U of X, because it is reduced to the case X proper, and follows from semisimplicity of the category of polarizable Hodge structures.

REMARK. Most of the arguments in this paper (except for (2.3), (2.10), etc.) would hold also for mixed motives defined by families of realizations (for varieties over a field k embeddable in $\mathbb{C}$) **[8, 9, 12, 16]** (using **[28]**), where the category of mixed Hodge structure is replaced by that of mixed motives. In this case $H^{p,p}(X, \mathbb{Q})$ is replaced by the group of absolute Hodge cycles.

1.2. Cycle maps. For X as in subsection 1.1, we have a cycle class map (cf., for example, **[12]**):

$$cl : CH^p(X)_{\mathbb{Q}} \to H^{p,p}(X, \mathbb{Q}), \tag{1.2.1}$$

where $CH^p(X)_{\mathbb{Q}}$ is the Chow group of codimension p cycles on X with $\mathbb{Q}$-coefficients. If X is not pure dimensional, it is better to use $CH_d(X)_{\mathbb{Q}}$ instead of $CH^p(X)_{\mathbb{Q}}$, where d is the dimension of the cycles.

The *Hodge conjecture* states the surjectivity of the cycle map (1.2.1). We will denote by $\mathrm{HC}(X, p)$ the Hodge conjecture for codimension p cycles on X. If X is not pure dimensional, we use the notation $\mathrm{HC}(X, \dim X - d)$, where d is the dimension of the cycle and $\dim X$ depends on each connected component of X.

1.3. REMARKS. (i) It is well known that the push-forward and pull-back of cycles are compatible with the Gysin (or trace) and restriction morphisms by the cycle map. (This can also be shown using the argument as in Remark (iv) below; see **[26, II]**.) In particular, if X is not proper, $\mathrm{HC}(X, p)$ is reduced to $\mathrm{HC}(\overline{X}, p)$ for any compactification $\overline{X}$ by (1.1.3).

(ii) Assume X smooth projective. Let l be the Chern class of a very ample line bundle, and Y a smooth hyperplane section whose cohomology class is l. Then the intersection with the cycle $[Y]$ corresponds to the cup product with l by the cycle map, because l is equal to the composition of the restriction and Gysin morphisms. We have the Lefschetz isomorphism

$$(1.3.1)\qquad l^k : H^{\dim X-k}(X, \mathbb{Q}) \xrightarrow{\sim} H^{\dim X+k}(X, \mathbb{Q})(k) \qquad (k > 0),$$

and one of the standard conjectures **[19]** states that it induces a bijection of their subgroups consisting of algebraic cycle classes.

If $p > \dim X/2$, $\mathrm{HC}(X, p)$ can be reduced to $\mathrm{HC}(Y, p-1)$ with Y any smooth hyperplane section of X using the Gysin morphism and the weak Lefschetz theorem. If $p < \dim X/2$, we have to use the Lefschetz pencil and the Leray spectral sequence, and $\mathrm{HC}(X, p)$ can be reduced to $\mathrm{HC}(Y, p)$ and $\mathrm{HC}(Y, p-1)$ for a generic hyperplane section Y of X using an argument similar to (1.10), (ii) below. See also **[15, 32]**. Here we need $\mathrm{HC}(Y, p-1)$ because $L_{2p-2}H^{2p}(\overline{X}, \mathbb{Q})$ is given by the image of the Gysin morphism by $X_s \to \overline{X}$ for $s \in S$ in the notation of (2.8), (ii) below.

(iii) For a smooth connected variety X, $\mathrm{HC}(X, p)$ is true for $p \leq 1$ or $p \geq \dim X - 1$. The case $p = 0$ or $p = \dim X$ is trivial, and $p = 1$ is due to Lefschetz-Hodge. Then, using the Remark (i), the remaining case $p = \dim X - 1$ is reduced to the projective case by Chow's lemma (because the composition of the restriction and trace morphisms is the identity) and follows from the divisor case, because (1.3.1) induces an isomorphism of Hodge cycles.

(iv) The cycle map can be defined also using mixed Hodge modules as in **[25, 26, 28, 29]**. In fact, $H^{2p}(X, \mathbb{Q})$ can be defined by $H^{2p}(a_X)_*\mathbb{Q}_X^H$ with $\mathbb{Q}_X^H := (a_X)^*\mathbb{Q}_{\mathrm{pt}}^H \in D^b\mathrm{MHM}(X)$ (the bounded derived category of mixed Hodge modules on X), where $a_X : X \to \mathrm{pt} := \mathrm{Spec}\,\mathbb{C}$ is a natural morphism. Here $\mathrm{MHM}(\mathrm{pt})$ is naturally identified with the category of polarizable (i.e., graded pieces are polarizable) mixed $\mathbb{Q}$-Hodge structures **[7]** (see **[25]**), and $\mathbb{Q}_{\mathrm{pt}}^H$ denotes the object of $\mathrm{MHM}(\mathrm{pt})$ corresponding to the Hodge structure of rank one and weight zero $\mathbb{Q}$. For X smooth, we can construct naturally a cycle map

$$(1.3.2)\qquad cl^H : CH^p(X)_{\mathbb{Q}} \to \mathrm{Hom}_{D^b\mathrm{MHM}(\mathrm{pt})}(\mathbb{Q}_{\mathrm{pt}}^H, (a_X)_*\mathbb{Q}_X^H(p)[2p]),$$

which induces (1.2.1) taking the cohomology of $(a_X)_*\mathbb{Q}_X^H(p)[2p]$.

The following is a more precise statement of **[29,** (1.16.1)**]** and is the basis of inductive argument.

1.4. PROPOSITION. *Let X be as above, U a Zariski-open subset of X, and $\pi: Y \to Z := X\backslash U$ a proper surjective morphism such that Y is smooth. Then* $\mathrm{HC}(X, \dim X-d)$ *is true if* $\mathrm{HC}(U, \dim U-d)$ *and* $\mathrm{HC}(Y, \dim Y-d)$ *are true.*

PROOF. We may assume X connected. Let $\zeta \in H^{p,p}(X, \mathbb{Q})$ for $p = \dim X - d$. Replacing ζ if necessary, we may assume the image of ζ in

$H^{p,p}(U, \mathbb{Q})$ is zero, because $\mathrm{HC}(U, p)$ is true and $CH^p(X)_{\mathbb{Q}} \to CH^p(U)_{\mathbb{Q}}$ is surjective. Let Y_i be the connected components of Y, and $r_i = \dim X - \dim Y_i$. Then we have natural morphisms of mixed Hodge structures:

$$\bigoplus_i H^{k-2r_i}(Y_i, \mathbb{Q})(-r_i) \to H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q}), \tag{1.4.1}$$

where the first morphism is the Gysin morphism, and is compatible with the push-forward of cycles (cf. Remark (1.3), (i)). So it is enough to show that Gr_k^W of (1.4.1) is exact, because it induces the exact sequence of Hodge cycles

$$\bigoplus_i H^{p-r_i, p-r_i}(Y_i, \mathbb{Q}) \to H^{p,p}(X, \mathbb{Q}) \to H^{p,p}(U, \mathbb{Q}), \tag{1.4.2}$$

using semisimplicity of polarizable Hodge structures. We have a natural exact sequence of mixed Hodge structures:

$$H^k_Z(X, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q}). \tag{1.4.3}$$

Let $\mathbb{D}_X^H$ $(:= (a_X)^! \mathbb{Q}^H_{\mathrm{pt}})$ denote the dualizing complex of X in $D^b\mathrm{MHM}(X)$ (same for Y, Z). Then $\mathbb{D}_X^H = \mathbb{Q}_X^H(\dim X)[2\dim X]$ (same for Y, where $\dim Y$ depends on each connected component). We have a trace morphism

$$\pi_* \mathbb{D}_Y^H \to \mathbb{D}_X^H, \tag{1.4.4}$$

and it induces the Gysin morphism $H^k(Y, \mathbb{D}_Y^H) \to H^k(X, \mathbb{D}_X^H)$. Let $i : Z \to X$ denote a natural inclusion. Since $i^! \mathbb{D}_X^H = \mathbb{D}_Z^H$, the trace morphism (1.4.4) is naturally factorized by $\mathbb{D}_Z^H$, and the trace morphism $\mathbb{D}_Z^H \to \mathbb{D}_X^H$ induces an isomorphism of mixed Hodge structures

$$H^k(Z, \mathbb{D}_Z^H) = H^k_Z(X, \mathbb{D}_X^H). \tag{1.4.5}$$

So it is enough to show the surjectivity of the trace morphism

$$\mathrm{Gr}_k^W H^k(Y, \mathbb{D}_Y^H) \to \mathrm{Gr}_k^W H^k(Z, \mathbb{D}_Z^H) \quad \text{for any} \quad k. \tag{1.4.6}$$

Let $M = C(\pi_* \mathbb{D}_Y^H \to \mathbb{D}_Z^H)$. It is enough to show

$$M \text{ has weights } > 0, \tag{1.4.7}$$

because it implies

$$H^k(Z, M) \text{ has weights } > k. \tag{1.4.8}$$

Let $i_x : \{x\} \to X$ denote a natural inclusion for $x \in X$. By [**25**, (4.6.1)], (1.4.7) is equivalent to

$$i_x^! M \text{ has weights } > 0 \text{ for any } x. \tag{1.4.9}$$

Let $Y_x = \pi^{-1}(x)$. Then $i_x^! M = C((a_{Y_x})_* \mathbb{D}_{Y_x}^H \to \mathbb{Q}^H_{\{x\}})$ by the base change theorem [**25**, (4.4.3)], and the assertion is reduced to the surjectivity of

$$H^0(a_{Y_x})_* \mathbb{D}_{Y_x}^H \to \mathbb{Q}^H_{\{x\}}, \tag{1.4.10}$$

using the long exact sequence, because $(a_{Y_x})_*\mathbb{D}^H_{Y_x}$, $\mathbb{Q}^H_{\{x\}}$ have weights ≥ 0. But the dual of (1.4.10) is the natural inclusion $\mathbb{Q}^H_{\{x\}} \to H^0(a_{Y_x})_*\mathbb{Q}^H_{Y_x}$, and the assertion follows. See also [**27**, (1.4)].

REMARKS. (i) The argument in the proof seems to be related with Grothendieck's theory of coniveau filtration and would be essentially known to specialists. See also [**7, 16, 32**], etc.

(ii) If $\dim X - d \leq 2$, then the condition on $\mathrm{HC}(Y, \dim Y - d)$ in (1.4) can be omitted, because we can take for Y a desingularization of each irreducible component of Z so that $\dim Y - d \leq 1$. So (1.4) implies

1.5. COROLLARY. *Let X and Y be smooth connected varieties birational to each other. Then* $\mathrm{HC}(X, 2)$ *and* $\mathrm{HC}(Y, 2)$ *are equivalent.*

1.6. COROLLARY. *If X has an increasing sequence of closed subvarieties X_i such that $U_i := X_i \setminus X_{i-1}$ is smooth, and* $\mathrm{HC}(U_i, \dim U_i - d)$ *holds for* $\dim U_i \geq d+2$ *on each connected component of U_i, then* $\mathrm{HC}(X, \dim X - d)$ *is true.*

PROOF. It is enough to apply (1.4) inductively to $X \setminus X_i$.

1.7. PROPOSITION. *Let $\pi : X \to Y$ be a dominant rational map of smooth connected varieties of the same dimension. Then* $\mathrm{HC}(Y, 2)$ *follows from* $\mathrm{HC}(X, 2)$.

PROOF. We may assume π proper surjective by (1.5). Then this is a special case of (1.4) where U is empty.

1.8. REMARK. As a corollary, $\mathrm{HC}(X, 2)$ is true for a unirational variety X. In particular, the Hodge conjecture is verified again for unirational fourfolds [**23**]. If X is uniruled (i.e., there is a generically finite dominant rational map of a ruled variety to it) and $\dim X = 4$, the Hodge conjecture is verified again using (1.7). In fact, $\mathrm{HC}(Z \times \mathbb{P}^1, p)$ for a smooth variety Z is reduced to $\mathrm{HC}(Z, p)$ and $\mathrm{HC}(Z, p-1)$ by Künneth decomposition (see also [**6**]). For the list of varieties for which the Hodge conjecture is verified, the reader is referred to [**31**].

1.9. REMARK. Let X be a smooth connected variety, and $\mathbb{C}(X)$ the field of rational functions on X. We define

$$H^{p,p}(\mathbb{C}(X), \mathbb{Q}) = \varinjlim H^{p,p}(U, \mathbb{Q}), \tag{1.9.1}$$

where the inductive limit is taken over nonempty open subvarieties U of X (cf. [**5**]). A weaker form of the Hodge conjecture $\mathrm{HC}(\mathbb{C}(X), p)$ (which may be called the generic Hodge conjecture) is defined by

$$H^{p,p}(\mathbb{C}(X), \mathbb{Q}) = 0. \tag{1.9.2}$$

By (1.1.3) this means $H^{p,p}(U, \mathbb{Q}) = 0$ for a sufficiently small nonempty Zariski-open subset U of X. Note that $\mathrm{HC}(X, p)$ implies $\mathrm{HC}(\mathbb{C}(X), p)$ by (1.1.3), and conversely, $\mathrm{HC}(X, \dim X - d)$ is true by (1.4) if

$\mathrm{HC}(\mathbb{C}(Y), \dim Y-d)$ is true for any smooth connected Zariski-locally closed subvariety Y of X. In particular, $\mathrm{HC}(X, 2)$ and $\mathrm{HC}(\mathbb{C}(X), 2)$ are equivalent.

1.10. Remarks. (i) Let k be an algebraic number field embedded in $\mathbb{C}$, and X a smooth projective variety over k. It is known that the Hodge type conjecture (see (0.1)) for X is reduced to the Hodge conjecture for $X \otimes_k \mathbb{C}$. In fact, the image of an absolute Hodge cycle in $H^{2p}(X(\mathbb{C}), \mathbb{Q}(p))$ belongs to the image of an algebraic cycle defined over a field K which is finitely generated over k. Then the assertion is reduced to the case K is algebraic over k, taking a finitely generated k-algebra R in K, which generates K, and using the reduction at a sufficiently general closed point of $\operatorname{Spec} R$. In this case, we may assume K is a Galois extension. Then the assertion is proved by taking the average of the images of the algebraic cycle by the action of the Galois group. In fact, the algebraic cycle defines a cycle class in $H^{2p}(X(\mathbb{C}), \mathbb{Q}(p))$ and $H^{2p}_{\text{ét}}(X \otimes_k \overline{k}, \mathbb{Q}_l(p))$ compatible with the comparison isomorphism, and its étale part is invariant by the action of $\mathrm{Gal}(\overline{k}/k)$, because it coincides with the original absolute Hodge cycle by its coincidence on $H^{2p}(X(\mathbb{C}), \mathbb{Q}(p))$. Compare to **[23]**.

By a similar argument, we see that the natural morphism

$$CH^p(X)_{\mathbb{Q}} \to CH^p(X \otimes_k K)_{\mathbb{Q}} \tag{1.10.1}$$

is injective for a field extension $k \to K$.

(ii) Let X be an irreducible and reduced complex algebraic variety. Let k be a subfield of $\mathbb{C}$ such that X is defined over k; i.e., there is a k-variety X_k with an isomorphism $X = X_k \otimes_k \mathbb{C}$. We say that a closed point x of X is *generic* with respect to X_k if x determines a geometric generic point of X_k (i.e., for any nonempty affine open subvariety $\operatorname{Spec} A$ of X_k, x belongs to $\operatorname{Spec} A \otimes_k \mathbb{C}$ and the morphism $A \to A \otimes_k \mathbb{C} \to \mathbb{C}$ defined by x is injective). The condition is equivalent to that x is not contained in any proper closed subvariety of X defined over k, and we can also consider this notion in the sense of Weil taking $\mathbb{C}$ as the universal domain.

Let $f : X \to S$ be a smooth projective morphism of complex algebraic varieties such that S is reduced and irreducible. Let k be a subfield of $\mathbb{C}$ such that f is defined over k; i.e., f is the base change of a morphism of k-varieties $f_k : X_k \to S_k$. Assume S has a generic closed point s' with respect to S_k (e.g., k is finitely generated over $\mathbb{Q}$). Let ξ be a section of $R^{2p} f_* \mathbb{Q}_X(p)$ on S. Its stalk at $s \in S$ is $\xi_s \in H^{2p}(X_s, \mathbb{Q}(2p))$, which is invariant by the action of the monodromy group. Let s' be a generic closed point of S with respect to S_k. Assume $\xi_{s'} = cl(\zeta')$ for $\zeta' \in CH^p(X_{s'})_{\mathbb{Q}}$. Then there exists $\zeta \in CH^p(X)_{\mathbb{Q}}$ such that $\xi_s = cl(\zeta_s)$ for any $s \in S$ where ζ_s is the restriction of ζ to X_s (but $\zeta_{s'}$ may be different from ζ'). For the proof, we may replace S with an open subvariety. By hypothesis, there exists a surjective morphism $S' \to S$ with a cycle on $X' \times_S S'$ such that the assertion is satisfied for the pull-back of ξ to S'. Replacing S' with a subvariety (and S with an open subvariety), we may assume S' finite over S. Then the assertion follows taking the direct image of the cycle and dividing it by the degree of the morphism. See also **[32]**.

2. Fibration

2.1. Let $f : X \to S$ be a proper smooth morphism of smooth connected varieties over $\mathbb{C}$, and r the relative dimension. Then we have the Leray spectral sequence

$$E_2^{i,j} = H^i(S, R^j f_* \mathbb{Q}_X) \Rightarrow H^{i+j}(X, \mathbb{Q}) \tag{2.1.1}$$

which degenerates at E_2 by the decomposition theorem for $\mathbf{R}f_*\mathbb{Q}_X$. (See **[2]** and also **[10]** in the projective case.) It is a spectral sequence in the category of mixed Hodge structures, because it is induced by the canonical filtration τ on the direct image $f_*\mathbb{Q}_X^M$ in $D^b\mathrm{MHM}(S)$ in the notation of (1.3), (iv), up to a shift of indices (see also **[7]**). This fact was shown in **[34]** if $\dim S = 1$. We denote by L the filtration induced by τ so that

$$\mathrm{Gr}_j^L H^n(X, \mathbb{Q}) = H^{n-j}(S, R^j f_* \mathbb{Q}_X). \tag{2.1.2}$$

Note that L splits in the category of mixed Hodge structures using the decomposition theorem of $f_*\mathbb{Q}_X^M$ (see **[25]**).

Remark. If $f : X \to S$ is projective with a relative ample line bundle L, then, restricting S to a dense Zariski-open subset, a multiple of L is very ample, and is associated with a smooth divisor D, which will be called a *smooth relative hyperplane section* of f.

2.2. Definition. Let $\mathbf{H} = (H_{\mathbb{C}}, F, H_{\mathbb{Q}})$ be a pure Hodge structure of weight w. The *level* of $\mathbf{H}$ is defined by

$$\max\{i : \mathrm{Gr}_F^i H_{\mathbb{C}} \neq 0\} - \min\{i : \mathrm{Gr}_F^i H_{\mathbb{C}} \neq 0\},$$

and the *geometric level* by the minimum of the dimension of a smooth projective variety Y such that $\mathbf{H}$ is isomorphic to a direct factor of $H^k(Y, \mathbb{Q})(-m)$ as Hodge structures, where $w = k + 2m$. We have

$$k = \dim Y \tag{2.2.1}$$

by the hard and weak Lefschetz theorems (see **[26]**). Note that a Hodge structure is called *geometric* if its geometric level is finite (see **[26]**).

Similarly, we can define the level and geometric level of a variation of Hodge structure $\mathbf{H}$ of weight w on a smooth connected variety S. The *level* is defined by the sum of the dimension of the variety and the level of the Hodge structure of a fiber. The *geometric level* is defined by the minimum of the dimension of a smooth variety Y with a smooth projective morphism π onto a dense Zariski-open subset U of S, such that $\mathbf{H}|_U$ is isomorphic to a direct factor of $R^k\pi_*\mathbb{Q}_Y(-m)$ as variation of Hodge structure (see **[26]**), where $w = k + 2m$. We also have $k = \dim Y - \dim S$. (The level and geometric level of a pure (i.e., polarizable) Hodge module with strict support are defined by those of the variation of Hodge structure which is obtained by restricting the Hodge module to a smooth open subvariety of the support.)

2.3. REMARK. The level of a Hodge structure is smaller than or equal to the geometric level. Grothendieck's *generalized Hodge conjecture* [**15**] implies

(2.3.1) The level and the geometric level of a geometric Hodge structure coincide.

In fact, it asserts that, for a smooth projective variety X and a sub-Hodge structure $\mathbf{H}$ of $H^n(X, \mathbb{Q})$ with level $n-2m$, there is a smooth projective variety Y' of dimension $\dim X - m$ with a morphism $\pi : Y' \to X$ such that $\mathbf{H}$ is contained in the image of the Gysin morphism.

Actually, Grothendieck's generalized Hodge conjecture is equivalent to (2.3.1) if the Hodge conjecture is true. In fact, let X, $\mathbf{H}$ be as above, and assume (2.3.1) for $\mathbf{H}$. Let k be the geometric level of $\mathbf{H}$ so that $\mathbf{H}$ is isomorphic to a direct factor of $H^k(Y, \mathbb{Q})(-m)$ for a k-dimensional smooth projective variety Y, where $k = n - 2m$, and (2.2.1) is used. We have a natural inclusion

$$\begin{aligned}(2.3.2)\quad &\mathrm{Hom}_{\mathrm{MHS}}(H^k(Y, \mathbb{Q})(-m), H^n(X, \mathbb{Q}))\\ &\qquad \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^{2n-2m}(X \times Y, \mathbb{Q})(n-m))\end{aligned}$$

using the Künneth decomposition and the self duality

$$\mathbb{D}H^k(Y, \mathbb{Q}) = H^k(Y, \mathbb{Q})(k).$$

Assume $\mathrm{HC}(X \times Y, n-m)$ is true. Then the composition

$$(2.3.3) \qquad H^k(Y, \mathbb{Q})(-m) \to \mathbf{H} \to H^n(X, \mathbb{Q})$$

is induced by an algebraic cycle on $X \times Y$ with dimension $\dim X - m$. Let Z be a desingularization of the support of the cycle. Since the action of the group of correspondences on the cohomology is expressed by the composition of the restriction and Gysin morphisms, the image of (2.3.3) is contained in the image of the Gysin morphism by $Z \to X$. So Grothendieck's generalized Hodge conjecture follows. (See also [**26**].)

2.4. THEOREM. *With the notation of* (2.1), *assume f is projective, and let d be an integer such that $0 < d \le \dim S$. Assume*

(i) *The variation of Hodge structure $R^r f_* \mathbb{Q}_X$ has geometric level $< \dim X$ so that $R^r f_* \mathbb{Q}_X|_U$ is isomorphic to a direct factor of $R^{r-2m} \pi_* \mathbb{Q}_Y(-m)$ for $\pi : Y \to U$ and m as in* (2.2), *and $\mathrm{HC}(Y, \dim Y - d + m)$ is true, where* $\dim Y = \dim X - 2m < \dim X$.

(ii) *There exists a smooth relative hyperplane section Y' of f defined over a dense Zariski-open subvariety U' of S (see Remark after* (2.1)) *such that* $\mathrm{HC}(Y', \dim Y' - d)$ *and* $\mathrm{HC}(Y', \dim Y' - d + 1)$ *are true.*

(iii) *For any reduced divisor Z' on S, there exists a proper surjective morphism $Z \to Z'$ such that Z is smooth and* $\mathrm{HC}(X \times_S Z, \dim Z + r - d)$ *is true.*

Then $\mathrm{HC}(X, \dim X - d)$ *is true. Furthermore, if* S *is affine and* $2d \le \dim X$, *then* $\mathrm{HC}(Y', \dim Y' - d + 1)$ *in the assumption* (ii) *can be omitted.*

Proof. By (1.4) and the assumption (iii), we may assume $U = U' = S$, replacing S by the complement of a divisor, which is contained in U and U' (using (1.1.3)). Let f' be the restriction of f to Y'. We have the restriction (resp. Gysin) morphism

$$\begin{gathered}\alpha_i : H^i(X, \mathbb{Q}) \to H^i(Y', \mathbb{Q}) \\ (\text{resp. } \beta_i : H^i(Y', \mathbb{Q}) \to H^{i+2}(X, \mathbb{Q})(1))\end{gathered} \tag{2.4.2}$$

which preserves L (resp. preserves L up to shift). Moreover,

$$\begin{gathered}\mathrm{Gr}_j^L \alpha_{i+j} : H^i(S, R^j f_* \mathbb{Q}_X) \to H^i(S, R^j f'_* \mathbb{Q}_{Y'}) \\ (\text{resp. } \mathrm{Gr}_j^L \beta_{i+j} : H^i(S, R^j f'_* \mathbb{Q}_{Y'}) \to H^i(S, R^{j+2} f_* \mathbb{Q}_X)(1))\end{gathered} \tag{2.4.3}$$

is bijective for $j < r-1$ (resp. $j > r-1$) and injective for $j = r-1$ (resp. surjective for $j = r-1$) by the weak Lefschetz theorem (on each fiber of f). Here the injective and surjective morphisms split by semisimplicity of polarizable variations of pure Hodge structures.

Let $\xi \in H^{p,p}(X, \mathbb{Q})$ for $p = \dim X - d$. We denote also by L the filtration on $H^{p,p}(X, \mathbb{Q})$ induced by L in (2.1.2). We first reduce to the case $\xi \in L_r H^{p,p}(X, \mathbb{Q})$ by modifying ξ with the image of the cycle map and then show that ξ vanishes by restricting S. This implies the first assertion by the assumption (iii). See the proof of (1.4).

Assume $\xi \in L_j H^{p,p}(X, \mathbb{Q})$ for $j > r$, and consider $\mathrm{Gr}_j^L \xi$ in

$$\mathrm{Gr}_j^L H^{p,p}(X, \mathbb{Q}) = \mathrm{Hom}(\mathbb{Q}, H^{2p-j}(S, R^j f_* \mathbb{Q}_X)(p)).$$

By (2.4.3), there exists

$$\xi' \in \mathrm{Gr}_{j-2}^L H^{p-1,p-1}(Y', \mathbb{Q}) = \mathrm{Hom}(\mathbb{Q}, H^{2p-j}(S, R^{j-2} f'_* \mathbb{Q}_{Y'})(p-1))$$

such that $\mathrm{Gr}_j^L \xi = \mathrm{Gr}_{j-2}^L \beta_{2p-2}(\xi')$. By assumption on $\mathrm{HC}(Y', \dim Y' - d)$, there is a cycle $\zeta' \in CH^{p-1}(Y')_{\mathbb{Q}}$ such that its cycle class belongs to $L_{j-2} H^{p-1,p-1}(Y', \mathbb{Q})$, and coincides with ξ' in $\mathrm{Gr}_{j-2}^L H^{p-1,p-1}(Y', \mathbb{Q})$. Let $i' : Y' \to X$ denote the natural inclusion. By the compatibility of push-forward with the Gysin morphism, the cycle class of $\zeta = i'_* \zeta'$ belongs to $L_j H^{p,p}(X, \mathbb{Q})$, and coincides with ξ modulo $L_{j-1} H^{p,p}(X, \mathbb{Q})$. So we can proceed by induction.

Now assume $\xi \in L_r H^{p,p}(X, \mathbb{Q})$, and show the vanishing of ξ by restricting S. By induction, it is enough to show that any

$$\xi \in \mathrm{Hom}(\mathbb{Q}, H^{2p-j}(S, R^j f_* \mathbb{Q}_X)(p))$$

vanishes by restricting S to a dense open subvariety for $j \le r$.

If $j = r$, we may replace $R^r f_* \mathbb{Q}_X$ with $R^{r-2m}\pi_* \mathbb{Q}_Y(-m)$ by the assumption (i). Then the assertion follows from $\mathrm{HC}(Y, \dim Y - d + m)$, because the dimension of the cycle is $d - m$ and is smaller than $\dim S$.

The argument proceeds similarly for $j < r$. The assertion is reduced to $\mathrm{HC}(Y', \dim Y' - d + 1)$ using the injectivity of the restriction morphisms in (2.4.3). This completes the proof of the first assertion.

The last assertion follows from Artin's theorem [**2**]:

$$H^{2p-j}(S, R^j f_* \mathbb{Q}_X) = 0 \quad \text{for } 2p - j > \dim S,$$

because the assumption $2d \leq \dim X$ implies $2p \geq \dim X$ and $2p > \dim S + j$ for $j < r$.

2.5. Remarks. (i) For the moment, we do not know any method that enables us to avoid the assumption (i) on the geometric level of $R^r f_* \mathbb{Q}_X$. So the induction is not complete, and the assumption on the Hodge conjecture for lower-dimensional varieties cannot be verified using this method. Also, we have to delete the union of singular fibers to get a fibration, and we do not know anything about the Hodge conjecture for the deleted subset in general. So it would be better used for the generic Hodge conjecture $\mathrm{HC}(\mathbb{C}(X), p)$ (see (1.9)), for which the condition (iii) can be neglected.

(ii) In the case of codimension two cycles, the assumptions on the Hodge conjecture in (2.4) are satisfied except for $\mathrm{HC}(Y', \dim Y' - d + 1)$, and we get Theorem (0.7). In fact, the assumption (i) on the geometric level of $R^2 f_* \mathbb{Q}_X$ for $r = 2$ follows from the condition on the geometric genus using the lemma below (applied to $R^2 f_* \mathbb{Q}_X(1)$), and the smooth relative hyperplane section Y' used in the proof of (2.4) for $j < r$ can be replaced by S if $H^1(X_s, \mathbb{Q}) = 0$ and $H^0(X_s, \mathbb{Q}) = \mathbb{Q}$.

2.6. Lemma. *Let* $\mathbf{H}$ *be a polarizable variation of Hodge structure of type* $(0, 0)$ *on a smooth connected variety. If the underlying local system is defined over* $\mathbb{Z}$, *then the geometric level of* $\mathbf{H}$ *is equal to the dimension of* S.

Proof. It is enough to show that the monodromy group of $\mathbf{H}$ is finite (taking the associated covering). But it is a discrete subgroup of the orthogonal group (which is compact), and the assertion is clear.

2.7. Example. Let X be a nonsingular hypersurface on $\mathbb{P}^{n+1}$, and $PH^i(X, \mathbb{Q})$ denote its primitive cohomology. Then $PH^i(X, \mathbb{Q}) = 0$ for $i \neq 0, n$, and $\mathrm{HC}(X, p)$ is trivial except for $n = 2p$. Assume the degree of X is 3 and $n = 2p$. We have a fibration $f : X' \to S$ induced by a generic projection $\mathbb{C}^{n+2} \to \mathbb{C}^{n-1}$ so that X' is a Zariski-open subset of the blow-up of X along the intersection of X with a two-dimensional plane, S is a Zariski-open subset of $\mathbb{P}^{n-2}$, and each fiber X_s of f is the intersection of X with a three-dimensional plane of $\mathbb{P}^{n+1}$ (i.e., a cubic surface in $\mathbb{P}^3$). Then X_s is the blow-up of $\mathbb{P}^2$ along six points, and we have $H^1(X_s, \mathscr{O}_{X_s}) = H^2(X_s, \mathscr{O}_{X_s}) = 0$ and $\dim PH^2(X_s, \mathbb{Q}) = 6$. So the Hodge conjecture is clear for $n = 4$ by (0.7). (This case was checked by many people **[6, 34]**.)

Assume $n = 6$, $p = 3$. Then the generic Hodge conjecture $\mathrm{HC}(\mathbb{C}(X), 3)$ (see (1.9)) is reduced to $\mathrm{HC}(\mathbb{C}(S'), 2)$ for the covering S' of S associated with the monodromy group of the local system $R^2 f_* \mathbb{Q}_{X'}$ (cf. the proof of (2.6)). In fact, the smooth relative hyperplane section Y' in the condition (ii) is induced by a projection $\mathbb{C}^{n+1} \to \mathbb{C}^{n-1}$, and is an open subvariety of the blow-up of a hyperplane section of X along finite points, and the Hodge conjecture for Y' is easily checked using (1.1.3). However, it is not easy to verify $\mathrm{HC}(\mathbb{C}(S'), 2)$.

2.8. Remarks. (i) Let $f : X \to S$ be as in (2.1), and assume S is a dense open subvariety of $\overline{S} = \mathbb{P}^1$. Let

$$H^j(X_s, \mathbb{Q})^{\mathrm{van}} = \mathrm{Coker}(H^j(X, \mathbb{Q}) \to H^j(X_s, \mathbb{Q})),$$
$$H^j(X, \mathbb{Q})^0 = \mathrm{Ker}(H^j(X, \mathbb{Q}) \to H^j(X_s, \mathbb{Q})).$$

Then, for $s \in S$, we have a short exact sequence

$$(2.8.1) \qquad 0 \to H^{2p-1}(X_s, \mathbb{Q})^{\mathrm{van}} \to H^{2p}(X, X_s; \mathbb{Q}) \to H^{2p}(X, \mathbb{Q})^0 \to 0$$

by the long exact sequence associated with relative cohomology.

Let $J^p(X)_{\mathbb{Q}}$ be the intermediate Jacobian **[14]** tensored with $\mathbb{Q}$, i.e.,

$$(2.8.2) \qquad J^p(X)_{\mathbb{Q}} = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^{2p-1}(X, \mathbb{Q})(p)).$$

Same for $J^p(X_s)_{\mathbb{Q}}$. Let $\xi \in H^{p,p}(X, \mathbb{Q})(p)$ such that its restriction to each X_s is zero. Then it induces the normal function

$$(2.8.3) \qquad \xi_s \in J^p(X_s)^{\mathrm{van}}_{\mathbb{Q}} := \mathrm{Coker}(J^p(X)_{\mathbb{Q}} \to J^p(X_s)_{\mathbb{Q}}) \quad \text{for } s \in S,$$

taking the pull-back of the short exact sequence (2.8.1) by the morphism

$$(2.8.4) \qquad \mathbb{Q}(-p) \to H^{2p}(X, \mathbb{Q})^0$$

induced by ξ. Let

$$(2.8.5) \qquad CH^p(X_s)^0_{\mathbb{Q}} = \mathrm{Ker}(CH^p(X_s)_{\mathbb{Q}} \to H^{2p}(X_s, \mathbb{Q})(p)),$$

and consider the composition of the Abel-Jacobi map with a natural projection:

$$(2.8.6) \qquad CH^p(X_s)^0_{\mathbb{Q}} \to J^p(X_s)_{\mathbb{Q}} \to J^p(X_s)^{\mathrm{van}}_{\mathbb{Q}}.$$

Then, to solve the Hodge conjecture (using induction on dimension), it is enough to construct a cycle $\zeta \in CH^p(X)_{\mathbb{Q}}$ such that its restriction to X_s belongs to $CH^p(X_s)^0_{\mathbb{Q}}$ and its image by (2.8.6) coincides with ξ_s for any $s \in S$ (restricting S if necessary). See **[26, I]**. So we have to show first that each ξ_s belongs to the image of (2.8.6).

(ii) Assume f is associated with a (generic) Lefschetz pencil of a smooth projective variety X' embedded in $\mathbb{P}^m$ so that f is extended to $\overline{f} : \overline{X} \to \overline{S} =$

$\mathbb{P}^1$ and $\overline{X}_s = X'_s := X' \cap H_s$ for a family of hyperplanes $\{H_s\}$ of $\mathbb{P}^m$ (where $\overline{X}$ is the blow up of X' along the intersection of two sufficiently general hyperplane sections). See **[14, 17, 33, 35]**. Then X, X_s in the above remark can be replaced by X', X'_s respectively using the restriction morphism, and we have $H^{2p}(X', X'_s; \mathbb{Q}) = H_c^{2p}(X' \backslash X'_s, \mathbb{Q})$. If the embedding of X' into $\mathbb{P}^m$ is sufficiently ample, $R^j \overline{f}_* \mathbb{Q}_{\overline{X}}$ is an intersection complex up to a shift (i.e., it has no nontrivial sub or quotient object supported on points as shifted perverse sheaves) by **[17]**. Note that this holds if $H^{\dim X - 1}(X_s, \mathbb{Q})^{\mathrm{van}} \neq 0$ by the Picard-Lefschetz formula **[11]**. In this case, we do not have to consider the Hodge conjecture for a desingularization of singular fibers.

2.9. Remark. Besides the problem mentioned in the introduction, there are two other problems to realize the idea on the proof of the Hodge conjecture using the restriction to fibers. One is that, for a given family of cycles $\{\zeta_s\}$ with $\zeta_s \in CH^p(X_s)_{\mathbb{Q}}$, it is not easy to construct a cycle $\zeta \in CH^p(X)_{\mathbb{Q}}$ whose restriction to each fiber coincides with ζ_s (by restricting S if necessary). The family of cycles must be parametrized algebraically in some sense, but we do not have such a notion in general.

In the case of the Lefschetz pencil in (2.8), we have a similar (but much less serious) problem. Even if we can show that each ξ_s belongs to the image of (2.8.6), we have to construct a cycle whose restriction to X_s induces ξ_s by (2.8.6). Since the intermediate Jacobian is defined only analytically, we would have to construct a compactification of the family of intermediate Jacobians, and use GAGA on the Hilbert scheme of the compactified family $\overline{f} : \overline{X} \to \overline{S}$ of $f : X \to S$.

The second is that we have to show a statement for $\mathrm{Ext}^{2p}(\mathbb{Q}_X^H, \mathbb{Q}_X^H(p))$ corresponding to the remark below.

2.10. Remark. Let $f : X \to S$ be as in (2.1). If the restriction of $\zeta \in CH^p(X)_{\mathbb{Q}}$ to any fiber is zero, then there exists a dense Zariski-open subset U of S such that the restriction of ζ to $f^{-1}(U)$ is zero. (The following proof is due to S. Bloch.) In fact, we may assume f and ζ are defined over a subfield k of $\mathbb{C}$, which is finitely generated over $\mathbb{Q}$. We have an inclusion $k(S) \to \mathbb{C}$, and it determines a $\mathbb{C}$-valued point of S. By assumption, the base change of ζ over this point is zero. Then there exists a subfield K of $\mathbb{C}$ containing $k(S)$ such that K is finitely generated over $k(S)$, and the base change of ζ over K is zero. We may assume K is algebraic over $k(S)$, and then $K = k(S)$ by the same argument as in (1.10).

2.11. Remark. In the notation of (2.4), assume $d < \dim S$ (i.e., $r < p$) so that the assumption of (2.10) is satisfied. Let $\mathbf{H} = R^r f_* \mathbb{Q}_X$ as a polarizable variation of Hodge structure of weight r. Then, for $i + r = 2p$, we have

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^i(U, \mathbf{H})(p)) = 0 \tag{2.11.1}$$

for $U \subset S$ sufficiently small, if $\mathrm{HC}(X, p)$ is true. By definition of geometric level, the assumption $r < p$ may be replaced with

$$\text{(2.11.2)} \qquad \mathbf{H} \text{ has geometric level } < \dim S + i\,.$$

So (2.11.1) is closely related with **[27,** (2.11)] using Yoneda extension in the category of mixed Hodge modules of geometric origin.

In the case $d = \dim S - 1$, the image of a cycle to S is a divisor. But we lose too much information if we use the push-forward of cycles, and it is not easy to detect the divisor by a cohomological method.

If $\mathbf{H}$ is constant with fiber H, then (2.11.1) is equivalent to

$$\text{(2.11.3)} \qquad \mathrm{Hom}_{\mathrm{MHS}}(H, H^i(U, \mathbb{Q})(p-r)) = 0$$

for U sufficiently small, where the dual of H is identified with $H(r)$ using a polarization. This situation is similar to Grothendieck's generalized Hodge conjecture **[15]** (see also (2.3)), and (2.11.1) may be viewed as an extension of Grothendieck's generalized Hodge conjecture to the case of variation of Hodge structure.

2.12. Remark. Let k be a field finitely generated over $\mathbb{Q}$, and consider the Hodge type conjecture over the field k. In this case, the first problem in (2.9) might be solved by considering the scheme-theoretic generic fiber Y of π and using

$$\text{(2.12.1)} \qquad \mathscr{M}_K(Y) = \varinjlim \mathscr{M}(\pi^{-1}(U))$$

where the inductive limit is taken over dense open subvarieties U of S, and $K = k(S)$. See **[28,** (1.9)].

If $d < \dim S$, (2.10–11) would be related with a conjecture

$$\text{(2.12.2)} \qquad \mathrm{Ext}^{2p}(\mathbb{Q}_Y^M, \mathbb{Q}_Y^M(p)) = 0 \quad \text{for } p > \dim Y,$$

where Ext is taken in $D^b\mathscr{M}_K(Y)^{\mathrm{go}}$. In the case of Example (v) in **[28,** (1.8)], we can show the equivalence of categories

$$\text{(2.12.3)} \qquad \mathscr{M}_K(Y)^{\mathrm{go}} \to \mathscr{M}(Y)^{\mathrm{go}},$$

where $\mathscr{M}(Y)$ is defined as in **[28]** for the variety Y over K.

References

1. A. Beilinson, *Height pairing between algebraic cycles*, Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–26.
2. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque, vol. 100, Soc. Math. France, Paris, 1982.
3. S. Bloch, *Lectures on algebraic cycles*, Duke Univ. Math. Ser. **4**, Durham, NC, 1980.
4. ——, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.
5. S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201.
6. A. Conte and J. P. Murre, *The Hodge conjecture for fourfolds admitting a covering by rational curves*, Math. Ann. **238** (1978), 79–88.
7. P. Deligne, *Théorie de Hodge.* I, Proc. Internat. Congr. Math. I (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 425–430; II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–58; III, ibid. **44** (1974), 5–77.

8. ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 313–346.
9. ——, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over $\mathbb{Q}$, Springer-Verlag, New York and Berlin, 1989, pp. 79–297.
10. ——, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 107–126.
11. ——, *La formule de Picard-Lefschetz*, Lecture Notes in Math., vol. 340, Springer-Verlag, Berlin, 1973, pp. 165–196.
12. P. Deligne, J. Milne, A. Ogus, and K. Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math., vol. 900, Springer, Berlin, 1982.
13. W. Fulton, *Intersection Theory*, Springer, Berlin, 1984.
14. Ph. Griffiths, *Some transcendental methods in the study of algebraic cycles*, Lecture Notes in Math., vol. 185, Springer, Berlin, 1971, pp. 1–46.
15. A. Grothendieck, *Hodge's general conjecture is false for trivial reasons*, Topology **8** (1969), 299–303.
16. U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, Berlin, 1990.
17. N. Katz, *Etude cohomologique des pinceaux de Lefschetz*, Lecture Notes in Math., vol. 340, Springer, Berlin, 1973, pp. 254–327.
18. S. Kleiman, *Motives*, Algebraic Geometry, Oslo, 1970; Wolters-Noordhoff, Holland, 1972, pp. 53–82.
19. ——, *Algebraic cycles and the Weil conjecture*, Dix Exposés sur la Cohomologie des Schémas, Masson, Paris and North-Holland, Amsterdam, 1968, pp. 359–386.
20. S. Lefschetz, *L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1924.
21. J. Manin, *Correspondences, motifs and monoidal transformations*, Math. USSR-Sb. **6** (1968), 439–470.
22. J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.
23. ——, *On the Hodge conjecture for unirational fourfolds*, Indag. Math. **80** (1977), 230–232.
24. M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), 849–995.
25. ——, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.
26. ——, *Hodge conjecture and mixed motives*. I, Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, pp. 283–303; II, Lecture Notes in Math., vol. 1479, Springer, Berlin, 1991, pp. 196–215.
27. ——, *On the injectivity of cycle maps*, Publ. Res. Inst. Math. Sci. **28** (1992), 99–127.
28. ——, *On the formalism of mixed sheaves*, preprint, RIMS-784, 1991.
29. ——, *Introduction to mixed Hodge modules*, Astérisque, no. 179–180, 1989, pp. 145–162.
30. ——, *On the bijectivity of some cycle maps*, these Proceedings, vol. 1, pp. 315–333.
31. T. Shioda, *What is known about the Hodge conjecture*? Algebraic Varieties and Analytic Varieties, Adv. Stud. Pure Math., vol. 1, Kinokuniya, Tokyo, 1983, pp. 55–68.
32. J. Steenbrink, *Some remarks about the Hodge conjecture*, Hodge Theory, Lecture Notes in Math., vol. 1246, Springer, Berlin, 1987, pp. 165–175.
33. S. Zucker, *The Hodge conjecture for cubic fourfolds*, Compositio Math. **34** (1977), 199–209.
34. ——, *Variation of mixed Hodge structures*. II, Invent. Math. **80** (1985), 543–565.
35. ——, *Generalized intermediate Jacobians and the theorem on normal functions*, Invent. Math. **33** (1976),185–222.
36. ——, *Leray filtrations, mixed cycle classes and a new example of the Hodge conjecture*, preprint, 1979.

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Independence of ℓ and Weak Lefschetz

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The problem of independence of ℓ for smooth affines

This paper grew out of an (as yet unsuccessful) attempt to deal with a special case of *the problem of "independence of ℓ"* of characteristic polynomials of Frobenius. Let k be a finite field of characteristic p, X/k a separated k-scheme of finite type. It is conjectured (cf. **[Ka-Review]**) that for each i in $[0, 2\dim X]$, we have **Indep**$(i, X/k)$: the characteristic polynomials

$$P_{i,\ell}(T) := \det(1 - TF|H^i_c(X \otimes_k \overline{k}, \mathbb{Q}_\ell))$$

have $\mathbb{Q}$-coefficients (or equivalently $\mathbb{Z}$-coefficients, since they are algebraic integers), independent of the auxiliary choice of $\ell \neq p$.

For brevity, let us say that X/k is "independent of ℓ" if **Indep**$(i, X/k)$ holds for all i.

An elementary but useful observation is this. In order to prove that X/k is independent of ℓ, i.e., that **Indep**$(i, X/k)$ holds for all i, it suffices to prove **Indep**$(i, X/k)$ for all but one value of i. For the alternating product of the $P_{i,\ell}$ is the zeta function of X/k, which is independent of ℓ.

Consider henceforth the special case in which X/k is affine, smooth, and geometrically connected, of some dimension $n \geq 0$. We will discuss the problem of proving that such an X is independent of ℓ. For such an X, the groups $H^i_c(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ vanish unless i is in the interval $[n, 2n]$: this is the Poincaré dual (X is smooth) of the "affine Lefschetz" vanishing (X is affine) of the ordinary groups $H^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ for $i > n$. Moreover, if we denote by

$$P_{i,\ell,*}(T) := \det(1 - TF|H^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell))$$

the characteristic polynomial of Frobenius on *ordinary* cohomology, then for each i the polynomials $P_{i,\ell,*}(T)$ and $P_{2n-i,\ell}(T)$ determine each other. For if the reciprocal roots of the first are $\{\alpha_j\}$, those of the second are

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$\{q^n/\alpha_j\}$. Thus **Indep**$(2n-i, X/k)$ is equivalent to **Indep**$(i, *, X/k)$: the characteristic polynomials

$$P_{i,\ell,*}(T) := \det(1 - TF|H^i(X \otimes_k \overline{k}, b\mathbb{Q}_\ell))$$

have $\mathbb{Q}$-coefficients, independent of the auxiliary choice of $\ell \neq p$. So for X/k affine, smooth, geometrically connected, of dimension n, in order to prove that X/k is independent of ℓ, it suffices to prove **Indep**$(i, *, X/k)$ for $i = 0, 1, \ldots, n-1$. Now as X is geometrically connected, **Indep**$(0, *, X/k)$ holds trivially. So when $\dim X \leq 1$, X is indeed independent of ℓ. To treat the case of $\dim X \geq 2$, it is natural to try to proceed by induction on n. And for this, it is very natural to try to bring to bear the affine weak Lefschetz theorem.

In the appendix, we give a "new" proof of affine weak Lefschetz in a fairly general form. For our present purposes, we need only the following special case: Let k be a field, ℓ a prime number $\neq \mathrm{char}(k)$, and X/k a smooth, affine, geometrically connected k-scheme of dimension $n \geq 2$. Fix a closed k-immersion $i\colon X \to \mathbb{A}^d$ into an affine space of some dimension d, defined by d functions $x_1, \ldots, x_d$ on X. For each $(d+1)$-tuple $(A, b) := (a_1, \ldots, a_d, b)$ in $\mathbb{A}^{d+1} := \mathrm{AffineMaps}(\mathbb{A}^d, \mathbb{A}^1)$, denote by $H_{A,b}$ the affine hyperplane in $\mathbb{A}^d$ of equation $\sum_i a_i x_i + b = 0$. There exists an open dense set $U_{d,i,\ell}$ of $\mathrm{AffineMaps}(\mathbb{A}^d, \mathbb{A}^1)$ such that

(1) for (A, b) in $U_{d,i,\ell}(\overline{k})$, the restriction map on cohomology

$$H^i(X \otimes \overline{k}, \mathbb{Q}_\ell) \to H^i((X \cap H_{A,b}) \otimes \overline{k}, \mathbb{Q}_\ell)$$

is bijective for $i < n-1$, and injective for $i = n-1$. (This is affine weak Lefschetz per se.)

At the expense of shrinking $U_{d,i,\ell}$, we will have in addition:

(2) for (A, b) in $U_{d,i,\ell}(\overline{k})$, $X \cap H_{A,b}$ is smooth, affine, and geometrically connected, of dimension $n-1$, cf. [**Ka-ACT,** 3.4.3].

(3) Over $U_{d,i,\ell}$, the smooth morphism π whose fibre over (A, b) is $X \cap H_{A,b}$ has all its compact cohomology sheaves $R^i\pi_! Q_\ell$ lisse on $U_{d,i,\ell}$ (by the constructibility of $R^i\pi_!\mathbb{Q}_\ell$) of formation compatible with arbitrary change of base (proper base change).

(4) Over $U_{d,i,\ell}$, the ordinary cohomology sheaves $R^i\pi_*\mathbb{Q}_\ell$ are lisse on $U_{d,i,\ell}$ and of formation compatible with arbitrary change of base (once 3) holds, by "fancy" Poincaré duality for π, both before and after an arbitrary base change, cf. [**SGA 4 1/2,** Arcata, VI, 4, Corollary]).

Now we return to our X/k, a smooth, affine, geometrically connected k-scheme of dimension $n \geq 2$. We wish to prove it is independent of ℓ. By induction on $\dim X$, we may assume that for any finite extension K of k, any smooth, affine, geometrically connected K-scheme Y/K of dimension $\leq n-1$ is independent of ℓ. It suffices to prove that for any two prime numbers ℓ_1 and ℓ_2 which are both $\neq \mathrm{char}(k)$, and for each $i \leq n-1$, the ℓ_1-adic and ℓ_2-adic characteristic polynomials

$$P_{i,\ell_1,*}(T) := \det(1 - TF|H^i(X \otimes_k \overline{k}, \mathbb{Q}_{\ell_1})),$$
$$P_{i,\ell_2,*}(T) := \det(1 - TF|H^i(X \otimes_k \overline{k}, \mathbb{Q}_{\ell_2}))$$

each lie in $\mathbb{Q}[T]$, and are equal in $\mathbb{Q}[T]$.

Fix two prime numbers ℓ_1 and ℓ_2 that are both $\neq \mathrm{char}(k)$, and fix a closed k-immersion $i: X \to \mathbb{A}^d$. Denote by U_{d,i,ℓ_1,ℓ_2} the open dense set $U_{d,i,\ell_1} \cap U_{d,i,\ell_2}$ of AffineMaps$(\mathbb{A}^d, \mathbb{A}^1)$.

Lemma 1. *Under the induction hypothesis,* **Indep**$(i, *, X/k)$ *holds for* $i \leq n-2$.

Proof. If $i \leq n-2$, and if U_{d,i,ℓ_1,ℓ_2} contains a k-rational point, say (A, b), then by (1) we have

$$\det(1 - TF \mid H^i(X \otimes_k \overline{k}, \mathbb{Q}_{\ell_\alpha})) = \det(1 - TF \mid H^i((X \cap H_{A,b}) \otimes_k \overline{k}, \mathbb{Q}_{\ell_\alpha}))$$

for $\alpha = 1, 2$. By the induction hypothesis, the right-hand side lies in $\mathbf{Q}[T]$, and is independent of $\alpha = 1, 2$.

If $i \leq n-2$, but U_{d,i,ℓ_1,ℓ_2} does not contain a k-rational point, we argue as follows. Inside $\overline{k}$, denote by k_r the unique extension of k of degree r. Since U_{d,i,ℓ_1,ℓ_2} is a dense open set of an affine space of dimension ≥ 1, the set $U_{d,i,\ell_1,\ell_2}(k_r)$ of its k_r-valued points is nonempty for all sufficiently large r, say for all $r \geq r_0$.

Fix embeddings of $\mathbf{Q}_{\ell_\alpha}$, $\alpha = 1, 2$, into $\mathbb{C}$. For any polynomial $f(T)$ in $\mathbb{C}[T]$ of the form

$$f(T) = \prod_j (1 - a_j T), \qquad \text{with all } a_j \neq 0,$$

and any integer $r \geq 1$, we denote by $f^{(r)}(T)$ the polynomial

$$f^{(r)}(T) := \prod_j (1 - (a_j)^r T).$$

Then using (1), and picking for each $r \geq r_0$ a k_r-valued point (A_r, b_r) of U_{d,i,ℓ_1,ℓ_2}, we find that for each $i \leq n-2$, the polynomials

$$P_{i,\ell_\alpha,*}(T) := \det(1 - TF \mid H^i(X \otimes_k \overline{k}, \mathbb{Q}_{\ell_\alpha}))$$

satisfy

$$P^{(r)}_{i,\ell_\alpha,*}(T) = \det(1 - TF_r \mid H^i(((X \otimes_k k_r) \cap H_{A_r,b_r}) \otimes_{k_r} \overline{k}, \mathbb{Q}_{\ell_\alpha}))$$

for $\alpha = 1, 2$. Thus for fixed $i \leq n-2$ and all $r \geq r_0$, the two polynomials $P^{(r)}_{i,\ell_\alpha,*}(T)$, $\alpha = 1, 2$, lie in $\mathbb{Q}[T]$ and are equal. We wish to conclude from this that the two polynomials $P_{i,\ell_\alpha,*}(T)$, $\alpha = 1, 2$, lie in $\mathbb{Q}[T]$ and are equal. This follows from the following lemma.

LEMMA 2. *Let* $f(T)$ *and* $g(T)$ *in* $\mathbb{C}[T]$ *be of the form*

$$f(T) = \prod_j (1 - a_j T), \quad \text{with all } a_j \neq 0,$$

$$g(T) = \prod_k (1 - b_k T), \quad \text{with all } b_k \neq 0.$$

Let E *be a subfield of* $\mathbb{C}$. *Suppose there exists an integer* R *such that, for all* $r \geq R$, $f^{(r)}(T)$ *and* $g^{(r)}(T)$ *lie in* $E[T]$ *and* $f^{(r)}(T) = g^{(r)}(T)$ *in* $E[T]$. *Then* $f(T)$ *and* $g(T)$ *lie in* $E[T]$, *and* $f(T) = g(T)$ *in* $E[T]$.

PROOF. Comparing the coefficients of $-T$ in $f^{(r)}(T)$ and $g^{(r)}(T)$, we find that the Newton sums $N_r(f) := \sum_j (a_j)^r$ and $N_r(g) := \sum_k (b_k)^r$ both lie in E, and $N_r(f) = N_r(g)$ in E, for all $r \geq R$. Therefore we have an equality of generating series in $E[[T]]$,

$$\sum_{r \geq R} N_r(f) T^r = \sum_{r \geq R} N_r(g) T^r.$$

Both sides are rational functions in $E(T)$ (indeed, $f(T)$ and $g(T)$ serve as respective denominators). More explicitly, we have an equality of rational functions

$$\sum_j (a_j)^R / (1 - a_j T) = \sum_k (b_k)^R / (1 - b_k T).$$

If we denote by n_j (resp. m_k) the multiplicity of a_j (resp. b_k) as a reciprocal root of $f(T)$ (resp. $g(T)$), we may rewrite this as

$$\sum_{\text{distinct } a_j\text{'s}} n_j (a_j)^R / (1 - a_j T) = \sum_{\text{distinct } b_k\text{'s}} m_k (b_k)^R / (1 - b_k T).$$

This is an equality of rational functions with simple poles at finite distance, and no pole at zero. Write this rational function as

$$\sum_{\text{distinct finite poles } \omega_i} c_i (\omega_i)^R / (1 - \omega_i T), \quad \text{all } \omega_i \neq 0, \ \text{all } c_i \neq 0.$$

We see, by comparing principal parts at the finite poles, that there are the same number of distinct a's as distinct b's as distinct ω's, and that after renumbering we have

$$a_i = b_i = \omega_i \quad \text{and} \quad n_i = m_i = c_i.$$

Since this rational function lies in $E(T)$, it is invariant under the action of the group $\mathrm{Aut}(\mathbb{C}/E)$ on its coefficients. Therefore, the set of the distinct ω_i is stable by $\mathrm{Aut}(\mathbb{C}/E)$, and the multiplicities c_i are constant on the orbits. Therefore the expression

$$\prod_{\text{distinct poles } \omega_i} (1 - \omega_i T)^{c_i}$$

lies in $E[T]$. This expression is equal both to $f(T)$ and to $g(T)$. Q.E.D. for Lemmas 2 and 1.

We have now proven **Indep**$(i, *, X/k)$ for $i \leq n-2$, provided we know that all smooth, geometrically connected affines of dimension $\leq n-1$ over finite fields of the same characteristic are independent of ℓ. It remains to prove **Indep**$(n-1, *, X/k)$. The naive idea is to fix a single closed k-immersion $i: X \to \mathbb{A}^d$, and to recover $P_{n-1,*}(T)$ for X/k as a suitable "gcd", in the style of **[Ka-Me]** and **[De-Weil II,** 4.5], of the $P_{n-1,*}(T)$'s for *all sufficiently general affine hyperplane sections* $X \cap H_{A,b}$ *over all finite* extensions of k.

We will see that this naive hope is overly optimistic, as Deligne pointed out. Before explaining why this is so, and discussing what might be true, we need some preliminaries on various notions of "divisibility".

Interlude on "divisibility"

DEFINITION. Let k be a finite field, U/k a smooth, connected k-scheme, ℓ a prime number, E_λ a finite extension of $\mathbb{Q}_\ell$, $\mathscr{F}$ and $\mathscr{G}$ lisse E_λ-sheaves on U. We say that $\mathscr{F}$ divides $\mathscr{G}$ on U if for every integer $r \geq 1$, and every point x in $U(k_r)$, we have a divisibility of "reversed" characteristic polynomials of Frobenius, i.e.,

$$\det(1 - TF_{x,k_r}|\mathscr{F}) \text{ divides } \det(1 - TF_{x,k_r}|\mathscr{G}),$$

or equivalently (since $F_{x,K}$ is an automorphism), if we have a divisibility of usual characteristic polynomials of Frobenius, i.e.,

$$\det(T - F_{x,k_r}|\mathscr{F}) \text{ divides } \det(T - F_{k,x_r}|\mathscr{G}).$$

This notion of divisibility is birational:

LEMMA 3. *Let k be a finite field, U/k a smooth, connected k-scheme, ℓ a prime number, E_λ a finite extension of $\mathbb{Q}_\ell$, $\mathscr{F}$ and $\mathscr{G}$ lisse E_λ-sheaves on U. Let V be a dense open set of U, and suppose that $\mathscr{F}|V$ divides $\mathscr{G}|V$ on V. Then $\mathscr{F}$ divides $\mathscr{G}$ on U.*

PROOF. Let D and N be the ranks of $\mathscr{F}$ and $\mathscr{G}$ respectively. Consider the universal monic polynomials of degrees D and N respectively,

$$F(T) := T^D + \sum_{0 \leq i < D-1} X_i T^i, \qquad G(T) := T^N + \sum_{0 \leq j \leq N-1} Y_j T^j,$$

over the ring $\mathbb{Z}[X_1, \dots, X_D, Y_1, \dots, Y_N]$. By the Euclidean algorithm, we may write

$$G(T) = F(T)Q(T) + R(T)$$

with a unique "remainder" $R(T)$ of degree $\leq D-1$, with coefficients in the ring $\mathbb{Z}[X_1, \dots, X_D, Y_1, \dots, Y_N]$. In words: over any ring A, the remainder when dividing a monic G in $A[T]$ of degree N by a monic F in $A[T]$ of degree D is a polynomial in $A[T]$ of degree $\leq D-1$, each of whose coefficients is a universal $\mathbb{Z}$-polynomial in the coefficients of F and of G.

Let us apply this universality to the ring R of continuous E_λ-valued central functions on $\pi_1(U)$, and to the (usual, not "reversed") characteristic polynomials of the representations corresponding to $\mathcal{F}$ and $\mathcal{G}$ respectively. The coefficients of the remainder polynomial, being universal $\mathbb{Z}$-polynomials in the coefficients of the two characteristic polynomials, are themselves continuous E_λ-valued central functions on $\pi_1(U)$. We wish to show $\mathcal{F}$ divides $\mathcal{G}$ on U, i.e., that these remainder coefficients vanish on all Frobenius conjugacy classes $F_{x,K}$ in $\pi_1(U)$. By Chebotarev, it is the same to show that these remainder coefficients vanish identically as central functions on $\pi_1(U)$. Because the natural map $\pi_1(V) \to \pi_1(U)$ is surjective (U being normal and connected), it suffices to show that these remainder coefficients vanish identically as central functions on $\pi_1(V)$. For this, by Chebotarev on V, it suffices to show that they vanish on all Frobenius conjugacy classes $F_{x,K}$ in $\pi_1(V)$. But this is precisely the hypothesis that $\mathcal{F}|V$ divides $\mathcal{G}|V$ on V. □

DEFINITION. Let k be a finite field, U/k a smooth, connected k-scheme, ℓ a prime number, E_λ a finite extension of $\mathbb{Q}_\ell$, $\mathcal{G}$ a lisse E_λ-sheaf on U, and $f(T)$ in $E_\lambda[T]$ a polynomial with constant term 1. We say that $f(T)$ divides $\mathcal{G}$ on U if for every integer $r \geq 1$, and every point x in $U(k_r)$,

$$f^{(r)}(T) \text{ divides } \det(1 - TF_{x,k_r}|\mathcal{G}).$$

[Notice that all the reciprocal roots a_j of $f(T) = \prod(1 - a_jT)$ are λ-adic units, simply because all the eigenvalues of any $F_{x,k_r}|\mathcal{G}$ are λ-adic units.]

For any polynomial $f(T) = \prod(1-a_jT)$ in $E_\lambda[T]$ with all a_i λ-adic units, there is a lisse E_λ-sheaf $\mathcal{F}_f$ on $\operatorname{Spec}(k)$ for which

$$f(T) = \det(1 - TF_k|\mathcal{F}_f|).$$

[This sheaf $\mathcal{F}_1$ is not unique, but its semisimplification is unique up to (nonunique) isomorphism.] Via the structural morphism

$$p\colon U \to \operatorname{Spec}(k),$$

we may form the pull-back $p^*\mathcal{F}_f$, a lisse, geometrically constant E_λ-sheaf on U. For every integer $r \geq 1$, and every point x in $U(k_r)$, we have the tautological identity

$$f^{(r)}(T) = \det(1 - TF_{x,k_r}|p^*\mathcal{F}_f).$$

Thus we have the equivalence

$$f(T) \text{ divides } \mathcal{G} \text{ on } U \Leftrightarrow p^*\mathcal{F}_i \text{ divides } \mathcal{G} \text{ on } U.$$

From Lemma 3, we obtain immediately that "$f(T)$ divides $\mathcal{G}$" is a birational notion:

LEMMA 4. *Let k be a finite field, U/k a smooth, connected k-scheme, ℓ a prime number, E_λ a finite extension of $\mathbb{Q}_\ell$, $\mathcal{G}$ a lisse E_λ-sheaf on U, and $f(T)$ in $E_\lambda[T]$ a polynomial with constant term* 1. *Let V be a dense open* *set of* U*, and suppose that* $f(T)$ *divides* $\mathscr{G}|V$ *on* V*. Then* $f(T)$ *divides* $\mathscr{G}$ *on* U.

DEFINITION. Let k be a finite field, U/k a smooth, connected k-scheme, ℓ a prime number, E_λ a finite extension of $\mathbb{Q}_\ell$, and $\mathscr{G}$ a lisse E_λ-sheaf on U. We define $\gcd(\mathscr{G})$ in $\overline{\mathbb{Q}}_\ell[T]$ to be the l.c.m. of all polynomials $f(T)$ in $\overline{\mathbb{Q}}_\ell[T]$ such that, for some finite extension F_λ of E_λ containing the coefficients of f, $f(T)$ divides $\mathscr{G} \otimes_{E_\lambda} F_\lambda$ on U.

REMARK. To see that this definition makes sense, fix a point x in $U(k_r)$ for some $r \geq 1$. Then any $f(T)$ which divides $\mathscr{G}$ has $f^{(r)}(T)$ a divisor of the fixed polynomial $\det(1 - TF_{x,k_r}|\mathscr{G})$, which has only a finite number (at most $2^{\mathrm{rank}(\mathscr{G})}$) of divisors in $\overline{\mathbb{Q}}_\ell[T]$ with constant term 1. Therefore $f^{(r)}(T)$ is on a finite list of possibilities, and $f(T)$ is determined up to finitely many ($r^{\deg(f)}$) possibilities by $f^{(r)}(T)$. So in fact we are taking the l.c.m. of a finite set of polynomials.

LEMMA 5. *Let* k *be a finite field,* U/k *a smooth, connected* k*-scheme,* ℓ *a prime number,* E_λ *a finite extension of* $\mathbb{Q}_\ell$*, and* $\mathscr{G}$ *a lisse* E_λ*-sheaf on* U.

(1) *Let* E *be any subfield of* E_λ *that contains all the coefficients of all the characteristic polynomials* $\det(1 - TF_{x,k_r}|\mathscr{G})$ *for all integers* $r \geq 1$ *and all points* x *in* $U(k_r)$*. Then* $\gcd(\mathscr{G})$ *lies in* $E[T]$.

(2) *For any dense open set* V *in* U*,* $\gcd(\mathscr{G}|V) = \gcd(\mathscr{G})$.

PROOF. For (1), if $f(T)$ divides $\mathscr{G}$, then for any σ in $\mathrm{Aut}(\overline{\mathbb{Q}}_\ell/E)$, $f^\sigma(T)$ also divides $\mathscr{G}$. Therefore the l.c.m. of all such $f(T)$ is invariant by $\mathrm{Aut}(\overline{\mathbb{Q}}_\ell/E)$, and hence lies in $E[T]$. For (2), just apply Lemma 4 to each $f(T)$ that divides $\mathscr{G}|V$. □

REMARK. The notation $\gcd(\mathscr{G})$ is slightly misleading, in that the polynomial $\gcd(\mathscr{G})$ need *not* divide $\mathscr{G}$. Here is a simple example. Let k be a finite field of odd characteristic, $\chi: k^\times \to \{\pm 1\}$ the quadratic character, and α in $k^\times$ a nonsquare. On $\mathbb{G}_m \otimes k := \mathrm{Spec}(k[z, z^{-1}])$, consider, for any prime number ℓ, the lisse $\mathbb{Q}_\ell$-sheaf $\mathscr{G}$ of rank 3 which is defined as the direct sum of the three lisse "Kummer" sheaves of rank one

$$\mathscr{G} := \mathbb{Q}_\ell \oplus \mathscr{L}_{\chi(z)} \oplus \mathscr{L}_{\chi(\alpha z)}.$$

For any finite extension K of k, and any point x in $\mathbb{G}_m(K) := K^\times$, we have

$$\det(1 - TF_{x,K}|\mathscr{G}) = \begin{cases} (1-T)^3, \\ \qquad \text{if both } x \text{ and } \alpha x \text{ are squares in } K^\times, \\ (1-T)(1+T)^2, \\ \qquad \text{if neither } x \text{ nor } \alpha x \text{ is a square in } K^\times, \\ (1-T)^2(1+T), \\ \qquad \text{if exactly one of } x \text{ and } \alpha x \text{ is a square in } K^\times. \end{cases}$$

If K is an extension of k of odd degree, then α is a nonsquare in K, and so whatever x in $K^\times$ we choose, exactly one of x and αx is a square in $K^\times$. So for K of odd degree over k, we have

$$\det(1 - TF_{\chi,K}|\mathscr{G}) = (1-T)^2(1+T) \quad \text{for every } \chi \text{ in } K^x.$$

If K is an extension of k of even degree, then α is a square in K, and so

$$\det(1 - TF_{x,K}|\mathscr{G}) = \begin{cases} (1-T)^3, & \text{if } x \text{ is a square in } K^\times, \\ (1-T)(1+T)^2, & \text{if } x \text{ is not a square in } K^\times. \end{cases}$$

Therefore each of the two linear polynomials $1 \pm T$ separately divides $\mathscr{G}$ [for each finite extension K of k, and each x in $K^\times$, it is separately true that each of $(1-T)$ and $(1-(-1)^{\deg(K/k)}T)$ divides $\det(1 - TF_{x,K}|\mathscr{G})$]. But $(1-T)^2$ does not divide $\mathscr{G}$ (check at x a nonsquare in an extension of even degree). Hence $\gcd(\mathscr{G}) = 1 - T^2$. Again checking at x a nonsquare in an extension of even degree, we see that $\gcd(\mathscr{G})$ does not divide $\mathscr{G}$.

Return to the problem of independence of ℓ for smooth affines

We now return to our finite field k, and to X/k, a smooth, affine, geometrically connected k-scheme of dimension $n \geq 2$. We assume that for any finite extension K of k, any smooth, affine, geometrically connected K-scheme Y/K of dimension $\leq n-1$ is independent of ℓ. For each closed k-immersion $i\colon X \to \mathbb{A}^d$ into an affine space of some dimension d, and each prime number $\ell \neq \mathrm{char}(k)$, we have introduced the dense open set $U_{d,i,\ell}$ of AffineMaps($\mathbb{A}^d$, $\mathbb{A}^1$) over which (1) through (4) hold. Let us denote by

$$\mathscr{G}_{d,i,\ell} := \text{the ordinary cohomology sheaf } R^{n-1}\pi_*\mathbb{Q}_\ell \text{ over } U_{d,i,\ell}.$$

By affine weak Lefschetz, we know that

$P_{n-l,\ell,*}(T) := \det(1 - TF|H^{n-1}(X \otimes_k \overline{k}, \mathbb{Q}_\ell))$ divides $\mathscr{G}_{d,i,\ell}$ over $U_{d,i,\ell}$.

Therefore

$$P_{n-1,\ell,*}(T) \text{ divides } \gcd(\mathscr{G}_{d,i,\ell}).$$

Because of the birational nature of the gcd, for any dense open set V of $U_{d,i,\ell}$, $\gcd(\mathscr{G}_{d,i,\ell}) = \gcd(\mathscr{G}_{d,i,\ell}|V)$. Take for V a finite intersection of $U_{d,i,\ell}$ for various ℓ's in some finite set S of primes. Over this V, the induction hypothesis shows that the characteristic polynomials of Frobenius on $\mathscr{G}_{d,i,\ell}$ at each $(K,\ x \text{ in } V(K))$ are in $\mathbb{Q}[T]$, independent of ℓ in S. Therefore we find that $\gcd(\mathscr{G}_{d,i,\ell})$, lies in $\mathbb{Q}[T]$, and is independent of $\ell \neq \mathrm{char}(k)$. We denote by $\gcd(i, d)$ in $\mathbb{Q}[T]$ the common value of $\gcd(\mathscr{G}_{d,i,\ell})$ for $\ell \neq \mathrm{char}(k)$.

At the conference, I suggested that one try to prove that for any fixed closed k-immersion $i\colon X \to \mathbb{A}^d$, one has

$$P_{n-1,\ell,*}(T) = \gcd(i, d).$$

Deligne later explained to me why this could not be true, already for X of dimension 2. Here is his argument. Suppose we take a projective smooth surface $\overline{X}/k$ in some surjective space $\mathbb{P}^d/k$, say with projective coordinates $(Z_0, \ldots, Z_d)$, in such a way that $\overline{X} \cap (Z_0 = 0)$ is a union of proper smooth geometrically connected curves with normal crossings, say $D_1, \ldots, D_e$. Consider the affine surface

$$X := \overline{X}[1/Z_0] = \overline{X} - \bigcup D_i,$$

and its given embedding into

$$\mathbb{A}^d := \mathbb{P}^d[1/Z_0].$$

A piece of the excision sequence gives, for any $\ell \neq \mathrm{char}(k)$, an exact sequence (where we write $H_c^i(W)$ for $H_c^i(W \otimes_k \overline{k}, \mathbb{Q}_\ell)$)

$$H_c^2(\overline{X}) \to H_c^2\left(\bigcup D_i\right) \to H_c^3\left(\overline{X} - \bigcup D_i\right) \to H_c^3(\overline{X}) \to 0.$$

Since the D_i are curves, we have $H_c^2(\bigcup D_i) = \bigoplus H_c^2(D_i)$, and the Poincaré dual "residue sequence" is an exact sequence

$$0 \to H^1(\overline{X}) \to H^1\left(\overline{X} - \bigcup D_i\right) \to \bigoplus H^0(D_i)(-\mathbf{1}) \to H^2(\overline{X}).$$

The map $\bigoplus H^0(D_i)(-\mathbf{1}) \to H^2(\overline{X})$ is the $(-\mathbf{1})$-Tate twist of the direct sum of the Gysin maps $H^0(D_i) = \mathbb{Q}_\ell \to H^2(\overline{X})(\mathbf{1})$, $1 \mapsto \mathrm{class}(D_i)$. Thus we have a three-term exact sequence

$$0 \to H^1(\overline{X}) \to H^1\left(\overline{X} - \bigcup D_i\right) \to \mathrm{Ker}\left[\bigoplus H^0(D_i)(-\mathbf{1}) \to H^2(\overline{X})\right] \to 0.$$

[We might remark parenthetically here that for surfaces, one knows that this kernel $\mathrm{Ker}[\bigoplus H^0(D_i)(-\mathbf{1}) \to H^2(\overline{X})]$ is in fact independent of ℓ, and hence $\overline{X} - \bigcup D_i$ is independent of ℓ.]

Now let us consider the extreme case in which the e classes $\mathrm{class}(D_i)$ are linearly independent in $H^2(\overline{X})(\mathbf{1})$, and $e \geq 2$. Then we will have

$$H^1(\overline{X}) \cong H^1\left(\overline{X} - \bigcup D_i\right),$$

and consequently $H^1(\overline{X} - \bigcup D_i)$ will be pure of weight one. On the other hand, for any sufficiently general hyperplane H in $\mathbb{P}^d$, say transverse to $\overline{X}$, to each D_i, and not passing through any $D_i \cap D_j$, we have a "residue" short exact sequence

$$0 \longrightarrow H^1(\overline{X} \cap H) \to H^1\left(\overline{X} \cap H - \bigcup D_i \cap H\right)$$
$$\longrightarrow \mathrm{Ker}\left[\bigoplus H^0(D_i \cap H)(-\mathbf{1}) \to H^2(\overline{X} \cap H)\right] \to 0.$$

Each $D_i \cap H$ is a finite nonempty k-scheme, so a finite nonempty disjoint union of closed points, and hence the "reversed" characteristic polynomial of Frobenius on its $H^0(-\mathbf{1})$ is a finite nonempty product of terms $(1 - (qT)^f)$, one such factor for each closed point of degree f. The key observation is that the polynomial $(1-(qT)^f)$ is divisible by $(1-qT)$. Since $H^2(\overline{X}\cap H)$ is $\mathbb{Q}_\ell(-\mathbf{1})$, we see that the polynomial $(1-qT)^{e-1}$ divides $\gcd(i, d)$ for this embedding. But $H^1(\overline{X}-\bigcup D_i)$ is pure of weight one, so its characteristic polynomial of Frobenius cannot be $\gcd(i, d)$, which has at least $e-1$ reciprocal roots of weight two.

[In slightly greater generality, any time we have $e \geq 2$ D_i's whose classes in $H^2(\overline{X})(\mathbf{1})$ have a span of dimension at least 2, the weight two part of $H^1(\overline{X}-\bigcup D_i)$ has dimension at most $e-2$, while the $\gcd(i, d)$ is divisible by $(1-qT)^{e-1}$.]

Here is the simplest example of this phenomenon: $\overline{X}$ is $\mathbb{P}^1\times\mathbb{P}^1$, embedded as $ST=UV$ in $\mathbb{P}^3$, H is $S=0$, X is $\mathbb{A}^2$. There are two $\mathbb{P}^1$'s at ∞, with independent classes in $H^2(\overline{X})(\mathbf{1})$. Concretely, we are embedding $\mathbb{A}^2$ in $\mathbb{A}^3$ as $t=uv$. A general affine hyperplane involves all three variables, so may be taken of the form $t=au+bv+c$. The corresponding affine hyperplane section is the affine curve

$$au+bv+c=vu.$$

For general a, b, c, this is a smooth quadric in $\mathbb{P}^2$, hence in $\mathbb{P}^1$, minus two distinct rational points at ∞ ($uv=0$ in $\mathbb{P}^1$). So its H^1 is a $\mathbb{Q}_\ell(-\mathbf{1})$, and the $\gcd(i, d)$ in this case is $(1-qT)$, despite the fact that $X=\mathbb{A}^2$ has $H^1=0$.

Is there any way the g.c.d. idea can be salvaged? One could ask if $P_{n-1,\ell,*}(T)$ can be recovered as the g.c.d. of *all* the polynomials $\gcd(i, d)$ for *all* the closed k-immersions $i: X\to\mathbb{A}^d$ into affine spaces of all dimensions d.

To see that this is plausible, consider the above counterexample, where instead of embedding $\mathbb{A}^2$ into $\mathbb{P}^2$ (which would have given the right g.c.d.), we viewed $\mathbb{A}^2$ as $t=f(u, v)$ in $\mathbb{A}^3$. When we took $f(u, v)=uv$, we got the "wrong" g.c.d., namely $(1-qT)$. If we take $f(u, v)=u^2-\alpha v^2$, with α in $k^\times$, and char(k) odd, then the g.c.d. will be $(1-qT)$ if α is a square in $k^\times$, $(1+qT)$ if α is a nonsquare in $k^\times$. Since there are both squares and nonsquares in k, the g.c.d. of *all* the $\gcd(i, d)$'s, even just over the wrong-headed embeddings of type $t=f(u, v)$, will still be correct, namely 1.

A more conservative approach would be to allow all *locally closed* k-immersions $i: X\to\mathbb{A}^d$ into affine spaces of all dimensions d.

Still more conservative would be to allow all quasi-finite k-morphisms $i: X\to\mathbb{A}^d$ to affine spaces of all dimensions d with the property that for a general affine hyperplane H in $\mathbb{A}^d$, $X\cap i^{-1}(H)$ is smooth. [Exercise for the reader: this last property holds if the coherent sheaf $\Omega := \Omega^1_{X/\mathbb{A}^d}$ on X satisfies the following condition: for every integer $r\geq 1$, the closed set

Z_r of X consisting of the points x in X where $\dim_{\kappa(x)} \Omega/\mathfrak{m}_x\Omega \geq r$ has $\operatorname{cdim}_X(Z_r) \geq r$.]

Much remains to be done.

Appendix: Affine weak Lefschetz via character sums

In this appendix, we give a new characteristic p proof of an "affine" version of the weak Lefschetz theorem on hyperplane sections. In its version II below, its statement is reminiscent of [**G-M,** 6.10]. The reader should compare this proof to the one given in [**Ka-ACT,** 3.4] and with Deligne's proof given in [**Ka-ACT,** Appendix]. Our proof is based heavily on some of the main results of [**Ka-ACT**], which we make no attempt to summarize here.

We work over an algebraically closed field k of characteristic $p > 0$. We fix a prime $\ell \neq p$, and work with $\overline{\mathbb{Q}}_\ell$-cohomology. The data we are given is this:

V, a separated k-scheme of finite type that is smooth and connected of dimension $d \geq 1$,

$n \geq 1$ an integer,

$x_1, \dots, x_n$, n functions on V that define a quasifinite map $V \to \mathbb{A}^n$,

$f\colon V \to \mathbb{A}^1$ a function on V,

K on V a $\overline{\mathbb{Q}}_\ell$-perverse sheaf on V (e.g., $K = \overline{\mathbb{Q}}_\ell[d]$ if V is smooth and everywhere of dimension d).

For each $(n+1)$-tuple $(A, b) := (a_1, \dots, a_n, b)$ in $\mathbb{A}^{n+1}(k)$, we denote by $f_{A,b}$ the function on V defined as

$$f_{A,b} := f + \sum_i a_i x_i + b.$$

"AFFINE" WEAK LEFSCHETZ THEOREM—VERSION I. *There exists a dense open set U in $\mathbb{A}^{n+1}$ such that if (A, b) lies in U, then for all but at most finitely many values of c in k, the restriction map*

$$H^i(V, K) \to H^i(V \cap (f_{A,b} = c), K)$$

is bijective for $i < -1$, and injective for $i = -1$.

We will deduce this from

"AFFINE" WEAK LEFSCHETZ THEOREM—VERSION II. *There exists a dense open set U in $\mathbb{A}^{n+1}$ such that if (A, b) lies in U, then the sheaves $R^i(f_{A,b})_*K$ on $\mathbb{A}^1$ are constant for $i < -1$, and $R^{-1}(f_{A,b})_*K$ has no nonzero punctual sections.*

To see that version II implies version I, use the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathbb{A}^1, R^j(f_{A,b})_*K) \Rightarrow H^{i+j}(V, K).$$

It has $E_2^{i,j} = 0$ unless i is 0 or 1 (cohomological dimension of open curves), so it degenerates at E_2, and gives short exact sequences

$$0 \to H^1(\mathbb{A}^1, R^{j-1}(f_{A,b})_*K) \to H^j(V, K) \to H^0(\mathbb{A}^1, R^j(f_{A,b})_*K) \to 0.$$

Since $H^1(\mathbb{A}^1$, constant sheaf$)=0$, the constancy of $R^{j-1}(f_{A,b})_*K$ for $j \le -1$ shows

$$(*) \qquad H^j(V, K) \cong H^0(\mathbb{A}^1, R^j(f_{A,b})_*K) \quad \text{for } j \le -1.$$

For any k-valued point c of $\mathbb{A}^1$, the map "restriction to the stalk at c"

$$(**) \qquad \text{restr}_c \colon H^0(\mathbb{A}^1, R^j(f_{A,b})_*K) \to (R^j(f_{A,b})_*K)_c$$

is an isomorphism for $j < -1$ (constancy of $R^j(f_{A,b})_*K$ for $j < -1$), and is injective for $j = -1$ (because $R^{-1}(f_{A,b})_*K$ has no nonzero punctual sections).

By Deligne's generic base change result **[SGA 4 1/2,** Th. Fin., 1.9], there is a dense open set of c's in $\mathbb{A}^1$ for which the natural map

$$(***) \qquad (R^j(f_{A,b})_*K)_c \to H^j(V \cap (f_{A,b} = c), K)$$

is an isomorphism for all j. Combining these steps, we find version I, with the same dense open set U.

We next deduce version II from a special case of a generic vanishing theorem whose proof (see **[Ka-ACT]**) used the full mechanism of perversity and Fourier Transform. Fix *any* nontrivial $\overline{\mathbb{Q}}_\ell$-valued multiplicative character χ of a finite subfield of k, denote by $\mathscr{L}_\chi$ the corresponding Kummer sheaf on $\mathbf{G}_m$, and by $j_!\mathscr{L}_\chi$ its extension by zero to $\mathbb{A}^1$. According to the semiperversity result **[Ka-ACT,** 1.5], applied with DK on V and $j_!\mathscr{L}_\chi$ **[1]** on $\mathbb{A}^1$, we have

GENERIC VANISHING THEOREM. *There exists a dense open set $\mathscr{V}$ in $\mathbb{A}^{n+1}$ such that if (A, b) in $\mathscr{V}$, then the cohomology groups*

$$H^i_c(V, (DK) \otimes (f_{A,b})^*\mathscr{L}_\chi) = 0 \quad \textit{for } i > 0.$$

[In the notation of **[Ka-ACT,** 1.6], $\mathscr{V}$ is any dense open set of

$$\text{AffMaps}(\mathbb{A}^n, \mathbb{A}^1) \cong \mathbb{A}^{n+1}$$

over which the semiperverse object noted M has lisse cohomology sheaves.]

Denote by $\mathscr{V}^+$ the image of $\mathscr{V}$ by the automorphism

$$(A, b) \mapsto (A, b+1),$$

and by U the dense open set $\mathscr{V} \cap \mathscr{V}^+$ in $\mathbb{A}^{n+1}$. We will show that version II holds with this choice of U.

Thus let (A, b) in U. Since $U \subset \mathscr{V}$, we have the vanishing

$$H^i_c(V, (DK) \otimes (f_{A,b})^*\mathscr{L}_\chi) = 0 \quad \text{for } i > 0.$$

Because $(f_{A,b})^*\mathscr{L}_\chi$ vanishes outside $V[1/f_{A,b}]$, and we are dealing with H_c, we may rewrite this as

$$H^i_c(V[1/f_{A,b}], (DK) \otimes (f_{A,b})^*\mathscr{L}_\chi) = 0 \quad \text{for } i > 0.$$

The dual assertion is then

$$H^i(V[1/f_{A,b}], K \otimes (f_{A,b})^* \mathcal{L}_{\overline{\chi}}) = 0 \quad \text{for } i < 0.$$

Now use the Leray spectral sequence for

$$f_{A,b}\colon V[1/f_{A,b}] \to \mathbb{G}_m$$

to compute these groups. Because $\mathcal{L}_{\overline{\chi}}$ is lisse on $\mathbf{G}_m$, we have a "trivial" projection formula

$$R^i(f_{A,b})_*(K \otimes (f_{A,b})^* \mathcal{L}_{\overline{\chi}}) \cong \mathcal{L}_{\overline{\chi}} \otimes R^i(f_{A,b})_*(K).$$

Again Leray degenerates at E_2 because $\mathbb{G}_m$ has cohomological dimension one, so the vanishing of the abutment gives

$$H^j(\mathbb{G}_m, \mathcal{L}_{\overline{\chi}} \otimes R^i(f_{A,b})_*(K)) = 0 \quad \text{for } i + j < 0.$$

Taking $j = 0$, we find that, for $i < 0$, $\mathcal{L}_{\overline{\chi}} \otimes R^i(f_{A,b})_*(K)$ has $H^0 = 0$, so certainly it has no nonzero punctual sections on $\mathbb{G}_m$. Since $\mathcal{L}_{\overline{\chi}}$ is lisse on $\mathbb{G}_m$, it follows that $R^i(f_{A,b})_*(K)$ itself has no nonzero punctual sections on $\mathbb{G}_m$, for $i < 0$.

For $i < -1$, both $E_2^{0,i}$ and $E_2^{1,i}$ vanish, whence the Euler characteristic vanishes:

$$\chi(\mathbb{G}_m, \mathcal{L}_{\overline{\chi}} \otimes R^i(f_{A,b})_*(K)) = 0 \quad \text{for } i < -1.$$

Now for any sheaf $\mathcal{F}$ on $\mathbb{G}_m$ with no nonzero punctual sections, the Euler-Poincaré formula

$$\chi(\mathbb{G}_m, \mathcal{F}) = -\sum_{x \text{ in } \mathbb{G}_m} \operatorname{drop}_x(\mathcal{F}) - \sum_{x \text{ in } \mathbb{P}^1} \operatorname{swan}_x(\mathcal{F})$$

has each of its terms a nonpositive integer. So for such an $\mathcal{F}$, we have the equivalence:

$$\chi(\mathbb{G}_m, \mathcal{F}) = 0 \text{ iff } \mathcal{F} \text{ is lisse on } \mathbb{G}_m \text{ and tame at both 0 and } \infty.$$

Applying this to the sheaf $\mathcal{L}_{\overline{\chi}} \otimes R^i(f_{A,b})_*(K) = 0$ for $i < -1$, and remembering that $\mathcal{L}_{\overline{\chi}}$ is itself lisse on $\mathbb{G}_m$ and tame at both 0 and ∞, we conclude that for $i < -1$, $R^i(f_{A,b})_*(K)$ is lisse on $\mathbb{G}_m$ and tame at both 0 and ∞.

Now we use the fact that (A, b) was also assumed to lie in $\mathscr{V}^+$. This means that $(A, b-1)$ lies in $\mathscr{V}$. But $f_{A,b-1}$ is the function

$$f_{A,b} - 1.$$

The previous argument, applied to $f_{A,b} - 1$, now gives: $R^i(f_{A,b} - 1)_*(K)$ is lisse on $\mathbb{G}_m$, tame at 0 and ∞, for $i < -1$; $R^{-1}(f_{A,b} - 1)_*(K)$ has no nonzero punctual sections on $\mathbb{G}_m$. But the sheaves $R^i(f_{A,b} - 1)_*(K)$ on $\mathbb{A}^1$ are just the additive translates by $x \mapsto x + 1$ of the sheaves $R^i(f_{A,b})_*(K)$, so our new information is that the $R^i(f_{A,b})_*(K)$ are lisse on $\mathbb{A}^1 - \{1\}$ for

$i < -1$, and that $R^{-1}(f_{A,b})_*(K)$ has no nonzero punctual sections on $\mathbb{A}^1 - \{1\}$. For $i < -1$, the sheaves $R^i(f_{A,b})_*(K)$ are thus lisse on all of $\mathbb{A}^1$; being tame at ∞, they are constant. For $i = -1$, the sheaf $R^{-1}(f_{A,b})_*(K)$ on $\mathbb{A}^1$ has no nonzero punctual sections. □

References

[SGA 4 1/2] P. Deligne, *Cohomologie Étale* (SGA 4 1/2), Lecture Notes in Math., vol. 569, Springer-Verlag, New York and Berlin, 1977.

[De-Weil II] P. Deligne, *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 313–428.

[G-M] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Springer-Verlag, New York and Berlin, 1988.

[Ka-ACT] N. Katz, *Affine cohomological transforms, perversity, and monodromy*, J. Amer. Math. Soc., vol. 6, No. 1 (1993), 149–122.

[Ka-Me] N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

[Ka-Review] N. Katz, *Review of ℓ-adic cohomology*, these Proceedings, vol. 1, pp. 21–30.

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Décompositions dans la catégorie dérivée

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Soit $\mathscr{D}$ une catégorie triangulée munie d'une t-structure [**1**, §1.3.1] $\mathscr{C}$ son coeur (loc. cit.). Nous utiliserons les notations $\mathscr{D}^{\leq a}$, $\mathscr{D}^{\geq b}$, $\tau_{[a,b]}$, H^i de [**1**, §1.3]. Soit $K \mapsto K(1)$ une autoéquivalence de $\mathscr{D}$, transformant triangles distingués en triangles distingués et respectant la t-structure. On note $K \mapsto K(i)$ l'itéré i-ème $(i \in \mathbb{Z})$ de cette autoéquivalence. On appellera "torsion" les foncteurs $K \mapsto K(i)$.

On suppose les objets de $\mathscr{D}$ à cohomologie bornée: tout objet est dans un $\mathscr{D}^{\leq a}$ et dans un $\mathscr{D}^{\geq b}$. Si cette hypothèse n'était pas remplie, il y aurait lieu de supposer les objets considérés par la suite à cohomologie bornée.

Soit K un objet de $\mathscr{D}$, muni d'un morphisme

$$\eta: K \to K(1)[2]. \tag{0.1}$$

On notera encore η les morphismes déduits de η par torsion (i), décalage et troncation $\tau_{[a,b]}$, et on notera η^i un composé de i tels morphismes. Par exemple, on note encore η les morphismes dans la catégorie abélienne $\mathscr{C}$

$$\eta: H^i K \to H^{i+2} K(1) \tag{0.2}$$

déduits de (0.1).

Supposons que η vérifie

(L.V.) Pour tout $i \geq 0$, le $i^{\text{ème}}$ itéré de η, $\eta^i: H^{-i}(K) \to H^i(K)(i)$ est un isomorphisme.

Les arguments de [**2**], rappelés au paragraphe 1, montrent alors que K est isomorphe à la somme des $H^i(K)$, chacun placé en son degré cohomologique:

$$K \simeq \bigoplus H^i(K)[-i]. \tag{0.3}$$

Dans [**2**], j'annonce l'existence d'un isomorphisme (0.3) canonique. Le but de cette note est d'en définir un, et même plusieurs.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Rappel de la preuve de [2]

Pour $i \geq 0$, la partie primitive P^{-i} de $H^{-i}(K)$ est le noyau de η^{i+1}: $H^{-i}(K) \to H^{i+2}(K)(i+1)$. Il résulte de (L.V.) que les morphismes (0.2) itérés induisent une décomposition en somme directe:

$$\sum_{j \geq 0} \eta^j : \bigoplus P^{-i-2j}(-j) \xrightarrow{\sim} H^{-i}(K). \tag{1.1}$$

Pour $H^i(K)$, on en déduit un isomorphisme

$$\sum_{j \geq 0} \eta^{i+j} : \bigoplus P^{-i-2j}(-i-j) \xrightarrow{\sim} H^i(K). \tag{1.2}$$

Ces isomorphismes se rassemblent en

$$\sum \eta^j : \bigoplus_{i,j} P^{-i}(-j) \xrightarrow{\sim} H^*(K), \tag{1.3}$$

la somme étant sur les i, j avec $0 \leq j \leq i$. Dans (1.3), $P^{-i}(-j)$ est facteur direct de $H^{2j-i}(K)$.

Rappelons qu'un foncteur cohomologique F de $\mathscr{D}$ dans une catégorie abélienne $\mathscr{A}$ est un foncteur additif tel que pour tout triangle distingué $K \to L \to M \to K[1]$, la suite $F(K) \to F(L) \to F(M)$ soit exacte [**6**, §3.1]. On pose $F^i(K) = F(K[i])$. Faisant tourner le triangle distingué $K \to L \to M \to K[1]$, on voit qu'un foncteur cohomologique F donne lieu à suite exacte longue

$$\cdots \to F^i(K) \to F^i(L) \to F^i(M) \to F^{i+1}(K) \to F^{i+1}(L) \to \cdots .$$

Pour K dans $\mathscr{D}$, les tronqués $\tau_{[a,b]}K$ de K donnent lieu à des triangles distingués

$$\to \tau_{[a,b]}K \to \tau_{[a,c]}K \to \tau_{[b+1,c]}K \to .$$

Le système de suites exactes longues correspondantes est un "objet spectral" au sens de J. L. Verdier et fournit une suite spectrale

$$E_2^{pq} = F^p H^q K \Rightarrow F^{p+q} K.$$

Voir l'appendice. Soit ${}^iE_2^{pq}$ la suite spectrale analogue pour $K(i)$. Un morphisme (0.1) fournit des morphismes de suites spectrales

$${}^iE_2^{pq} \to {}^{i+1}E_2^{p,q+2}.$$

Montrons que (L.V.) force la dégénérescence de ces suites spectrales. Procédant par récurrence sur $r \geq 2$, supposons que les d_s sont nuls pour $2 \leq s < r$ et prouvons que $d_r = 0$. Pour $q \leq 0$, soit ${}^iP^{pq}$ le facteur direct $F^p P^q(i)$ de ${}^iE_2^{pq} = {}^iE_r^{pq}$.

Par (1.1) et (1.2), il suffit de vérifier que d_r est nul sur les ${}^iP^{pq}$. Par définition, ${}^iP^{pq}$ est annulé par η^{-q+1}. Le morphisme d_r envoie ${}^iP^{pq}$ dans ${}^iE_2^{p+r,q-r+1}$, sur lequel η^{-q+1} est injectif. La nullité de d_r en résulte.

Appliquant la dégénérescence de (1.3) à un foncteur cohomologique $\operatorname{Hom}(X, K)$, pour X dans $\mathscr{C}$, on trouve que tout morphisme $X \to H^iK$ se relève en un morphisme de degré i de X dans K. En particulier, pour $X = H^iK$, l'application identique de H^iK se relève en un morphisme de degré i de H^iK dans K. La somme de ces morphismes

$$\bigoplus H^iK[-i] \xrightarrow{\sim} K$$

est un isomorphisme en cohomologie, donc un isomorphisme.

2. Un isomorphisme (0.3)

Soient K, η vérifiant (L.V.).

LEMME 2.1. *Soit F un foncteur cohomologique de $\mathscr{D}$ dans la catégorie des groupes abéliens. Soient $i \geq 0$ et $x \in F(\tau_{\geq -i}K)$ tel que $\eta^{i+1}(x) \in F^{2(i+1)}(\tau_{\geq i+2}K(i+1))$ soit nul. Alors, x admet un unique relèvement y dans $F(\tau_{\geq -i-1}K)$ tel que $\eta^{i+1}(y) \in F^{2(i+1)}(\tau_{\geq i+1}K(i+1))$ soit nul.*

PREUVE. Nous avons vu au paragraphe 1 que K est isomorphe à la somme de ses $H^i(K)[-i]$. Il en résulte que les suites

$$0 \to F^aH^aK \to F\tau_{\geq a}K \to F\tau_{\geq a+1}K \to 0$$

sont exactes, et de même pour les $K(i)$. Le morphisme η^{i+1} induit un morphisme de suites exactes courtes

$$\begin{array}{ccccccccc} 0 \longrightarrow & F^{i+1}H^{-i-1}K & \longrightarrow & F\tau_{\geq -i-1}K & \longrightarrow & F\tau_{\geq -i}K & \longrightarrow 0 \\ & \downarrow \wr & & \downarrow & & \downarrow & \\ 0 \longrightarrow & F^{i+1}H^{i+1}K(i+1) & \longrightarrow & F^{2(i+1)}\tau_{\geq i+1}K(i+1) & \longrightarrow & F^{2(i+1)}\tau_{\geq i+2}K(i+1) & \longrightarrow 0 \end{array}$$

Il reste à appliquer le lemme du serpent.

L'élément y de 2.1 vérifie l'hypothèse de 2.1 pour $i+1$ et on conclut par récurrence:

LEMME 2.2. *Sous les hypothèses 2.1, il existe un unique relèvement $y \in F(K)$ tel que pour chaque $s > i$, $\eta^s y$ ait une image nulle dans $F^{2s}(\tau_{\geq s}K(s))$.*

2.3. Appliquons le lemme 2.2 au foncteur cohomologique $F(L) := \operatorname{Hom}(P^{-i}[i], L)$ et à $x : P^{-i}[i] \to \tau_{\geq -i}K$ défini par l'inclusion de P^{-i} dans $H^{-i}K$. L'hypothèse $\eta^{i+1}x = 0$: $P^{-i}[-i-2] \to \tau_{\geq i+2}K(i+1)$ est vérifiée car $\eta^{i+1}: P^{-i} \to H^{i+2}K(i+1)$ est nul. On conclut

PROPOSITION 2.4. *Soit $i \geq 0$. Il existe un unique morphisme f_i de $P^{-i}[i]$ dans K tel que*

(i) *$H^{-i}(f_i)$ est l'inclusion de P^{-i} dans $H^{-i}K$,*
(ii) *Pour chaque $s > i$, le morphisme induit par $\eta^s f_i$, de $P^{-i}[i]$ dans $(\tau_{\geq s}K)(s)[2s]$, est nul.*

2.5. Des morphismes f_i de 2.4 et de (1.3), on déduit des morphismes $g_i: H^i(K)[-i] \to K$, par

(2.5.1) sur le facteur direct $P^{-i}(-j)$ de H^{2j-i} $(0 \leq j \leq i)$, g_i est $\eta^j f_i$.

La somme de ces morphismes est un isomorphisme (0.3).

2.6. Soit φ un isomorphisme

$$\bigoplus H^i(K)[-i] \xrightarrow{\sim} K$$

induisant l'identité sur la cohomologie. Posons $Q^{i,j} = P^{-i}(-j)[i-2j]$: $P^{-i}(-j)$ en degré cohomologique $2j-i$. Combinant φ avec l'isomorphisme (1.3), on obtient

$$\varphi_1 : \bigoplus_{0 \leq j \leq i} Q^{i,j} \xrightarrow{\sim} K. \tag{2.6.1}$$

Nous noterons $\eta_{ij}^{k\ell}$ les éléments de matrice de $\eta: K \to K(1)[2]$ relatifs à cette décomposition:

$$\eta_{i,j}^{k,\ell} : Q^{i,j} \to Q^{k,\ell}(1)[2].$$

Ces éléments de matrice sont nuls pour $2j-i < 2\ell-k-2$ puisque, pour A, B dans $\mathscr{C}$, $\mathrm{Hom}(A[-n], B[-m]) = 0$ pour $n < m$. Pour $2j-i = 2\ell-k-2$, c'est le morphisme induit par $\eta: H^{2j-i}(K) \to H^{2j-i+2}(K)(1)$: l'identité de $P^{-i}(-j)$ si $k = i$ et $\ell = j+1$, 0 sinon.

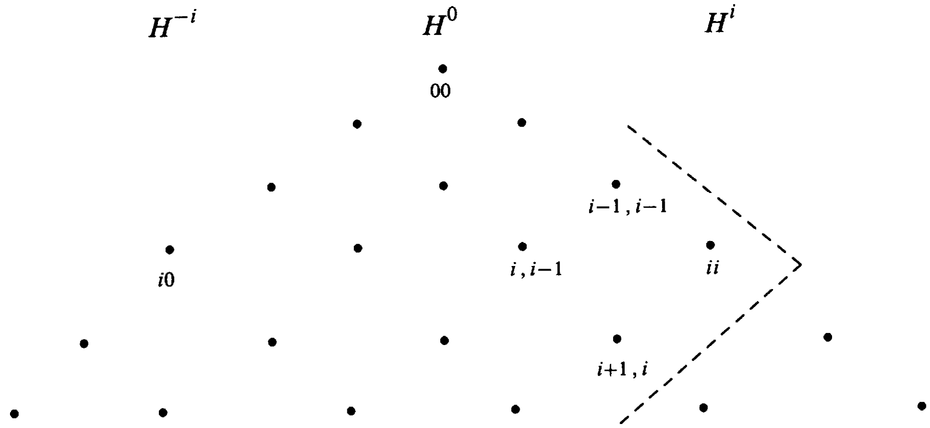
Proposition 2.7. *Pour qu'un isomorphisme φ comme en 2.6 soit celui défini en 2.5, il faut et il suffit que*

(i) *pour $j < i$, le seul $\eta_{ij}^{k\ell}$ non nul est le morphisme évident (identique) $\eta_{i,j}^{i,j+1}$;*

(ii) *$\eta_{ii}^{k\ell}$ n'est non nul que pour $\ell \leq i$.*

Le dessin suivant peut aider à suivre le raisonnement. Chaque • y représente un facteur direct $Q^{i,j}$. Dans chaque ligne, i est constant. Chaque colonne correspond à un H^n, $Q^{i,j}$ contribuant à H^{2j-i}. La condition (i) détermine η, sauf sur le côté droit du triangle. La condition (ii) dit que $\eta_{ii}^{k\ell}$

n'est non nul que pour k, ℓ dans la région à gauche de la ligne pointillée



Notations: On allègera les notations en appelant a-morphisme de K dans L un morphisme de K dans $L(a)[2a]$. Par exemple, $\eta_{i,j}^{k,\ell}$ est un 1-morphisme de $Q^{i,j}$ dans $Q^{k,\ell}$.

PREUVE. Identifions K à la somme des $H^i(K)[-i]$ par φ. Par définition, l'isomorphisme φ de 2.5 est caractérisé par (i) et la condition

($*$) Pour chaque $i \geq 0$ et $s > i$, le s-morphisme η^s envoie $Q^{i,0}$ dans la somme des $H^n(K)[-n]$ pour $n < s$.

Supposons (i): η évident sur $Q^{i,j}$ pour $j < i$. La condition ($*$) équivaut alors à

($\substack{*\\*}$) Pour $t > 0$, le t-morphisme η^t envoie $Q^{i,i}$ dans la somme des $H^n(K)[-n]$ pour $n < i + t$.

Preuve de ($\substack{*\\*}$) $\Rightarrow$ (ii). Prouvons par récurrence sur $k - \ell$ que $\eta_{i,i}^{k,\ell} = 0$ si $\ell > i$. Supposons donc l'assertion prouvée pour $k - \ell < a$, et prouvons-la pour $k - \ell = a$. La restriction de η^{a+1} à $Q^{i,i}$ est la somme

$$\sum \eta^a \circ \eta_{i,i}^{k,\ell}.$$

Les termes avec $\ell > i$ sont contrôlés par l'hypothèse de récurrence et (i): ils sont nuls si $k - \ell < a$; pour $k - \ell \geq a$, ils sont à valeur dans les $Q^{k,\ell+a}$, tous distincts, et la nullité de $\eta^a \eta_{i,i}^{k,\ell}$ implique celle de $\eta_{i,i}^{k,\ell}$. Noter que $Q^{k,\ell+a} \subset H^n[-n]$ pour $n = 2\ell + 2a - k$. Pour $k - \ell = a$, on tombe donc dans H^k et l'hypothèse $\ell = k - a > i$ donne $k > i + a$, une valeur de k interdite par ($\substack{*\\*}$). Il reste à montrer que ces termes ne peuvent pas se simplifier avec ceux pour $\ell \leq i$. Il suffit de montrer que si $\ell \leq i$, sur $Q^{k,\ell}$, le a-morphisme η^a est à valeur dans la somme des $H^n[-n]$ pour $n \leq i + a$.

1[er] cas: $a \leq k - \ell$: on tombe dans $H^{-k+2\ell+2a}$ et $-k + 2\ell + 2a = (-k+\ell+a)+\ell+a \leq 0+i+a = i+a$.
2[ème] cas: $a > k-\ell$: on écrit $\eta^a = \eta^{a-(k-\ell)}\eta^{k-\ell}$ et on a à considérer $\eta^{a-(k-\ell)}$ sur $Q^{k,k}$.

L'hypothèse $\binom{*}{*}$ assure qu'on tombe dans les H^n pour $n < k + (a-(k-\ell)) = a+\ell \leq a+i$.

Preuve de (ii) $\Rightarrow \binom{*}{*}$. Prouvons par récurrence sur $a \geq 1$ que le a-morphisme η^a restreint à $Q^{i,i}$ est à valeurs dans les $Q^{k,\ell}$ pour $\ell \leq i+a-1$. Pour $a = 1$, c'est l'hypothèse (ii). Supposons-le pour a, et considérons $\eta^{a+1} = \eta \circ \eta^a$. Il suffit de voir que si $\ell \leq i+a-1$, la restriction du 1-morphisme η à $Q^{k,\ell}$ est à valeur dans la somme des $Q^{m,n}$ pour $n \leq i+a$. Si $\ell < k$, cela résulte de (i). Si $\ell = k$, de (ii).

Enfin, $Q^{-k,\ell}$ est dans $H^n[-n]$ pour $n = 2\ell - k$, et, si $\ell \leq i+a-1$, on a $n = 2\ell - k = \ell - (k-\ell) \leq \ell = i+a-1$.

3. Un autre isomorphisme (0.3)

3.1. La catégorie $\mathscr{D}^0$ opposée à $\mathscr{D}$ est encore une catégorie triangulée. Son foncteur de translation est

$$K^0 \longmapsto (K[-1])^0.$$

On la munit de l'auto-équivalence (1) pour laquelle

$$K^0(1) = K(-1)^0.$$

La t-structure de $\mathscr{D}$ en fournit une sur $\mathscr{D}^0$, $\eta: K \to K(1)[2]$ se transpose en

$$^t\eta: K^0 \to K^0(1)[2]$$

et pour que η vérifie (L.V.), il faut et suffit que $^t\eta$ vérifie (L.V.).

La construction (2.5) n'est pas autoduale, ainsi qu'on le voit sur sa caractérisation 2.7. La construction duale, i.e., déduite de 2.4 appliqué à $^t\eta$, fournit des morphismes

$$f'_i: K \to P^{-i}(-i)[-i]$$

tels que $H^i(f'_i)$ soit la projection de $H^i(K)$ sur $P^{-i}(-i)$, et que pour chaque $s > i$, le morphisme induit par $f'_i\eta^s$, de K dans $P^{-i}(-i)[-i](s)[2s]$, soit nul sur $\tau_{\leq -s}K$. Procédant comme en 2.5, on en déduit des morphismes $g'_i: K \to H^i(K)[-i]$ de somme un isomorphisme de K avec la somme des $H^i(K)[-i]$. Cet isomorphisme n'est en général pas l'inverse de celui construit en 2.5.

Dans ce paragraphe, nous supposons être en caractéristique 0, i.e., que la catégorie $\mathscr{D}$ est $\mathbb{Q}$-linéaire. Soient K, η vérifiant (L.V.). Notre but est la construction d'un isomorphisme (0.3) autodual.

3.2. La graduation de H^*K définit un endomorphisme

$$h: H^*K \to H^*K: \text{ multiplication par } n \text{ sur } H^nK.$$

Par passage à la cohomologie, η fournit un morphisme de degré 2

$$e: H^*K \to H^*K(1).$$

Parce qu'on est en caractéristique 0, on peut compléter h et e par un morphisme de degré -2

$$f: H^*K \to H^*K(-1)$$

tel que h, e, et f vérifient les relations entre les générateurs standard de sl(2) :

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Pour donner un sens à ces formules, on désigne de même un morphisme et ceux qui s'en déduisent par une torsion (n).

En caractéristique 0, l'existence de f équivaut à (L.V.). Dans la décomposition (1.3), chaque somme sur j, $\bigoplus P^{-i}(j)$, est stable par h, e, f et, du facteur dans H^{n+1} vers celui dans H^{n-1}, f est l'inverse de e, multiplié par $\frac{(i+1)^2}{4} - \frac{n^2}{4}$.

Soit F l'un des bifoncteurs

$$\mathrm{Ext}^a(A, B) = \mathrm{Hom}_{\mathscr{D}}(A, B[a])$$

sur $\mathscr{C}$. A isomorphisme canonique près le foncteur $F(A(i), B(j))$ ne dépend que de $n := j - i$. On le note $F_n(A, B)$. De h, e, f on déduit encore une action de sl(2) sur $F_*(H^*K, H^*K)$. Plus précisément, des morphismes

$$h\colon\ F_j(H^*K, H^*K) \to F_j(H^*K, H^*K),$$
$$e\colon\ F_j(H^*K, H^*K) \to F_{j+1}(H^*K, H^*K),$$
$$f\colon\ F_j(H^*K, H^*K) \to F_{j-1}(H^*K, H^*K)$$

vérifiant les relations de sl(2). L'endomorphisme h est la multiplication par n en degré n et e, de degré 2, est le crochet avec $\eta: H^*K \to H^*K(1)$. Notons F_j^d la partie de degré d de $F_j(H^*K, H^*K)$: la somme des $F_j(H^pK, H^qK)$ pour $q - p = d$. Si $H^iK = 0$ pour $|i| > N$, $F_j^d = 0$ pour $|d| > 2N$. Sur la somme des $\mathbb{Q}$-espaces vectoriels $\bigoplus_d F_{j+d/2}^d$ (somme sur d de parité fixée, j entier pour d pair, demi-entier pour d impair), (h, e, f) définit une action de sl(2) et e vérifie donc (L.V.). Sur la décomposition (1.3) correspondante, on lit:

Lemme 3.3. *Avec les notations précédentes, pour* $d \leq 1$,

(i) $e: F_{j-1}^{d-2} \to F_j^d$ *est injectif,*

(ii) F_j^d *est somme directe de l'image de* e *et du noyau de* $e^{1-d}: F_j^d \to F_{j+1-d}^{2-d}$.

3.4. Soit φ un isomorphisme

$$\bigoplus H^i(K)[-i] \xrightarrow{\sim} K \tag{3.4.1}$$

induisant l'identité sur le cohomologie. Comme en 2.7 (notations), regardons η comme une 1-application de K dans K et soit $\eta^{(d)}$ sa partie homogène de degré d, dans la décomposition (3.4.1). Avec les notations de 3.2, $\eta^{(d)}$ est dans F_1^d pour F le foncteur Ext^{2-d}. Pour $d > 2$, $\eta^{(d)}$ est donc nul, et pour $d = 2$, $\eta^{(2)} \subset \mathrm{Hom}(H^*K, H^*K(1))$ est l'endomorphisme e de 3.2.

PROPOSITION 3.5. *Supposons $\mathscr{D}$ $\mathbb{Q}$-linéaire et que (K, η) vérifie (L.V.). Il existe alors un unique isomorphisme* (3.4.1) *pour lequel on ait*
(3.5.1) *pour* $d \leq 1$, $e^{1-d}(\eta^{(d)}) = 0$.

Noter que pour $d = 1$, (3.5.1) signifie que $\eta^{(1)} = 0$.

PREUVE. Par récurrence sur $n \geq -1$, nous allons monter que les conditions (3.5.1) pour $-n < d \leq 1$ déterminent φ uniquement modulo composition avec un endomorphisme $1 + \psi$ de la somme des $H^i(K)[-i]$, avec ψ de degré $\leq -n-2$. Noter que, avec les notations de 3.2, la composante homogène $\psi^{(d)}$ de degré d de ψ est dans F_0^d pour F le foncteur Ext^{-d}.

Pour $n = -1$, l'hypothèse est vide et la conclusion exprime que φ, supposé être l'identité sur la cohomologie, est défini modulo composition avec $1 + \psi$, ψ de degré ≤ -1.

Prouvons l'assertion pour $n \geq 0$, en la supposant vraie pour $n - 1$. Changer φ en $\varphi \circ (1 + \psi)$ conjugue le 1-morphisme η (plutôt, $\varphi^{-1}\eta\varphi$) de la somme des $H^iK[-i]$ en $(1+\psi)^{-1}\eta(1+\psi)$. Pour ψ de degré $\leq -n-1$, puisque η est de degré ≤ 2, on a

$$\begin{aligned}(1+\psi)^{-1}\eta(1+\psi) &\equiv \eta + [\eta, \psi] \\ &\equiv \eta + e(\psi^{(-n-1)})\end{aligned}$$

modulo degré $\leq -n$. Par 3.3, un unique choix de $\psi^{(-n-1)}$ fournit un nouveau η vérifiant (3.5.1) pour $d > -n$, comme promis.

PROPOSITION 3.6. *Soit φ l'isomorphisme* (3.4.1) *de* 3.5. *Alors, sur $P^{-i}[i]$, φ coïncide avec le morphisme f_i de* 2.4.

PREUVE. Identifions la somme des $H^i[-i]$ à K par φ. Par la caractérisation 2.4 de f_i, il suffit de vérifier que pour $s > i$, la restriction à $P^{-i}[i]$ du s-morphisme η^s est de degré $< s+i$. On a une décomposition en composantes homogènes

$$\eta = \eta^{(2)} + \sum_{a \leq 0} \eta^{(a)}.$$

Posons $e = \eta^{(2)}$. Par 3.5, on a

$$(\mathrm{ad}\, e)^{-a+1}(\eta^{(a)}) = 0.$$

Développons η^s, et réordonnant les facteurs par application itérée de la règle

$$ex = xe + \operatorname{ad} e(x).$$

On trouve que η^s est somme de termes des types suivant:

(a) e^s

(b) $(\operatorname{ad} e)^{r_k}(\eta^{(a_k)}) \cdots (\operatorname{ad} e)^{r_1}(\eta^{(a_1)})e^{r_0}$ avec $\sum_0^k r_k < s$.

Si $s > i$, e^s s'annule sur $P^{-i}[i]$. De même, les termes (b) avec $r_0 > i$ s'annulent sur $P^{-i}[i]$. Le facteur $(\operatorname{ad} e)^{r_k}(\eta^{(a_k)})$ est de degré $2r_k + a_k$, et nul si $r_k > -a_k$. S'il est non nul, il est de degré

$$2r_k + a_k = r_k + (r_k + a_k) \leq r_k.$$

Chaque terme (b) non nul sur $P^{-i}[i]$ est donc de degré $\leq 2r_0 + \sum_1^k r_k = r_0 + \sum_0^k r_i < i + s$, comme requis.

4. Cohomologie ℓ-adique

4.1. Soient k un corps, ℓ un nombre premier premier à la caractérisque et prenons pour $\mathcal{D}$ la catégorie dérivée ℓ-adique avec sa t-structure naturelle [5]. Le coeur $\mathcal{C}$ de $\mathcal{D}$ est la catégorie des représentations ℓ-adiques (V, ρ) de $\operatorname{Gal}(\bar{k}/k)$: V de dimension finie sur $\mathbb{Q}_\ell$ et $\rho: \operatorname{Gal}(\bar{k}/k) \to \operatorname{GL}(V)$ continu. Les $\operatorname{Ext}^a(V, W)$ sont les

$$H^a(\operatorname{Gal}(\bar{k}/k), \operatorname{Hom}(V, W))$$

calculés en terme de cochaînes continues.

Soit X projectif et lisse sur k, purement de dimension N. Soit $a: X \to \operatorname{Spec}(k)$ le morphisme structural. L'image directe $Ra_*\mathbb{Q}_\ell$ est alors dans $\mathcal{D}$. Fixons un faisceau invertible ample $\mathcal{O}(1)$ et soit $\eta: Ra_*\mathbb{Q}_\ell \to Ra_*\mathbb{Q}_\ell(1)[2]$ le produit par $c_1\mathcal{O}(1)$. On sait que $(Ra_*\mathbb{Q}_\ell[N], \eta)$ vérifie (L.V.) (P. Deligne, [4, Théorème 4.1.1]).

Le décomposition 2.5 de $Ra_*\mathbb{Q}_\ell[N]$ fournit par translation une décomposition de $Ra_*\mathbb{Q}_\ell$. Changeant de notation, nous noterons P^i le facteur direct de $H^i Ra_*\mathbb{Q}_\ell$ noyau de

$$\eta^{N-i+1}: H^i Ra_*\mathbb{Q}_\ell \to H^{2N-i+2} Ra_*\mathbb{Q}_\ell.$$

Dans la notation du §1 pour $Ra_*\mathbb{Q}_\ell[N]$, c'est P^{i-N}. La décomposition 2.6.1 déduite de 2.5 se translate en une décomposition

$$\bigoplus_{0 \leq j \leq N-i} P^i(-j)[-i-2j] \xrightarrow{\sim} Ra_*\mathbb{Q}_\ell. \tag{4.1.1}$$

D'après 2.7, les seules composantes intéressantes du 1-morphisme η dans cette décomposition sont les $\eta_{i,N-i}^{k,\ell}$ pour $\ell \leq N - i$. On a

$$\eta_{i,N-i}^{k,\ell} \in \operatorname{Ext}^{(2N-i)-(k+2\ell)+2}(P^i(-(N-i)), P^k(-\ell)(1)).$$

On aimerait regarder ces classes comme étant motiviques: elles ont été définies par un procédé uniforme en ℓ. Par ailleurs, des conjectures (optimistes) sur la relation entre cycles algébriques et la catégorie dérivée motivique impliquent [3, §§3.7, 3.8] que pour M un motif effectif de poids w, et $i \geq 0$, on a

$$\mathrm{Ext}^n(1, \check{M}(i)) = 0 \quad \text{pour } n > w + i.$$

Le cas particulier qui nous importe est l'annulation de cet Ext pour n égal à l'opposé du poids de $\check{M}(i)$, M effectif et $i > 0$.

Dans le cas qui nous importe, par (L.V.) et la dualité de Poincaré, P^k est isomorphe à $\check{P}^k(-k)$ et

$$\eta^{k,\ell}_{i,N-i} \in \mathrm{Ext}^n(1, V)$$

avec n l'opposé du poids de V et

$$V = \check{P}^i \otimes \check{P}^n(N - i - k - \ell + 1)$$

Si les $\eta^{k,\ell}_{i,N-i}$ ℓ-adiques sont motiviques, la conjecture implique donc une réponse positive à la

QUESTION. A-t-on $\eta^{k,\ell}_{i,N-i} = 0$ pour $N - i - k - \ell \geq 0$?

4.2. Dans la décomposition (4.1.1), le cup-produit

$$Ra_*\mathbb{Q}_\ell \otimes Ra_*\mathbb{Q}_\ell \to Ra_*\mathbb{Q}_\ell$$

a des composantes

$$c^{m,n}_{i,j;k,\ell} \in \mathrm{Ext}^A(P^i(-j) \otimes P^k(-\ell), P^m(-n))$$
$$= \mathrm{Ext}^A(1, V)$$

avec A l'opposé du poids de V et

$$M = \check{P}^i \otimes \check{P}^k \otimes \check{P}^m(j + k - m - n).$$

On veut donc nullité pour $j + k - m - n > 0$.

4.3. Soit Z un cycle algébrique de codimension d. Il a une classe

$$cl(Z): \mathbb{Q}_\ell \to Ra_*\mathbb{Q}_\ell(d)[2d],$$

de composantes

$$cl(Z)^{i,j} \in \mathrm{Ext}^A(1, V),$$

où A est l'opposé du poids de V et où

$$V = P^i(-j + d) = \check{P}^i(d - i - j).$$

On espère donc la nullité des composantes $cl(Z)^{i,j}$ pour $i + j < d$.

Appendice. Objets spectraux, d'après J. L. Verdier

J'ai appris le formalisme exposé dans cet appendice de J. L. Verdier, qui disait s'être inspiré du traitement des suites spectrales dans Cartan-Eilenberg (vol. 1, pp. 318–319).

A.1. Soit $\mathscr{D}$ une catégorie triangulée. Un *objet spectral* dans $\mathscr{D}$ est la donnée de

(1) une famille X_{pq} d'objets de $\mathscr{D}$, indexée par les paires d'entiers $p \leq q$;
(2) pour $p' \leq p$, $q' \leq q$, un morphisme $X_{pq} \to X_{p'q'}$;
(3) pour $p \leq q \leq r$, un morphisme degré un, appelé *cobord*, $X_{pq} \to X_{qr}$.
On exige que
(a) les morphismes (2) définissent un foncteur contravariant de l'ensemble ordonné des paires (p, q) avec $p \leq q$ dans $\mathscr{D}$;
(b) pour $p \leq q \leq r$, $p' \leq q' \leq r'$ et $p' \leq p$, $q' \leq q$, $r' \leq r$, le diagramme

$$\begin{array}{ccc} X_{pq} & \xrightarrow{\partial} & X_{qr} \\ \downarrow & & \downarrow \\ X_{q'q'} & \xrightarrow{\partial} & q'r' \end{array}$$

de morphismes (2) et (3) est commutatif;
(c) pour $p \leq q \leq r$, le triangle

$$X_{qr} \to X_{p,r} \to X_{p,q} \xrightarrow{1} X_{q,r}$$

de morphismes (2) et (3) est distingué.

Exemple. Pour $\mathscr{D}$ la catégorie dérivée d'une catégorie abélienne $\mathscr{A}$ et (K, F) un complexe filtré, les $X_{pq} = F^p(K)/F^q(K)$ forment un objet spectral.

Remarque. Un objet spectral X sera dit d'amplitude dans $[a, b]$ si les morphismes (1): $X_{pq} \to X_{p'q'}$ sont des isomorphismes pour $p' \leq p \leq a$ et $q' = q$, ainsi que pour $b < q' \leq q$ et $p' = p$. Il revient au même d'imposer la nullité des $X_{p,p+1}$ pour $p \notin [a, b]$. Un objet spectral d'amplitude $[0, n]$ avec $n = 1$ (resp. 2) s'identifie à un triangle distingué (resp. à un diagramme de l'octaèdre), cf. [**1**, §§1.1.6, 1.1.7, 1.1.14].

A.2. Soit $\mathscr{A}$ une catégorie abélienne, munie d'un "foncteur de translation" $X \mapsto X[1]$ (une autoéquivalence). On note $X \mapsto X[n]$ l'itéré $n^{\text{ième}}$ du foncteur de translation et on appelle morphisme de degré n de X dans Y un morphisme de X dans $Y[n]$. Une suite $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$, avec α (resp. β) de degré n (resp. m) est dite exacte si la suite $X \to Y[n] \to Z[n+m]$ l'est.

Un *objet spectral* de $\mathscr{A}$ est la donnée $(1')$, $(2')$, $(3')$ d'objets et de morphismes de $\mathscr{A}$, comme en A.1, vérifiant les conditions (a'), (b') de même formulation que A.1 (a), (b), et la condition

(c') pour $p \leq q \leq r$, la suite

$$\cdots \to X_{qr} \to X_{pr} \to X_{p,q} \xrightarrow{\partial} X_{q,r} \to X_{pr} \to \cdots$$

de morphismes $(2')$ et $(3')$ est exacte.

Soit X un objet spectral de $\mathscr{A}$. D'après (b') le morphisme $(2')$: $X_{p,r} \to X_{p,r}$ est l'identité. D'après (c') pour $p = q$, on a $X_{pp} = 0$. Si $p \leq q'$, le morphisme $X_{pq} \to X_{p'q'}$ est nul, car il si factorise par X_{pp} . Pour $p \leq q \leq r \leq s$, le composé des morphismes cobord

$$X_{pq} \to X_{qr} \to X_{rs}$$

est nul: le premier se factorise par le morphisme cobord vers X_{qs} , suivi de $X_{qs} \to X_{qr}$, et le composé $X_{qs} \to X_{qr} \to X_{rs}$ est nul par (c') .

Si $\mathscr{A}$ est la catégorie des objets gradués d'une catégorie abélienne $\mathscr{B}$, avec le foncteur de translation défini par

$$(X[1])^n = X^{n+1} ,$$

la donnée $(1')$ est celle d'objets $X^n_{p,q}$ de $\mathscr{B}$, $(2')$ celle de morphismes $X^n_{pq} \to X^n_{p'q'}$, $(3')$ celle de morphismes $X^n_{pq} \to X^{n+1}_{qr}$ et (c$'$) devient une suite exacte longue

$$\cdots \longrightarrow X^n_{pr} \to X^n_{pq} \xrightarrow{\partial} X^{n+1}_{qr} \to X^{n+1}_{pr} \to \cdots .$$

Exemple. Avec les notations précédentes, si F est un foncteur cohomologique [**6**, §3.1] de $\mathscr{D}$ dans $\mathscr{B}$, et $F^\bullet$ le foncteur de $\mathscr{D}$ dans $\mathscr{A}$ défini par

$$F^n(X) = F(X[n]) ,$$

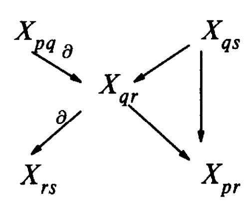
alors, $F^\bullet$ transforme objets spectraux de $\mathscr{D}$ en objets spectraux de $\mathscr{A}$.

Remarque. Comme en A.1, si X est un objet spectral de $\mathscr{A}$, pour que les morphismes $(2')$ soient des isomorphismes pour $p' \leq p' \leq a$ et $q' = q$, ainsi que pour $b < q' \leq q$ et $p' = p$, il faut et il suffit que $X_{p,p+1} = 0$ pour $p \notin [a, b]$. On dit alors que X est d' amplitude dans $[a, b]$. On notera alors $X_{-\infty,q}$ (resp. $X_{p,\infty}$) un quelconque $X_{p,q}$ avec $p \leq a$ (resp. $b < q$).

A.3. Soit X un objet spectral de $\mathscr{A}$. Pour $p \leq q \leq r \leq s$, posons

(A.3.1) $$E(pqrs) := \mathrm{Im}(X_{qs} \to X_{pr}).$$

Le morphisme $X_{qs} \to X_{pr}$ se factorise par X_{qr} , et cette factorisation donne lieu à une croix de suites exactes



On en déduit qu'on a aussi

(A.3.2) $$E(pqrs) = H(X_{pq} \to X_{qr} \to X_{rs}).$$

Pour $k \le \ell \le m \le n \le p \le q$, le diagramme commutatif

$$\begin{array}{ccc} X_{\ell n} & \xrightarrow{\partial} & X_{nq} \\ \downarrow & & \downarrow \\ X_{km} & \xrightarrow{\partial} & X_{mp} \end{array}$$

fournit par passage aux images par les flèches $(2')$ verticales un morphisme de degré un $\partial: E(k\ell mn) \to E(mnpq)$. Pour $k \le \ell \le m \le n \le p \le q \le r \le s$, le composé

A.3.3) $$E(k\ell mn) \xrightarrow{\partial} E(mnpq) \xrightarrow{\partial} E(pqrs)$$

est nul, car les lignes du diagramme

$$\begin{array}{ccccc} X_{\ell n} & \xrightarrow{\partial} & X_{nq} & \xrightarrow{\partial} & X_{qs} \\ \downarrow & & \downarrow & & \downarrow \\ X_{km} & \xrightarrow{\partial} & X_{mp} & \xrightarrow{\partial} & X_{pr} \end{array}$$

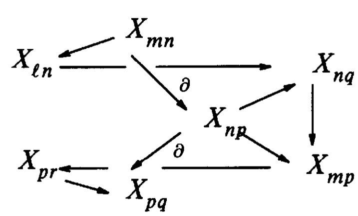
ont un composé nul (voir A.2).

Construction. *La cohomologie de la suite* (A.3.3) *est* $E(\ell npr)$.

La cohomologie de (A.3.3) est aussi celle de

$$X_{\ell n} \xrightarrow{\partial} \mathrm{Im}(X_{nq} \to X_{mp}) \xrightarrow{\partial} X_{pr}.$$

Considérons le diagramme commutatif



La cohomologie cherchée est encore celle de

$$X_{\ell n} \to H(X_{mn} \to X_{np} \to X_{pq}) \to X_{pr},$$

égale à

$$H(X_{\ell n} \to X_{np} \to X_{pr}),$$

ce qui fournit la construction cherchée par (A.3.2).

A.4. Pour $r \ge 1$, posons

$$E_r^p = E(p - 1 + 1, p, p + 1, p + r).$$

Le morphisme ∂ de A.3, pour $(p-r+1, p, p+1, p+r, p+r+1, p+2r)$ est une flèche de degré un: $d_r: E_r^p \to E_r^{p+r}$. On a $d_r^2 = 0$ et A.3 fournit

$$E_{r+1}^p = H(E_r^{p-r} \to E_r^p \to E_r^{p+r}):$$

les E_r^p forment une suite spectrale.

Si X est d'amplitude finie, i.e., dans un intervalle $[a, b]$ (A.2), cette suite spectrale converge vers $X_{-\infty,\infty}$: E_r^p est indépendant de r pour r assez grand $(r > b-a)$ et s'identifie au gradué de $X_{-\infty,\infty}$ filtré par les images des $X_{p\infty}$. Si on note F cette filtration, la suite exacte

$$X_{p,\infty} \to X_{-\infty,\infty} \to X_{-\infty,p}$$

montre en effet que

$$X_{p,\infty} \twoheadrightarrow F^p \qquad \text{et} \qquad X_{-\infty,\infty}/F^p \hookrightarrow X_{-\infty,p},$$

de sorte que $F^p/F^{p+1} \sim \mathrm{Im}(X_{p,\infty} \to X_{-\infty,p+1}) = E_\infty^p$.

La suite spectrale ainsi construite part de E_1. Dans le cas où $\mathscr{A}$ est la catégorie des objets gradués de $\mathscr{B}$, si on veut une suite spectrale partant de E_2, il suffit de renuméroter:

$$E_2^{pq} := (E_1^{-q})^{p+q}.$$

Bibliographie

1. A. A. Beilinson, J. Bernstein, et P. Deligne, *Faisceaux pervers,* Astérisque **100** (1983) pp. 1–172.
2. P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales,* Inst. Hautes Études Sci. Publ. Math. **35** (1968) pp. 107–126.
3. ——, *A quoi servent les motifs?* ce volume, pp. 143–161.
4. ——, *La conjecture de Weil.* II, Inst. Hautes Études Sci. Publ. Math. **52** (1980), pp. 137–252.
5. T. Ekedahl, *On the adic formalism,* Grothendieck Festschrift, vol. II, pp. 197–218.
6. J. L. Verdier, *Catégories dérivées, état zéro,* Séminaire de Géométrie Algébrique 4 1/2, pp. 262–311, Springer Lecture Notes, no. 569, Springer-Verlag, New York and Berlin, 1977.

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Arithmetic Analogs of the Standard Conjectures

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The standard conjectures on algebraic cycles [G] are central in Grothendieck's theory of motives. They state that the groups of cycles modulo homological equivalence of any smooth projective variety over an algebraically closed field behave like the cohomology groups of a Kähler manifold; i.e., they should satisfy the hard Lefschetz theorem and the Hodge index theorem, and have an algebraic star operator. An origin of these conjectures is the analogy between complex geometry and algebraic geometry, which has been strongly advocated by Weil.

Arakelov theory of arithmetic surfaces [A] offered a new way to combine these two geometries. It led Faltings [F] and Hriljac [H] to prove a Hodge index theorem for arithmetic surfaces. In this paper, we propose that all standard conjectures have analogs for the "arithmetic Chow groups" (see [GS1] and §1.1) of arbitrary arithmetic varieties.

After recalling the arithmetic intersection formalism of [GS1], we conjecture that the arithmetic intersection pairing is nondegenerate (Conjecture 1). This fits nicely with conjectures of Bloch and Beilinson about heights and regulators (2.2). We propose also a hard Lefschetz and a Hodge index type statement (Conjecture 2). These conjectures are true for arithmetic surfaces (Theorem 1).

We then give a definition of arithmetic correspondences (3.2.1) based on the notion of "regular kernels", due to Schwartz [S1]. It is interesting to notice that the arithmetic correspondences from X to Y do *not* consist only of arithmetic cycles on $X \times Y$. For instance, the identity operator from X to X is induced by a correspondence but not by a cycle.

However, our notion of correspondences is still too restrictive, and we have no analog of the assertion that the different operators coming from Lefschetz decomposition (e.g., the star operator) are induced by correspondences. The

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reason is that, in the absence of a base point, the product $X \times Y$ has to be taken over a base of dimension one, and our correspondences are relative to this base. It is our hope, however, that the attempt to combine Kähler geometry with intersection theory will help in gaining some insight on these questions. For instance, the case of abelian varieties would deserve more study (3.3.4).

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1. Arithmetic intersection theory

1.1. We first recall the definition of the arithmetic Chow groups and their basic properties (see **[GS1]** for more details). Let X be a regular scheme that is projective and flat over $\mathbb{Z}$. For any integer $p \geq 0$, denote by $A^{pp}(X_{\mathbb{R}})$ (resp. $D^{pp}(X_{\mathbb{R}})$) the real vector space of real differential forms (resp. currents, i.e., forms with distribution coefficients) α that are of type (p, p) over $X(\mathbb{C})$ (the set of complex points of X) and such that $F_\infty^*(\alpha) = (-1)^p \alpha$, where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ denotes complex conjugation. Since $X(\mathbb{C})$ is oriented, $A^{pp}(X_{\mathbb{R}})$ is embedded into $D^{pp}(X_{\mathbb{R}})$. We also denote by $\widetilde{A}^{pp}(X_{\mathbb{R}})$ (resp. $\widetilde{D}^{PP}(X_{\mathbb{R}})$) the quotient of $A^{pp}(X_{\mathbb{R}})$ (resp. $D^{pp}(X_{\mathbb{R}})$) by elements of the form $\partial u + \bar{\partial} v$.

A *cycle* of codimension p on X (with real coefficients) is a finite formal sum $Z = \sum_\alpha r_\alpha Z_\alpha$, where $r_\alpha \in \mathbb{R}$ and Z_α is a closed irreducible subset of codimension p in X. Such a cycle defines a current of integration $\delta_Z \in D^{pp}(X_{\mathbb{R}})$, whose value on a form η of complementary degree is

$$\delta_Z(\eta) = \sum_\alpha r_\alpha \int_{\widetilde{Z_\alpha(\mathbb{C})}} \pi_\alpha^*(\eta),$$

where $\pi_\alpha : \widetilde{Z_\alpha}(\mathbb{C}) \to Z_\alpha(\mathbb{C})$ is any resolution of the singularities of $Z_\alpha(\mathbb{C})$.

A *Green current* for Z is any current $g \in D^{p-1,p-1}(X_{\mathbb{R}})$ such that

$$dd^c g + \delta_Z = \omega, \tag{1}$$

where ω is a smooth form in $A^{pp}(X_{\mathbb{R}}) \subset D^{pp}(X_{\mathbb{R}})$ (recall that $d^c = (i/4\pi) \times (\bar{\partial} - \partial)$ and $dd^c = (i/2\pi)\partial\bar{\partial}$).

The (real) *arithmetic Chow group* of codimension p is the real vector space $\widehat{CH}^p(X)_{\mathbb{R}}$ generated by pairs (Z, g), where Z is a real cycle of codimension p on X and g a Green current for Z, the addition being defined componentwise, with the following relations. First any pair $(0, \partial u + \bar{\partial} v)$ is trivial in $\widehat{CH}^p(X_{\mathbb{R}})$. Second, if $Y \subset X$ is a closed irreducible subscheme of codimension $p-1$ in X, $f \in k(Y)^*$ a nonzero rational function on Y, and $\operatorname{div}(f)$ its divisor, the pair $(\operatorname{div}(f), -\log|f|^2)$ is zero in $\widehat{CH}^p(X_{\mathbb{R}})$, where

$-\log|f|^2 \in D^{p-1,p-1}(X_{\mathbb{R}})$ is the current defined by the formula

$$(2) \qquad (-\log|f|^2)(\eta) = -\int_{\widetilde{Y(\mathbb{C})}} \log|f|^2 \pi^*(\eta),$$

where $\pi : \widetilde{Y(\mathbb{C})} \to Y(\mathbb{C})$ is any resolution of singularities.

This real vector space $\widehat{CH}^p(X)_{\mathbb{R}}$ is in general infinite dimensional. It fits in an exact sequence **[GS1**, 3.3.5]

$$(3) \qquad CH^{p,p-1}(X)_{\mathbb{R}} \xrightarrow{\rho} \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X)_{\mathbb{R}} \xrightarrow{z} CH^p(X)_{\mathbb{R}} \to 0,$$

where $CH^p(X)_{\mathbb{R}} = CH^p(X)\otimes_{\mathbb{Z}}\mathbb{R}$ (resp. $CH^{p,p-1}(X)_{\mathbb{R}} = CH^{p,p-1}(X)\otimes_{\mathbb{Z}}\mathbb{R}$), $CH^p(X)$ being the usual (algebraic) Chow group of codimension p of the scheme X (resp. $CH^{p,p-1}(X)$ being a K_1-analog of $CH^p(X)$), ρ is the Beilinson regulator, z maps the class of (Z, g) to the class of Z, and $a(\eta)$ denotes the class of $(0, \eta)$. Since the image of ρ lies in the cohomology group $H^{p-1,p-1}(X_{\mathbb{R}}) = \operatorname{Ker}(dd^c) \subset \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$, the group $\widehat{CH}^p(X_{\mathbb{R}})$ is infinite dimensional when $1 \le p \le d$, where d is the relative dimension of X over $\mathbb{Z}$.

Consider the space of closed forms $Z^{pp}(X_{\mathbb{R}}) = \operatorname{Ker}(d) \subset A^{pp}(X_{\mathbb{R}})$, and let $\omega : \widehat{CH}^p(X)_{\mathbb{R}} \to Z^{pp}(X_{\mathbb{R}})$ be the map sending the class of (Z, g) to $\omega(Z, g) = dd^c g + \delta_Z$. If $cl : CH^p(X)_{\mathbb{R}} \to H^{pp}(X_{\mathbb{R}})$ sends an algebraic cycle to its cohomology class and if $h: Z^{pp}(X_{\mathbb{R}}) \to H^{pp}(X_{\mathbb{R}})$ is the obvious projection, we get another exact sequence **[GS1**, 3.3.5], namely,

$$CH^{p,p-1}(X)_{\mathbb{R}} \xrightarrow{\rho} H^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X)_{\mathbb{R}} \xrightarrow{(z,-\omega)} CH^p(X)_{\mathbb{R}} \oplus Z^{pp}(X_{\mathbb{R}}) \xrightarrow{cl+h} H^{pp}(X_{\mathbb{R}}).$$

Notice that $\omega \circ a = dd^c$.

1.2. As was shown in **[GS1]** there is an intersection product

$$\widehat{CH}^p(X)_{\mathbb{R}} \otimes \widehat{CH}^q(X)_{\mathbb{R}} \to \widehat{CH}^{p+q}(X)_{\mathbb{R}}$$

and a direct image

$$\widehat{\deg} : \widehat{CH}^{d+1}(X)_{\mathbb{R}} \to \widehat{CH}^1(\operatorname{Spec}\mathbb{Z})_{\mathbb{R}} = \mathbb{R}$$

(mapping (Z, g) to $\log(\#\Gamma(Z, \mathcal{O}_Z)) + \frac{1}{2}\int_{X(\mathbb{C})} g$). Note the following useful formula. For any $x \in \widehat{CH}^p(X)_{\mathbb{R}}$ and $\eta \in \widetilde{A}^{q-1,q-1}(X_{\mathbb{R}})$,

$$(5) \qquad xa(\eta) = a(\omega(x)\eta) \quad \text{in } \widehat{CH}^{p+q}(X)_{\mathbb{R}}.$$

In particular, we get an *intersection pairing*

$$(6) \qquad \widehat{CH}^p(X)_{\mathbb{R}} \otimes \widehat{CH}^{d+1-p}(X)_{\mathbb{R}} \to \mathbb{R}$$

by mapping $x \otimes y$ to $\widehat{\deg}(xy)$.

CONJECTURE 1. *The intersection pairing* (6) *is nondegenerate.*

1.3. Given a line bundle H on X, equipped with a smooth Hermitian metric, invariant under F_∞, on the corresponding holomorphic line bundle on $X(\mathbb{C})$, one gets a first Chern class $\hat{c}_1(\overline{H}) \in \widehat{CH}^1(X)_{\mathbb{R}}$, defined as the class $(\operatorname{div}(s), -\log\|s\|^2)$, for any nonzero rational section s of H over X (see **[D]** and **[GS2]**).

Denote by

$$L : \widehat{CH}^p(X)_{\mathbb{R}} \to \widehat{CH}^{p+1}(X)_{\mathbb{R}}$$

the product by $\hat{c}_1(\overline{H})$, i.e., $L(x) = x \cdot \hat{c}_1(\overline{H})$. The metric on H is called positive if the first Chern form $c_1(\overline{H})$ is a positive 1-1 form on $X(\mathbb{C})$.

CONJECTURE 2. *When H is ample on X, one can choose a positive metric on H in such a way that, if $2p \leq d+1$,*

(i) *the operator*

$$L^{d+1-2p} : \widehat{CH}^p(X)_{\mathbb{R}} \to \widehat{CH}^{d+1-p}(X)_{\mathbb{R}}$$

is an isomorphism.

(ii) *If $x \in \widehat{CH}^p(X)_{\mathbb{R}}$, $x \neq 0$, and $L^{d+2-2p}(x) = 0$, then*

$$(-1)^p \widehat{\deg}(xL^{d+1-2p}(x)) > 0.$$

REMARKS. Note that Conjecture 2 implies Conjecture 1. Concerning the choice of a metric on H in Conjecture 2, we refer to **[GS3]** and **[Z]** for a discussion of the notion of "arithmetic ampleness".

2. Remarks on the conjectures

2.1. THEOREM 1. (i) *Conjectures* 1 *and* 2 *are true for any X when $p = 0$.*
(ii) *When $d = 1$, Conjectures* 1 *and* 2 *hold for all p.*

PROOF. We may assume that X is irreducible since $\widehat{CH}^p$ is additive under disjoint union. Also, recall that Conjecture 2 implies Conjecture 1.

To prove Conjecture 2 when $p = 0$, notice that $\widehat{CH}^0(X)_{\mathbb{R}} = CH^0(X)_{\mathbb{R}} = \mathbb{R}$ by (3). On the other hand $CH^{d+1}(X)$ is torsion (it is even finite, see **[Bl1]** and **[KS]**). Therefore, when $p = d+1$, we obtain from (3) that

$$\widehat{CH}^{d+1}(X)_{\mathbb{R}} = H^{d,d}(X_{\mathbb{R}})/\rho(CH^{d,d-1}(X)_{\mathbb{R}}).$$

The ring $\Gamma(X, \mathcal{O}_X)$ of regular functions on X is the ring of integers $\mathcal{O}_F$ in a number field F. According to **[So1]**, $CH^{d,d-1}(X)_{\mathbb{R}}$ is equal to $\Gamma(X, \mathcal{O}_X)^* \otimes_{\mathbb{Z}} \mathbb{R}$, and, by **[GS1**, 3.4.3]. together with the naturality of (3), the map ρ coincides with the regulator of Dirichlet

$$\rho : \mathcal{O}_F^* \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^{r_1+r_2}$$

(where r_1 and r_2 are the number of real and complex places of F). Its cokernel $\widehat{CH}^{d+1}(X)_{\mathbb{R}}$ is thus isomorphic to $\mathbb{R}$ by the degree map $\widehat{\deg}$ (see 1.2). The intersection pairing

$$\widehat{CH}^0(X)_{\mathbb{R}} \otimes \widehat{CH}^{d+1}(X)_{\mathbb{R}} \to \mathbb{R}$$

is the multiplication $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \to \mathbb{R}$, hence it is nondegenerate. Finally, given any ample line bundle H on X, we may find a positive metric on H such that $\widehat{\deg}(\hat{c}_1(\overline{H})^{d+1}) > 0$. Indeed, when we multiply the norm on H by $\exp(\lambda)$, this number gets changed by the addition of $-\lambda \deg_H(X)$, and the algebraic degree $\deg_H(X)$ of H on X_F is positive since H is ample. So, given any positive metric on H, a small scalar multiple of this metric will do. This proves Conjecture 2 when $p = 0$.

Assume now that $d = 1$ and let H be an ample line bundle on X. Choose any positive metric H such that $\widehat{\deg}(\hat{c}_1(\overline{H})^2) > 0$. We want to show that the Hodge index theorem holds for $\widehat{CH}^1(X)_{\mathbb{R}}$. The positive $(1,1)$ form $\mu = c_1(\overline{H})$ defines a Kähler structure on $X(\mathbb{C})$, and we denote by $CH^1(\overline{X})_{\mathbb{R}} \subset \widehat{CH}^1(X)_{\mathbb{R}}$ the subspace of those x such that $\omega(x)$ is harmonic, i.e., equal to a multiple of μ.

Let $x \in \widehat{CH}^1(X)_{\mathbb{R}}$ be such that $\widehat{\deg}(x \cdot \hat{c}_1(\overline{H})) = 0$ and $x \neq 0$. We have to show that $\widehat{\deg}(x^2) < 0$. Let $\sigma(x) \in CH^1(\overline{X})_{\mathbb{R}}$ be its projection in $CH^1(\overline{X})_{\mathbb{R}}$ (see **[GS1**, 5.1.2] and §3.3.3). By definition of σ, we have $x - \sigma(x) = a(\varphi)$ where $\int_{X(\mathbb{C})} \varphi\mu = 0$.

Therefore, $\widehat{\deg}(x \cdot \hat{c}_1(\overline{H})) = \widehat{\deg}(\sigma(x)\hat{c}_1(\overline{H})) = 0$. By the Hodge index theorem for $CH^1(\overline{X})$, proved by Faltings **[F]** and Hriljac **[H]**, it follows that $\widehat{\deg}(\sigma(x)^2) \leq 0$, with equality if and only if $\sigma(x) = 0$. Since $x = \sigma(x) + a(\varphi)$, we get, using (5)

$$\widehat{\deg}(x^2) = \widehat{\deg}(\sigma(x)^2) + \int_{X(\mathbb{C})} \varphi\omega(\sigma(x)) + \frac{1}{2}\int_{X(\mathbb{C})} \varphi dd^c(\varphi).$$

Since $\omega(\sigma(x))$ is a multiple of μ, the integral $\int_{X(\mathbb{C})} \varphi\omega(\sigma(x))$ vanishes. Furthermore,

$$\int_{X(\mathbb{C})} \varphi dd^c(\varphi) \leq 0$$

with equality if and only if $\varphi = 0$. □

2.2. Let us now examine how Conjecture 1 relates to several statements which were proposed by Grothendieck **[G]**, Bloch **[Bl2]**, and Beilinson **[B1, B2]**. So let $x \in \widehat{CH}^p(X)_{\mathbb{R}}$ be such that, for all $y \in \widehat{CH}^{d+1-p}(X)_{\mathbb{R}}$, $\widehat{\deg}(xy) = 0$. Taking $y = a(\eta)$ we find, using (5),

$$\widehat{\deg}(xy) = \frac{1}{2}\int_{X(\mathbb{C})} \omega(x)\eta = 0$$

for any $\eta \in A^{d-p,d-p}(X_{\mathbb{R}})$. This implies $\omega(x) = 0$. In particular, the cycle class $z(x) \in CH^p(X)_{\mathbb{R}}$ is homologically trivial (see (4)).

If we assume, as proposed by Bloch [**Bl1**] and by Beilinson [**B2**] (who also stated in [**B2**] other "standard conjectures" for homologically trivial cycles), that the height pairing on cycles over $X_{\mathbb{Q}}$ that are homologically trivial is nondegenerate, we conclude that the image of $z(x)$ in $CH^p(X_{\overline{\mathbb{Q}}})_{\mathbb{R}}$ is trivial.

It follows, using (4), that x may be written $x = a(h) + x_0$, where $h \in H^{p-1,p-1}(X_{\mathbb{R}})$ and x_0 is the class in $\widehat{CH}^p(X)_{\mathbb{R}}$ of the pair $(Z, 0)$, where Z is a cycle on X whose support does not meet the generic fiber $X_{\mathbb{Q}}$.

According to another conjecture of Beilinson [**B1**], there exists an element $\alpha \in CH^{p,p-1}(X)_{\mathbb{R}}$ and a cycle $u \in CH^{p-1}(X)_{\mathbb{R}}$ such that

$$h = \rho(\alpha) + cl(u) \quad \text{in } H^{p-1,p-1}(X_{\mathbb{R}}).$$

Since $a \circ \rho = 0$ by (3), we get

$$x = a(cl(u)) + x_0.$$

Finally, if we know that $x_0 = 0$, we get, for all $y \in \widehat{CH}^{d+1-p}(X)_{\mathbb{R}}$,

$$\widehat{\deg}(xy) = \int_{X(\mathbb{C})} cl(u)\omega(y) = 0;$$

i.e., $cl(u) \in H^{p-1,p-1}(X_{\mathbb{R}})$ is orthogonal to $cl(v)$ for any cycle $v \in CH^{d+1-p}(X)_{\mathbb{R}}$. By a transfer argument, it follows that $cl(u)$ is orthogonal to $cl(v)$ for any cycle $v \in CH^{d+1-p}(X_{\overline{\mathbb{Q}}})_{\mathbb{R}}$. The standard conjecture that numerical equivalence implies homological equivalence [**G**] then gives $x = cl(u) = 0$.

(Conversely, Conjecture 1 together with the assertion that $\mathrm{Im}(\rho) \cap \mathrm{Im}(cl) = 0$, see [**B1**], implies that numerical equivalence coincides with homological equivalence for cycles on $X_{\mathbb{Q}}$.)

2.3. The groups $\widehat{CH}^p(X)_{\mathbb{R}}$ that we considered above are in several respects analogous to the Chow groups $CH^p(Y)_{\mathbb{Q}} = CH^p(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, where Y is a smooth projective variety over a finite field. Similar standard conjectures might also be true for $CH^p(Y)_{\mathbb{Q}}$. For instance, when Y is a product of curves and abelian varieties, it is shown in [**So2**, Theorem 7] that the following hard Lefschetz theorem holds: if $\xi \in CH^1(Y)_{\mathbb{Q}}$ is the class of an ample line bundle on Y, for any $p \geq 0$ with $2p \leq d = \dim(Y)$, the product by ξ^{d-2p} is an isomorphism from $CH^p(Y)_{\mathbb{Q}}$ to $CH^{d-p}(Y)_{\mathbb{Q}}$.

Notice that we take cycles on Y modulo linear equivalence, when the standard conjectures for varieties over algebraically closed fields [**G**] consider cycles modulo homological equivalence. Indeed, one expects that, up to torsion, for smooth projective varieties over a finite field, homological equivalence coincides with linear equivalence. As was pointed out to us by Deligne, this assertion would follow from Beilinson's conjectures on motivic sheaves.

When X is regular, projective, and flat over $\mathbb{Z}$, one may ask whether there exists an (infinite-dimensional) complex cohomology for X playing a role similar to that of ℓ-adic cohomology for varieties over finite fields (see **[De]**). If it was the case, the discussion above suggests that $\widehat{CH}^p(X)_{\mathbf{R}}$ would inject into such cohomology.

3. Relative arithmetic correspondences

3.1. Regular kernels.

3.1.1. Let M and N be smooth compact complex manifolds. Denote by $A(M) = \bigoplus_{n\geq 0} A^n(M)$ (resp. $D(M) = \bigoplus_{n\geq 0} D^n(M)$) the total space of complex differential forms (resp. currents) on M. By definition, $D(M)$ is the topological dual of $A(M)$ for the Schwartz topology. A *kernel* K from M to N is a $\mathbb{C}$-linear map $K : A(M) \to D(N)$ which is continuous for the Schwartz topology on $A(M)$ and the (weak) dual topology on $D(N)$.

Given such a kernel K there exists a unique current $T \in D(M \times N)$ such that, for any $\eta \in A(M)$,

$$K(\eta) = p_{2*}(p_1^*(\eta)T), \tag{7}$$

where $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ are the two projections. To prove the existence of T, choose local open charts $U_i \simeq \mathbb{C}^m$ and $V_j \simeq \mathbb{C}^n$ on M and N, and partitions of unity ρ_i and σ_j supported in U_i and V_j respectively. We may then decompose K as

$$K = \sum_{i,j} \sigma_j K \rho_i ,$$

where $\sigma_j K \rho_i$ is a continuous linear map from $A(\mathbb{C}^m)$ to $D(\mathbb{C}^n)$, with compact support on both $\mathbb{C}^m$ and $\mathbb{C}^n$. Using the standard basis on $A(\mathbb{C}^m)$ and $D(\mathbb{C}^n)$, we may decompose $\sigma_j K \rho_i$ and view each component as a continuous linear map from smooth functions on $\mathbb{C}^m$ to distributions on $\mathbb{C}^n$. According to Schwartz's representation theorem (Theorems II and IV in **[S1]**; see also **[S2]**), such a map comes from a distribution with compact support on $\mathbb{C}^m \times \mathbb{C}^n$; hence, $\sigma_j K \rho_i$ is obtained from a current T_{ij} with compact support on $U_i \times V_j$. If $T = \sum_{i,j} T_{ij}$, K is given by formula (7). The uniqueness of T is easy to check.

In the sequel we shall usually not distinguish between a kernel K and the current T that represents K.

3.1.2. A kernel $K : A(M) \to D(N)$ is said to be *regular* if it maps $A(M)$ into $A(N) \subset D(N)$ and if its transpose $K^t : A(N) \to D(M)$ maps $A(N)$ into $A(M)$, where K^t is defined by the formula

$$K^t(\eta)(\omega) = K(\omega)(\eta) \quad \text{for all } \omega \in A(N) \text{ and } \eta \in A(M).$$

This notion is the global analog of the notion of regular kernels considered in **[S1**, §7]. Notice that, if K is regular, the transpose of $K^t : A(N) \to A(M)$

defines an extension of K to a continuous map $K : D(M) \to D(N)$. Smooth currents $T \in A(M \times N)$ are regular, but the converse does *not* hold.

Given k smooth compact complex manifolds $M_1, \dots, M_k$ and kernels

$$K_i : A(M_i) \to D(M_{i+1}), \qquad i = 1, \dots, k-1,$$

the composite kernel

$$K_{k-1} \circ K_{k-2} \circ \cdots \circ K_1 : A(M_1) \to D(M_k)$$

is defined as soon as all the K_i's, except maybe one of them, are regular. This composition law is associative (compare §8). For example, if K_1 (resp. K_2) is regular, $K_2 \circ K_1$ is the composite

$$A(M_1) \xrightarrow{K_1} A(M_2) \xrightarrow{K_2} D(M_3)$$

$$(\text{resp. } A(M_1) \xrightarrow{K_1} D(M_2) \xrightarrow{K_2} D(M_3)).$$

3.1.3. The operators of Kähler geometry are regular kernels. Namely, if $\omega \in A^{1,1}(M)$ is a Kähler form on M, which we assume to be compact and equidimensional of dimension d, the operator

$$L : A^n(M) \to A^{n+2}(M)$$

mapping x to $x \cup \omega$, the star operator

$$* : A^n(M) \to A^{d-n}(M)$$

(such that $\eta \cup (*\eta) = \omega^d/(d!)$), the adjoint $\Lambda = \pm * L *$ of L, the differentials ∂, $\bar\partial$, their adjoints, the Laplace operator, the Green operator, the harmonic projection (see for instance [**GH**, Chapter 0]) are all regular. The Kähler identities are thus relations in the algebra of regular kernels from M to M, which plays the role of the ring of correspondences for differential forms.

3.1.4. LEMMA 1. *Let $T \in D(M \times N)$ be a kernel from M to N. Assume that $dd^c(T)$ is regular. Then, for any form $\omega \in A(M)$ such that $\partial_M \omega = \bar\partial_M \omega = 0$, we may write $T\omega = \eta + \partial_N u + \bar\partial_N v$, with $\eta \in A(N)$.*

PROOF. Since the total differentials on $M \times N$ are $d = d_M + d_N$ and $d^c = d^c_M + d^c_N$, we get $(dd^c(T))(\omega) = d_N d^c_N(T(\omega))$. Since this current is smooth, the conclusion follows from the de Rham regularity theorem (e.g., [**GS1**, Theorem 1.2.2 (i)]).

3.2. Correspondences.

3.2.1. Let F be a number field, $\mathcal{O}_F$ its ring of integers, X and Y two projective smooth schemes over $S = \operatorname{Spec}(\mathcal{O}_F)$, and $X \times Y$ their product over S. The set of complex points of X is then the disjoint union $(X \times Y)(\mathbb{C}) = \coprod_\sigma X_\sigma \times Y_\sigma$, where σ runs over all complex embeddings of F and X_σ (resp. Y_σ) is the complex variety defined by X (resp. Y). A current on $(X \times Y)(\mathbb{C})$ is said to be *regular* if, for any σ, its restriction to $X_\sigma \times Y_\sigma$ defines a regular kernel from X_σ to Y_σ.

A (*relative*) *arithmetic correspondence* from X to Y over S consists of a pair (C, γ), where C is a (real) cycle on $X \times Y$ and $\gamma \in D((X \times Y)_{\mathbb{R}}) := \bigoplus_{p\geq 0} D^{p,p}((X \times Y)_{\mathbb{R}})$ is a current such that $dd^c(\gamma) + \delta_C$ is regular on $(X\times Y)(\mathbb{C})$. We denote by $C(X, Y)$ the real vector space of arithmetic correspondences, modulo the relations $(0, \partial u + \overline{\partial} v) = 0$ and $(\operatorname{div}(f), -\log|f|^2) = 0$ for any $f \in k(W)^*$, $W \subset X \times Y$ (see 1.1). It contains as a subspace $\widehat{CH}(X \times Y)_{\mathbb{R}} := \bigoplus_{p\geq 0} \widehat{CH}(X \times Y)_{\mathbb{R}}$, which consists of those pairs (C, γ) such that $\omega(C, \gamma) := dd^c(\gamma) + \delta_C$ is not only regular but smooth.

Any arithmetic correspondence $c \in C(X, Y)$ may be written

$$c = u + a(r) \tag{8}$$

where $u \in \widehat{CH}(X \times Y)_{\mathbb{R}}$ and $r \in D((X \times Y)_{\mathbb{R}})$ is such that $dd^c(r)$ is regular (here $a(r)$ denotes the class of $(0, r)$). Indeed, let (C, γ) be any representative of c and $g \in \widetilde{D}((X \times Y)_{\mathbb{R}})$ a Green current for C. Then, if $u = (C, g)$ and $r = \gamma - g$, it is clear that (8) holds and $dd^c(r) = \omega(C, \gamma) - \omega(C, g)$ is regular.

3.2.2. An arithmetic correspondence $c \in C(X, Y)$ defines as follows a morphism

$$c_* : \widehat{CH}(X)_{\mathbb{R}} \to \widehat{CH}(Y)_{\mathbb{R}}.$$

Let $p_1 : X\times Y \to X$ and $p_2 : X\times Y \to Y$ be the projections, and $c = u+a(r)$ as in (8). For any $x \in \widehat{CH}(X)_{\mathbb{R}}$ we let

$$c_*(x) = p_{2*}(p_1^*(x)u) + a(r(\omega(x))) \in \widehat{CH}(Y)_{\mathbb{R}}. \tag{9}$$

Here the product $p_1^*(x)u$ is the intersection product in $\widehat{CH}(X \times Y)_{\mathbb{R}}$ and the class of $r(\omega(x))$ is well defined in $A(Y_{\mathbb{R}})$ by Lemma 1. Note that

$$\omega(c_*(x)) = \omega(c)(\omega(x))$$

in $A(Y_{\mathbb{R}})$ (where the right-hand side is the image of the form $\omega(x)$ by the regular kernel $\omega(c)$).

We have to show that (9) is independent of the decomposition $c = u+a(r)$. First notice that, since $\partial_X(\omega(x)) = \overline{\partial}_X(\omega(x)) = 0$, adding $\partial u+\overline{\partial} v$ to r does not change the class of $r(\omega(x))$ in $\widetilde{A}(Y_{\mathbb{R}})$. Furthermore, if $c = u + a(r) = u' + a(r')$, there is an element $t \in \tilde{A}((X \times Y)_{\mathbb{R}})$ such that

$$u' - u = a(t) = a(r - r');$$

therefore, for any x in $\widehat{CH}(X)_{\mathbb{R}}$, using (5) and (7), we get

$$\begin{aligned} p_{2*}(p_1^*(x)u') - p_{2*}(p_1^*(x)u) &= p_{2*}(p_1^* a(t)) \\ &= a(p_{2*}(p_1^*\omega(x))t) = a(r(\omega(x))) - a(r'(\omega(x))). \quad \square \end{aligned}$$

3.2.3. Let X, Y, and Z be smooth projective varieties over S. We define as follows a composition of arithmetic correspondences

$$C(X, Y) \otimes C(Y, Z) \to C(X, Z).$$

If $c = u + a(r) \in C(X, Y)$ and $c' = u' + a(r') \in C(Y, Z)$ we let

$$c' \circ c = p_{13*}(p_{12}^*(u) p_{23}^*(u')) + a(r' \circ \omega(u)) + a(\omega(u') \circ r) + a(r' \circ dd^c(r)).$$

Here p_{13}, p_{12}, p_{23} are the projections from $X \times Y \times Z$ to $X \times Z$, $X \times Y$, and $Y \times Z$ respectively, and the compositions of kernels $r' \circ \omega(u)$, $\omega(u') \circ r$, and $r' \circ dd^c(r)$ make sense since, in each case, at least one of the factors is regular (see 3.1.2). By arguments similar to 3.2.2, one checks that this definition of $c' \circ c$ is independent of the decompositions $c = u + a(r)$ and $c' = u' + a(r')$. Note that $\omega(c' \circ c) = \omega(c') \circ \omega(c)$.

LEMMA 2. (i) *The morphism* $(c' \circ c)_*$ *from* $\widehat{CH}(X)_{\mathbb{R}}$ *to* $\widehat{CH}(Z)_{\mathbb{R}}$ *is equal to* $c'_* \circ c_*$.

(ii) *The composition of correspondences is associative.*

PROOF. to prove (i) we may assume that $c = u$ or $a(r)$ and $c' = u'$ or $a(r')$. When $c = u$ and $c' = u'$, the proof follows as usual from the projection formula. When $c = u$ and $c' = a(r')$, we get

$$(c' \circ c)_*(x) = a(r' \circ \omega(u))_*(x) = a((r' \circ \omega(u))(\omega(x))),$$

while

$$c'_* \circ c_*(x) = a(r'(\omega(c_*(x)))) = a(r'(\omega(u)(\omega(x))));$$

therefore, $(c' \circ c)_* = c'_* \circ c_*$. When $c = a(r)$ and $c' = a(r')$, we get

$$(c' \circ c)_*(x) = a(r' \circ dd^c(r))_*(x) = a((r' \circ dd^c(r))(\omega(x))),$$

while

$$(c' \circ c)_*(x) = a(r'(\omega(c_*(x))))$$

with $\omega(c_*(x)) = (\omega \circ a)(r(\omega(x))) = d_Y d_Y^c(r(\omega(x))) = dd^c(r)(\omega(x))$; therefore, $(c' \circ c)_* = c'_* \circ c_*$ in that case also. We leave to the reader the proof of the case $c = a(r)$ and $c' = u'$ as well as the proof of (ii).

3.3. Examples.

3.3.1. If $\Delta \subset X \times X$ is the diagonal, the class of $(\Delta, 0)$ is the unit in the ring $C(X, X)$ and acts as identity on $\widehat{CH}(X)_{\mathbb{R}}$.

For any $u = [(Z, g)] \in \widehat{CH}(X)_{\mathbb{R}}$, the map sending $x \in \widehat{CH}(X)_{\mathbb{R}}$ to the product $x \cdot u$ is induced by the correspondence $pr_1^*(u) \cdot [(\Delta, 0)] = [(\Delta_*(Z), \Delta_*(g))]$. In particular, the operator L in §1.3 is given by a correspondence.

Let $f : X \to Y$ be a morphism between smooth projective varieties over S, whose restriction to the generic fiber X_F is smooth, and $\Gamma_f \subset X \times Y$ the graph of f. Then, for any smooth form ω on $Y(\mathbb{C})$ (resp. $X(\mathbb{C})$), $(\delta_{\Gamma_f})_*(\omega) = f_*(\omega)$ (resp. $(\delta_{\Gamma_f^t})_*(\omega) = f^*(\omega)$) is smooth. Therefore, the kernel δ_{Γ_f} is regular. The arithmetic correspondence $(\Gamma_f, 0)$ (resp. its transpose $(\Gamma_f^t, 0)$) acts by f_* (resp. f^*) on $\widehat{CH}(X)_{\mathbb{R}}$ (resp. $\widehat{CH}(Y)_{\mathbb{R}}$) where f_* and f^* are defined in **[GS1]**.

Several examples of correspondences lying in $\widehat{CH}(X \times Y)_{\mathbb{R}}$ were given in [**GS1**, §5.2].

3.3.2. Let X be a smooth projective variety over S. We denote by π_p : $\widehat{CH}(X)_{\mathbb{R}} \to \widehat{CH}^p(X)_{\mathbb{R}}$ the obvious projection.

When Conjecture 2(i) holds, we get a Lefschetz decomposition of any element $x \in \widehat{CH}^p(X)_{\mathbb{R}}$, namely,

$$x = \sum_{j \geq i_0} L^j x_j,$$

with $i_0 = \max(2p - d - 1, 0)$ and $L^{d+1+2j-2p} x_j = 0$. We can define

$$\Lambda x = \sum_{j \geq i_1} L^{j-1} x_j, \qquad i_1 = \max(2p - d - 1, 1)$$

and

$$*x = \sum_{j \geq i_0} (-1)^{p-j} L^{d+1+j-2p} x_j.$$

Note that $*^2 = 1$ and $\Lambda = *L*$ (compare [**K**]).

By analogy with the geometric case [**G**], one would like the operators π_p, Λ, and $*$ to be induced by arithmetic correspondences from X to itself. However, since the product $X \times X$ has been taken over the one-dimensional base S, these operators cannot be implemented by the action of $C(X, X)$.

For instance we get from Theorem 1(i) an isomorphism

$$* : \widehat{CH}^{d+1}(X)_{\mathbb{R}} \to \widehat{CH}^0(X)_{\mathbb{R}}.$$

It is not given by the action of an element of $C(X, X)$ on $\widehat{CH}(X)_{\mathbb{R}}$ since such an action does not shift down the degree by more than d.

3.3.3. Let X and Y be as in 3.2.1, and assume we have chosen Kähler metrics on $X(\mathbb{C})$ and $Y(\mathbb{C})$, invariant under F_∞. Let $CH(\overline{X})_{\mathbb{R}}$ be the subspace of $\widehat{CH}(X)_{\mathbb{R}}$ made of elements x such that $\omega(x)$ is harmonic. Define similarly $CH(\overline{Y})_{\mathbb{R}}$ and $CH(\overline{X \times Y})_{\mathbb{R}}$ (for the product of the two Kähler structures). Denote by H the orthogonal projection of currents onto harmonic forms. As shown in [**GS1**, 5.1.2], one gets a splitting $\sigma : \widehat{CH}(X)_{\mathbb{R}} \to CH(\overline{X})_{\mathbb{R}}$ of the inclusion $CH(\overline{X})_{\mathbb{R}} \subset \widehat{CH}(X)_{\mathbb{R}}$ by mapping (Z, g) to (Z, g') where $dd^c g' + \delta_Z = H(\delta_Z)$ and $H(g') = H(g)$. The same recipe defines a projection σ: $C(X, Y) \to CH(\overline{X \times Y})_{\widehat{\mathbb{R}}}$ and one can check that $\sigma(c_*(x)) = \sigma(c)_*(x)$ for any x in $CH(\overline{X})_{\mathbb{R}}$. In particular, when the product of two harmonic forms on $X(\mathbb{C})$ is harmonic, the ring $CH(\overline{X \times Y})_{\mathbb{R}}$ is the natural ring of correspondences on $CH(\overline{X})_{\mathbb{R}}$.

3.3.4. It would be interesting to study arithmetic correspondences (in the sense of §§3.2.1 to 3.3.3) on abelian schemes over $\mathcal{O}_F$ and to mimic techniques from the algebraic case [**K, DM, Ku**] (this question was also raised by

Bloch and Deninger). Notice that the Fourier transform of Hermitian vector bundles on abelian varieties has already been used in complex geometry (see for instance [**D-K**, Chapter 3]).

References

[A] S. J. Arakelov, *Intersection theory of divisors on an arithmetic surface*, Math. USSR Izv. **8** (1974), 1167–1180.

[B1] A. A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet Math. **30** (1985), 2036–2070.

[B2] ——, *Height pairings between algebraic cycles*, Contemp. Math., no. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 1–24.

[Bl1] S. Bloch, *Algebraic K-theory and class-field theory for arithmetic surfaces*, Ann. of Math. (2) **114** (1981), 229–265.

[Bl2] ——, *Height pairings for algebraic cycles*, Proc. Luminy Conf. on Algebraic K-theory, Luminy 1983, J. Pure Appl. Algebra **34** (1984), 119–145.

[D] P. Deligne, *Le déterminant de la cohomologie*, Current Trends in Arithmetical Algebraic Geometry, Contemp. Math., no. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 93–178.

[De] C. Deninger, *Local L-factors of motives and regularized determinants*, Invent. Math. **107** (1992), 135–150; and *Motivic L-functions and regularized determinants*, these Proceedings, vol. 1, pp. 707–743.

[DM] C. Deninger and J. Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.

[D-K] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Math. Monographs, New York, 1990.

[F1] G. Faltings, *Calculus on arithmetic surfaces*, Ann. of Math. (2) **119** (1984), 387–424.

[GS2] H. Gillet and C. Soulé, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. **72** (1990), 94–174.

[GS3] ——, *Characteristic classes for algebraic vector bundles with Hermitian metrics*. I, II, Ann. of Math. (2) **131** (1990), 163–203, 205–238.

[GS7] ——, *An arithmetic Riemann-Roch theorem*, Invent. Math. **110** (1992), 473–543.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[G] A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic Geometry, Bombay 1968, Oxford Univ. Press, Oxford, 1969, pp. 193–199.

[H] P. Hriljac, *Heights and Arakelov's intersection theory*, Amer. J. Math. **107** (1985), 23–38.

[KS] K. Kato and S. Saito, *Global class field theory of arithmetic schemes*, Contemp. Math., no. 51, Amer. Math. Soc., Providence, RI, 1986, pp. 255–331.

[K] S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Cohomologie des Schémas (J. Giraud, ed.), North-Holland, Amsterdam, 1968, pp. 359–386.

[Ku] K. Künnemann, *On the Chow motive of an abelian scheme*, these Proceedings, vol. 1, pp. 189–205.

[S1] L. Schwartz, *Théorie des noyaux*, Proc. Internat. Congr. Math., Cambridge, MA, 1950, pp. 220–230.

[S2] ——, *Distributions à valeurs vectorielles*. I, Ann. Inst. Fourier (Grenoble) **7** (1957), 1–142.

[So1] C. Soulé, *Opérations en K-théorie algébrique*, Canad. J. Math. **37** (1985), 488–550.

[So2] ——, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), 317–345.

[Z] S. Zhang, *Positive line bundles on arithmetic surfaces*, Ann. of Math. **136** (1992), 569–587.

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Chow Groups, *K*-theory, and Motivic Cohomology

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A quoi servent les motifs?

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La première des "conjectures standard" (Grothendieck [19], Kleiman [20]), celle de type Lefschetz, demande que certaines classes de cohomologie soient algébriques. De toute façon, si les motifs sont les "facteurs directs" de variétés algébriques X définis par des projecteurs, cycles algébriques sur $X \times X$, leur définition n'est raisonnable que si on dispose d'assez de cycles algébriques. Sur ce problème: la construction de cycles algébriques intéressants, les progrès ont été maigres.

Grothendieck a tenté de dresser un catalogue de constructions projectives de cycles (contours apparents ...), cf. [19, p. 197]. Pour ceux dont on a calculé la classe de cohomologie, cette classe s'exprime en terme de classes de Chern de fibrés vectoriels évidents. Bien que je ne connaisse pas de contre-exemple, il me semble peu probable que l'anneau des classes de cycles sur $X \times X$ engendré par la diagonale, les diviseurs et les images inverses des classes de Chern de X par les pr_i $(i = 1, 2)$ contienne toujours les cycles requis par la première conjecture standard, par exemple les composantes de Künneth de la diagonale.

Plutôt que de chercher à construire des cycles, on peut chercher à construire des fibrés vectoriels, pour ensuite prendre leurs classes de Chern. C'est ainsi en fait que K. Kodaira et D. C. Spencer (1953) prouvent la conjecture de Hodge pour les diviseurs (un théorème de Lefschetz), le groupe $\mathrm{Pic}(X) = H^1(X, \mathscr{O}^*)$ étant accessible, sur $\mathbb{C}$, par la suite exacte exponentielle

$$0 \to 2\pi i\mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0.$$

De même, sur une variété abélienne, il est plus facile d'écrire l'équation fonctionnelle des fonctions Θ (un cocycle pour un fibré en droites) que de définir une fonction Θ, elle-même fournissant le diviseur $\Theta = 0$.

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Malheureusement, en rang supérieur, on ne sait pas plus construire des fibrés vectoriels dont les classes de Chern soient intéressantes qu'on ne sait construire des cycles intéressants.

Sur $\mathbb{C}$, la conjecture de Hodge fournirait les cycles voulus. En direction de cette conjecture, deux difficultés ont été découvertes. La première: Atiyah et Hirzebruch ont montré qu'elle ne pouvait être vraie que rationnellement (cf. Atiyah-Hirzebruch [**1**]).

La seconde: du point de vue de la théorie de Hodge, la classe d'un cycle Z de codimension d sur X projectif et lisse vit naturellement non dans $H^{d,d}_{\mathbb{Z}} := H^{d,d} \cap H_{\mathbb{Z}}$, mais dans une extension E_d de ce groupe par la jacobienne intermédiaire

$$J^d(X) := H^{2d-1}(X, \mathbb{C})/F^d + H^{2d-1}(X, \mathbb{Z}),$$

où F^d désigne le d-ième terme de la filtration de Hodge de $H^{2d-1}(X, \mathbb{C})$. Dans la catégorie des structures de Hodge mixtes, il s'agit respectivement de $\mathrm{Hom}(\mathbb{Z}(d), H^{2d}(X))$ et de $\mathrm{Ext}^1(\mathbb{Z}(d), H^{2d-1}(X))$. La conjecture de Hodge demande que le groupe des classes de cycles s'envoie sur un sous-groupe d'indice fini du quotient $H^{d,d}_{\mathbb{Z}}$. Par contre, on n'a aucune idée de ce que devrait être l'image dans E_d. Outre l'inconfort esthétique que cause cette ignorance, elle rend généralement inapplicable la méthode de P. A. Griffiths pour prouver la conjecture de Hodge par récurrence sur la dimension de la variété X. Cette méthode est inspirée de celle que Lefschetz utilise pour les surfaces, Lefschetz [**24**]. L'idée est que, si $(H_t)_{t\in\mathbb{P}^1}$ est un pinceau de sections hyperplanes, construire le cycle Z sur X revient à construire les cycles $Z_t = Z \cap H_t$. Dans des bons cas (a) la classe de cohomologie $c \in H^{d,d}_{\mathbb{Z}}(X)$ qu'on veut être celle d'un cycle Z a une restriction nulle sur les H_t, (b) si Z existe, sa classe de cohomologie c détermine la classe des Z_t dans les jacobiennes intermédiaires $J^d(H_t)$, et (c) la construction de Z revient à celle de cycles Z_t de classes définissant une section donnée de la famille des $J^d(H_t)$. On peut alors conclure si tout élément de $J^d(H_t)$ est la classe d'un cycle de codimension d cohomologue à zéro. Par cette méthode, Zucker [**32**] prouve la conjecture de Hodge pour les hypersurfaces cubiques dans $\mathbb{P}^5$. Noter toutefois que, si tout élément de $J^d(H_t)$ est la classe d'un cycle cohomologue à zéro, alors $H^{2d-1}(H_t)$ est de type de Hodge $\{(d, d-1)(d-1, d)\}$ (voir §1.6), la réciproque étant une conséquence de la conjecture de Hodge appliquée au produit de H_t par une variété abélienne convenable. De plus, cette surjectivité implique l'existence d'une courbe C_t et d'un cycle algébrique W_t sur $H_t \times C_t$ envoyant $H^1(C_t)$ sur $H^{2d-1}(H_t)$. On peut alors s'attendre à ce que $H^{2d}(X)$ soit contrôlé par le H^2 de la surface fibrée sur $\mathbb{P}^1$ de fibres les C_t, et pour un H^2 la conjecture de Hodge est de toute façon disponible.

Le but de ces notes est de montrer que, malgré cette absence de progrès sur le problème de la construction de cycles, la philosophie des motifs est un outil puissant.

Je remercie S. Bloch de ses commentaires sur une première version de ces notes.

1. Motifs

Selon ce qu'on peut et veut faire, on dispose de plusieurs définitions des motifs—ou d'aucune. Il y a lieu de distinguer les motifs purs, typiquement fournis par la cohomologie de variétés projectives non singulières, et les motifs mixtes, où des variétés ouvertes et singulières sont permises. La notion de motif sur S (famille de motifs paramétrée par S) pose d'autres problèmes encore.

1.1. Pour k un corps, on voudrait dans tous les cas que la catégorie des motifs sur k soit une catégorie abélienne $\mathbb{Q}$-linéaire $\mathscr{M}(k)$ avec des groupes Hom de dimension finie. Dans le cas pur, on la voudrait semi-simple et graduée (par le poids). Dans le cas mixte, chaque motif M devrait admettre une filtration finie croissante W, avec $\mathrm{Gr}_n^W(M)$ pur de poids n. Autres structures essentielles:

(a) Chaque variété algébrique X sur k doit avoir des groupes de cohomologie motiviques $H^i_{\mathrm{mot}}(X)$, objets de $\mathscr{M}(k)$. Dans le cas pur, on se limite aux variétés X projectives non singulières, et $H^i_{\mathrm{mot}}(X)$ est un motif pur de poids i.

(b) Pour chacune des théories de cohomologie usuelles H, on doit disposer d'un foncteur "réalisation" real et d'isomorphismes
$$H^i(X) = \mathrm{real}\, H^i_{\mathrm{mot}}(X).$$

(c) Un produit tensoriel $\otimes$, auquels les foncteurs de réalisation sont compatibles.

1.2. La structure (c) permet d'appliquer la théorie des catégories tannakiennes, inventée par Grothendieck pour étudier le formalisme des motifs. Références: Saavedra **[28]**, Deligne–Milne **[12]**, Deligne **[15]**.

Pour k de caractéristique 0, cette théorie dit que la catégorie des motifs sur k, avec son produit tensoriel, doit être équivalente à celle des représentations linéaires d'un schéma en groupes affine G sur $\mathbb{Q}$. Noter que deux groupes G_1 et G_2 formes intérieures l'un de l'autre ont mêmes catégories de représentations. Le *groupe de Galois motivique* G n'est donc pas uniquement déterminé par $\mathscr{M}(k)$. La théorie indique que le choix de G et d'une équivalence $\mathscr{M}(k) \simeq \mathrm{Rep}(G)$ équivaut à celui d'un foncteur exact compatible au produit tensoriel de $\mathscr{M}(k)$ dans les $\mathbb{Q}$-espaces vectoriels: un *foncteur fibre* ω. A ω correspond $G := \underline{\mathrm{Aut}}^{\otimes}(\omega)$, le schéma en groupes des $\otimes$-automorphismes de ω. On peut voir ω, ou plutôt $\omega \circ H_{\mathrm{mot}}$, comme une théorie de cohomologie à valeurs dans les $\mathbb{Q}$-espaces vectoriels.

Pour k muni d'un plongement dans $\mathbb{C}$, une théorie de cohomologie possible est: "cohomologie singulière de l'espace topologique des points complexes". Pour $k = \mathbb{Q}$, un autre choix possible est: "cohomologie de de Rham".

En caractéristique p, un exemple de Serre montre qu'il ne peut pas y avoir de théorie de cohomologie raisonnable à valeurs dans les $\mathbb{Q}$-espaces vectoriels: si E est une courbe elliptique supersingulière, l'algèbre $\mathrm{End}(E)\otimes\mathbb{Q}$ est une algèbre de quaternions, donc n'a pas de représentation linéaire de dimension deux sur $\mathbb{Q}$. De l'existence d'une théorie de cohomologie à valeurs dans les espaces vectoriels sur une extension de $\mathbb{Q}$ résulte que $\mathscr{M}(k)$ doit toutefois être la catégorie des représentations d'une gerbe convenable.

La théorie tannakienne est un analogue *linéaire* d'une théorie *ensembliste*, celle du π_1-profini, également due à Grothendieck (SGA1). Analogie: si $\mathscr{R}$ est la catégorie des revêtements étales finis d'un schéma connexe S (resp. une catégorie tannakienne sur K) et ω un foncteur fibre, à valeurs dans les ensembles finis (resp. dans les K-espaces vectoriels de dimension finie), $\mathscr{R}$ est naturellement équivalente à la catégorie des G-ensembles finis (resp. des représentations de G), pour G le groupe profini $\mathrm{Aut}(\omega)$ (resp. le schéma en groupes affine $\underline{\mathrm{Aut}}^{\otimes}(\omega)$). De là la terminologie "groupe de Galois motivique", la théorie du π_1 devenant dans le cas des spectres de corps celle de Galois.

1.3. En général, le groupe de Galois motivique G est énorme. Sauf pour k algébrique sur un corps fini, on s'attend par exemple à ce qu'il admette comme quotient $\mathrm{PGL}(2)^{\mathrm{card}(k)}$. Il ne "sait pas" que des motifs M_t peuvent former une famille dépendant algébriquement d'un paramètre t.

Bien que G soit énorme, son existence facilite le maniement d'objets "motiviques".

EXEMPLE. La catégorie $\mathrm{ind}\mathscr{M}(k)$ des ind-objets de la catégorie $\mathscr{M}(k)$ des motifs sur k hérite du produit tensoriel de $\mathscr{M}(k)$. Ceci permet de définir une algèbre de Hopf commutative dans $\mathrm{ind}\mathscr{M}(k)$ comme étant un objet H muni d'un produit et coproduit: $H\otimes H \to H$ et $H \to H\otimes H$, ainsi que d'unité, counité et antipode: $1 \to H$, $H \to 1$, $H \to H$ vérifiant des axiomes convenables. On définit la catégorie des *schémas en groupes affines motiviques* comme étant l'opposée de la catégorie des algèbres de Hopf commutatives de $\mathrm{ind}\mathscr{M}(k)$. Si ω est un foncteur fibre, à valeurs dans les espaces vectoriels sur $\mathbb{Q}$, de groupe de Galois motivique $G := \underline{\mathrm{Aut}}^{\otimes}(\omega)$, le foncteur ω induit une équivalence de $\mathrm{ind}\mathscr{M}(k)$ avec la catégorie des représentations linéaires—non nécessairement de dimension finie—de G, et $H \mapsto \mathrm{Spec}\,\omega(H)$ est une équivalence de la catégorie des schémas en groupes affines motiviques avec celle des schémas en groupes affines sur $\mathbb{Q}$, munis d'une action de G.

Voici trois exemples de tels objets.

1.3.1. La version motivique de $\pi_1(X, x)$ rendu unipotent, cf. Deligne **[14]**, spécialement les §§5,7, et 10 à 13.

1.3.2. Le groupe de Galois motivique: il admet une version motivique G_{mot} telle que pour tout foncteur fibre ω, le groupe de Galois motivique correspondant soit déduit de G_{mot} par application de ω. Il s'agit d'une construction valable dans toute catégorie tannakienne, cf. Deligne [**15**, §8]. Dans l'analogie avec la théorie du π_1 profini, son analogue est le suivant: pour x variable, les $\pi_1(X, x)$ forment un système local sur X, ou plus précisément un objet en groupes dans la catégorie des pro-objets de celle des revêtements étales de X. Le π_1 en x s'obtient en en prenant la fibre en x.

1.3.3. Soit $\mathscr{M}$ une sous-catégorie pleine de celle des motifs mixtes sur k, stable par $\otimes$, dual et sous-quotients, et soit $\mathscr{M}_{\mathrm{pur}}$ la sous-catégorie pleine des sommes directes de motifs purs dans $\mathscr{M}$. On dispose du foncteur exact et compatible au produit tensoriel: $M \mapsto \mathrm{Gr}^W(M)$, de $\mathscr{M}$ dans $\mathscr{M}_{\mathrm{pur}}$. Il existe dans $\mathscr{M}_{\mathrm{pur}}$ un schéma en groupes pro-unipotent motivique U, agissant fonctoriellement sur $\mathrm{Gr}^W(M)$ pour M dans $\mathscr{M}$, et tel que $M \mapsto \mathrm{Gr}^W(M)$ soit une équivalence de $\mathscr{M}$ avec la catégorie des représentations (dans $\mathscr{M}_{\mathrm{pur}}$) de U. C'est une conséquence de Deligne [**15**, 8.17].

On peut de même définir et manier schémas, torseurs,... motiviques.

1.4. Le groupe de Galois motivique rendu abélien G^{ab} a une taille plus raisonnable. Il est indépendant du foncteur réalisation choisi (et de son existence). Conjecturalement, si $k_1 \subset k_2$ sont deux corps algébriquement clos, G^{ab} est le même pour k_1 et k_2 et la sous-catégorie correspondante de motifs est engendrée par les H^1_{mot} de variétés abéliennes de type CM.

En caractéristique 0, si on prend les motifs au sens des cycles de Hodge absolus et qu'on se limite à la sous-catégorie engendrée par les H^1 de variétés abéliennes, on sait calculer G^{ab}. Pour $\mathscr{M}(\mathbb{Q})$, on sait même calculer le quotient de G par le groupe dérivé de $\mathrm{Ker}(G \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$. Voir Deligne [**13**] ou l'exposé de Schappacher à cette conférence.

Pour $\mathscr{M}(\mathbb{F}_q)$, tout motif devrait être somme directe de motifs purs et le groupe de Galois motivique G devrait contenir un élément F, le Frobenius, dont les puissances soient denses dans G. En particulier, G devrait être commutatif. Le calcul conjectural de $\mathscr{M}(\mathbb{F}_q)$ a été fait par Grothendieck (non publié). Voir Langlands–Rapoport [**23**] et l'exposé de Milne à cette conférence.

1.5. Une source de 1.1.(a), (b) est l'exemple suivant. Une courbe projective et lisse X définit sa jacobienne $J(X) = \mathrm{Pic}^0(X)$. Les variétés abéliennes à isogénie près sur k forment une catégorie abélienne semi-simple avec des $\mathrm{Hom}(A, B)$ de dimension finie sur $\mathbb{Q}$ et pour chaque théorie de cohomologie usuelle H, le dual $H_1(X)$ de $H^1(X)$, se déduit de $J(X)$ par application d'un foncteur "réalisation". Noter que $H_1(X)$ est aussi le twist à la Tate $H^1(X)(1)$ de $H^1(X)$. Pour la cohomologie ℓ-adique, $\mathrm{real}_\ell(A)$ est le module de Tate $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}_\ell$. Pour k de caractéristique 0 et la théorie de de Rham, $\mathrm{real}_{\mathrm{DR}}(A)$ est l'algèbre de Lie de l'extension additive universelle de A.

On veut donc pouvoir identifier les variétés abéliennes à des motifs de poids -1 particuliers. Pour certaines applications (par exemple l'étude des variétés de Shimura), on serait déjà content de disposer d'une catégorie de motifs contenant les variétés abéliennes et stable par $\otimes$. En caractéristique 0, la théorie des cycles de Hodge absolus fournit une telle théorie (Deligne–Brylinski **[11]**, Deligne–Milne **[12]**, ou l'exposé de Panchishkin à cette conférence). Si on ne se limite pas aux motifs pur, un autre exemple clé est celui des courbes lisses non nécessairement complètes. Plus généralement, on peut considérer une courbe connexe projective et lisse $\overline{X}$, S et T disjoints dans $\overline{X}$ et la cohomologie $H^1(\overline{X}-S, \mathrm{rel}\, T)$. Un *1-motif* $K^\bullet$ est un complexe de schémas en groupes réduit aux degrés -1 et 0, avec, sur la clôture algébrique, K^{-1} un $\mathbb{Z}$-module libre de type fini et K^0 une extension d'une variété abélienne par un tore. Supposons pour simplifier k algébriquement clos. Pour chaque théorie de cohomologie usuelle, le H^1 considéré, ou plutôt son twist à la Tate $H^1(\overline{X}-S, \mathrm{rel}\, T)(1)$, se déduit alors du 1-motif suivant par application d'un foncteur réalisation.

Soit $J_T(\overline{X})$ la jacobienne généralisée classifiant les faisceaux inversibles de degré 0 sur $\overline{X}$ trivialisés sur T. C'est une extension de la variété abélienne $\mathrm{Pic}^0(\overline{X})$ par le tore de groupe de caractères $\mathrm{Ker}(\mathbb{Z}^T \xrightarrow{\Sigma} \mathbb{Z})$. Chaque $s \in S$ définit le faisceau inversible $\mathscr{O}(s)$, trivialisé sur T et de degré 1, d'où

$$\mathrm{Ker}(\mathbb{Z}^S \to \mathbb{Z}) \to J_T(\overline{X}). \tag{1.5.1}$$

C'est le 1-motif promis. Voir Deligne **[9,** §10].

REMARQUE. Pour $S, T \neq \varnothing$, le 1-motif (1.5.1) détermine aussi la catégorie des faisceaux inversibles sur $\overline{X}-S$, trivialisés sur T: un point $x \in J_T(\overline{X})$ définit un tel faisceau inversible $\mathscr{L}_x$, et un isomorphisme de $\mathscr{L}_x$ avec $\mathscr{L}_y$ qui soit l'identité sur T s'identifie à $k \in \mathrm{Ker}(\mathbb{Z}^S \to \mathbb{Z})$, avec $y-x=\delta k$.

A nouveau, on veut pouvoir identifier les 1-motifs à des motifs (mixtes) particuliers. Pour certaines applications (l'étude à l'infini des variétés de Shimura) on serait déjà content de disposer d'une catégorie de motifs contenant les 1-motifs et stable par $\otimes$. En caractéristique 0, on sait le faire (Brylinski [7]).

1.6. Les variétés abéliennes ont des espaces de modules et ceux-ci permettent d'interpréter algébriquement certains quotients d'espaces hermitiens symétriques par des groupes arithmétiques. Le cas des motifs est plus compliqué. Si S est un schéma sur $\mathbb{C}$ et M une famille de motifs purs paramétrée par S, M fournit en réalisation de Hodge une variation de structures de Hodge M_h sur $S(\mathbb{C})$, polarisée si M l'est. De là, une application φ de $S(\mathbb{C})$ dans l'espace $\mathscr{C}$ classifiant les structures de Hodge polarisées de nombres de Hodge ceux de M.

La variation M_h fournit en particulier un système local d'espaces vectoriels complexes $M_{\mathbb{C}}$ muni d'une filtration de Hodge F variant continûment.

L'holomorphie de F et la condition de "transversalité" découvertes par Griffiths **[17]** (voir aussi Griffiths [**18**]) peuvent s'exprimer comme suit. Soit T le fibré tangent de S, S étant vue comme une variété C^∞. La structure complexe de S fournit une structure complexe sur T. Soit $\varphi \colon T \otimes_{\mathbb{R}} \mathbb{C} \to T$ le prolongement $\mathbb{C}$-linéaire de l'identité de T et soit $T'' \subset T \otimes_{\mathbb{R}} \mathbb{C}$ le noyau de φ. Définissons le filtration de Hodge de $T \otimes \mathbb{C}$ par

$$F^{-1} = T \otimes \mathbb{C}, \qquad F^0 = T'', \qquad F^1 = 0.$$

Pour t et m des sections C^∞ de $T \otimes \mathbb{C}$ et de $M_{\mathbb{C}}$, la structure plate de $M_{\mathbb{C}}$ permet de définir $\nabla_t m$. La condition est que $t, m \mapsto \nabla_t m$ est compatible aux filtrations de Hodge:

t dans T'', m dans F^i, $\nabla_t m$ dans F^i: holomorphie;
t dans T, m dans F^i, $\nabla_t m$ dans F^{i-1}: transversalité.

Sur l'espace classifiant $\mathscr{C}$ il existe une structure complexe et une distribution holomorphe $\tau \subset T$ telles que holomorphie et transversalité se traduisent comme holomorphie de l'application classifiante $\varphi \colon S(\mathbb{C}) \to \mathscr{C}$ et tangence de $\varphi(S)$ à τ.

La distribution τ est en général non intégrable. L'ensemble des points de $\mathscr{C}$ qui correspondent à un facteur direct d'un $H^i(X)$, X projectif non singulier, est la réunion dénombrable de sous-variétés tangentes à τ. Pour τ différent du fibré tangent tout entier, on n'en a aucune description, même conjecturale.

Un argument analogue, également dû à Griffiths, explique pourquoi les points de la jacobienne intermédiaire $J^d(X)$ ne peuvent pas tous être classes de cycles algébriques, sauf lorsque $H^{2d-1}(X)$ est de type $\{(d-1, d), (d, d-1)\}$. Soit $f \colon X \to S$ une famille de variétés projectives non singulières. On pose $X_s = f^{-1}(s)$. Les jacobiennes intermédiaires $J^d(X_s) = H^{2d-1}(X_s, \mathbb{C})/H^{2d-1}(X_s, \mathbb{Z}) + F^d$ forment un fibré holomorphe J^d en tores complexes et l'axiome de transversalité assure que la connexion de Gauss–Manin (exprimant que les $H^{2d-1}(X_s, \mathbb{C})$ forment un système local) passe au quotient pour définir un opérateur différentiel D, défini sur le faisceau de sections de J^d et à valeur dans celui des 1-formes à valeur dans le fibré holomorphe des $H^{2d-1}(X_s, \mathbb{C})/F^{d-1}$. Une famille algébrique $(Z_s)_{s \in S}$ de cycles algébriques cohomologues à zéro sur les X_s définit une section holomorphe z de J^d. Le résultat de Griffiths est que $Dz = 0$. La section z définit une famille holomorphe de structures de Hodge mixtes extension de $\mathbb{Z}$ (type $(0, 0)$) par $H^{2d-1}(X)(d)$, et $Dz = 0$ équivaut à ce que cette famille vérifie l'axiome de transversalité.

1.7. Quand la distribution τ est le fibré tangent tout entier, $\mathscr{C}$ est un quotient arithmétique d'un domaine hermitien symétrique. Cette description des points complexes des variétés de Shimura comme modules de structures de Hodge suggère que ces variétés sont des espaces de modules de motifs.

Soit S le $\mathbb{R}$-groupe algébrique $\mathbb{C}^*$: $S = R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. Se donner une action de S sur l'espace vectoriel réel V revient à se donner une décomposition de $V \otimes \mathbb{C}$ en somme directe de $V^{p,q}$ avec $\overline{V^{p,q}} = V^{q,p}$, $z \in S(\mathbb{R}) = \mathbb{C}^*$ agissant sur $V^{p,q}$ par multiplication par $z^{-p}\overline{z}^{-q}$. On définit $w: \mathbb{G}_m \to S$ (resp. $\mu: \mathbb{G}_m \to S$, défini sur $\mathbb{C}$) par la condition que pour tout V, $w(\lambda)$ (resp. $\mu(\lambda)$) agisse sur $V^{p,q}$ par multiplication par λ^{p+q} (resp. λ^p).

Les données définissant une variété de Shimura $\mathrm{Sh}_K(G, X)$ sont un groupe réductif G sur $\mathbb{Q}$, une $G(\mathbb{R})$-classe de conjugaison de morphismes $h: S \to G_{\mathbb{R}}$, avec $w_X := h \circ w$ central (donc indépendant de h) et un sous-groupe compact ouvert K de $G(\mathbb{A}^f)$. Plaçons-nous dans le cas où w_X est défini sur $\mathbb{Q}$ et où $\operatorname{int} h(i)$ est une involution de Cartan du quotient $G/w_X(\mathbb{G}_m)$. Le *corps dual* $E(G, X)$ est le corps de définition de la classe de conjugaison de $h\mu$ (h quelconque dans X). C'est un sous-corps de $\mathbb{C}$. La variété de Shimura est définie sur $E(G, X)$, avec pour points complexes $K\backslash X \times G(\mathbb{A}^f)/G(\mathbb{Q})$. Sa construction dans le cas général est due à Borovoï **[6]**.

Un point de la variété de Shimura sur un corps F contenant $E(G, X)$ devrait correspondre à

(a) un $\otimes$-foncteur exact x de la catégorie $\mathrm{Rep}(G)$ des représentations de G dans celle des motifs purs sur F;

(b) une structure entière. En terme de la réalisation adélique finie, produit restreint des réalisations ℓ-adiques, on peut la décrire comme un isomorphisme de $\otimes$-foncteurs

$$x(V)_{\mathbb{A}^f} \xrightarrow{\sim} V \otimes \mathbb{A}^f,$$

donné à composition près avec un élément de K.

La condition suivante doit être vérifiée. On y suppose pour simplifier F plongeable dans $\mathbb{C}$.

(c) Soit ι un plongement de F dans $\mathbb{C}$, prolongeant le plongement identique de $E(G, X)$ dans $\mathbb{C}$. Il doit exister $h \in X$ tel que les $\otimes$-foncteurs suivants de $\mathrm{Rep}(G)$ dans les structures de Hodge soient isomorphes:

(1) $(V, \rho) \mapsto V$ muni de la structure de Hodge définie par $\rho \circ h$,

(2) $(V, \rho) \mapsto$ réalisation de Hodge de $x(V)$, après extension de corps de base à $\mathbb{C}$ par ι.

Que cette condition (c) soit indépendante du plongement complexe ι choisi—supposé prolonger l'inclusion de $E(G, X)$ dans $\mathbb{C}$—n'est pas évident. Parfois elle résulte toutefois de (b) et des conditions (d), (e) plus algébriques, conséquence de (c), qui suivent.

(d) La réalisation de de Rham définit un foncteur fibre sur $\mathrm{Rep}(G)$: $V \mapsto x(V)_{\mathrm{DR}}$, correspondant à un G-torseur P sur F. La filtration de Hodge des $X(V)_{\mathrm{DR}}$ est exacte et compatible au produit tensoriel, donc provient d'un parabolique Q de G^P et de $\mu_{\mathrm{DR}}: \mathbb{G}_m \to$ centre de $Q/\mathscr{R}_u Q$, se relevant en une classe de conjugaison de morphismes de $\mathbb{G}_m$ dans Q, Saavedra [**28,** IV, 2.4, p. 229]. Parce que F contient $E(G, X)$, il a un sens de demander que la classe de conjugaison correspondante d'applications de $\mathbb{G}_m$ dans G coincide avec celle des $h \circ \mu$ $(h \in X)$. La condition (c) l'implique. Ceci explique l'apparition du corps dual.

(e) Pour (V, ρ) une représentation de poids 0: $\rho \circ w$ trivial et $V \otimes V \to \mathbb{Q}$ une forme bilinéaire invariante symétrique telle que sur $V_{\mathbf{R}}$, $B(v, h(i)w)$ soit symétrique défini positif, on demande que $x(V) \otimes x(V) \to 1$ soit positive, pour la polarisation (loc. cit. V, 2.4, p. 276) espérée de la catégorie des motifs.

1.8. Cette interprétation motivique des variétés de Shimura a été un guide pour l'élaboration des axiomes qui les caractérisent, ainsi que pour la détermination de leur conjuguées (Borovoï, [**6**]).

Soit p un nombre premier tel que $G_{\mathbb{Q}_p}$ se prolonge en un groupe réductif sur $\mathbb{Z}_p$, avec K produit de $G(\mathbb{Z}_p)$ par un sous-groupe du produit restreint des $G(\mathbb{Q}_\ell)$, $\ell \neq p$. Le corps dual $E(G, X)$ est alors non ramifié en p. On espère que la variété de Shimura a une réduction mod p naturelle, et on peut essayer de paraphraser la description conjecturale motivique qui précède pour conjecturer la structure de l'ensemble de ses points sur un corps fini de caractéristique p. Il y a des difficultés: comment interpréter (c) et la p-partie de (b), à traiter conjointement avec (d)? Voir dans cette direction, Langlands–Rapoport [**23**].

2. Théories de cohomologie

2.1. Principle. Une construction géométrique, possible dans l'une des théories de cohomologie usuelles, doit avoir un sens motivique et donc avoir un analogue dans les autres théories usuelles.

Ce principe a été crucial pour développer la théorie de Hodge mixte. Grothendieck a vu que chaque $H^i(X)$ doit se dévisser en terme de sous-quotients de cohomologie de variétés projectives non singulières. Le motif mixte $H^i_{\mathrm{mot}}(X)$ doit donc avoir une classe dans le groupe de Grothendieck des motifs purs. En caractéristique 0, le nombre de Hodge h^{pq} de la réalisation de Hodge d'un motif pur défini un homomorphisme de ce groupe de Grothendieck dans $\mathbb{Z}$. Des nombres de Hodge $h^{pq}(H^i_{\mathrm{mot}}(X))$ doivent donc avoir un sens.

Pour aller plus loin, il fallait se convaincre que tout motif a une filtration par le poids W, croissante, avec $\mathrm{Gr}^W_i(M)$ pur de poids i (= facteur direct de $H^i_{\mathrm{mot}}(X)$ pour X projectif non singulier). C'est sur le H^1 des courbes, i.e., sur les 1-motifs, que je m'en suis convaincu, et le premier test qu'a dû passer la définition des structures de Hodge mixtes est qu'elles redonnent comme cas particulier les 1-motifs sur $\mathbb{C}$ (Deligne [**9,** §10]).

Le même principe de transfert a permis de conjecturer le comportement asymptotique d'une variation de structures de Hodge sur un disque épointé, ou un produit de disques épointés.

Mise en garde. Soit (S, η, s) un trait, avec S spectre d'un anneau de valuation discrète complet. D'après Raynaud, une variété abélienne sur le point générique η, à réduction semi-stable, admet une description rigide analytique comme conoyau d'une flèche définissant un 1-motif. Parallèlement, en théorie de Hodge, si $\mathscr{H}$ est une variation de structures de Hodge polarisée de type $\{(1, 0), (0, 1)\}$ sur un disque épointé D^*, la monodromie fournit sur $\mathscr{H}_Q$ une filtration par le poids W qui, jointe à la filtration de Hodge originelle, fait de $\mathscr{H}$ une variation de structures de Hodge mixtes au voisinage épointé de 0.

Il ne faut pas s'attendre à un comportement analogue pour les motifs, lorsque la condition de transversalité est non triviale en théorie de Hodge. En général, un objet décrivant le comportement asymptotique doit exister seulement sur l'espace tangent de Zariski épointé (on suppose réduction semi-stable), et il ne doit pas suffire à reconstruire l'objet de départ. Par exemple, pour $\mathscr{H}$ une variation de structures de Hodge polarisées sur $D^* = D - \{0\}$, l'orbite nilpotente asymptotique de W. Schmid est une variation de structures de Hodge mixtes sur l'espace tangent épointé en 0 et, pour $\mathscr{H}$ se prolongeant sur D, c'est la variation constante de valeur la fibre en 0 du prolongement.

La théorie de Morihiko Saito (Saito **[29]**) qui fournit les six opérations (et les cycles évanescents) en théorie de Hodge mixte est en partie inspirée par la théorie ℓ-adique et le point de vue motivique. Inversement, elle suggère que si on veut considérer des motifs sur une base S, plutôt que de vouloir disposer de motifs M de réalisations ℓ-adiques des faisceaux ℓ-adiques sur S, il peut être préférable de disposer de motifs de réalisations des faisceaux pervers. En tout cas, ce n'est que dans un tel cadre qu'on peut espérer une filtration par le poids.

C'est encore la philosophie des motifs qui a amené Grothendieck à conjecturer l'existence du foncteur "mystérieux" reliant la cohomologie étale p-adique d'une variété définie sur $\mathbb{Q}_p$, supposée à bonne réduction, et sa cohomologie de de Rham: existant pour le H^1, i.e., pour les motifs que sont les variétés abéliennes, il devait exister toujours, de façon compatible au produit tensoriel.

2.2. Soit X_0 une variété algébrique sur k et X déduit de X_0 par extension des scalaires à une clôture algébrique $\overline{k}$ de k. Pour $x \in X_0(k)$, le groupe fondamental profini $\pi_1(X, x)$ est muni d'une action de $\mathrm{Gal}(\overline{k}/k)$. Supposons pour simplifier X normal. Le foncteur "fibre en x" est alors une équivalence de catégories: ($\mathbb{Q}_\ell$-faisceaux lisses sur X_0) $\to$ (représentations ℓ-adiques de $\mathrm{Gal}(\overline{k}/k)$, munies d'une action équivariante de $\pi_1(X, x)$).

Je ne connais pas de sens en lequel $\pi_1(X, x)$ soit motivique. La situation change pour son pro-ℓ-complété $\pi_1(X, x)^{\wedge}_{\ell}$. Ce groupe rendu abélien est $H_1(X, \mathbb{Z}_\ell)$, et son H_2 est contenu dans celui de X. Il devrait exister un schéma en groupes motiviques $\pi_1(X, x)_{\mathrm{un}}$ sur k tel que

(a) Pour chaque théorie de cohomologie usuelle H, la réalisation correspondante de $\pi_1(X, x)_{\mathrm{un}}$ sur $\overline{k}$ a pour représentations linéaires les "systèmes locaux" (au sens de H) sur X, qui sont unipotents.
(b) Le foncteur "fibre en x" est une équivalence de catégories: (motifs lisses sur X_0, extensions itérées d'images universes de motifs sur k) $\to$ (motifs sur k, munis d'une action de $\pi_1(X, x)_{\mathrm{un}}$).

Dans (a), les systèmes locaux considérés ne sont pas supposés motiviques. D'après (b), ceux qui le sont forment toutefois un système fidèle de représentations.

Cette philosophie a inspiré la théorie de Hodge du π_1: voir Steenbrink–Zucker **[30]**.

Plus généralement, supposons donné un ensemble A de motifs purs sur X_0. Ce qui précède correspond à $A = \varnothing$ ou, ce qui revient au même, A réduit au motif unité. Il devrait exister sur k un schéma en groupes motivique $\pi_1(X, x)_A$, le π_1 motivique relatif à A, tel que

(a′) Sur $\overline{k}$, sa réalisation relative à la théorie H a pour représentations les "systèmes locaux" sur X qui sont extensions itérées de sous-quotients de produits tensoriels de réalisations d'objets de A et de leurs duaux.
(b′) Le foncteur "fibre en x" est une équivalence: (motifs lisses sur X_0, extensions itérées de sous-quotients (motiviques) de produits tensoriels d'objets de A, de leurs duaux et d'images inverses de motifs sur k) $\to$ (motifs sur k, munis d'une action de $\pi_1(X, x)_A$).

Dans (a′), les sous-quotients sont à prendre sur X et ne sont pas supposés motiviques.

En théorie de Hodge, ceci suggère la question suivante. Soit X lisse et connexe sur $\mathbb{C}$ et $x \in X$. Soit A un ensemble de variations de structures de Hodge polarisables sur X. Soit $\mathscr{A}$ la catégorie tannakienne de $\mathbb{Q}$-systèmes locaux sur X engendrée (par $\otimes$, dual, sous-quotients et extensions), par les systèmes locaux sous-jacents aux objets de A. Soit $\mathscr{A}_H$ la catégorie tannakienne des variations de structures de Hodge mixtes sur X, admissibles au sens de Steenbrink–Zucker **[30]**, engendrée (par $\otimes$, dual, sous-quotients et extensions) par A et les variations constantes de gradué par le poids polarisable.

Sur $\mathscr{A}$, on dispose du foncteur fibre "fibre en x" ω_x. Soit $G = \mathrm{Aut}^{\otimes}(\omega_x)$. C'est l'adhérence de Zariski de $\pi_1(X, x)$ dans le produit des $\mathrm{GL}(H_x)$, H objet de $\mathscr{A}$. Soit $\mathscr{H}(G)$ l'algèbre de Hopf dont G est le spectre.

Pour M dans $\mathscr{A}_H$, le système local sous-jacent $M_{\mathbb{Q}}$ est dans $\mathscr{A}$, de sorte que G agit sur sa fibre $(M_Q)_x$ en x: cette fibre est un $\mathscr{H}(G)$-comodule:

(2.2.21) $$(M_Q)_x \to (M_Q)_x \otimes \mathscr{H}(G).$$

L'algèbre de Hopf $\mathscr{H}(G)$ devrait admettre une structure de ind-structure de Hodge mixte, i.e., G devrait être un schéma en groupes affine dans la catégorie tannakienne des structures de Hodge mixtes, de sorte que (a) les flèches (2.2.1) sont des morphismes de Hodge mixte, (b) le foncteur $M \mapsto M_x$ est une équivalence de $\mathscr{A}$ avec la catégorie des structures de Hodge mixtes H à gradué polarisable, munies d'une structure de comodule $H \to H \otimes \mathscr{H}(G)$ qui soit un morphisme de Hodge mixte.

3. Cohomologie absolue

3.1. Dans chacune des théories de cohomologie usuelles, les foncteurs $H^i(X)$ proviennent d'un foncteur $R\Gamma(X)$ à valeurs dans une catégorie triangulée $\mathscr{D}$. Celle-ci est munie d'une t-structure (la donnée de sous-catégories $\mathscr{D}^{\leq 0}$ et $\mathscr{D}^{\geq 0}$, vérifiant des axiomes convenables: Beilinson–Bernstein–Deligne **[3**, 1.3]), de coeur (la sous-catégorie abélienne intersection de $\mathscr{D}^{\leq 0}$ et $\mathscr{D}^{\geq 0}$) la catégorie dans laquelle H^* prend ses valeurs, et

$$H^i(X) = H^i R\Gamma(X)$$

au sens de cette t-structure.

On peut espérer que ces foncteurs $R\Gamma$ soient les "réalisations" d'un foncteur $R\Gamma$ motivique: on espère l'existence d'une catégorie triangulée $\mathscr{D}(k)$, avec t-structure, de coeur la catégorie $\mathscr{M}(k)$ des motifs mixtes sur k, et celle d'un foncteur $R\Gamma_{\mathrm{mot}}$ à valeurs dans $\mathscr{D}(k)$, dont les $R\Gamma$ dans les diverses théories se déduisent par application de foncteurs réalisations.

Une catégorie triangulée $\mathscr{D}$ avec t-structure n'est pas toujours la catégorie dérivée de son coeur $\mathscr{C}$. Exemple: soient X un CW-complexe fini connexe et $\mathscr{D} \subset D^b(X, \mathbb{Q})$ la sous-catégorie de la catégorie dérivée formée des complexes à faisceaux de cohomologie localement constants. Son coeur est la catégorie abélienne des faisceaux localement constants de $\mathbb{Q}$-espaces vectoriels sur X. On dispose d'un foncteur naturel $D^b(\mathscr{C}) \to \mathscr{D}$, mais ce foncteur n'est une équivalence que si X est un $K(\pi, 1)$.

Dans ce paragraphe, nous supposerons l'existence de $\mathscr{D}(k)$ et celle de foncteurs "réalisations". Nous supposerons aussi l'existence dans $\mathscr{D}(k)$ d'un "twist à la Tate" qui soit une auto-équivalence de catégorie, et corresponde par les foncteurs réalisations au twist à la Tate usuel. Nous supposerons enfin que la catégorie $\mathscr{M}(k)$ est tannakienne et que la cohomologie ℓ-adique est un foncteur fibre. Ceci implique que si $\varphi: M_1 \to M_2$ est un morphisme de motifs dont la réalisation ℓ-adique est un isomorphisme, alors φ est un isomorphisme.

Pour M_1, M_2 dans $\mathscr{M}(k)$, posons

$$\mathrm{Ext}^i(M_1, M_2) = \mathrm{Hom}_{\mathscr{D}(k)}(M_1, M_2[i]).$$

Ce sont les Hom et Ext^1 usuels pour $i = 0, 1$ mais pour $i > 1$, ils pourraient différer des Ext de Yoneda, qui sont les $\mathrm{Hom}(M_1, M_2[i])$ dans $D^b(\mathcal{M}(k))$.

Pour chacune des théories de cohomologie usuelles, ces Ext^i motiviques doivent s'envoyer dans les Ext^i entre les réalisations correspondantes de M_1 et M_2.

3.2. Pour X une variété algébrique sur k, appelons groupes de cohomologie absolue de X les

$$H^i_{\mathrm{abs}}(X) := \mathrm{Hom}_{\mathcal{D}(k)}(1, R\Gamma_{\mathrm{mot}}(X)[i]),$$

où 1 est le motif unité. Nous aurons aussi à considérer les

$$H^i_{\mathrm{abs}}(X, \mathbb{Q}(j)) := \mathrm{Hom}_{\mathcal{D}(k)}(1, R\Gamma_{\mathrm{mot}}(X)(j)[i]).$$

Ces groupes sont souvent appelés groupes de cohomologie motiviques. Ce sont des espaces vectoriels sur $\mathbb{Q}$. On veillera à ne pas les confondre avec les $H^i_{\mathrm{mot}}(X)$, qui sont des motifs sur k. Les deux sont liés par une suite spectrale

$$E_2^{pq} = \mathrm{Ext}^p(1, H^q_{\mathrm{mot}}(X)) \Rightarrow H^{p+q}_{\mathrm{abs}}(X). \tag{3.2.1}$$

Pour M un motif, on écrira encore $H^p_{\mathrm{abs}}(M)$ pour $\mathrm{Ext}^p(1, M) = \mathrm{Hom}_{\mathcal{D}(k)}(1, M[p])$. Avec cette notation, la suite spectrale (3.2.1) se récrit

$$E_2^{pq} = H^p_{\mathrm{abs}}(H^q_{\mathrm{mot}}(X)) \Rightarrow H^{p+q}_{\mathrm{abs}}(X). \tag{3.2.2}$$

L'analogue ℓ-adique est le suivant: une variété X sur k, de clôture algébrique $\overline{k}$, définit des groupes de cohomologie ℓ-adique $H^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$, qu'on espère réalisations de motifs $H^i_{\mathrm{mot}}(X)$. La variété X a aussi une cohomologie étale absolue $H^i(X, \mathbb{Q}_\ell)$, qui est l'aboutissement d'une suite spectrale

$$H^p(\mathrm{Gal}(\overline{k}/k), H^q(X \otimes_k \overline{k}, \mathbb{Q}_\ell)) \Rightarrow H^i(X, \mathbb{Q}_\ell).$$

Pour X projectif et lisse sur k, on espère que $R\Gamma_{\mathrm{mot}}(X)$ est (non canoniquement) isomorphe à la somme directe dans $\mathcal{D}(X)$ des $H^i_{\mathrm{mot}}(X)[-i]$. Cela résulterait de l'hypothèse suivante:

(∗) Pour $\mathcal{L}$ un faisceau inversible sur X, on dispose de $c_1(\mathcal{L})$: $R\Gamma_{\mathrm{mot}}(X) \to R\Gamma_{\mathrm{mot}}(X)(1)[2]$ qui, en réalisation ℓ-adique, fournit le cup-produit avec la classe de Chern $c_1(\mathcal{L})$ de $\mathcal{L}$: $H^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) \to H^{i+2}(X \otimes_k \overline{k}, \mathbb{Q}_\ell(1))$.

Prenons en effet $\mathcal{L}$ ample. Le foncteur réalisation ℓ-adique étant un foncteur fibre, il résulte de (∗) et du théorème de Lefschetz difficile que pour X purement de dimension N, les

$$c_1(\mathcal{L})^i : H^{N-i}_{\mathrm{mot}}(X) \to H^{N+i}_{\mathrm{mot}}(X)(i)$$

sont des isomorphismes. On peut alors appliquer Deligne **[8]**.

Analysant la démonstration, on peut en déduire, pour chaque faisceau inversible ample $\mathscr{L}$, une décomposition canonique (et même plusieurs)

$$\varphi_{\mathscr{L}}: R\Gamma_{\text{mot}}(X) \leftrightarrow \bigoplus H^i_{\text{mot}}(X)[-i].$$

Voir Deligne, "Décompositions dans la catégorie dérivée", dans ce volume.

Mise en garde. Dépendant du théorème de Lefschetz difficile, ces arguments sont essentiellement rationnels. Si on disposait d'une variante entière de $\mathscr{D}(X)$, on devrait s'attendre à ce que dans ce cadre $R\Gamma(X)$ ne soit pas toujours somme des $H^i(X)[-i]$.

3.3. Soit X lisse sur k. Les diverses "classes" d'un cycle algébrique Z de codimension d doivent provenir d'une classe motivique

(3.3.1) $$cl(Z) \in \text{Hom}(1, R\Gamma_{\text{mot}}(X)[2d](d)) =: H^{2d}_{\text{abs}}(X, \mathbb{Q}(d)).$$

La classe (3.3.1) ne doit dépendre que de la classe d'équivalence linéaire de Z.

Si X est projectif et lisse et qu'on a choisi une décomposition

$$R\Gamma_{\text{mot}}(X) \simeq \bigoplus H^i_{\text{mot}}(X)[-i]$$

(cf. 3.2), la classe (3.3.1) équivaut à une série de classes

(3.3.2) $$cl_n(Z) \in \text{Ext}^n(1, H^{2d-n}_{\text{mot}}(X)(d)) = H^n_{\text{abs}}(H^{2d-n}_{\text{mot}}(X)(d)).$$

La première de ces classes qui est non nulle ne dépend pas de la décomposition choisie.

Exemple 3.4. En théorie de Hodge, Z définit tout d'abord une classe de cohomologie entière de type (d, d), i.e., un morphisme de structures de Hodge

$$\mathbb{Z} \to H^{2d}(X)(d).$$

Si cette classe est nulle, Z a une classe dans la jacobienne intermédiaire

$$J_d(X) = H^{2d-1}(X, \mathbb{C})/F^d \oplus H^{2d-1}(X, \mathbb{Z}),$$

groupe qui n'est autre que

$$\text{Ext}^1(\mathbb{Z}, H^{2d-1}(X)(d))$$

dans la catégorie des structures de Hodge mixte. Là s'arrête l'histoire car la catégorie abélienne des $\mathbb{Q}$-structures de Hodge mixtes est de dimension cohomologique 1.

Exemple 3.5. Soient X projectif et lisse purement de dimension N sur k algébriquement clos, $\mathscr{O}(1)$ un faisceau inversible ample et montrons comment on peut attacher à un diviseur D une classe $cl(D)$ dans $\text{Pic}^0(X) \otimes \mathbb{Q}$, groupe que la théorie des motifs interprète comme étant $H^1_{\text{abs}}(X)(1))$.

D'après un théorème de A. Weil [**31,** Théorème 7], l'application des classes d'équivalence linéaire de diviseurs dans les classes d'équivalences linéaires de 0-cycles:

$$E \to E \cdot c_1(\mathcal{O}(1))^{N-1}$$

induit une isogénie

$$\varphi\colon \mathrm{Pic}^0(X) \to J(X),$$

donc une bijection de $\mathrm{Pic}^0(X) \otimes \mathbb{Q}$ avec $J(X) \otimes \mathbb{Q}$. On prend

$$cl_1(D) = \varphi^{-1} cl \left(D \cdot c_1(\mathcal{O}(1))^{N-1} - \frac{\deg(D \cdot c_1(\mathcal{O}(1))^{N-1})}{\deg(c_1(\mathcal{O}(1))^N)} \cdot c_1(\mathcal{O}(1))^N \right)$$

EXEMPLE 3.6. Plaçons-nous non plus sur le spectre d'un corps mais sur une base S, lisse sur un corps algébriquement clos k. Soit $f\colon X \to S$ projectif et lisse sur S et Z un cycle de codimension d. Travaillons en cohomologie ℓ-adique, ℓ inversible sur S. Le cycle Z a une classe dans

$$H^{2d}(X, \mathbb{Z}_\ell(d)) = \mathrm{Hom}_{D(S)}(\mathbb{Z}_\ell, Rf_*\mathbb{Z}_\ell[2d](d)).$$

Si on choisit une décomposition

$$Rf_*\mathbb{Q}_\ell \simeq \sum R^i f_* \mathbb{Q}_\ell[-i], \tag{3.6.1}$$

cette classe donne lieu comme une série de classes

$$cl_n(Z) \in \mathrm{Ext}^n(\mathbb{Z}, R^{2d-n} f_* \mathbb{Q}_\ell(d)) = H^n(S, R^{2d-n} f_* \mathbb{Q}_\ell(d)). \tag{3.6.2}$$

Lorsque X est de la forme $X_0 \times_{\mathrm{Spec}(k)} S$, et que la décomposition (3.6.1) provient de l'unique décomposition analogue de la cohomologie de X_0, ces classes s'identifient aux composantes de Künneth de la classe de cohomologie de Z.

3.7. Supposons X lisse sur un corps. Décomposons le groupe de K-théorie $K_n(X) \otimes \mathbb{Q}$ par les valeurs propres des opérations d'Adams:

$$K_n(X) \otimes \mathbb{Q} = \bigoplus K_n(X)^{(j)}$$

où sur $K_n(X)^{(j)}$, l'opération d'Adams Ψ_k agit par multiplication par k^j. Rappelons que $K_0(X)^{(j)}$ est le groupe de Chow des cycles de codimension j, modulo équivalence rationnelle, tensorisé avec $\mathbb{Q}$.

L'exemple des théories de cohomologie usuelles incite à espérer des classes de Chern

$$ch^j\colon K_n(X) \otimes \mathbb{Q} \to H^{2j-n}_{\mathrm{abs}}(X, \mathbb{Q}(j))$$

se factorisant par $K_n(X)^{(j)}$.

Une conjecture optimiste est que les morphismes "classe de Chern" soient des isomorphismes

$$ch^j\colon K_n(x)^{(j)} \xrightarrow{\sim} H^{2j-n}_{\mathrm{abs}}(X, \mathbb{Q}(j)) := \mathrm{Hom}(1, R\Gamma(X)(j)[2j-n]). \tag{3.7.1}$$

Cette conjecture sous-tend la terminologie "groupe de cohomologie motivique absolue" pour les $K_n(X)^{(j)}$, et la notation

$$H^{2j-n}_{\mathcal{M}}(X, \mathbb{Q}(j)) := K_n(X)^{(j)}.$$

Bloch [4] a proposé une interprétation des $K_n(X)^{(j)}$ comme "groupes de Chow supérieurs" $\mathrm{Ch}^j(X; n)$. Son argument à un trou (moving lemma insuffisant) qu'il promet de réparer. Cette interprétation rend plus naturelle la définition de ch^j, mais l'interprétation en terme de K-théorie reste nécessaire pour le calcul explicite dans le cas des corps de nombres. Les groupes de Chow supérieurs ont l'avantage d'avoir une structure entière naturelle.

3.8. D'après Beilinson, la conjecture (3.7.1) peut se déduire d'hypothèses dont les essentielles sont que (a) le foncteur $X \mapsto K.(X)$ se factorise par $R\Gamma_{\mathrm{mot}}$, (b) la catégorie $\mathcal{D}(k)$ est la catégorie dérivée de celle des motifs. La conjecture a des conséquences frappantes pour les deux membres de (3.7.1). Ils s'annulent pour des raisons triviales dans des régions différentes du plan (n, j):

(a) Pour M un motif, $\mathrm{Ext}^i(1, M) = 0$ pour $i < 0$, et $\mathrm{Ext}^0(1, M) = \mathrm{Hom}(1, M)$. De là, $\mathrm{Hom}(1, R\Gamma(X)(j)[N]) = 0$ pour $N < 0$ ou pour $N = 0$ et $j \neq 0$. En K-théorie cela donne la conjecture de Beilinson et Soulé que $K_n(X)^{(j)} = 0$ pour $2j < n$ ainsi que pour $2j \leq n$ si $j \neq 0$.

(b) Le groupe de Chow $\mathrm{Ch}^j(X)$ des cycles algébriques de codimension j est nul pour $j > \dim X$ et, de même, $\mathrm{Ch}^j(X; n)$ est nul pour $j > \dim(X) + n$. De plus, $\mathrm{Ch}^j(X; n) = 0$ pour $n < 0$.

Appliquant (3.7.1), on en déduirait les

NULLITÉS 3.8.1.
(i) *Pour M un motif pur de poids w,* $\mathrm{Hom}^i(1, M) = 0$ *pour* $i > -w$.
(ii) *Pour M effectif de poids w, et $b > 0$,* $\mathrm{Hom}^i(1, M(w+b)) = 0$ *pour* $i > w + b$.

Pour (i), on suppose M facteur direct de $H^a_{\mathrm{mot}}(X)(c)$, avec $c \geq 0$, X une variété projective et lisse. Le $\mathrm{Ext}^i(1, M)$ est alors facteur direct de $\mathrm{Ch}^c(X; 2c-i-a)$, et, s'il est non nul, on a $2c - i - a \geq 0$, i.e., $i \leq 2c - a = -w$.

Pour (ii), on utilise de plus le théorème de Lefschetz faible (ℓ-adique, donc motivique si la cohomologie ℓ-adique est un foncteur fibre) pour supposer que $\dim(X) \leq a$. On suppose $c \geq a$ et il faut prouver la nullité de $\mathrm{Ext}^i(1, H^a_{\mathrm{mot}}(X)(c))$ pour $i > (c-a) + a = c$. Cette inégalité équivaut à $c > a + (2c - i - a) \geq \dim(X) + (2c - i - a)$.

Que 3.7.1 soit un isomoprhisme impliquerait aussi des théorèmes de Lefschetz faibles pour les groupes de Chow, et les conjectures sur les groupes de Chow présentées à cette conférence par J. P. Murre.

3.9. Pour M le motif unité, (3.8.1) (ii) demande que

(3.9.1) $$H^n_{\text{abs}}(\text{Spec}(k), \mathbb{Q}(j)) = 0 \quad \text{pour } n > j.$$

Il n'est pas clair pour moi si on doit espérer un analgoue sur $\mathbb{Z}$ à (3.9.1). Si on définit les $H^n_{\text{abs}}(\text{Spec}(k), \mathbb{Z}(j))$ comme étant des groupes de Chow supérieurs, (3.9.1) est trivialement vrai.

Une question plus concrète : supposons qu'on ait pu définir de façon "indépendante de ℓ", des classes de cohomologie galoisienne dans les $H^n(\text{Gal}(\overline{k}/k), \mathbb{Z}_\ell(j))$. Si $n > j$, (3.9.1) demande que ces classes soient de torsion. Doit-on même les espérer nulles?

EXEMPLE 3.10. Soit G un groupe algébrique absolument simple et simplement connexe sur un corps k. Sur la clôture algébrique $\overline{k}$ de k, les relations entre cohomologie de G, BG, et BT, pour T un tore maximal, montrent dans chacune des théories de cohomologie usuelles que $H^i(G_{\overline{k}}) = 0$ pour $i = 1, 2$ et que $H^3(G_{\overline{k}})(2)$ a un générateur naturel a. Plaçons-nous en cohomologie ℓ-adique. Sur k, la cohomologie ℓ-adique absolue de G est somme directe de celle du point et d'une cohomologie réduite $\widetilde{H}$, liée à celle de $G_{\overline{k}}$ par une suite exacte d'Hochschild–Serre. On a donc $\widetilde{H}^i(G, \mathbb{Z}_\ell(2)) = 0$ pour $i \leq 2$ et $\widetilde{H}^3(G, \mathbb{Z}_\ell(2))$ est isomorphe à $\mathbb{Z}_\ell$, avec un générateur naturel a. Ce générateur est primitif, donc définit un homomorphisme

$$G(k) \to H^3(\text{Spec}(k), \mathbb{Z}_\ell(2)).$$

On peut vérifier que, comme prévu par 3.9, cette flèche est nulle: par Mercurev-Suslin **[27]** le H^3 est sans torsion, et un argument de trace montre que l'image est de torsion: G se déploie sur une extension finie k' de k et $G(k')$ est éqal à son groupe dérivé, du moins si $|k'| \geq 4$.

Le même cercle d'idée permet de construire une extension centrale canonique de $G(k)$ par $H^2(\text{Spec}(k), \mathbb{Z}/n(2))$, égal à $K_2(k)/nK_2(k)$ par Mercurjev-Suslin **[27]**. Rappelons que pour G déployé, une extension centrale canonique de $G(k)$ par $K_2(k)$ a été construite par Matsumoto **[25**, 5.11**]**.

3.11. Pour M un motif sur $\mathbb{Q}$, $H^1_{\text{abs}}(M)$ apparaît dans la conjecture de Beilinson sur le comportement en $s = 0$ de la fonction L de M: Beilinson **[2]** et les exposés de Soulé, Nekovar et Scholl à cette conférence. Je renvoie à ces exposés pour une description de la conjecture et de ce qui en est connu.

3.12. Si F est un corps de nombres, (3.7.1) prédit que dans la catégorie des motifs sur F, on a

(3.12.1) $$\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(j)) = K_{2j-1}(F) \otimes \mathbb{Q},$$

avec nullité des Ext^n, $n \geq 2$. L'assertion (3.12.1) impose des restrictions sévères à la catégorie des motifs extensions itérées de motifs de Tate, et pour les systèmes de réalisations attachés à un objet de cette catégorie. Voir Deligne **[14]** où un exemple considéré en détail est celui de l'algèbre de Lie du π_1 rendu unipotent de la droite projective moins trois points.

Voici un programme pour prouver des conséquences de (3.12.1) pour de tels systèmes de réalisations.

(A) Définir une catégorie tannakienne $\mathcal{T}$, avec pour objets simples les puissances tensorielles $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, d'un objet de rang un $\mathbb{Q}(1)$, pour laquelle (3.12.1) soit vrai "par définition".

L'idée est que les groupes de Chow supérieurs de Bloch $\mathrm{Ch}^n(\mathrm{Spec}(F), i)$ sont donnés par un complexe simplicial gradué par la codimension n des cycles et que le produit est presque donné par une structure d'algèbre commutative simpliciale graduée. "Presque" car l'intersection de deux cycles n'est pas toujours définie, seulement "en général". Rationnellement, il devrait être possible de remplacer cette presque algèbre par une vraie algèbre différentielle bigraduée commutative, avec d de degré $(1, 0)$, et

$$A^{*,m} = 0 \text{ si } m < 0,$$
$$A^{*,0} \text{ réduit à } \mathbb{Q} \text{ en degré } 0,$$
$$H^*(A^{*,n}) \text{ réduit à } K_{2n-1}(F) \otimes \mathbb{Q} \text{ en degré } 1.$$

On peut alors définir $\mathcal{T}$ comme une catégorie convenable de modules différentiels bigradués et de morphismes à homotopie près.

Cette partie du programme a été essentiellement réalisée par May, **[26]**. Pour une variante cubique, voir l'exposé de Bloch dans ce volume.

(B) Définir, pour chacune des théories cohomologiques usuelles, un foncteur fibre "réalisation" sur la catégorie tannakienne $\mathcal{T}$ de (A),

(C) Montrer que les motifs de Tate mixtes auxquels on s'intéresse, par exemple, Lie $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$—ou plutôt les systèmes de leurs réalisations—proviennent d'un objet de $\mathcal{T}$.

Bibliographie

1. M. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–85.
2. A. A. Beilinson, *Higher regulators and values of L-functions*, Current Problems in Mathematics **24** (1984), 191–238.
3. A. A. Beilinson, J. Bernstein, et P. Deligne, *Faisceaux pervers*, Analyse et Topologie sur les Espaces Singuliers, Astérisque **100** (1982).
4. S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), 267–304.
5. ——, *Algebraic cycles and higher K-theory, A correction*, preprint (1989).
6. M. Borovoĭ, *Langlands conjecture concerning conjugation of connected Shimura varieties*, Selecta Math. Soviet. **31** (1983–84), 3–59.
7. J. L. Brylinski, 1-*motifs et formes automorphes*, Journées SMF de Dijon (1981), Publ. Math. de l'Univ. de Paris VII **15**, 43–106.
8. P. Deligne, *Théorèmes de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 107–126.
9. ——, *Théorie de Hodge* III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.
10. ——, *Variétés de Shimura: Interprétation modulaire et techniques de construction de modèles canoniques*, Proc. Sympos. Pure Math., **33**, Amer. Math. Soc., Providence, RI, 1979, pp. 247–290.
11. ——(rédigé par J. L. Brylinski), *Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes*, Mémoires de la Soc. Math. France **2**, 1980, pp. 23–33.

12. ——— (notes by J. Milne), *Hodge cycles on abelian varieties*, SLN **900** (1982), 9–100.
13. ———, *Motifs et groupe de Taniyama*, SLN **900** (1982), 261–279.
14. ———, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$, MSRI publications no. 16, Springer-Verlag, 1989, pp. 79–297.
15. ———, *Catégories tannakiennes*, Grothendieck Festschrift, vol. 2, Birkhaüser, 1990, pp. 111–195.
16. P. Deligne and J. Milne, *Tannakian categories*, SLN **900** (1982), 101–228.
17. P. A. Griffiths, *Periods of integrals on algebraic manifolds*. II, Amer. J. Math. **90** 3 (1968), 805–865.
18. ———, *Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems*, Bull. Amer. Math. Soc. **76** 2 (1970), 228–296.
19. A. Grothendieck, *Standard conjectures on algebraic cycles*, Bombay Coll. on Algebraic Geometry (1968), Oxford, 1969, pp. 193–199.
20. S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Théorie des Schémas, 359–386, Adv. Stud. Pure Math. **3**, North-Holland Publ. Co., Amsterdam, 1968.
21. K. Kodaira and D. C. Spencer, *Divisor classes on algebraic varieties*, Proc. Nat. Acad. Sci. **39** (1953), 872–877.
22. I. Kriz and P. May, *An addendum to "Notes toward Bloch—Deligne motives"*, preprint, 1991.
23. R. P. Langlands and M. Rapoport, *Shimura Varietäten und Gerben*, Crelle **378** (1987), 113–220.
24. S. Lefschetz, *L'Analysis Situs et la Géométrie Algébrique*, Gauthier-Villars—reproduit dans Selected Papers, Chelsea, Paris, 1924.
25. H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62.
26. P. May, *Notes toward Block—Deligne motives*, preprint, 1991.
27. A. S. Merkurjev and A. A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphisms*, Izv. Akad. Sci. URSS **46** 5 (1968), 1011–1046. (Traduction Anglaise: Math. USSR Izv. **21** (1983), 307–340)
28. N. Saavedra, *Catégories Tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin and New York, 1972.
29. Morihiko Saito, *Mixed Hodge Modules*, Res. Inst. Math. Sci. (July 1987).
30. J. Steenbrink and S. Zucker, *Variations of Mixed Hodge Structures*. I, II, Invent. Math. **80** (1985), 489–542, 543–565.
31. A. Weil, *Critères d'équivalence en géométrie algébrique*, Math. Ann. **128** (1954), 95–127, Oeuvres Vol. II.
32. S. Zucker, *The Hodge conjecture for cubic fourfolds*, Comp. Math. **34** (1977), 199–209.

[SGA] *Séminaire de Géométrie Algébrique du Bois-Marie*, dirigé par A. Grothendieck.

[SGA1] *Revêtements étales et groupe fondamental*, Lectures Notes in Math., vol. 224, Springer-Verlag, Berlin and New York, 1971.

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Classical Motives

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Introduction

This paper is based on a talk given on the first day of the conference, entitled "Examples". Its purpose was to give an elementary survey of some examples of classical motives from a purely geometric standpoint (that is, without recourse to any cohomological methods). This report has the same aims and consequent shortcomings. There are four main parts:

(i) An account of the definitions and basic properties of motives. This is included for completeness; it can all be found, in a somewhat different form but in greater detail, in the definitive accounts of Kleiman **[12]** and Manin **[19]**.

(ii) The relation between the motive of a curve and its Jacobian variety, due in essence to Weil.

(iii) The motives $h^1(X)$ and $h^{2d-1}(X)$ for a variety X of dimension d. Here we follow Murre **[23]**, with minor modifications.

(iv) An elementary proof of the canonical decomposition of the motive of an abelian variety, inspired by the work of Deninger and Murre **[6]** and Künnemann **[15, 16]**.

There is therefore little that is original contained in these pages. I have given a more or less complete proof of Murre's result in §4, so as to make the comparison between the different decompositions in Theorem 5.3. Also included is a proof (Corollary 3.5) of the unsurprising fact that the category of motives constructed using rational equivalence of cycles is in general not an abelian category. Otherwise proofs have been sketched or omitted.

A word about the notion of motive used in this paper is appropriate. Grothendieck's definition of a motive involves replacing the category of va-

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rieties by a category with the same objects, but whose morphisms are correspondences, modulo a suitable equivalence relation. Depending on the equivalence relation chosen, one gets rather different theories (see §6.2 for a discussion).

It is usual to take numerical or homological equivalence, obtaining motive categories $\mathcal{M}_k^{\text{num}}$ and $\mathcal{M}_k^{\text{hom}}$, but in this report we concentrate more on $\mathcal{M}_k^{\text{rat}}$, the category of motives for rational equivalence (sometimes called Chow motives). One reason to do this is that rational equivalence is the finest adequate equivalence on cycles, so that $\mathcal{M}_k^{\text{rat}}$ is in some sense universal.

Another reason is that many interesting cohomology-like functors factor through $\mathcal{M}_k^{\text{rat}}$ but not in general through $\mathcal{M}_k^{\text{num}}$ or $\mathcal{M}_k^{\text{hom}}$. Examples of such functors include absolute ℓ-adic cohomology $H^*(X_{/k}, \mathbf{Q}_\ell(n))$, Deligne-Beilinson (absolute Hodge) cohomology, and motivic cohomology (see Nekovář's paper in these Proceedings for details of both of these). In particular, to formulate Beilinson's conjectures for motives it is at present necessary to work in $\mathcal{M}_k^{\text{rat}}$.

The obvious disadvantage of using rational equivalence is that of not being in an abelian (or even conjecturally abelian) category. Arguments that are trivial when one uses numerical or homological equivalence can become cumbersome. Philosophically, one has to study $\mathcal{M}_k^{\text{rat}}$ as well as $\mathcal{M}_k^{\text{hom}}$ for the same reasons that in cohomology it is often necessary to work with complexes (i.e., in the derived category) rather than simply with cohomology groups. Unfortunately we do not yet have the whole derived category of motives to play with.

1. Formal properties of motives

1.1. We fix a base field k. Let $\mathcal{V}_k$ denote the category of smooth and projective k-schemes. We refer to the objects of $\mathcal{V}_k$ as simply *varieties*—note that they are not necessarily irreducible or even equidimensional. However, we will only consider connected varieties from §3 onward. If $\phi: Y \to X$ is a morphism in $\mathcal{V}_k$, we denote its graph by $\Gamma_\phi \subset X \times Y$.

1.2. For a variety X and an integer d, the cycle group $\mathcal{Z}^d(X)$ is the free abelian group generated by irreducible subvarieties of X of codimension d. Central to the definition of motives is the choice of an *adequate equivalence relation* $\sim$ on cycles. For a precise definition of what that entails we refer to **[12]**; there are three important examples of adequate equivalence relations:

(i) rational equivalence;
(ii) homological equivalence, with respect to a (fixed) Weil cohomology theory H^*;
(iii) numerical equivalence.

We write (with apologies for the notation) $A^d(X) = \mathcal{Z}^d(X) \otimes \mathbf{Q}/\sim$, where $\sim$ is a fixed adequate equivalence relation. If Z is a cycle on X, we write $[Z]$ for its class in $A^d(X)$. The definition of adequate implies that the groups

$A^d(X)$ enjoy a number of functorial properties:

- For a morphism $\phi: X \to Y$ there are pull-back and push-forward maps $\phi^*: A^*(Y) \to A^*(X)$, $\phi_*: A^*(X) \to A^{*+\dim Y - \dim X}(Y)$.
- There is a product structure $A^d(X) \otimes A^e(X) \to A^{d+e}(X)$ given by intersection theory.

In this report we will be mostly concerned with the case when $\sim$ is rational equivalence, in which case $A^d(X)$ is the usual codimension d Chow group tensored with $\mathbf{Q}$. The motives arising from this choice are sometimes called *Chow motives.* For further comments on the effect of choosing a different equivalence relation, see §§3.5 and 6.2.

1.3. Let X, Y be in $\mathscr{V}_k$. Define $\operatorname{Corr}^r(X, Y)$, the group of correspondences of degree r from X to Y, as follows. If X is purely d-dimensional, then

$$\operatorname{Corr}^r(X, Y) = A^{d+r}(X \times Y).$$

In general, let $X = \coprod X_i$ where each X_i is a connected variety, and set

$$\operatorname{Corr}^r(X, Y) = \bigoplus \operatorname{Corr}^r(X_i, Y) \subset A^*(X \times Y).$$

For varieties X, Y, and Z there is a composition of correspondences

$$\operatorname{Corr}^r(X, Y) \otimes \operatorname{Corr}^s(Y, Z) \to \operatorname{Corr}^{r+s}(X, Z)$$

defined by

$$f \otimes g \mapsto g \circ f = p_{13*}(p_{12}^* f \cdot p_{23}^* g)$$

where p_{ij} are the projections of $X \times Y \times Z$ onto products of factors.

1.4. The category $\mathscr{M}_k$ of k-motives is now defined as follows: an object of $\mathscr{M}_k$ is a triple (X, p, m) where X is a k-variety, m is an integer, and $p = p^2 \in \operatorname{Corr}^0(X, X)$ is an idempotent. If (X, p, m) and (Y, q, n) are motives, then

$$\operatorname{Hom}_{\mathscr{M}_k}((X, p, m), (Y, q, n)) = q \operatorname{Corr}^{n-m}(X, Y) p \subset \operatorname{Corr}^*(X, Y)$$

and composition is given by composition of correspondences.

This is the definition of the category of motives as given, for example, in **[9]**. It is equivalent to the definition found in other places (for example **[19]** and **[12]**) because of the following elementary fact:

LEMMA 1.5. *Let p, q be commuting endomorphisms of an abelian group B, which are idempotents. Then the map $x \mapsto pqx$ gives an isomorphism*

$$\frac{\ker(p-q)}{\ker p \cap \ker q} \xrightarrow{\sim} pqB.$$

THEOREM 1.6. *$\mathscr{M}_k$ is an additive, $\mathbf{Q}$-linear category, which is pseudo-abelian.*

If (X, p, m) and (Y, q, n) are motives with $m = n$ then their direct sum is defined to be

$$(X, p, m) \oplus (Y, q, m) \stackrel{\text{def}}{=} (X \coprod Y, p \oplus q, m).$$

It is immediate that this satisfies the necessary properties. The general construction of the direct sum is given in §1.14; the reader can check that the proof is not circular.

Now recall that an additive category such as $\mathcal{M}_k$ is pseudoabelian if every projector $f \in \operatorname{End} M$ has an image, and the canonical map $\operatorname{Im}(f) \oplus \operatorname{Im}(1-f) \to M$ is an isomorphism. In this case it follows formally from the definition that there is a decomposition

$$M = (X, pfp, m) \oplus (X, p - pfp, m)$$

if $M = (X, p, m)$ is a motive and $f = pfp \in \operatorname{End} M$ is a projector.

REMARK 1.7. One should bear in mind that the category $\mathcal{M}_k$ is in general *not* abelian (see Corollary 3.5), and some caution must therefore be exercised when discussing kernels of arbitrary morphisms. In view of this it is worth making the following trivial observation:

> If $f: M \to N$ is a morphism in $\mathcal{M}_k$ (or any pseudoabelian category) that has a left inverse, then it has an image, which is (noncanonically!) a direct factor of N, and $M \to \operatorname{Im} f$ is an isomorphism.

In fact, if g is any such retract of f, then fg is an idempotent, so $fgN \subset N$ exists, and $f: M \xrightarrow{\sim} fgN$. Dually, if f has a section, then N is canonically a quotient object of M and noncanonically a direct factor of M.

1.8. There is a functor

$$h: \mathcal{V}_k^{\text{opp}} \to \mathcal{M}_k,$$

which on objects is given by $h(X) = (X, \text{id}, 0)$ and on morphisms $\phi: Y \to X$ by

$$h(\phi) = [\Gamma_\phi] \in \operatorname{Corr}^0(X, Y) = \operatorname{Hom}(h(X), h(Y))$$

(usually one writes ϕ^* for $h(\phi)$).

1.9. There is a tensor product on $\mathcal{M}_k$, defined on objects by

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

and on morphisms by

$$q_1 f_1 p_1 \otimes q_2 f_2 p_2 = (q_1 \otimes q_2)(f_1 \otimes f_2)(p_1 \otimes p_2) \in \operatorname{Corr}^*(X_1 \times X_2, Y_1 \times Y_2)$$
$$\text{if } q_i f_i p_i : (X_i, p_i, m_i) \to (Y_i, q_i, n_i).$$

One writes $\mathbf{1} = (\operatorname{Spec} k, \text{id}, 0)$ (the unit motive) and $\mathbb{L} = (\operatorname{Spec} k, \text{id}, -1)$ (the Lefschetz motive). Then $\mathbf{1}$ is the identity for the tensor product, and every motive is a direct factor of $h(X) \otimes \mathbb{L}^{\otimes n}$ for suitable X and n; in fact, if $p \in \operatorname{Corr}^0(X, X) = \operatorname{End} h(X)$ is a projector, then

$$(X, p, m) = ph(X) \otimes \mathbb{L}^{\otimes -m} \subset h(X) \otimes \mathbb{L}^{\otimes -m}.$$

In the future we write simply $\mathbb{L}^n$ for $\mathbb{L}^{\otimes n}$, and for a morphism $f: M \to N$ of motives we will write f also for the tensor product $M \otimes \mathbb{L}^{\otimes n} \to N \otimes \mathbb{L}^{\otimes n}$.

The diagonal $\Delta: X \to X \times X$ defines a product structure on $h(X)$, given by the composite

$$m_X: h(X) \otimes h(X) = h(X \times X) \xrightarrow{\Delta^*} h(X).$$

1.10. If $\phi: Y \to X$ and X and Y are purely d- and e-dimensional, respectively, then the transpose $[{}^t\Gamma_\phi] \in A^d(Y \times X)$ is a correspondence of degree $d-e$ from Y to X and so defines a morphism

$$\phi_*: h(Y) \to h(X) \otimes \mathbb{L}^{e-d}.$$

Suppose that $d = e$ and that ϕ is generically finite, of degree r. Then the composite $\phi_* \circ \phi^* \in \operatorname{End} h(X)$ is multiplication by r. In fact,

$$\phi_* \circ \phi^* = p_{12*}(p_{13}^*[\Gamma_\phi] \cdot p_{23}^*[{}^t\Gamma_\phi]) = p_{13*}(\phi, \mathrm{id}, \phi)_*[Y] = r[\Delta_X]$$

where $\Delta_X \subset X \times X$ is the diagonal.

If $f \in \operatorname{Corr}^0(X, Y) = \operatorname{Hom}(h(X), h(Y))$ and $\phi: X \to X'$, $\psi: Y \to Y'$ are morphisms in $\mathscr{V}_k$ then the formula for the composite map of motives is

$$\psi_* \circ f \circ \phi^* = (\phi \times \psi)_* f.$$

Similarly if $\phi: X' \to X$ and $\psi: Y' \to Y$ then

$$\psi^* \circ f \circ \phi_* = (\phi \times \psi)^* f.$$

Because of the inherent confusion in formulae of this type, we will attempt to distinguish between operations on cycles (direct and inverse image and intersection) and on correspondences by frequent use of the usual symbols $\cdot$ and $\circ$. In particular, the notation c^2 will generally denote $c \cdot c$ and not $c \circ c$.

1.11. Suppose that X is irreducible of dimension d and that there is a k-rational point $x \in X(k)$. Denote by $\alpha: X \to \operatorname{Spec} k$ the structural morphism. Then $x^* \circ \alpha^* = \mathrm{id}$, so by the remark in §1.7, the map

$$\alpha^*: \mathbf{1} \to h(X)$$

is a subobject of $h(X)$. Similarly, since $\alpha_* \circ x_* = \mathrm{id}$, the map

$$\alpha_*: h(X) \to \mathbb{L}^d$$

is a quotient object.

More generally, let X be irreducible, and write $k' = \Gamma(X, \mathscr{O}_X)$ and $\alpha: X \to \operatorname{Spec} k'$ for the structural morphism. Let k''/k' be a finite separable extension such that there exists $x \in X(k'')$. Write γ: $\operatorname{Spec} k'' \to \operatorname{Spec} k'$ for the natural map. Then by §1.10, the composite

$$h(\operatorname{Spec} k') \xrightarrow{\alpha^*} h(X) \xrightarrow{x^*} h(\operatorname{Spec} k'') \xrightarrow{\gamma_*} h(\operatorname{Spec}(k')$$

is multiplication by $[k'': k']$. Therefore,

$$\alpha^*: h(\operatorname{Spec} k') \to h(X)$$

defines a subobject of $h(X)$, denoted $h^0(X)$. It is well defined as a subobject up to unique isomorphism.

We denote the quotient of $h(X)$ by $h^0(X)$ as $h^{\geq 1}(X)$. The quotient exists because $h^0(X)$ is (noncanonically) a direct factor of $h(X)$. The choice of a point x determines a splitting $h(X) = h^0(X) \oplus h^{\geq 1}(X)$.

REMARK. When $\sim$ is homological or numerical equivalence, the class of Γ_x in $A^*(X \otimes_{k'} k'')$ is independent of x, so the splitting is in this case canonical.

In particular, if X is absolutely irreducible, then $h^0(X) = \mathbf{1}$ is a direct summand of $h(X)$. We can use this to eliminate the nuisance of dealing with varieties with components of different dimensions as follows:

PROPOSITION 1.12. *Any motive M can be expressed as a direct factor of some $h(X') \otimes \mathbb{L}^n$, with X' equidimensional.*

PROOF. It is enough to show this for $M = h(X)$. Let $X = \coprod X_i$ be the decomposition of X into its components. Choose integers $d_i \geq 0$ such that $\dim X_i + d_i$ does not depend on i. We then have

$$h(X) = \bigoplus h(X_i) = \bigoplus (h(X_i) \otimes h^0(\mathbf{P}^{d_i})),$$

and we have seen that this is a direct factor of $\bigoplus h(X_i \otimes \mathbf{P}^{d_i}) = h(\coprod X_i \times \mathbf{P}^{d_i})$.

1.13. Continuing with the assumptions of §1.11, let $d = \dim X$. Then the composite

$$h(\operatorname{Spec} k') \otimes \mathbb{L}^d \xrightarrow{x_* \gamma^*} h(X) \xrightarrow{\alpha_*} h(\operatorname{Spec} k') \otimes \mathbb{L}^d$$

is multiplication by $[k'' : k']$, so

$$\alpha_* : h(X) \to h(\operatorname{Spec} k') \otimes \mathbb{L}^d$$

is a quotient object of $h(X)$, denoted $h^{2d}(X)$.

If X is irreducible and has a rational point $x \in X(k)$ then

$$h^0(X) \simeq (X, \{x\} \times X, 0) \simeq \mathbf{1},$$
$$h^{2d}(X) \simeq (X, X \times \{x\}, 0) \simeq \mathbb{L}^d.$$

Assuming only that X is irreducible, let Z be any zero-cycle on X whose degree d is positive. Then $p_0 = (1/d)[Z \times X] \in A^d(X \times X)$ is an idempotent and gives rise to canonical isomorphisms

$$h^0(X) \simeq (X, p_0, 0), \qquad h^{2d}(X) \simeq (X, p_{2d}, 0) \quad \text{where } p_{2d} = {}^t p_0.$$

For example, consider $X = \mathbf{P}^1$. Since the cycles $\mathbf{P}^1 \times \{x\}$, $\{x\} \times \mathbf{P}^1$ on $\mathbf{P}^1 \times \mathbf{P}^1$ do not depend on the choice of a rational point $x \in \mathbf{P}^1(k)$, and since their sum is rationally equivalent to the diagonal, we have canonically

$$h(\mathbf{P}^1) = h^0(\mathbf{P}^1) \oplus h^2(\mathbf{P}^1) = \mathbf{1} \oplus \mathbb{L}.$$

In most treatments this is taken as the definition of $\mathbb{L}$.

1.14. We can now construct arbitrary direct sums in $\mathscr{M}_k$. Let $M = (X, p, m)$ and $N = (Y, q, n)$ be motives. Assume that $m < n$. Then

$$M = (X, p, n) \otimes L^{n-m} = (X, p, n) \otimes h^2(\mathbf{P}^1)^{n-m} = (X \times (\mathbf{P}^1)^{n-m}, p', n)$$

for a suitable projector p', and the direct sum of M and N is then

$$\left(X \times (\mathbf{P}^1)^{n-m} \coprod Y, p' \oplus q, n\right)$$

as in Theorem 1.6.

1.15. There is an involution $^\vee : \mathscr{M}_k^{\mathrm{opp}} \to \mathscr{M}_k$, defined on objects by

$$(X, p, m)^\vee = (X, {}^t p, d - m) \quad \text{if } X \text{ is purely } d\text{-dimensional}$$

and on morphisms as the transpose of correspondences. In particular, $h(X)^\vee = h(X) \otimes \mathbb{L}^{-d}$ ("Poincaré duality"). Clearly $M^{\vee\vee} = M$ for every M, and the standard formula

$$\mathrm{Hom}(M \otimes N, P) = \mathrm{Hom}(M, N^\vee \otimes P)$$

is trivially seen to hold. Then one can define an internal Hom in $\mathscr{M}_k$ by the formula $\mathrm{Hom}(M, N) = M^\vee \otimes N$. These constructions give $\mathscr{M}_k$ the structure of a rigid additive tensor category, once commutativity and associativity constraints are defined. (This tensor structure gives what in the Tannakian setting is usually called the false category of motives—see **[5**, pp. 200ff].)

1.16. It is sometimes convenient to construct a decomposition of motives over an extension field (this is needed in §4, for instance). Because of what was said in §1.10, the groups A^* satisfy Galois descent: if k'/k is a finite Galois extension then $A^*(X)$ is the subspace of invariants of $A^*(X \otimes k')$ under $\mathrm{Gal}(k'/k)$. Therefore, Galois-invariant decompositions of motives descend to the ground field; more precisely, we have the following easy result.

LEMMA 1.17. *Let $X \in \mathscr{V}_k$ be purely d-dimensional, and let k'/k be a finite Galois extension of degree m. Write $X' = X \otimes_k k'$, and denote by β the canonical map $X' \times_{k'} X' \to X \times_k X$. Suppose that $p'_1, \ldots, p'_r \in A^d(X' \times_{k'} X')$ is a complete system of orthogonal idempotents, which are invariant under $\mathrm{Gal}(k'/k)$. Then the correspondences $p_i = (1/m)\beta_*(p'_i)$ form a complete system of orthogonal idempotents in $A^d(X \times_k X)$.*

2. Cycles and Manin's identity principle

2.1. An immediate consequence of the definition of motives is that the cycle class groups A^* can be interpreted in terms of $\mathscr{M}_k$; in fact,

$$A^d(X) = \mathrm{Hom}(\mathbb{L}^d, h(X)).$$

If $\xi \in A^d(X)$, then we write $\xi_* : \mathbb{L}^d \to h(X)$ for the corresponding mapping of motives and $\xi^* : h(X) \to \mathbb{L}^{\dim X - d}$ for its transpose. (If ξ is the class of a rational point $x \in X(k)$, then this agrees with the notation x_* and x^*.)

Then there is a morphism $\bar{\xi}\colon h(X)\otimes \mathbb{L}^d \to h(X)$ ("cup-product with ξ") defined by

$$h(X)\otimes \mathbb{L}^d \overset{\mathrm{id}\otimes\xi_*}{\to} h(X)\otimes h(X) \overset{m_X}{\to} h(X).$$

The morphism $\bar{\xi}$ is represented by the cycle class

$$\Delta_*(\xi)\in \mathrm{Corr}^d(X, X)\subset A^*(X\times X).$$

If η is another cycle class then

$$\bar{\xi}\circ\bar{\eta} = \overline{\xi\cdot\eta} \quad\text{and}\quad \bar{\xi}\circ\eta_* = (\xi\cdot\eta)_*. \tag{2.1.1}$$

In possible conflict with the convention of §1.10 the notation $\bar{\xi}^i$ will mean the correspondence induced by ξ^i, which is the ith iterate of $\bar{\xi}$ and not the i-fold intersection. We define for any motive M and $d\in\mathbf{Z}$ the cycle groups of M by

$$A^d(M) \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{L}^d, M).$$

Then $A^*(-)$ is a $\mathbf{Z}$-graded additive functor from $\mathscr{M}_k$ to $\mathrm{Vect}_{\mathbf{Q}}$, the category of $\mathbf{Q}$-vector spaces. For $M = h(X)$ this therefore agrees with the previous notation.

2.2. If $M, N\in\mathscr{M}_k$ then

$$\mathrm{Hom}(N, M) = \mathrm{Hom}(\mathbf{1}, M\otimes N^\vee) = A^0(M\otimes N^\vee).$$

Therefore by the Yoneda lemma the functor that attaches to $M\in\mathscr{M}_k$ the functor $A^0(M\otimes -)\colon \mathscr{M}_k\to\mathrm{Vect}_{\mathbf{Q}}$ is fully faithful.

Now any $N\in\mathscr{M}_k$ is a direct factor of some $h(Y)\otimes\mathbb{L}^n$ with $Y\in\mathscr{V}_k$ and $n\in\mathbf{Z}$, and $A^0(M\otimes h(Y)\otimes\mathbb{L}^n) = A^{-n}(M\otimes h(Y))$. Therefore, if we denote by $\omega_M\colon \mathscr{V}_k^{\mathrm{opp}}\to\mathrm{Vect}_{\mathbf{Q}}$ the functor

$$\omega_M(Y) = A^*(M\otimes h(Y))$$

then $M\mapsto\omega_M$ is fully faithful. From this one can deduce (using for example, [20, II.7.1]):

MANIN'S IDENTITY PRINCIPLE 2.3. (i) *Let $f, g\colon M\to N$ be morphisms of motives. Then f is an isomorphism if and only if the induced map*

$$\omega_f(Y)\colon A^*(M\otimes h(Y))\to A^*(N\otimes h(Y))$$

is an isomorphism for every $Y\in\mathscr{V}_k$; and $f = g$ if and only if $\omega_f(Y) = \omega_g(Y)$ for every Y.

(ii) *A sequence $0\to M'\overset{f}{\to} M\overset{g}{\to} M''\to 0$ in $\mathscr{M}_k$ is exact if and only if, for every $Y\in\mathscr{V}_k$, the sequence*

$$0\to A^*(M'\otimes h(Y))\overset{\omega_f(Y)}{\to} A^*(M\otimes h(Y))\overset{\omega_g(Y)}{\to} A^*(M''\otimes h(Y))\to 0$$

is exact.

2.4. As a first example we calculate the motive of a projective bundle following Manin [19]. Let $S\in\mathscr{V}_k$, and let $\mathscr{E}$ be a locally free sheaf on S

of constant rank $r+1 \geq 1$. Let $X = \mathbf{P}_S[\mathscr{E}] \xrightarrow{\pi} S$ be the associated projective bundle, and let $\xi = c_1(\mathscr{O}_X(1)) \in A^1(X)$ be the divisor class of the tautological line bundle on X.

Recall that $A^*(X)$ is a free module over $A^*(S)$ (via π^*) with basis 1, ξ, $\dots$, ξ^r and that the multiplication is given by

$$\xi^{r+1} = \sum_{j=0}^{r}(-1)^{r-j}c_{r-j+1}(\mathscr{E})\xi^j$$

where $c_i(\mathscr{E}) \in A^i(S)$ are the Chern classes of $\mathscr{E}$. For any integer $n \geq 0$, write $\xi^n = \sum_{j=0}^r \theta_{n,j}\xi^j$. Then $\theta_{n,j} \in A^{n-j}(S)$ are given by certain universal polynomials in the Chern classes.

THEOREM 2.5. *The map*

$$\sum_{i=0}^{r} \bar{\xi}^i \circ \pi^* : \bigoplus_{i=0}^{r} h(S) \otimes \mathbb{L}^i \to h(X)$$

is an isomorphism of motives. In terms of this isomorphism, the product structure on $h(X)$ *is given by*

$$\left(h(S)\otimes\mathbb{L}^i\right)\otimes\left(h(S)\otimes\mathbb{L}^{n-i}\right) \xrightarrow{m_S} h(S)\otimes\mathbb{L}^n \xrightarrow{(\theta_{n,j})} \bigoplus_{j=0}^{r} h(S)\otimes\mathbb{L}^j .$$

PROOF. Manin's identity principle implies that it is equivalent to know that the corresponding mapping at the level of cycle class groups

$$\bigoplus_{i=0}^{r} A^{*-i}(S\times Y) \to A^*(X\times Y),$$

$$(z_i) \mapsto \sum_{i=0}^{r}\xi^i \cdot (\pi\times \mathrm{id}_Y)^*(z_i)$$

is a ring isomorphism for every $Y \in \mathscr{V}_k$. But since the Chern classes of $pr_1^*\mathscr{E}$ on $S\times Y$ are simply $pr_1^*c_i(\mathscr{E})$, this follows from the facts recalled in §2.4 (replacing X/S by $X\times Y/S\times Y$).

2.6. Similar results hold for other varieties which admit cellular decompositions relative to a base (grassmanians, etc.). Suppose that $\pi\colon X \to S$ is a flat morphism of pure relative dimension n and that X admits a filtration by closed subschemes $X = X_0 \supset X_1 \supset \cdots$ such that $X_i - X_{i+1}$ is S-isomorphic to the affine space $\mathbf{A}_S^{n-d_i}$, for some $d_i \in \mathbf{Z}$. Then there is an isomorphism

$$\bigoplus_i A^{*-d_i}(S) \to A^*(X) \tag{2.6.1}$$

which is functorial with respect to Cartesian squares

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi} \\ S' & \longrightarrow & S \end{array}$$

and so by Manin's identity principle $h(X) \simeq \bigoplus h(S) \otimes \mathbb{L}^{d_i}$. See [14], particularly the appendix, for a proof of (2.6.1) and further examples. Calculations for many of the classical varieties of this type can be found already in SGA6.

2.7. The identity principle can also be used to calculate the motive of a blowup. Let $Y \subset X$ be a nonsingular subvariety that is purely of codimension $r+1 > 1$, and let

$$\begin{array}{ccc} Y' & \overset{i'}{\hookrightarrow} & X' \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$

be the blowup of X along Y. Then $Y' \to Y$ is a projective bundle; let $\xi \in A^1(Y')$ be the class of the tautological line bundle. Consider the sequence of motives:

$$(1) \quad 0 \to h(Y) \otimes \mathbb{L}^{r+1} \xrightarrow{(i_*, \xi^r \cdot \pi^*)} h(X) \oplus (h(Y) \otimes \mathbb{L}) \xrightarrow{\pi'^* \oplus (-i'_*)} h(X') \to 0.$$

There is a retract of the first arrow given by

$$0 \oplus \pi_* : h(X) \oplus (h(Y) \otimes \mathbb{L}) \to h(Y) \otimes \mathbb{L}^{r+1}.$$

THEOREM 2.8 (MANIN). *The sequence* (1) *is split exact.*

The proof relies on the identity principle and the behaviour of A^* under blowups. For details we refer the reader to [19]. This result was used there by Manin to prove the Weil conjectures for unirational varieties of dimension three.

3. Curves and abelian varieties (I)

3.1. One reason for Grothendieck's introduction of motives was to serve as analogues of the Jacobian of a curve in higher dimensions. Here we explain the precise relationship between the motive of a curve and its Jacobian. *For the rest of the paper all varieties will be assumed connected.*

3.2. Let $X \in \mathscr{V}_k$ be a curve, with field of constants k'. As explained in §§1.11–1.13 there is a submotive $h^0(X) \simeq h(\operatorname{Spec} k')$ and a quotient motive $h^2(X) \simeq h(\operatorname{Spec} k') \otimes \mathbb{L}$ of $h(X)$. The choice of a zero-cycle Z on X of positive degree determines projectors p_0, $p_2 \in \operatorname{Corr}^0(X, X) = \operatorname{End} h(X)$ with $h^i(X) \simeq (X, p_i, 0)$ for $i = 0, 2$ as in §1.13.

Let $p_1 = 1 - p_0 - p_2 \in \operatorname{Corr}^0(X, X)$, and write $h^1(X) = (X, p_1, 0) \in \mathscr{M}_k$. Then there is a direct sum decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$$

which in general depends on the choice of Z (more specifically, its class in $A^1(X)$). However, $h^1(X)$ is well defined up to unique isomorphism; in fact, it is the kernel of the composite map $h^{\geq 1}(X) \to h(X) \to h^2(X)$, so is well defined as a subquotient of $h(X)$. The theory of the motives $h^1(X)$ is essentially that of Jacobian varieties:

PROPOSITION 3.3. *If X, X' are curves with Jacobian varieties J, J', then*

$$\mathrm{Hom}(h^1(X), h^1(X')) = \mathrm{Hom}(J, J') \otimes \mathbf{Q}$$

and

$$\mathrm{Hom}(L, h^1(X)) = \begin{cases} 0 & \text{if } \sim \text{ is numerical (or homological) equivalence,} \\ J(k) \otimes \mathbf{Q} & \text{if } \sim \text{ is rational equivalence.} \end{cases}$$

(Some explanation for the dichotomy will be given at the end of the paper.)

PROOF. By [27] we have $A^1(X \times X') = A^1(X) \oplus A^1(X') \oplus \mathrm{Hom}(J, J') \otimes \mathbf{Q}$, giving the first formula. For the second, we have $\mathrm{Hom}(L, h(X)) = A^1(X)$ and $\mathrm{Hom}(L, h^0(X)) = 0$, while $\mathrm{Hom}(L, h^2(X)) = \mathbf{Q}$ is generated by the class of a closed point in $A^1(X)$. Hence $\mathrm{Hom}(L, h^1(X)) = \ker(\deg\colon A^1(X) \to \mathbf{Q})$, giving the second formula.

COROLLARY 3.4. *Let $\mathscr{C}_k$ be the full subcategory of $\mathscr{M}_k$ whose objects are direct summands of motives of the form $h^1(X)$, where X is of dimension one. Then $\mathscr{C}_k$ is equivalent to the category of abelian varieties over k up to isogeny.*

This follows from the fact that every abelian variety is an abelian subvariety of a Jacobian and Poincaré's complete reducibility theorem. (Note that the result is independent of the particular equivalence relation chosen to define $\mathscr{M}_k$.)

COROLLARY 3.5. *Assume that k is not contained in the algebraic closure of a finite field and that $\sim$ is rational equivalence. Then $\mathscr{M}_k^{\mathrm{rat}}$ is not an abelian category.*

PROOF. Under the hypothesis on k, one can find an elliptic curve E/k and a point $P \in E(k)$ of infinite order. Let $\xi \in A^1(E)$ be the class of the divisor $(P) - (0)$. Then the morphism $\xi_*\colon \mathbb{L} \to h^1(E)$ is nonzero by 3.3. The composite $\xi_* \circ \xi^*\colon h^1(E) \otimes \mathbb{L} \to h^1(E)$ is represented by the class of the zero-cycle $\eta = (P, P) + (0, 0) - (P, 0) - (0, P)$ on $E \times E$. If $P = 2Q$ for $Q \in E(\bar{k})$, then in $\mathrm{CH}^2(E \times E_{/\bar{k}})$ we can write

$$\eta = [(P, P) + (0, 0) - 2(Q, Q)] + [2(Q, Q) - (P, 0) - (0, P)],$$

and this is rationally equivalent to zero. Therefore, η is a torsion class in $\mathrm{CH}^2(E \times E)$, whence $\xi_* \circ \xi^* = 0$. Thus ξ_* is not a monomorphism. If $\mathscr{M}_k^{\mathrm{rat}}$ were abelian then $\ker \xi_*$ would be a proper subobject of $\mathbb{L}$. Tensoring by $\mathbb{L}^{-1}$ this would imply that $\mathbb{1}$ had a nontrivial subobject. But the unit object in a rigid abelian tensor category is completely decomposable, by [5, Proposition 1.17], so as $\mathrm{End}\,\mathbb{1} = \mathbf{Q}$ this is impossible.

3.6. In the next section we will give a generalisation of the above to higher-dimensional varieties. First we recall the bare essentials of the theory of Albanese and Picard varieties (see for example [17]).

3.7. Let X be a variety over k. The Albanese variety J_X is an abelian variety over k equipped with a morphism $\nu: X \times X \to J_X$, satisfying the following universal property: any morphism $\phi: X \times X \to A$ to an abelian variety such that $\phi(x, y) + \phi(y, z) = \phi(x, z)$ (or equivalently, $\phi(x, x) = 0$) factors uniquely as $\phi = \beta \circ \nu$ for some homomorphism $\beta: J_X \to A$.

If X has a k-rational point x_0 and $\gamma: X \to J_X$ is the morphism $\gamma(x) = \nu(x, x_0)$, then γ satisfies the usual universal property for morphisms from X to abelian varieties. If $\psi: X \to Y$ is a morphism, then the universal property defines a homomorphism $J_\psi: J_X \to J_Y$, so that J is a covariant functor.

3.8. Let $\mathscr{P}_X$ be the functor on varieties given by

$$\mathscr{P}_X(S) = \frac{\left\{\begin{array}{l}\text{isomorphism classes of line bundles } \mathscr{L} \text{ on } X \times S \text{ such that} \\ \mathscr{L}|_{X\times s} \text{ is algebraically equivalent to } 0 \text{ for all } s \in S(\bar{k})\end{array}\right\}}{\{pr_2^*\mathscr{G} \text{ for } \mathscr{G} \in \operatorname{Pic} S\}}.$$

Recall that the Picard variety P_X is an abelian variety over k such that there are functorial injections

$$P_X(S) \hookrightarrow \mathscr{P}_X(S)$$

for varieties S, which are bijections whenever $X(S)$ is nonempty. (See for example §5(d) in [22, Chapter 0].) P_X is contravariant in X and is (functorially) isomorphic to the dual abelian variety of J_X.

THEOREM 3.9. *Let X and $Y \in \mathscr{V}_k$ be varieties of dimensions d and e. Then*

(i)

$$\operatorname{Hom}(J_X, P_Y) \otimes \mathbf{Q} \simeq \frac{A^1(X \times Y)}{pr_1^*A^1(X) + pr_2^*A^1(Y)}.$$

(ii) *Let $\xi \in A^d(X)$, $\eta \in A^e(Y)$ be 0-cycles of positive degree. Then there is an isomorphism*

$$\Omega: \operatorname{Hom}(J_X, P_Y) \otimes \mathbf{Q} \xrightarrow{\sim} \{c \in A^1(X \times Y) \mid c \circ \xi_* = 0 \textit{ and } \eta^* \circ c = 0\}.$$

Recall the proof. First assume that $X(k)$ and $Y(k)$ are nonempty, and let $x_0 \in X(k)$. Then by the universal properties we have

$$\begin{aligned}\operatorname{Hom}(J_X, P_Y) &= \{\phi: X \to P_Y \mid \phi(x_0) = 0\} \\ &= \frac{\left\{\begin{array}{c}\text{isomorphism classes of line bundles } \mathscr{L} \text{ on} \\ X \times Y \text{ such that } \mathscr{L}|_{\{x_0\}\times Y} \simeq \mathscr{O}_Y\end{array}\right\}}{\{pr_1^*\mathscr{G} \text{ for } \mathscr{G} \in \operatorname{Pic} X\}} \\ &= \frac{\operatorname{Pic} X \times Y}{pr_1^* \operatorname{Pic} X + pr_2^* \operatorname{Pic} Y}\end{aligned}$$

by the seesaw theorem. In general this will hold when k is replaced by a finite Galois extension k'/k. Tensoring with $\mathbf{Q}$ and taking invariants under

$\mathrm{Gal}(k'/k)$ then yields (i). The isomorphism (ii) is obtained by combining (i) and seesaw.

The following easy proposition gives the functorial behaviour of Ω.

PROPOSITION 3.10. *Let $\phi: X' \to X$, $\psi: Y' \to Y$ be morphisms of varieties. Choose positive zero-cycles ξ', η' on X', Y' with direct images ξ, η on X and Y. If $\beta: J_X \to P_Y$ is a homomorphism, then*

$$\Omega(P_\psi \circ \beta) = \psi^* \circ \Omega(\beta) \quad \text{and} \quad \Omega(\beta \circ J_\phi) = \Omega(\beta) \circ \phi_*$$

(*where Ω denotes the isomorphism of* 3.9(ii) *with respect to the chosen* 0-*cycles*).

4. The motives $h^1(X)$ and $h^{2d-1}(X)$

4.1. Let X/k be a variety of dimension d, and fix a projective embedding of X of degree m. Let $\xi \in A^1(X)$ be the class of a hyperplane section. Following Murre **[23]** we will define projectors p_1, $p_{2d-1} \in A^d(X \times X)$ such that the corresponding motives $h^i(X) = (X, p_i, 0)$ satisfy the analogue of the hard Lefschetz theorem:

$$\xi^{d-1}: h^1(X) \xrightarrow{\sim} h^{2d-1}(X) \otimes \mathbb{L}^{1-d}$$

and are closely related to the Picard and Albanese varieties of X. This generalises work of Grothendieck, Kleiman, and Lieberman, who proved this when $\sim$ is homological equivalence (see **[11,** Appendix to §2]).

4.2. Assume from now on that there is a 1-dimensional linear section C of X that is a smooth (and connected) curve. This does not involve loss of generality, since one can always find such a section after a finite base extension k'/k. Then the projectors p_i may be constructed over k' and descended to k by Lemma 1.17.

Let Z be a 0-cycle on C that is cut out by a hyperplane, and let $\zeta \in A^d(X)$ be the class of Z. Write $i: C \hookrightarrow X$ for the embedding. Then

$$\xi^{d-1} = i_* \circ i^*: h(X) \to h(X) \otimes \mathbb{L}^{1-d}.$$

4.3. The functoriality of the Picard and Albanese varieties defines a composite homomorphism

$$\alpha: P_X \xrightarrow{P_i} P_C = J_C \xrightarrow{J_i} J_X.$$

The construction relies upon the theorem of Weil **[28,** Corollary 1 to Theorem 7]) that α is an isogeny and does not depend on the choice of section C. Choose $n \geq 1$ and $\beta: J_X \to P_X$ such that $\alpha \circ \beta = [\times n]$. Since duality interchanges the functors P and J, one has $\widehat{\alpha} = \alpha$ and $\widehat{\beta} = \beta$.

By Theorem 3.9, β corresponds to a cycle

$$\widetilde{\beta} := \Omega(\beta) \in A^1(X \times X) = \mathrm{Hom}(h(X) \otimes \mathbb{L}^{1-d}, h(X))$$

satisfying $\widetilde{\beta} \circ \zeta_* = 0$ and $\zeta^* \circ \widetilde{\beta} = 0$. Since $\widehat{\beta} = \beta$ we have ${}^t\widetilde{\beta} = \widetilde{\beta}$. Normalise the projectors p_0, p_{2d} of §1.13 by $p_0 = \frac{1}{m}[Z \times X] = {}^t p_{2d}$, and define

$$p_1^? = \frac{1}{n}\widetilde{\beta} \circ \xi^{d-1} = \frac{1}{n}\widetilde{\beta} \cdot [C \times X] = \frac{1}{n}\widetilde{\beta} \circ i_* \circ i^*,$$
$$p_{2d-1}^? = {}^t p_1^? = \frac{1}{n} i_* \circ i^* \circ \widetilde{\beta}.$$

Define $p_1 = p_1^? \circ (1 - \frac{1}{2}p_{2d-1}^?)$ and $p_{2d-1} = {}^t p_1$. (We will show that $p_1 = p_1^?$ if $d > 2$; see the proof of Theorem 4.4(i).) Let $h^i(X) = (X, p_i, 0) \in \mathcal{M}_k$.

THEOREM 4.4. (i) *p_0, p_1, p_{2d-1}, p_{2d} are orthogonal idempotents.*

(ii) *The composite morphism*

$$h^1(X) \hookrightarrow h(X) \xrightarrow{\xi^{d-1}} h(X) \otimes \mathbb{L}^{1-d} \twoheadrightarrow h^{2d-1}(X) \otimes \mathbb{L}^{1-d}$$

is an isomorphism.

(iii) *Assume $\sim$ is rational equivalence. Let $A^r(X)^0 \subset A^r(X)$ be the subgroup of cycle classes numerically equivalent to zero, and let* Alb: $A^d(X)^0 \to J_X(k) \otimes \mathbf{Q}$ *be the Albanese map. The cycle class groups of $h^i(X)$ are given as*

$$A^r(h^1(X)) = \begin{cases} 0 & \text{if } r \neq 1, \\ A^1(X)^0 & \text{if } r = 1; \end{cases}$$
$$A^r(h^{2d-1}(X)) = \begin{cases} 0 & \text{if } r \neq d, \\ A^d(X)^0/\ker(\mathrm{Alb}) & \text{if } r = d. \end{cases}$$

PROOF. (i) We first consider the $p_i^?$. We have

$$n^2 p_1^? \circ p_1^? = \widetilde{\beta} \circ i_* \circ i^* \circ \widetilde{\beta} \circ i_* \circ i^*.$$

Now if $f = i^* \circ \widetilde{\beta} \circ i_* \in A^1(C \times C)$ and $g = \widetilde{\beta} \circ i_* \in A^1(C \times X)$ then $\Omega^{-1}(f) = P_i \circ \beta \circ J_i$ and $\Omega^{-1}(g) = \beta \circ J_i$, by Proposition 3.10. Therefore,

$$\Omega^{-1}(g \circ f) = \beta \circ J_i \circ P_i \circ \beta \circ J_i = [\times n] \circ \beta \circ J_i \in \mathrm{Hom}(J_C, P_X).$$

Since $\zeta^* \circ f = \zeta^* \circ g \circ f = 0$ and $f \circ \zeta_* = g \circ f \circ \zeta_* = 0$, from Theorem 3.9 we deduce that $g \circ f = ng$. Therefore

$$p_1^? \circ p_1^? = p_1^?. \tag{4.4.1}$$

Next, $\widetilde{\beta} \circ \zeta_* = 0 = \zeta^* \circ \widetilde{\beta}$ implies $pr_{2*}(\widetilde{\beta} \cdot [Z \times X]) = pr_{1*}(\widetilde{\beta} \cdot [X \times Z]) = 0$, and so we get

$$p_1^? \circ p_{2d} = 0 = p_0 \circ p_1^?. \tag{4.4.2}$$

Also $p_1^? \circ p_0 = (1/mn)[Z] \times pr_{2*}(p_1^?) = ap_0$ for some $a \in \mathbf{Q}$. Squaring gives $a^2 p_0 = (p_1^? \circ p_0)^2 = 0$, so $a = 0$, and

$$p_1^? \circ p_0 = 0. \tag{4.4.3}$$

A similar argument shows that

$$p_{2d} \circ \overset{?}{p_1} = 0 \tag{4.4.4}$$

and, by transposition,

$$\overset{?}{p}_{2d-1} \circ p_i = p_i \circ \overset{?}{p}_{2d-1} = 0 \qquad (i = 0,\ 2d). \tag{4.4.5}$$

Now consider

$$n^2 \overset{?}{p}_{2d-1} \circ \overset{?}{p_1} = i_* \circ i^* \circ \widetilde{\beta} \circ \widetilde{\beta} \circ i_* \circ i^*.$$

We have $\widetilde{\beta} \circ \widetilde{\beta} \in A^{2-d}(X \times X)$. So if $d > 2$ then $\widetilde{\beta} \circ \widetilde{\beta}$ vanishes, and if $d = 2$ then $\widetilde{\beta} \circ \widetilde{\beta} = a[X \times X]$ for some $a \in \mathbf{Q}$. Now $\widetilde{\beta} \circ \zeta_* = 0$, whereas $[X \times X] \circ \zeta_* = m[X]$, so in fact $a = 0$. Thus in every case $\widetilde{\beta} \circ \widetilde{\beta} = 0$, and

$$\overset{?}{p}_{2d-1} \circ \overset{?}{p_1} = 0. \tag{4.4.6}$$

Combining (4.4.1)–(4.4.6) with the definition of p_1 and p_{2d-1} we get all the required orthogonalities after a series of trivial computations.

For completeness we finally calculate $\overset{?}{p_1} \circ \overset{?}{p}_{2d-1}$. We have

$$n^2 \overset{?}{p_1} \circ \overset{?}{p}_{2d-1} = \widetilde{\beta} \circ (i_* \circ i^*) \circ (i_* \circ i^*) \circ \widetilde{\beta}.$$

If $d > 2$ then $(i_* \circ i^*) \circ (i_* \circ i^*) = 0$ since $[C]^2 = 0$ in $A^*(X)$. But if $d = 2$ then we have

$$(i_* \circ i^*) \circ (i_* \circ i^*) = [\Delta_*(Z)].$$

If it happens that

$$n[\Delta_*(Z)] = [Z \times Z] \in A^4(X \times X)$$

then we will have $\overset{?}{p_1} \circ \overset{?}{p_3} = n^{-3} \widetilde{\beta} \circ \zeta_* \circ \zeta^* \circ \widetilde{\beta} = 0$. However, in general it appears that this need not hold (the situation is rather similar to Remark (iv) in §4.6), and the correcting terms in the definition of p_1 and p_3 are needed.

(ii) Consider the morphisms

$$h^1(X) \underset{p_1 \circ \widetilde{\beta} \circ p_{2d-1}}{\xrightarrow{p_{2d-1} \circ \xi^{d-1} \circ p_1}} h^{2d-1}(X) \otimes \mathbb{L}^{1-d}.$$

Now

$$p_1 \circ \widetilde{\beta} \circ p_{2d-1} = \overset{?}{p_1} \circ (1 - \frac{1}{2} \overset{?}{p}_{2d-1}) \circ \widetilde{\beta} \circ (1 - \frac{1}{2} \overset{?}{p_1}) \circ \overset{?}{p}_{2d-1}.$$

Since $\widetilde{\beta} \circ \widetilde{\beta} = 0$ (cf. the above proof of (i)) we get $\overset{?}{p}_{2d-1} \circ \widetilde{\beta} = 0 = \widetilde{\beta} \circ \overset{?}{p_1}$ and so

$$p_1 \circ \widetilde{\beta} \circ p_{2d-1} = \overset{?}{p_1} \circ \widetilde{\beta} \circ \overset{?}{p}_{2d-1} = \overset{?}{p_1} \circ \widetilde{\beta} = \widetilde{\beta} \circ \overset{?}{p}_{2d-1}.$$

Therefore

$$p_1 \circ \widetilde{\beta} \circ p_{2d-1} \circ \xi^{d-1} \circ p_1 = \overset{?}{p_1} \circ \widetilde{\beta} \circ \xi^{d-1} \circ p_1 = n p_1.$$

Similarly,

$$p_{2d-1} \circ \xi^{d-1} \circ p_1 \circ \widetilde{\beta} \circ p_{2d-1} = p_{2d-1} \circ \xi^{d-1} \circ \widetilde{\beta} \circ \overset{?}{p}_{2d-1} = n p_{2d-1}$$

and thus the arrows are isomorphisms.

(iii) See **[23]** for details of this part.

Proposition 4.5. *Let X, $X' \in \mathscr{V}_k$ with chosen projective embeddings. Then*

$$\mathrm{Hom}(h^1(X), h^1(X')) = \mathrm{Hom}(P_X, P_{X'}) \otimes \mathbf{Q}$$

and

$$\mathrm{Hom}(h^{2d-1}(X), h^{2d-1}(X')) = \mathrm{Hom}(J_{X'}, J_X) \otimes \mathbf{Q}.$$

Proof. Use $'$ to denote the corresponding objects in the above construction when applied to X'. By 4.4(ii) we have

$$\begin{aligned}\mathrm{Hom}(h^1(X), h^1(X')) &\simeq \mathrm{Hom}(h^{2d-1}(X) \otimes \mathbb{L}^{1-d}, h^1(X')) \\ &= \{c \in A^1(X \times X') \mid p_1' \circ c \circ p_{2d-1} = c\},\end{aligned}$$

and the isogeny β gives an isomorphism

$$\begin{aligned}\mathrm{Hom}(P_X, P_{X'}) \otimes \mathbf{Q} &\simeq \mathrm{Hom}(J_X, P_{X'}) \otimes \mathbf{Q} \\ &\simeq \{c \in A^1(X \times X') \mid \zeta'^* \circ c = 0 = c \circ \zeta_*\}.\end{aligned}$$

Since $\zeta'^* \circ p_1' = 0 = p_{2d-1} \circ \zeta_*$ by 4.4(i), the first subspace of $A^1(X \times X')$ is contained in the second. To get an inclusion in the other direction, suppose that $\zeta'^* \circ c = 0 = c \circ \zeta_*$, so that $c = \Omega(\nu)$ for some $\nu \in \mathrm{Hom}(J_X, P_{X'}) \otimes \mathbf{Q}$. Then

$$\begin{aligned}c \circ p_{2d-1}^? &= \tfrac{1}{n}\Omega(\nu) \circ i_* \circ i^* \circ \Omega(\beta) \\ &= \tfrac{1}{n}\Omega(\nu \circ J_i) \circ \Omega(P_i \circ \beta) \\ &= \tfrac{1}{n}\Omega(\nu \circ J_i \circ P_i \circ \beta) = \tfrac{1}{n}\Omega([\times n] \circ \nu) = c\end{aligned}$$

and by transposition $p_1^{?\prime} \circ c = c$ also. This settles the case $d > 2$ completely. If $d = 2$, then consider $c \circ p_1^? = c \circ \widetilde{\beta} \circ i_* \circ i^*$. We have $c \circ \widetilde{\beta} \circ i_* \in A^0(C \times X')$, so $c \circ \widetilde{\beta} \circ i_* = a[C \times X']$ for some $a \in \mathbf{Q}$. Since $\zeta'^* \circ c = 0$ we have $a = 0$. Thus $c \circ p_1^? = 0$; hence, $c \circ p_3 = c \circ p_3^? = c$ and likewise $p_1 \circ c = c$.

The second equality in the proposition follows from the first by duality.

4.6. By 4.4(i) we can write

$$h(X) = h^0(X) \oplus h^1(X) \oplus M \oplus h^{2d-1}(X) \oplus h^{2d}(X)$$

for some M. Suppose now that X has dimension 2. We can then define $p_2 = 1 - p_0 - p_1 - p_3 - p_4$ and $h^2(X) = (X, p_2, 0) = M$, and there is a decomposition

$$h(X) = \bigoplus_{i=0}^{4} h^i(X).$$

When $\sim$ is rational equivalence the cycle groups $A^j(h^i(X))$ are given by the following table:

$M =$	$h^0(X)$	$h^1(X)$	$h^2(X)$	$h^3(X)$	$h^4(X)$
$A^0(M) =$	$A^0(X)$	0	0	0	0
$A^1(M) =$	0	$A^1(X)^0$	$NS(X) \otimes \mathbf{Q}$	0	0
$A^2(M) =$	0	0	$\ker(\mathrm{Alb})$	$A^2(X)^0/\ker(\mathrm{Alb})$	$\mathbf{Q}$

REMARKS. (i) Murre calls the motives $h^1(X)$ and $h^{2d-1}(X)$ the *Picard* and *Albanese* motives of X, respectively, in view of the previous result. Observe that p_1 factors through $h(C)$, and in fact $h^1(X)$ is a direct summand of $h^1(C)$.

(ii) The minor differences between the construction we give and that in **[23]** are in slightly different normalisations of the projectors p_i; we use the zero-cycle ζ rather than an auxiliary rational point to normalise p_0 and p_{2d}, and in the case $d = 2$ we have made a different choice of p_1 and p_3 to preserve Poincaré duality and the analogue of the hard Lefschetz theorem.

(iii) To complete the picture in Proposition 4.5 one would like to know for surfaces X, X' the nature of the group $\operatorname{Hom}(h^2(X), h^2(X'))$ and, in particular, that it did not depend on the choice of equivalence relation on cycles. This would be answered by enough knowledge about the conjectural filtration on Chow groups; see §6.2 and Jannsen's paper **[10]** in these proceedings for more information.

(iv) The reader is warned against reading too much into Theorem 4.4(ii). In particular, in $\mathcal{M}_k^{\mathrm{rat}}$ it will not generally be the case that $h(X)$ has a decomposition into primitive pieces that satisfy a naïve analogue of the hard Lefschetz theorem. The analogous situation in the derived category of $\mathbf{Q}_\ell$-sheaves is considered in **[3]** and especially **[4]**; here we give a simple example.

Consider a curve X of genus g over an algebraically closed field embedded in projective space by a multiple of $P + Q$, where P, Q are points on X. We take $\sim$ to be rational equivalence. Then cup-product with the hyperplane section ξ does of course give an isomorphism

$$h^0(X) \xrightarrow{\sim} h^2(X) \otimes \mathbb{L}^{-1}$$

where the projectors are taken to be

$$p_0 = \tfrac{1}{2}[(P+Q) \times X] = {}^{\mathrm{t}}p_2, \qquad p_1 = 1 - p_0 - p_2.$$

However, it also induces a morphism

$$h^1(X) \to h^1(X) \otimes \mathbb{L}^{-1}$$

which is represented by a nonzero multiple of the cycle

$$\eta = (P, P) + (Q, Q) - (P, Q) - (Q, P).$$

Using arguments of Bloch, Mumford, and Roitman one sees that for $g > 1$ and P, Q sufficiently generic, η does not vanish. (This can be deduced fairly easily from **[2,** Theorem 3.1(a)**]**, for instance.) So $h^1(X)$ is not killed by $\bar{\xi}$ in general.

5. Abelian varieties (II)

5.1. In this section we will consider an abelian variety X over k of dimension g. For an integer n, write $[\times n]\colon X \to X$ for multiplication by n. Also let $\mu\colon X \times X \to X$ be the group law, $\varepsilon \in X(k)$ the identity element, and $\sigma\colon X \to X$ multiplication by -1.

THEOREM 5.2.

(i) *There is a unique decomposition in* $\mathcal{M}_k$

$$h(X) = \bigoplus_{i=0}^{2g} h^i(X)^{\mathrm{can}}$$

which is stable under $[\times n]^*$ *and such that* $[\times n]^*|_{h^i(X)}$ *is multiplication by the scalar* n^i, *for every* $n \in \mathbf{Z}$.

(ii) *The iterated product maps*

$$h(X) \otimes \cdots \otimes h(X) = h(X \times \cdots \times X) \xrightarrow{\mathrm{diag}^*} h(X)$$

induce for every $i \geq 0$ *isomorphisms*

$$\bigwedge^i h^1(X)^{\mathrm{can}} \xrightarrow{\sim} h^i(X)^{\mathrm{can}}.$$

(iii) *Let* $\xi \in A^1(X)$ *be the class of an ample symmetric line bundle on* X. *Then there is a commutative diagram*

$$\begin{array}{ccc} h^i(X) & \hookrightarrow & h(X) \\ \downarrow \wr & & \downarrow \xi^{g-i} \\ h^{2g-i}(X) \otimes \mathbb{L}^{i-g} & \hookrightarrow & h(X) \otimes \mathbb{L}^{i-g} \end{array}$$

in which the horizontal arrows are the obvious inclusions.

(Recall that $\xi \in A^*(X)$ is symmetric if $\sigma^*\xi = \xi$.) There is also a relation between (i) and the decomposition of the previous section. Let $p_i^{\mathrm{can}} \in \operatorname{End} h(X)$ be the projectors for which $h^i(X)^{\mathrm{can}} = (X, p_i^{\mathrm{can}}, 0)$, and for $i = 0, 1, 2g-1, 2g$ let p_i be the projectors defined in §4, using the class ξ of a very ample line bundle on X.

THEOREM 5.3. *If* ξ *is symmetric, then* $p_i^{\mathrm{can}} = p_i$ *for* $i = 0, 1, 2g-1$ *and* $2g$.

5.4. For numerical (or homological) equivalence, Theorem 5.2 was proved by Grothendieck, Kleiman, and Lieberman; see [**11**, Appendix to §2]. For rational equivalence, the existence of a decomposition $h(X) = \bigoplus h^i(X)$ in $\mathcal{M}_k^{\mathrm{rat}}$ satisfying (ii) was first proved by Manin and Shermenev [**26**], using Jacobians. An elegant proof of (i) was recently found by Deninger and Murre [**6**], who used the *Fourier transform* on Chow groups [**1**]. Künnemann extended their ideas to prove (ii); see his paper [**15**] in these proceedings, where he gives an elegant explicit formula for the projectors p_i^{can} using the Pontryagin product on the Chow groups of X. The work of Deninger-Murre and Künnemann applies more generally to abelian schemes over any smooth variety. Finally, in his 1992 Ph.D. thesis Künnemann not only proves (iii) but also obtains a complete Lefschetz decomposition of $h(X)$, just as one has in cohomology. For details see [**16**].

Rather than reproduce any of these arguments here, we will give an elementary proof of 5.2(i) from which 5.2(iii) and 5.3 will be easy consequences, and which uses Fourier theory at just one point (§5.7).

5.5. We first introduce some simple notation. If $i \in \mathbf{Z}$ then write $A^*(X)^{(i)}$ for the subspace comprising all $c \in A^*(X)$ such

$$[\times n]^*(c) = n^i c \quad \text{for every } n \in \mathbf{Z}.$$

Likewise, if X' is a second abelian variety, of dimension g', write $A^*(X \times X')^{(i,j)}$ for the set of all $c \in A^*(X \times X')$ such that

$$([\times m] \times [\times n])^*(c) = m^i n^j c \quad \text{for all } m,\ n \in \mathbf{Z}.$$

From §1.10 it follows that $c \in A^*(X \times X')^{(i,j)}$ if and only if, for every $n \in \mathbf{Z}$, one has identities of correspondences

$$[\times n]^*_{X'} \circ c = n^j c \quad \text{and} \quad c \circ [\times n]^*_X = n^{2g-i} c.$$

Therefore if $c \in A^*(X \times X')^{(i,j)}$ and $d \in A^*(X' \times X'')^{(r,s)}$ one has $d \circ c = 0$ unless $j = 2g' - r$.

5.6. Recall ([**21**, §II.6]) that if $\mathscr{L}$ is any line bundle on X then

$$[\times n]^* \mathscr{L} \simeq \mathscr{L}^{\otimes n(n+1)/2} \otimes (\sigma^* \mathscr{L})^{\otimes n(n-1)/2}.$$

Therefore $\xi \in A^1(X)$ is symmetric if and only if $\xi \in A^1(X)^{(2)}$, or equivalently $[\times n]^* \circ \bar{\xi} = n^2 \bar{\xi} \circ [\times n]^*$. Likewise, ξ is antisymmetric if and only if $\xi \in A^1(X)^{(1)}$ (which holds if and only if ξ is algebraically equivalent to zero).

5.7. If $\xi \in A^1(X)$ is the class of an ample line bundle, we have the usual formula

$$\deg(\xi^g) = g!d$$

where d^2 is the degree of the polarisation determined by ξ. Using Fourier theory one can show that if ξ is symmetric then

$$\xi^g = g!d[\varepsilon] \in A^g(X)$$

(see for example [**1**, middle of p. 249]).

5.8. Now pick a symmetric ξ which is the class of an ample line bundle, and write $\lambda = \mu^*\xi - pr_1^*\xi - pr_2^*\xi \in A^1(X \times X)$. Then

$$\lambda \in A^1(X \times X)^{(1,1)}.$$

Indeed, the restrictions of λ to the fibres of pr_1 and pr_2 are algebraically equivalent to zero, so $([\times m] \times [\times n])^* \lambda - mn\lambda$ has zero restriction to the fibres by 5.6, hence is zero.

REMARK. If $\phi_{\mathscr{L}}\colon X \to \widehat{X}$ is the usual homomorphism $x \to T_x^* \mathscr{L} \otimes \mathscr{L}^\vee$ attached to a line bundle $\mathscr{L}$ and $\xi = c_1(\mathscr{L})$, then λ is the pull-back by

$\mathrm{id}\times\phi_{\mathscr{L}}$ of the class of the Poincaré bundle on $X\times\widehat{X}$. This is the link between the formulae given below and Fourier theory.

5.9. *Proof of* 5.2(i), (iii). Firstly, it is obvious that the decomposition (i) is unique if it exists. This shows in particular that the choice of ξ is unimportant.

If $0\le i\le 2g$, define

$$f_i=\sum_{\max(0,i-g)\le j\le i/2}\frac{1}{j!(g-i+j)!(i-2j)!}pr_1^*\xi^j\cdot pr_2^*\xi^j\cdot\lambda^{i-2j},$$

$$q_i=\sum_{\max(0,i-g)\le j\le i/2}\frac{1}{j!(g-i+j)!(i-2j)!}pr_1^*\xi^{g-i+j}\cdot pr_2^*\xi^j\cdot\lambda^{i-2j}.$$

If $0\le i\le g$ then

(5.9.1) $q_i=pr_1^*\xi^{g-i}\cdot f_i=f_i\circ\xi^{g-i}$ and $q_{2g-i}=pr_2^*\xi^{g-i}\cdot f_i=\xi^{g-i}\circ f_i$.

Also we have

$$f_i\in A^i(X\times X)^{(i,i)}\quad\text{and}\quad q_i\in A^g(X\times X)^{(2g-i,i)}.$$

In particular $q_i\circ q_{i'}=0$ if $i\ne i'$. Now

$$\begin{aligned}\sum_{i=0}^{2g}q_i&=\sum_{\substack{0\le i\le 2g\\ \max(0,i-g)\le j\le i/2}}\frac{1}{j!(g-i+j)!(i-2j)!}pr_1^*\xi^{g-i+j}\cdot pr_2^*\xi^j\cdot\lambda^{i-2j}\\ &=\frac{1}{g!}(pr_1^*\xi+pr_2^*\xi+\lambda)^g\\ &=\frac{1}{g!}\mu^*\xi^g=d\mu^*[\varepsilon]=d[\Gamma_\sigma].\end{aligned}$$

Define $p_i^{\mathrm{can}}=\frac{1}{d}\sigma^*\circ q_i=\frac{(-1)^i}{d}q_i\in A^g(X\times X)^{(2g-i,i)}$. Then by the above,

$$\sum p_i^{\mathrm{can}}=1\quad\text{and}\quad\sum p_i^{\mathrm{can}}\circ p_i^{\mathrm{can}}=\left(\sum p_i^{\mathrm{can}}\right)^2=1.$$

This forces $p_i^{\mathrm{can}}\circ p_i^{\mathrm{can}}=p_i^{\mathrm{can}}$, and then $h^i(X)^{\mathrm{can}}=(X,p_i^{\mathrm{can}},0)$ satisfies (i).

By §5.6 and (i) the cup-product ξ^{g-i} maps $h^i(X)^{\mathrm{can}}$ into $h^{2g-i}(X)^{\mathrm{can}}\otimes\mathbb{L}^{i-g}$. Then the formulae (5.9.1) show that it is an isomorphism, giving (iii).

Corollary 5.10. *The natural map*

$$\mathrm{Hom}(X,X')\otimes\mathbf{Q}\to\mathrm{Hom}(h^1(X')^{\mathrm{can}},h^1(X)^{\mathrm{can}})$$

is an isomorphism.

Proof. Composition with ξ^{g-1} gives

$$\begin{aligned}\mathrm{Hom}(h^1(X')^{\mathrm{can}},h^1(X)^{\mathrm{can}})&\simeq\mathrm{Hom}(h^{2g-1}(X')^{\mathrm{can}}\otimes\mathbb{L}^{1-g},h^1(X)^{\mathrm{can}})\\ &=A^1(X'\times X)^{(1,1)}\\ &\simeq A^1(X'\times X)/pr_1^*A^1(X')+pr_2^*A^1(X)\\ &\simeq NS(X'\times X)\otimes\mathbf{Q}\\ &\simeq\mathrm{Hom}(X',X)\otimes\mathbf{Q},\end{aligned}$$

the last isomorphism coming from the polarisation ξ. It is easily checked that this is inverse to the map of the corollary.

5.11. A formal consequence of 5.2(i) is the eigenspace decomposition of the Chow groups; we have

$$A^d(X) = \mathrm{Hom}(\mathbb{L}^d, h(X)) = \bigoplus_{i=0}^{2g} \mathrm{Hom}(\mathbb{L}^d, h^i(X)^{\mathrm{can}}) = \bigoplus_{i=0}^{2g} A^d(X)^{(i)}.$$

Since composition with $\bar{\xi}^{g-i}$ gives an isomorphism

$$A^d(X)^{(i)} \xrightarrow{\sim} A^{g+d-i}(X)^{(2g-i)},$$

one gets by dimensional considerations that $A^d(X)^{(i)} = 0$ unless $d \le i \le g+d$. Similarly, we have

$$A^d(X \times X') = \bigoplus_{(i,j)} A^d(X \times X')^{(i,j)}$$

where the sum is over pairs $(i, j) \in \mathbf{Z}^2$ such that $d - g' \le i \le d + g$, $d - g \le j \le d + g'$, and $d \le i + j \le d + g + g'$. These decompositions are of course direct consequences of Fourier theory **[1, 6]**, and it was by using them that Deninger and Murre proved Theorem 5.2(i).

The interpretation of motives for rational equivalence as complexes (see §6.2) suggests that a stronger vanishing result

$$A^d(X)^{(i)} = \mathrm{Hom}(\mathbb{L}^d, h^i(X)^{\mathrm{can}}) = 0 \quad \text{for } i > 2d$$

holds. See **[10]**, where the conjectural filtration on Chow groups (of which this vanishing is a part) is explained.

Proof of Theorem 5.3. We have $p_{2g-i}^{\mathrm{can}} = {}^{t}p_i^{\mathrm{can}}$ and $p_{2g-i} = {}^{t}p_i$, so it is enough to treat p_0 and p_1. We have $p_0 = \frac{1}{g!d}(\xi^g \times [X])$, so by §5.7 this is the same as p_0^{can}.

For p_1 we first observe that the difficulties arising in the previous section in dimension 2 disappear here; for by §5.7 we have $\xi^g = d!g[\varepsilon]$ so that (4.4.7) obviously holds. So in every case we have $p_1 = p_1^?$ (in the notation of §4.3).

Lemma 5.12. $[\times n]^* \circ p_1 = np_1 = p_1 \circ [\times n]^*$.

Proof. By definition $p_1 = (1/m)\widetilde{\beta}\circ\bar{\xi}^{g-1}$ for a certain isogeny $\beta: X \to \widehat{X}$. We therefore have, using 3.10,

$$[\times n]^* \circ \widetilde{\beta} = \Omega(P_{[\times n]} \circ \beta) = \Omega([\times n] \circ \beta) = n\Omega(\beta).$$

This gives the first equality. For the second we have

$$p_1 \circ [\times n]^* = \frac{1}{m}\widetilde{\beta} \circ \bar{\xi}^{g-1} \circ [\times n]^* = \frac{n^{2-2g}}{m}\widetilde{\beta} \circ [\times n]^* \circ \bar{\xi}^{g-1}.$$

Now $[\times n]^* \circ [\times n]_* = n^{2g} \in \operatorname{End} h(X)$, so if we write $\widetilde{\beta} \circ [\times n]^* = \Omega(\beta')$ for some $\beta' \in \operatorname{Hom}(X, \widehat{X}) \otimes \mathbf{Q}$ then

$$n^{2g}\widetilde{\beta} = \widetilde{\beta} \circ [\times n]^* \circ [\times n]_* = \Omega(\beta' \circ J_{[\times n]}) = \Omega(\beta' \circ [\times n]) = n\Omega(\beta').$$

Therefore, $\widetilde{\beta} \circ [\times n]^* = n^{2g-1}\widetilde{\beta}$ and thus $p_1 \circ [\times n]^* = np_1$, proving the lemma.

Now from 5.12 we see that p_1, p_1^{can} commute and that $p_1 \circ p_1^{\mathrm{can}} = p_1$. Therefore, $h^1(X) = (X, p_1, 0)$ is a direct factor of $h^1(X)^{\mathrm{can}} = (X, p_1^{\mathrm{can}}, 0)$. But by Proposition 4.5 and Corollary 5.10, $\operatorname{End} h^1(X) = \operatorname{End} h^1(X)^{\mathrm{can}} = \operatorname{End} X \otimes \mathbf{Q}$. Therefore, $h^1(X) = h^1(X)^{\mathrm{can}}$.

REMARK. One can also check that β can be chosen to be the isogeny $\phi_{\mathscr{L}} : X \to \widehat{X}$ determined by $\xi = c_1(\mathscr{L})$. This gives an alternative verification of Theorem 5.3.

6. Further topics

6.1. (*Relative motives*). Let S be a smooth (not necessarily projective) variety, and let $\mathscr{V}_S$ be the category of smooth projective S-schemes. Then one can introduce the notion of a relative correspondence between two objects X, Y of $\mathscr{V}_S$ as a cycle class of codimension $\dim(X/S)$ on $X \times_S Y$. In **[6]** this is used to define the category $\mathscr{M}_S$ of motives over the base S. Deninger and Murre show there that if X/S is an abelian scheme of relative dimension d, then the relative motive $h(X/S)$ has a canonical decomposition $h(X/S) = \bigoplus_{i=0}^{2d} h^i(X/S)$.

If S itself is projective, then the functor $\mathscr{V}_S \to \mathscr{V}_k$ induces a functor $h(S, -) : \mathscr{M}_S \to \mathscr{M}_k$. In other words, relative motives should be thought of as (complexes of) sheaves on S. For an abelian scheme X/S one can then form various motives $h(S, h^i(X/S))$.

In an ideal world one would like a more general notion of relative motive: for example, if $j : S \hookrightarrow S'$ is the smooth compactification of a curve S and $X/S \in \mathscr{V}_S$, then one would like to be able to give a meaning to $j_* h(X/S)$ and define motives such as $h^p(S', j_* h^q(X/S))$.

There is one simple case in which it is possible to do this. Let $S/\mathbf{Q}$ be the standard modular curve of level $n \geq 3$ (which parameterises elliptic curves together with a chosen basis for the subgroup of n-division points), and let $\pi : E \to S$ be the universal elliptic curve. Let $j : S \to S'$ be the smooth compactification of S. Write $M = h^1(E/S)$, a relative motive over S, and let M_ℓ be the corresponding ℓ-adic sheaf $R^1\pi_*\mathbf{Q}_\ell$ on S. The parabolic cohomology groups

$${}_kW_\ell = H^1(S' \otimes \overline{\mathbf{Q}}, j_* \operatorname{Sym}^k M_\ell)$$

are representations of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which are pure of weight $k+1$. Since $\operatorname{Sym}^k M$ is a submotive of $h(X)$, where X is the k-fold fibre product of E over S, ${}_kW_\ell$ is in fact a subquotient of the cohomology of X', a smooth compactification of X. One can in this case prove **[25]** by brutal construction that there is a submotive ${}_kW \subset h(X')$ (in the category of Chow motives) such that the ℓ-adic cohomology of ${}_kW$ is the parabolic cohomology. It is tempting to write this motive as $h^1(S', j_* \operatorname{Sym}^k M)$, but there is as yet no way to make similar constructions in a general situation. We do not give any details since the proof uses arguments that are rather dissimilar to what has gone before.

6.2. In this final section we will try to give some vague hints of how the results described above fit in to a general (mainly conjectural) picture. To begin with we discuss the effect of choosing different equivalence relations on cycles.

• Numerical equivalence. This is the coarsest adequate equivalence relation. In **[9]** Jannsen proves, without using the standard conjectures, that the category of motives over a field is abelian and semisimple if and only if the equivalence relation used is numerical equivalence.

• Homological equivalence (with respect to a fixed Weil cohomology theory H^*). According to the standard conjectures (see **[11, 13]**) $\mathscr{M}_k^{\text{num}}$ and $\mathscr{M}_k^{\text{hom}}$ should coincide. At present, it is only for $\mathscr{M}_k^{\text{hom}}$ that we can define the realisation functors—but see also **[9,** Remark (4)**]**.

The standard conjectures also predict that in $\mathscr{M}_k^{\text{hom}}$ the motive $h(X)$ has a direct sum decomposition

$$h(X) = \bigoplus_{i=0}^{2\dim X} h^i(X) \tag{6.2.1}$$

such that the cohomology functor H^i factors through the projection $h(X) \to h^i(X)$. Now in the usual cohomology theories any homomorphism $H^i(X) \to H^j(Y)$ induced by an algebraic cycle is zero unless $i = j$—in the ℓ-adic theory this is by Deligne's proof of the Weil conjectures. Therefore, the decomposition (6.2.1) is unique.

• Rational equivalence. In general $\mathscr{M}_k^{\text{rat}}$ is not abelian, by Corollary 3.5. One expects, however (for reasons explained below), that every $h(X)$ has in $\mathscr{M}_k^{\text{rat}}$ a direct sum decomposition (6.2.1) in which the motives $h^i(X)$ are well defined up to unique isomorphism. Moreover, the filtration on $h(X)$ by subobjects

$$h^{\leq i}(X) = \bigoplus_{j \leq i} h^j(X)$$

should be uniquely determined and functorial with respect to inverse image and duality. The direct sum decomposition itself will not, however, be uniquely determined. The corresponding filtration on the Chow groups $A^d(X) = \operatorname{Hom}(\mathbb{L}^d, h(X))$ by the subgroups $\operatorname{Hom}(\mathbb{L}^d, h^{\leq i}(X))$ would be the conjectural filtration discussed in **[10]**. See **[24]**, where the idea of using such a "Chow–Künneth" decomposition to study the filtration is introduced and elaborated in detail.

If $\xi \in A^1(X)$ is the class of an ample divisor, then the hard Lefschetz theorem should hold in the following sense: $\bar{\xi}^i: h(X) \otimes \mathbb{L}^i \to h(X)$ respects the filtration up to a shift by $2i$ and induces an isomorphism between the subquotients $h^{\dim X - i}(X) \otimes \mathbb{L}^i$ and $h^{\dim X + i}(X)$.

In (say) ℓ-adic cohomology the ring of correspondences $\mathrm{Corr}^0(X, X)$ acts not only on $H^*(\overline{X}, \mathbf{Q}_\ell)$ but also on the object $R\Gamma(\overline{X}, \mathbf{Q}_\ell)$ in the derived category $\mathscr{D}^b(\operatorname{Spec} k, \mathbf{Q}_\ell)$ of complexes of ℓ-adic representations of $\mathrm{Gal}(\bar{k}/k)$. In this category there is the canonical filtration by truncation. Moreover, a theorem of Deligne ([**3**, Proposition 2.4]; see also [**4**]) states that there is an isomorphism (which is not unique) in $\mathscr{D}^b(\operatorname{Spec} k, \mathbf{Q}_\ell)$

$$R\Gamma(\overline{X}, \mathbf{Q}_\ell) \xrightarrow{\sim} \bigoplus_{i=0}^{2\dim X} H^i(\overline{X}, \mathbf{Q}_\ell)[-i].$$

One should therefore think of the object $h^i(X)$ in $\mathscr{M}_k^{\mathrm{rat}}$ as a complex concentrated in degree i, and of $\mathrm{Hom}(h^i(X), h^j(Y))$ as an analogue of

$$\mathrm{Hom}(H^i(\overline{X}, \mathbf{Q}_\ell)[-i], H^j(\overline{Y}, \mathbf{Q}_\ell)[-j]) = \mathrm{Ext}^{i-j}(H^i(\overline{X}, \mathbf{Q}_\ell), H^j(\overline{Y}, \mathbf{Q}_\ell)).$$

The image of the Lefschetz motive $\mathbb{L}$ in the ℓ-adic setting is $\mathbf{Q}_\ell(-1)[-2]$, so $A^d(h^i(X)) = \mathrm{Hom}(\mathbb{L}^d, h^i(X))$ should be analogous to

$$\mathrm{Hom}(\mathbf{Q}_\ell(-d)[-2d], H^i(\overline{X}, \mathbf{Q}_\ell)[-i]) = \mathrm{Ext}^{2d-i}(\mathbf{Q}_\ell(-d), H^i(\overline{X}, \mathbf{Q}_\ell)).$$

This fits in well with the formalism of mixed motives and suggests that there exists a "derived category of mixed motives". It would contain $\mathscr{M}_k^{\mathrm{num}}$ as the (abelian, semisimple!) subcategory formed of direct sums of "pure complexes" (of any weight) concentrated in degree zero and would contain $\mathscr{M}_k^{\mathrm{rat}}$ as the subcategory of "pure complexes of weight zero". In particular, this would imply that $\mathrm{Hom}(h^i(X), h^i(Y))$ was the same in $\mathscr{M}_k^{\mathrm{num}}$ and $\mathscr{M}_k^{\mathrm{rat}}$. For further indications along these lines we refer to the papers of Jannsen [**10**] and Levine [**18**], as well as other papers in these Proceedings.

References

1. A. Beauville, *Sur l'anneau de Chow d'une variété abélienne*, Math. Ann. **273** (1986), 647–651.
2. S. Bloch, *Some elementary theorems about algebraic cycles on abelian varieties*, Invent. Math. **37** (1976), 215–228.
3. P. Deligne, *Théorèmes de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 107–126.
4. ——, *Décompositions dans la catégorie dérivée*, these Proceedings, vol. 1, pp. 115–128.
5. P. Deligne and J. S. Milne, *Tannakian categories*, Lecture Notes in Math., vol. 900, Springer-Verlag, New York, 1982, 101–228.
6. C. Deninger and J. P. Murre, *Motivic decomposition of abelian schemes and Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.
7. J.-M. Fontaine and B. Mazur, *Représentations ℓ-adiques potentiellement semi-stables* (to appear).
8. J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: Cohomologie galoisienne et valeurs de fonctions L*, these Proceedings, vol. 1, pp. 599–706.

9. U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **197** (1992), 447–452.
10. ___, *Motivic sheaves and filtrations on Chow groups*, these Proceedings, vol. 1, pp. 245–302.
11. S. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968.
12. ___, *Motives*, Algebraic Geometry, Oslo, 1970 (F. Oort, ed.), Walters-Noordhoff, Groningen, 1972, pp. 53–82.
13. ___, *The standard conjectures*, these Proceedings, vol. 1, pp. 3–20.
14. B. Köck, *Chow motive and higher Chow theory of G/P*, Manuscripta Math. **70** (1991), 363–372.
15. K. Künnemann, *On the Chow motive of an abelian scheme*, these Proceedings, vol. 1, pp. 189–205.
16. ___, *A Lefschetz decomposition for Chow motives of abelian schemes*, Invent. Math. **113** (1993), 85–102.
17. S. Lang, *Abelian Varieties*, Wiley, New York, 1959
18. M. Levine, *On the derived motivic category*, preprint 1991.
19. Yu. I. Manin, *Correspondences, motives and monoidal correspondences*, Math. USSR-Sb. **6** (1968), 439–470.
20. B. Mitchell, *Theory of Categories*, Academic Press, New York, 1965.
21. D. Mumford, *Abelian Varieties*, Oxford Univ. Press, London, 1970.
22. D. Mumford and J. Fogarty, *Geometric Invariant Theory*, Springer-Verlag, New York, 1982.
23. J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.
24. ___, *On a conjectural filtration on the Chow groups of algebraic varieties*, Indag. Math. New Series, vol. 4 (1993), 177–201.
25. A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.
26. A. M. Shermenev, *The motive of an abelian variety*, Funct. Anal. **8** (1974), 55–61.
27. A. Weil, *Sur les Courbes Algébriques et les Variétés qui s'en Déduisent*, Hermann, Paris, 1948.
28. ___, *Sur les critères d'équivalence en géométrie algébrique*, Math. Ann. **128** (1954), 95–127.

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On the Chow Motive of an Abelian Scheme

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Introduction

For the Chow motive $h(A)$ of an abelian variety A over an algebraically closed field Shermenev has shown in **[Sh]** an isomorphism

$$h(A) \cong \wedge^{\cdot} h^1(A). \tag{1}$$

This is the motivic analogue of the well-known formula $H^{\cdot}(A) \cong \wedge^{\cdot} H^1(A)$ for the cohomology algebra of A with respect to any Weil cohomology $H^{\cdot}$. In particular, Shermenev's result yields a decomposition

$$h(A) \cong \bigoplus_{i=0}^{2\dim A} h^i(A) \tag{2}$$

such that the realisation $H^{\cdot}(h^i(A))$ is $H^i(A)$. But for Chow motives a decomposition (2) is by no means unique. In his formulation and proof of (1) Shermenev uses the description of $h^1(A)$ via the Jacobians of curves. Hence this description, and therefore Shermenev's decomposition, depends on choices.

In their paper **[DeMu]** Deninger and Murre have established a canonical functorial decomposition (2) not only for abelian varieties but also for abelian schemes. They use Fourier theory for abelian schemes as developed by Mukai **[Muk]** and Beauville **[Beau1, Beau2]** for abelian varieties in order to give a quite explicit description of the projectors π_i in Chow theory yielding the decomposition (2).

In this paper we give an explicit closed formula for the projectors π_i using the Pontrjagin $*$-product. Namely, we have

$$\pi_i = \frac{1}{(2g-i)!} \log([\Gamma_{\mathrm{id}}])^{*(2g-i)}. \tag{3}$$

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Here the logarithmic series is taken with respect to the Pontrjagin product on the abelian A-scheme $A \times_S A \overset{\mathrm{pr}_1}{\to} A$. So, more explicitly, formula (3) reads

$$\pi_i = \frac{1}{(2g-i)!}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}([\Gamma_{\mathrm{id}}]-[\Gamma_e])^{*n}\right\}^{*(2g-i)}$$

where $[\Gamma_{\mathrm{id}}]$ (resp. $[\Gamma_e]$) is the class of the diagonal Δ_A (resp. $[A\times_S e(S)]$) in the Chow ring $CH^*(A\times_S A, \mathbb{Q})$. Note, however, that Fourier theory enters our motivic considerations in an essential way through the proof of Theorem (1.4.1), which assures the convergence of the logarithmic series in (3). Up to torsion, this is a generalisation of Bloch's theorem about zero cycles on abelian varieties.

Using our description of the projectors π_i, the proof of a functorial isomorphism

$$\wedge^{\cdot} R^1(A/S) \overset{\sim}{\to} R(A/S) \tag{4}$$

for an abelian scheme A/S follows by quite explicit calculations involving only the basic properties of the Pontrjagin product.

Finally I would like to thank C. Deninger for drawing my attention to these problems and for many helpful discussions.

Notation and assumptions. Let k be a field, and fix a smooth quasi-projective connected base scheme S over k. We consider the category $\mathcal{V}(S)$ of smooth projective S-schemes $\lambda: X \to S$. We denote by $d(X/S)$ the fibre dimension of λ and by d the dimension of S. Let $CH^*(X)$ be the Chow ring of algebraic cycles with respect to rational equivalence graded by codimension. Set $CH^*(X,\mathbb{Q}) = CH^*(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. To a map $f: X\to Y$ in $\mathcal{V}(S)$ we associate the class $[\Gamma_f] := \gamma_{f_*}([X])$ in $CH^*(X\times_S Y, \mathbb{Q})$ of its graph over S. Here $\gamma_f: X \to X\times_S Y$ is the graph morphism of f. For an abelian scheme $\lambda: A\to S$ in $\mathcal{V}(S)$ we denote the relative dimension of λ by g. To A/S belong the multiplication map $m = m_A: A\times_S A \to A$ and the zero section $e = e_A: S\to A$. For $n\in\mathbb{Z}$ the map $\mathrm{mult}(n): A\to A$ is n-multiplication on A, and for $n_1, \dots, n_i\in\mathbb{Z}$ we set $\mathrm{mult}(n_1,\dots,n_i) = \mathrm{mult}(n_1)\times_S\cdots\times_S \mathrm{mult}(n_i): A^i = A\times_S\cdots\times_S A\to A^i$. As usual S_i denotes the ith symmetric group, and $\mu_2 = \{1,-1\}$.

1. Fourier theory for abelian schemes

(1.1) Relative correspondences. Let X/S and Y/S be in $\mathcal{V}(S)$. The elements in $CH^*(X\times_S Y, \mathbb{Q})$ will be called relative correspondences between X and Y over S. For $\alpha\in CH^*(X\times_S Y,\mathbb{Q})$ and $\beta\in CH^*(Y\times_S Z,\mathbb{Q})$ the composition of α and β is defined as usual by

$$\beta\circ\alpha = p^{XYZ}_{XZ*}(p^{XYZ*}_{XY}(\alpha)\cdot p^{XYZ*}_{YZ}(\beta));$$

α has a transpose ${}^t\alpha \in CH^*(Y \times_S X, \mathbb{Q})$ defined by ${}^t\alpha = \tau_*(\alpha)$ where $\tau: X \times_S Y \to Y \times_S X$ reverses the factors. We will frequently use the following formulas:

PROPOSITION (1.1.1). *For X, Y, Z in $\mathscr{V}(S)$, morphisms $f: X \to Y$, $f': Y \to X$, $g: Y \to Z$, $g': Z \to Y$ over S and classes α in $CH^*(X \times_S Y, \mathbb{Q})$ and β in $CH^*(Y \times_S Z, \mathbb{Q})$ we have*

$$[\Gamma_g] \circ \alpha = (\mathrm{id}_X \times_S g)_*(\alpha), \qquad [{}^t\Gamma_{g'}] \circ \alpha = (\mathrm{id}_X \times_S g')^*(\alpha),$$
$$\beta \circ [\Gamma_f] = (f \times_S \mathrm{id}_Z)^*(\beta), \quad \beta \circ [{}^t\Gamma_{f'}] = (f' \times_S \mathrm{id}_Z)_*(\beta).$$

PROOF. [**DeMu**, Proposition 1.2.1; **Fult**, Proposition 16.1.1]. □

For later use we define for $\alpha \in CH^*(X, \mathbb{Q})$, $\beta \in CH^*(Y, \mathbb{Q})$, $\gamma \in CH^*(X \times_S Y, \mathbb{Q})$, and $\delta \in CH^*(W \times_S Z, \mathbb{Q})$

$$\alpha \times_S \beta := p_X^{XY*}(\alpha) \cdot p_Y^{XY*}(\beta) \in CH^*(X \times_S Y, \mathbb{Q}),$$
$$\gamma \otimes_S \delta := t^*(\gamma \times_S \delta) \in CH^*((X \times_S W) \times_S (Y \times_S Z), \mathbb{Q})$$

where $t: (X \times_S W) \times_S (Y \times_S Z) \to (X \times_S Y) \times_S (W \times_S Z)$ is the map permuting the factors.

(1.2) Pontrjagin product. We consider an abelian scheme $\lambda: A \to S$ in $\mathscr{V}(S)$. In this case we have the Pontrjagin product on $CH^*(A, \mathbb{Q})$ defined by

$$\alpha * \beta = m_{A*}(\alpha \times_S \beta)$$

for $\alpha, \beta \in CH^*(A, \mathbb{Q})$. We observe

LEMMA (1.2.1). *If $g: A \to B$ is a homomorphism of abelian schemes in $\mathscr{V}(S)$, then*

$$g_*(\alpha * \beta) = (g_*\alpha) * (g_*\beta)$$

for $\alpha, \beta \in CH^*(A, \mathbb{Q})$.

PROOF. The proof is straightforward. We denote the projection on the ith factor of $A \times_S A$, $B \times_S B$, and $B \times_S A$ by p_i, q_i, and r_i $(i = 1, 2)$. Using the homomorphism property $m_B \circ (g \times_S g) = g \circ m_A$ and base change with respect to the Cartesian diagrams

$$\begin{array}{ccc} A \times_S A & \xrightarrow{g \times_S \mathrm{id}} & B \times_S A \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle r_1} \\ A & \xrightarrow{g} & B \end{array} \qquad \begin{array}{ccc} B \times_S A & \xrightarrow{\mathrm{id} \times_S g} & B \times_S B \\ \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle q_2} \\ A & \xrightarrow{g} & B \end{array}$$

we obtain

$$\begin{aligned} g_*(\alpha * \beta) &= g_* m_{A*}(p_1^*\alpha \cdot p_2^*\beta) \\ &= m_{B*}(\mathrm{id} \times_S g)_*(g \times_S \mathrm{id})_*(p_1^*\alpha \cdot (g \times_S \mathrm{id})^* r_2^*\beta) \\ &= m_{B*}(\mathrm{id} \times_S g)_*((g \times_S \mathrm{id})_* p_1^*\alpha \cdot r_2^*\beta) \\ &= m_{B*}(\mathrm{id} \times_S g)_*((q_1 \circ (\mathrm{id} \times_S g))^* g_*\alpha \cdot r_2^*\beta) \\ &= m_{B*}(q_1^* g_*\alpha \cdot (\mathrm{id} \times_S g)_* r_2^*\beta) \\ &= (g_*\alpha) * (g_*\beta). \quad \square \end{aligned}$$

COROLLARY (1.2.2). *Let $f: A \to B$ be a homomorphism of abelian schemes in $\mathscr{V}(S)$, and consider $A \times_S A$ and $A \times_S B$ as abelian A-schemes via the projection on the first factor. Then for $\alpha, \beta \in CH^*(A \times_S A, \mathbb{Q})$*

$$[\Gamma_f] \circ (\alpha * \beta) = ([\Gamma_f] \circ \alpha) * ([\Gamma_f] \circ \beta).$$

PROOF. Use (1.1.1) and apply (1.2.1) with $g = (\mathrm{id}_A \times_S f)$. $\square$

(1.3) Fourier theory. For an abelian scheme $\lambda: A \to S$ in $\mathscr{V}(S)$ let $\mu: \widehat{A} \to S$ be the dual abelian scheme representing the relative Picard functor $\mathrm{Pic}^0_{A/S}$. There is an element in

$$\mathrm{Pic}_{A/S}(\widehat{A}) \cong \mathrm{Pic}(A \times_S \widehat{A})/p_2^*\mathrm{Pic}(\widehat{A})$$

corresponding to $\mathrm{id} \in \widehat{A}(\widehat{A})$. Since $(e_A \times_S \mathrm{id})^* p_2^* = \mathrm{id}$ this can be lifted uniquely to an element L in $\mathrm{Pic}(A \times_S \widehat{A})$ with $(e_A \times_S \mathrm{id})^* L = 0$. Now by functoriality $a \in \widehat{A}(S)$ yields a commutative square

$$(5) \qquad \begin{array}{ccc} \widehat{A}(\widehat{A}) & \longrightarrow & \mathrm{Pic}_{A/S}(\widehat{A}) \\ \big\downarrow{\scriptstyle \widehat{A}(a)} & & \big\downarrow{\scriptstyle (\mathrm{id}_A \times_S a)^*} \\ \widehat{A}(S) & \longrightarrow & \mathrm{Pic}_{A/S}(S) = \mathrm{Pic}(A)/\lambda^*\mathrm{Pic}(S). \end{array}$$

We have $\widehat{A}(a)(\mathrm{id}) = a \in \widehat{A}(S)$. For $a = e_{\widehat{A}}$ we obtain $(\mathrm{id}_A \times_S e_{\widehat{A}})^* L = \lambda^* N$ for some $N \in \mathrm{Pic}(S)$ as $\mathrm{id} \in \widehat{A}(\widehat{A})$ is mapped to the zero element in $\widehat{A}(S)$. But

$$N = e_A^* \circ \lambda^* N = e_A^* \circ (\mathrm{id}_A \times_S e_{\widehat{A}})^* L = e_{\widehat{A}}^* \circ (e_A \times_S \mathrm{id}_{\widehat{A}})^* L = 0;$$

hence, $(\mathrm{id}_A \times_S e_{\widehat{A}})^* L = 0$. We call a line bundle $\mathscr{L}$ in the class of L a Poincaré line bundle rigidified along the zero sections, and we denote the associated divisor class in $CH^1(A \times_S \widehat{A})$ by $l = c_1(\mathscr{L})$. As a consequence of (5), we see that under the homomorphism $\widehat{A}(S) \to \mathrm{Pic}_{A/S}(S)$ an element $a \in \widehat{A}(S)$ is mapped to the class of $(\mathrm{id}_A \times_S a)^* L$. So for $a, b \in \widehat{A}(S)$ there is an $N \in \mathrm{Pic}(S)$ with

$$(\mathrm{id}_A \times_S a)^* L \otimes (\mathrm{id}_A \times_S b)^* L = (\mathrm{id}_A \times_S (a+b))^* L \otimes \lambda^* N.$$

As above we obtain $N = 0$. By the biduality theorem for abelian schemes we see that for $a, b \in A(S)$

$$i_a^* l + i_b^* l = i_{a+b}^* l \qquad (6)$$

where i_a, i_b, and i_{a+b} denote $a \times_S \mathrm{id}_{\widehat{A}}$, $b \times_S \mathrm{id}_{\widehat{A}}$, and $(a+b) \times_S \mathrm{id}_{\widehat{A}}$, respectively.

DEFINITION. The Fourier transform F on A is the relative correspondence $F = F_A$ in $CH^*(A \times_S \widehat{A}, \mathbb{Q})$ given by

$$F = \mathrm{ch}(L) = \exp(l) = 1 + l + \frac{1}{2!} l^2 + \cdots .$$

The homomorphism $F_{CH}\colon CH^*(A, \mathbb{Q}) \to CH^*(\widehat{A}, \mathbb{Q})$ is associated with F by $F_{CH}(\alpha) = F \circ \alpha$ where $\alpha \in CH^*(A, \mathbb{Q})$ is regarded as a relative correspondence in $CH^*(S \times_S A, \mathbb{Q})$. By the biduality of A we can define $\widehat{F} = F_{\widehat{A}}$ and $\widehat{F}_{CH}\colon CH^*(\widehat{A}, \mathbb{Q}) \to CH^*(A, \mathbb{Q})$ in the same way.

The following equations of correspondences yield the inversion formula in Fourier theory and the compatibility of the Fourier transform with intersection and Pontrjagin product. They have been proved by Beauville **[Beau1]** for abelian varieties. The generalisation of (i) to abelian schemes has been done by Deninger and Murre.

THEOREM (1.3.1) (Mukai, Beauville, Deninger, Murre).
(i) $\widehat{F} \circ F = (-1)^g [{}^t\Gamma_{\mathrm{mult}(-1)}]$ *in* $CH^*(A \times_S A, \mathbb{Q})$.
(ii) $F \circ [\Gamma_m] = [{}^t\Gamma_\Delta] \circ (F \otimes_S F)$ *in* $CH^*(A \times_S A \times_S \widehat{A}, \mathbb{Q})$.
(iii) $F \circ [{}^t\Gamma_\Delta] = (-1)^g [\Gamma_m] \circ (F \otimes_S F)$ *in* $CH^*(A \times_S A \times_S \widehat{A}, \mathbb{Q})$.

PROOF. The proof of (i) is given in **[DeMu**, Corollary 2.22]. In this proof the following is shown as well. Consider the map $m_A \times_S \mathrm{id}\colon A \times_S A \times_S \widehat{A} \to A \times_S \widehat{A}$. Then

$$(m_A \times_S \mathrm{id})^* L = p_{13}^* L \otimes p_{23}^* L \qquad (7)$$

in $\mathrm{Pic}(A \times_S A \times_S \widehat{A})$ where p_{ij} denotes the canonical projections from $A \times_S A \times_S \widehat{A}$. Using (7) and (1.1.1) we obtain

$$\begin{aligned} F \circ [\Gamma_m] &= (m_A \times_S \mathrm{id})^* F \\ &= \exp(p_{13}^* l + p_{23}^* l) \\ &= p_{13}^* F \cdot p_{23}^* F \\ &= (q_1 \circ t' \circ (\mathrm{id}_{A \times_S A} \times_S \Delta_{\widehat{A}}))^* F \cdot (q_2 \circ t' \circ (\mathrm{id}_{A \times_S A} \times_S \Delta_{\widehat{A}}))^* F \\ &= [{}^t\Gamma_\Delta] \circ t'^* (F \times_S F) \\ &= [{}^t\Gamma_\Delta] \circ (F \otimes_S F) \end{aligned}$$

where $t'\colon (A \times_S A) \times_S (\widehat{A} \times_S \widehat{A}) \to (A \times_S \widehat{A}) \times_S (A \times_S \widehat{A})$ is given by $(a, b, c, d) \mapsto (a, c, b, d)$ and q_i is the ith projection $(i = 1, 2)$ from

$(A \times_S \widehat{A}) \times_S (A \times_S \widehat{A})$ to $A \times_S \widehat{A}$. To demonstrate (iii) we use (ii) for $\widehat{F}$. We apply F from the left and $F \otimes_S F$ from the right and obtain

$$F \circ \widehat{F} \circ [\Gamma_m] \circ (F \otimes_S F) = F \circ [{}^t\Gamma_\Delta] \circ ((\widehat{F} \circ F) \otimes_S (\widehat{F} \circ F)).$$

This yields by using (i)

$$[\Gamma_m] \circ (F \otimes_S F) = (-1)^g [{}^t\Gamma_{\mathrm{mult}(-1)}] \circ F \circ [{}^t\Gamma_\Delta] \circ [{}^t\Gamma_{\mathrm{mult}(-1,-1)}].$$

Since

$$[{}^t\Gamma_\Delta] \circ [{}^t\Gamma_{\mathrm{mult}(-1,-1)}] = [{}^t\Gamma_{\mathrm{mult}(-1)}] \circ [{}^t\Gamma_\Delta]$$

and

$$F \circ [{}^t\Gamma_{\mathrm{mult}(-1)}] = \exp(-l) = [{}^t\Gamma_{\mathrm{mult}(-1)}] \circ F,$$

the claim follows. □

COROLLARY (1.3.2). *For $\alpha, \beta \in CH^*(A, \mathbb{Q})$ we have*

(i) $\widehat{F}_{\mathrm{CH}} \circ F_{\mathrm{CH}}(\alpha) = (-1)^g \mathrm{mult}(-1)^*(\alpha)$,
(ii) $F_{\mathrm{CH}}(\alpha * \beta) = F_{\mathrm{CH}}(\alpha) \cdot F_{\mathrm{CH}}(\beta)$,
(iii) $F_{\mathrm{CH}}(\alpha \cdot \beta) = (-1)^g F_{\mathrm{CH}}(\alpha) * F_{\mathrm{CH}}(\beta)$.

PROOF. Apply (1.3.1) to α and $\alpha \times_S \beta$. □

DEFINITION. For $a \in A(S)$ we call $[\Gamma_a] := a_*[S] \in CH^*(A, \mathbb{Q})$ the graph of a.

Regard $A \times_S A$ as an abelian A-scheme via the projection on the first factor. If we identify $(A \times_S A)(A)$ and $\mathrm{Hom}_S(A, A)$, then this definition is compatible with the defintion of the graph of a morphism from A to A over S.

For $\alpha \in CH^*(A, \mathbb{Q})$ we set $\alpha^{*n} = \alpha * \cdots * \alpha$ (n times, $n \geq 0$).

The following proposition is the generalisation of **[Beau1**, Proposition 4, §3] to the relative case. The proof remains just the same.

PROPOSITION (1.3.3) (Beauville). *For $a \in A(S)$ we have*

(i) $F_{\mathrm{CH}}([\Gamma_a]) = \exp(i_a^* l)$,
(ii) $(-1)^{g+1} \widehat{F}_{\mathrm{CH}}(i_a^* l) = \sum_{n=1}^{g+d} \frac{(-1)^{n-1}}{n} ([\Gamma_2] - [\Gamma_e])^{*n}$.

PROOF. Base change with respect to the Cartesian diagram

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{i_a} & A \times_S \widehat{A} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle p_1} \\ S & \xrightarrow{a} & A \end{array}$$

yields

$$\begin{aligned} F_{CH}([\Gamma_a]) &= p_{2*}(p_1^* a_*[S] \cdot \exp(l)) \\ &= p_{2*}(i_{a*}\mu^*[S] \cdot \exp(l)) \\ &= p_{2*} i_{a*}([\widehat{A}] \cdot i_a^* \exp(l)) \\ &= \exp(i_a^* l). \end{aligned}$$

To prove the second assertion we consider $CH^*(A, \mathbb{Q})$ as a ring with respect to intersection product with unit element $1 = [A]$:

$$-i_a^* l = \log(1 + (\exp(-i_a^* l) - 1)) = \sum_{n=1}^{g+d} \frac{(-1)^{n-1}}{n} (\exp(-i_a^* l) - 1)^n.$$

Observe that the terms of the sum vanish for $n \geq \dim \widehat{A} = g + d$. By (6) we have $-i_a^* l = i_{-a}^* l$, $i_e^* l = 0$, and $F_{\mathrm{CH}}([\Gamma_e]) = \exp(i_e^* l) = 1$. Using (1.2.2) and (1.3.2) we obtain

$$\begin{aligned}\widehat{F}_{\mathrm{CH}}((\exp(-i_a^* l) - 1)^n) &= \widehat{F}_{\mathrm{CH}} \circ F_{\mathrm{CH}}(([\Gamma_{-a}] - [\Gamma_e])^{*n}) \\ &= (-1)^g [\Gamma_{\mathrm{mult}(-1)}]([\Gamma_{-a}] - [\Gamma_e])^{*n} \\ &= (-1)^g ([\Gamma_a] - [\Gamma_e])^{*n},\end{aligned}$$

which proves (ii). □

COROLLARY (1.3.4). $[\Gamma_a] * [\Gamma_b] = [\Gamma_{a+b}]$ *for all* $a, b \in A(S)$.

PROOF. $F_{\mathrm{CH}}([\Gamma_a] * [\Gamma_b]) = F_{\mathrm{CH}}([\Gamma_a]) \cdot F_{\mathrm{CH}}([\Gamma_b]) = \exp(i_a^* l) \cdot \exp(i_b^* l) = \exp(i_a^* l + i_b^* l) = \exp(i_{a+b}^* l) = F_{\mathrm{CH}}([\Gamma_{a+b}])$. Our claim follows from the inversion formula (1.3.2i). □

(1.4) Application to relative zero cycles. Up to torsion, we generalize Theorem (0.1) of [Bloch] about zero cycles on abelian varieties to the case of abelian schemes. Bloch's theorem is also proved in **[Beau1]** by using Fourier theory, and this proof still works for abelian schemes. $I(A/S)$ denotes the $\mathbb{Q}$-linear subspace of $CH^g(A, \mathbb{Q})$ generated by the elements $[\Gamma_a] - [\Gamma_e]$ for $a \in A(S)$. By Corollary (1.3.4) $I(A/S)$ is a subring of $CH^g(A, \mathbb{Q})$ with respect to the Pontrjagin product ring structure.

THEOREM (1.4.1). $I(A/S)^{*(g+d+1)} = 0$.

PROOF. For $a_1, \ldots, a_n \in A(S)$ we get

$$F_{CH}(([\Gamma_{a_1}] - [\Gamma_e]) * \cdots * ([\Gamma_{a_n}] - [\Gamma_e])) = (\exp(i_{a_1}^* l) - 1) \cdots (\exp(i_{a_n}^* l) - 1).$$

Clearly this term vanishes for $n > \dim \widehat{A} = g + d$. Now the assertion follows from the inversion formula. □

REMARKS (1.4.2). (i) This result justifies the notation

$$(-1)^{g+1} \widehat{F}_{CH}(i_a^* l) = \log([\Gamma_a])$$

for formula (1.3.1ii). Using (6) or the formal power series identity

$$\log(1+x) + \log(1+y) = \log((1+x)(1+y))$$

we see that

$$A(S) \to I(A/S), \qquad a \mapsto \log([\Gamma_a])$$

is a homomorphism of groups.

(ii) Comparing codimensions, we see that in order to prove (1.3.4) and (1.4.1) we need only the weak form of the inversion formula as given in **[DeMu**, Proposition 2.9].

2. Relative motives

In **[DeMu]** the category $\mathscr{M}^0(S)$ of relative Chow motives over the base S with respect to graded correspondences is introduced. This is a straightforward generalisation to the relative case of Grothendieck's theory of motives as described in **[Man]**. We give a different description of the category $\mathscr{M}^0(S)$ following Jannsen, **[Jann**, §4]. This is easily seen to be equivalent to the one constructed in **[DeMu]**.

(2.1) The category $\mathscr{M}(S)$. The category $\mathscr{M}(S)$ of relative Chow motives with respect to ungraded correspondences is given as follows: The objects of $\mathscr{M}(S)$ are pairs (λ, p), where $\lambda: X \to S$ is in $\mathscr{V}(S)$, and $p \in CH^*(X \times_S X, \mathbb{Q})$ is an idempotent in the ring of relative correspondences. We often abbreviate (λ, p) by (X, p). The morphisms are given by

$$\mathrm{Hom}_{\mathscr{M}(S)}((X, p), (Y, q)) = \{q \circ \alpha \circ p \mid \alpha \in CH^*(X \times_S Y, \mathbb{Q})\}$$

where composition of morphisms is induced by composition of correspondences. $\mathrm{Hom}_{\mathscr{M}(S)}$ carries a natural grading. For $i \in \mathbb{Z}$ the graded part $\mathrm{Hom}_{\mathscr{M}(S)}((X, p), (Y, q))$ consists of all elements $q \circ \alpha \circ p$ as above which satisfy

$$q \circ \alpha \circ p \in \bigoplus_j CH^{d(X_j/S)+i}(X_j \times_S Y, \mathbb{Q})$$

where $X = \coprod_j X_j$ is the decomposition of X into connected components. We have $\mathrm{Hom}_{\mathscr{M}(S)} = \bigoplus_i \mathrm{Hom}^i_{\mathscr{M}(S)}$ and $\alpha \circ \beta \in \mathrm{Hom}^{i+j}_{\mathscr{M}(S)}$ for $\alpha \in \mathrm{Hom}^i_{\mathscr{M}(S)}$ and $\beta \in \mathrm{Hom}^j_{\mathscr{M}(S)}$. $\mathscr{M}(S)$ is an additive, pseudoabelian (i.e., every projector has a kernel and a cokernel) $\mathbb{Q}$-category, which carries a natural $\otimes$-product

$$(X, p) \oplus_S (Y, q) = \left(X \coprod Y, p \coprod q\right),$$
$$(X, p) \otimes_S (Y, q) = (X \times_S Y, p \otimes_S q)$$

where $\coprod$ denotes disjoint union.

(2.2) The category $\mathscr{M}^0_+(S)$ of effective relative Chow motives. Objects of $\mathscr{M}^0_+(S)$ are pairs (X, p) in $\mathscr{M}(S)$ with the projector p of degree zero. Morphisms in $\mathscr{M}^0_+(S)$ are morphisms of degree zero in $\mathscr{M}(S)$, i.e., $\mathrm{Hom}_{\mathscr{M}^0_+(S)} = \mathrm{Hom}^0_{\mathscr{M}(S)}$. There is a canonical contravariant functor

$$R: \mathscr{V}(S) \to \mathscr{M}^0_+(S)$$

defined by $R(\lambda) = (X, [{}^t\Gamma_{\mathrm{id}}])$ for $\lambda: X \to S$ in $\mathscr{V}(S)$ and $R(f) = [{}^t\Gamma_f]$ for a morphism $f: X \to Y$ in $\mathscr{V}(S)$. We abbreviate $R(\lambda)$ and $R(X/S)$. A direct consequence of the definitions is the motivic Künneth formula

$$R((X\times_S Y)/S) = R(X/S)\otimes_S R(Y/S).$$

(2.3) The category $\mathscr{M}^0(S)$. The objects of $\mathscr{M}^0(S)$ are triples (λ, p, m) where (λ, p) is an object of $\mathscr{M}^0_+(S)$ and $m\in\mathbb{Z}$. As above we write (X,p,m) for $(\lambda\colon X\to S, p, m)$. The morphisms in $\mathscr{M}^0(S)$ are given by

$$\mathrm{Hom}_{\mathscr{M}^0(S)}((X, p, m), (Y, q, n)) = \mathrm{Hom}^{n-m}_{\mathscr{M}(S)}((X, p), (Y, q)).$$

Composition is induced by composition of correspondences. $\mathscr{M}^0(S)$ is an additive, pseudoabelian $\mathbb{Q}$-category that carries a canonical $\otimes$-product

$$(X, p, m)\otimes_S (Y, q, n) = (X\times_S Y, p\otimes_S q, m+n).$$

Furthermore $\mathscr{M}^0(S)$ is a rigid tensor category in the sense of **[DeMi]**. The functor

$$\cdot^\vee\colon \mathscr{V}(S)\to\mathscr{M}^0(S)$$

which maps an S-scheme X of pure relative dimension n to $R(X/S)^\vee = (X, [\Gamma_{\mathrm{id}}], n)$ and a morphism f to $[\Gamma_f]$ factors over $\mathscr{M}^0(S)$. So for every object M in $\mathscr{M}^0(S)$ we obtain a dual $M^\vee$ yielding a canonical isomorphism

$$\mathrm{Hom}(P\otimes M, N) = \mathrm{Hom}(P, M^\vee\otimes N)$$

functorial in P, M, and N. For an S-scheme X of pure relative dimension n in $\mathscr{V}(S)$ the dual of $M=(X, p, m)$ is given by $M^\vee = (X, {}^tp, n-m)$. In particular, we obtain the motivic Poincaré duality

$$R(X/S)^\vee = R(X/S)(n)$$

where twisting in $\mathscr{M}^0(S)$ is defined by $(X, p, m)(n) = (X, p, m+n)$. For a more precise result compare (3.1.2i).

Next we describe the direct sum in $\mathscr{M}^0(S)$. For this we have to recall the definition of the Lefschetz motive $L_S = (\mathbb{P}^1_S, \pi_2, 0)$ from **[DeMu**, 1.4]. We have

$$(X, p, 0)\oplus_S (Y, q, 0) = \left(X\coprod Y, p\coprod q, 0\right).$$

In general, we have to observe that the direct sum is compatible with twisting and that $(X, p, m)\cong (X, p, 0)\otimes_S L_S^{-m}$ for $m\le 0$. Then we obtain for $r\ge\max(m, n)$ that $(X, p, m)\oplus_S (Y, q, n)$ is isomorphic to

$$\left(X\times_S (\mathbb{P}^1_S)^{r-m}\coprod Y\times_S (\mathbb{P}^1_S)^{r-n}, p\otimes_S (\pi_2^{\otimes(r-m)})\coprod q\otimes_S (\pi_2^{\otimes(r-n)}), r\right).$$

Identifying (X, p) with $(X, p, 0)$ we can regard $\mathscr{M}^0_+(S)$ as a full subcategory of $\mathscr{M}^0(S)$.

(2.4) Relation with the derived category of $\mathbb{Q}_l$-sheaves. Let $D^b(S, \mathbb{Q}_l)$ be the bounded derived category of $\mathbb{Q}_l$-sheaves on S. As explained in **[DeMu]**, the functor $R_l\colon \mathcal{V}(S) \to D^b(S, \mathbb{Q}_l)$ mapping $\lambda\colon X \to S$ to $R\lambda_*\mathbb{Q}_l$ extends to a functor

$$R_l\colon \mathcal{M}^0(S) \to D^b(S, \mathbb{Q}_l).$$

We set $R_l^i = H^i \circ R_l$ and $R_l^* = \bigoplus_{i\in\mathbb{Z}} R_l^i$.

(2.5) Multiplicative structures. A multiplicative structure for a relative motive M in $\mathcal{M}(S)$ is by definition a morphism $M \otimes_S M \to M$ in $\mathcal{M}(S)$. We call a morphism α in $\mathcal{M}(S)$ compatible with the multiplicative structures if there is a commutative diagram

$$\begin{array}{ccc} M \otimes_S M & \longrightarrow & M \\ \big\downarrow{\scriptstyle \alpha\otimes_S\alpha} & & \big\downarrow{\scriptstyle \alpha} \\ N \otimes_S N & \longrightarrow & N. \end{array}$$

EXAMPLES (2.5.1). (i) For $\lambda\colon X \to S$ in $\mathcal{V}(S)$ the relative motive $R(X/S)$ carries a canonical multiplicative structure induced by the diagonal embedding $\Delta\colon X \to X \times_S X$:

$$R(X/S) \otimes_S R(X/S) \xrightarrow{[{}^t\Gamma_\Delta]} R(X/S).$$

This is called the canonical multiplicative structure on $R(X/S)$.

(ii) For an abelian scheme $\lambda\colon A \to S$ in $\mathcal{V}(S)$ we have the Pontrjagin multiplicative structure defined via the multiplication map m of A/S:

$$R(A/S) \otimes_S R(A/S) \xrightarrow{[\Gamma_m]} R(A/S).$$

(iii) For A/S as in (ii) the Fourier transform F can be regarded as a morphism in $\mathcal{M}(S)$. Then Theorem (1.3.1) states the following: F yields an isomorphism between $R(A/S)$ and $R(\widehat{A}/S)$ in $\mathcal{M}(S)$ with inverse $(-1)^g[{}^t\Gamma_{\mathrm{mult}(-1)}] \circ \widehat{F}$, and this isomorphism is compatible with the canonical multiplicative structure on $R(A/S)$ and the Pontrjagin multiplicative structure on $R(\widehat{A}/S)$.

(2.6) Exterior product. Let $M = (X, p, m)$ be a relative motive in $\mathcal{M}^0(S)$. Then we set for $i \geq 0$

$$s_i = \frac{1}{i!}\sum_{\sigma\in S_i}[{}^t\Gamma_\sigma] \in CH^g(X^i \times_S X^i, \mathbb{Q}),$$

$$\wedge^i M = (X^i, s_i \circ (p \otimes_S \cdots \otimes_S p), m \cdot i)$$

where $\sigma\colon X^i = X \times_S \cdots \times_S X \to X^i$ is the map permuting the factors. We call M finite dimensional if there is an i_0 such that $\wedge^i M = 0$ for all $i > i_0$.

In this case we define $\wedge^{\cdot} M = \bigoplus_{i=0}^{i_0} \wedge^i M$. Then $\wedge^{\cdot} M$ carries a canonical multiplicative structure induced by the canonical projections

$$s_{i+j} \colon \wedge^i M \otimes_S \wedge^j M \to \wedge^{i+j} M.$$

REMARK (2.6.1). Suppose our ground field is separably closed and M is a relative motive in $\mathscr{M}^0(S)$ with odd l-adic realisations, i.e., $R_l^i(M) = 0$ for i even. Then we have

$$R_l^*(\wedge^{\cdot} M) \cong \wedge^{\cdot} R_l^*(M).$$

This is a consequence of the Künneth formula (compare **[DeMu]**) and the graded anticommutativity of the cup product. Observe that for $\sigma \in S_i$ the following diagram commutes since $R_l^i(M) = 0$ for i even

$$\begin{array}{ccc} R_l^*(M)^{\otimes i} & \xrightarrow{\sigma} & R_l^*(M)^{\otimes i} \\ \downarrow & & \downarrow \\ R_l^*(M^{\otimes i}) & \xrightarrow{\mathrm{sgn}(\sigma) R_l^*(\sigma)} & R_l^*(M^{\otimes i}). \end{array}$$

3. On the Chow motive of an abelian scheme

(3.1) Motivic decomposition of abelian schemes. For an abelian scheme A/S in $\mathscr{V}(S)$ the class of the diagonal $[\Delta] = [\Delta(A/S)] = [{}^t\Gamma_{\mathrm{id}}]$ in $CH^g(A \times_S A, \mathbb{Q})$ can be decomposed into its Künneth components π_i. This was shown in **[DeMu]**. Here we are going to derive a new formula for the π_i which is better suited for the following computations in the proof of our Theorem (3.3.1).

THEOREM (3.1.1) (Deninger, Murre). *There is a unique decomposition*

$$[\Delta] = \sum_{i=0}^{2g} \pi_i \in CH^g(A \times_S A, \mathbb{Q})$$

such that $[{}^t\Gamma_{\mathrm{mult}(n)}] \circ \pi_i = n^i \pi_i$ *for all* $n \in \mathbb{Z}$. *We have* $\pi_i \circ \pi_j = 0$ *for* $i \neq j$, *and* $\pi_i^2 = \pi_i$. *Moreover,*

(i) $\pi_i \circ [{}^t\Gamma_{\mathrm{mult}(n)}] = n^i \pi_i$ *for integers* n;
(ii) *For* $f \colon A \to B$ *a homomorphism of abelian* S*-schemes*

$$[{}^t\Gamma_f] \circ \pi_{i,B} = \pi_{i,A} \circ [{}^t\Gamma_f];$$

(iii) ${}^t\pi_i = \pi_{2g-i}$.

PROOF. These are Theorem 3.1, Proposition 3.3, and Remark 3) before 3.3 in **[DeMu]**. Assertion (iii) is not proved there. It follows from the equalities

$$[\Delta] = \sum_{i=0}^{2g} {}^{\mathrm{t}}\pi_i,$$
$$[\Gamma_{\mathrm{mult}(n)}] \circ [{}^{\mathrm{t}}\Gamma_{\mathrm{mult}(n)}] = n^{2g}[\Delta],$$
$$[\Gamma_{\mathrm{mult}(n)}] \circ {}^{\mathrm{t}}\pi_i = {}^{\mathrm{t}}(\pi_i \circ [{}^{\mathrm{t}}\Gamma_{\mathrm{mult}(n)}]) = n^i\, {}^{\mathrm{t}}\pi_i. \quad \square$$

Let us give a different proof for the existence of the π_i based on Theorem (1.4.1). For this we regard $A \times_S A$ as an abelian A-scheme via the projection on the first factor and $CH^*(A \times_S A, \mathbb{Q})$ as a ring with respect to the Pontrjagin product. We set

$$\pi_i := \frac{1}{(2g-i)!} \log([\Gamma_{\mathrm{id}}])^{*(2g-i)} \tag{8}$$

for $i \leq 2g$. By Theorem (1.4.1) and Remark (1.4.2) this is well defined, and $\pi_i = 0$ for $i < -d$. From the formal power series identity $\exp(\log(1+x)) = 1 + x$ we obtain

$$[\Gamma_{\mathrm{id}}] = \sum_{i=-d}^{2g} \pi_i.$$

Observing (1.2.2), (1.3.4), and (1.4.2) we get

$$\begin{aligned}
[\Gamma_{\mathrm{mult}(n)}] \circ \pi_i &= \frac{1}{(2g-i)!} \log([\Gamma_{\mathrm{mult}(n)}])^{*(2g-i)} \\
&= \frac{1}{(2g-i)!} \log([\Gamma_{\mathrm{id}}]^{*n})^{*(2g-i)} \\
&= \frac{1}{(2g-i)!} (n \log([\Gamma_{\mathrm{id}}]))^{*(2g-i)} \\
&= n^{2g-i}\pi_i.
\end{aligned}$$

Using this equation one can directly prove the first statement of Theorem (3.1.1) by the Liebermann trick just as described in **[DeMu]**. Observe, in particular, that we obtain $\pi_i = 0$ for $i < 0$ from Theorem (3.1.1iii).

REMARKS (3.1.2). (i) We define $R^i(A/S)$ to be the relative motive $(A, \pi_i, 0)$. Then the theorem yields a canonical decomposition

$$R(A/S) = \bigoplus_{i=0}^{2g} R^i(A/S)$$

where $[{}^{\mathrm{t}}\Gamma_{\mathrm{mult}(n)}]$ acts on $R^i(A/S)$ as multiplication by n^i. In view of assertion (iii) we can improve the motivic Poincaré duality

$$R^{2g-i}(A/S)^\vee = R^i(A/S)(g).$$

(ii) For our Poincaré line bundle $\mathscr{L}$ we have

$$(\mathrm{id}_A \times_S \mathrm{mult}(n))^* \mathscr{L} \cong \mathscr{L}^{\otimes n}$$

and hence that $[{}^t\Gamma_{\mathrm{mult}(n)}] \circ l^i = n^i l^i$ in $CH^i(A \times_S \widehat{A}, \mathbb{Q})$. As a consequence we obtain for $i \in \{0, \dots, 2g\}$

$$\frac{1}{i!} l^i = \pi_i \circ \left(\frac{1}{i!} l^i\right) = \left(\frac{1}{i!} l^i\right) \circ \pi_{2g-i} = \pi_i \circ F = F \circ \pi_{2g-i}.$$

Furthermore F induces morphisms

$$F \circ \pi_i \colon R^i(A/S) \to R^{2g-i}(\widehat{A}/S)(g-i)$$

in $\mathscr{M}^0(S)$, and we see again

$$\widehat{F} \circ F = \sum_{i=0}^{2g} \left(\frac{1}{i!} \hat{l}^i\right) \circ \left(\frac{1}{(2g-i)!} l^{2g-i}\right) \in CH^g(A \times_S \widehat{A}, \mathbb{Q}).$$

(iii) Using Proposition (1.3.3i, ii) it is not difficult to see that the π_i given in (8) are precisely those constructed by Deninger and Murre.

(iv) By definition we have $\pi_{2g} = [A \times_S e(S)]$. From Theorem (3.1.1iii) we see $\pi_0 = [e(S) \times_S A]$ in agreement with the classical definition of π_0 and π_{2g}. The relation between our π_1 and the Picard motive π_1, defined via the Picard variety in **[Mur]**, is investigated in: A. Scholl, *Classical Motives*, these Proceedings.

(3.2) Description of $\wedge^i R^1(A/S)$. Fix $i \geq 0$, and let $\lambda \colon A \to S$ be an abelian scheme as before. We have $R^1(A/S) = (A, \pi_1)$, and $\wedge^i R^1(A/S) = (A \times_S \cdots \times_S A,\ s_i \circ (\pi_1 \otimes_S \cdots \otimes_S \pi_1))$ with $s_i = \frac{1}{i!} \sum_{\sigma \in S_i} [{}^t\Gamma_\sigma]$ as defined in (2.6). We give another description of $\pi_1 \otimes_S \cdots \otimes_S \pi_1$ using the operation of μ_2 on A.

We set $\chi(a_1, \dots, a_i) = a_1 \cdot \dots \cdot a_i \in \mu_2$, and

$$(9)\quad \lambda_i := \frac{1}{2^i} \sum_{(a_1, \dots, a_i) \in \mu_2^{\times i}} \chi(a_1, \dots, a_i)[{}^t\Gamma_{\mathrm{mult}(a_1, \dots, a_i)}] \in CH^{gi}(A^i \times_S A^i, \mathbb{Q}).$$

We collect some equalities needed in the following.

LEMMA (3.2.1).
(i) $s_i^2 = s_i$, $\lambda_i^2 = \lambda_i$, *and* $\lambda_i \circ s_i = s_i \circ \lambda_i$.
(ii) $\pi_{i,A^i} \circ s_i = s_i \circ \pi_{i,A^i}$ *and* $\pi_{i,A^i} \circ \lambda_i = \lambda_i \circ \pi_{i,A^i}$.
(iii) $\lambda_i \circ \pi_{i,A^i} = (\pi_{1,A} \otimes_S \cdots \otimes_S \pi_{1,A})$.
(iv) $\wedge^i R^1(A/S) = (A^i, \lambda_i \circ s_i \circ \pi_{i,A^i})$.

PROOF. (i) is easy, and (ii) is a consequence of the functoriality of the π_i. We prove (iii). The motivic Künneth formula yields

$$\lambda_i \circ \pi_{i,A^i} = \lambda_i \circ \left(\sum_{\substack{n_1, \dots, n_i = 0 \\ n_1 + \cdots + n_i = i}}^{2g} \pi_{n_1,A} \otimes_S \cdots \otimes_S \pi_{n_i,A} \right).$$

Since $\lambda_i \circ (\pi_{n_1,A} \otimes_S \cdots \otimes_S \pi_{n_i,A}) = 0$ if $n_j = 0$ for one j our assertion follows observing the condition $n_1 + \cdots + n_i = i$. (iv) is a direct consequence of (iii). □

(3.3) The main theorem. Next we are going to prove the isomorphism of motives with multiplicative structure

$$\wedge^{\cdot} R^1(A/S) \xrightarrow{\sim} R(A/S).$$

This is the motivic analogue of the well-known formula for the cohomology algebra of an abelian variety $H^{\cdot}(A) \cong \wedge^{\cdot} H^1(A)$ (**[Klei]** or **[Miln]**) identifying cup product and exterior product. Observe that our proof makes no use of Hopf's theorem (**[Klei**, 2A.4] or **[Miln**, 15.2]).

To state the theorem we define for $i \geq 0$

$$\Sigma_i \colon A^i \to A, \qquad (a_1, \ldots, a_i) \mapsto a_1 + \cdots + a_i,$$
$$\Delta_i \colon A \to A^i, \ a \mapsto (a, \ldots, a).$$

The formulas $[{}^t\Gamma_{\Delta_i}] \circ s_i = [{}^t\Gamma_{\Delta_i}]$ and $s_i \circ [{}^t\Gamma_{\Sigma_i}] = [{}^t\Gamma_{\Sigma_i}]$ are immediate.

THEOREM (3.3.1). *For $i \geq 0$ the morphisms*

$$\Phi_i = [{}^t\Gamma_{\Delta_i}] \circ \lambda_i \circ \pi_{i,A^i} \in \mathrm{Hom}_{\mathcal{M}^0(S)}(\wedge^i R^1(A/S), R^i(A/S)),$$
$$\Psi_i = \frac{1}{i!}\lambda_i \circ \pi_{i,A^i} \circ [{}^t\Gamma_{\Sigma_i}] \in \mathrm{Hom}_{\mathcal{M}^0(S)}(R^i(A/S), \wedge^i R^1(A/S))$$

are mutually inverse isomorphisms. In particular, $\wedge^i R^1(A/S) = 0$ for $i > 2g$, and the Φ_i induce an isomorphism

$$\Phi \colon \wedge^{\cdot} R^1(A/S) \xrightarrow{\sim} R(A/S),$$

which is compatible with the canonical multiplicative structures on both sides.

PROOF. The demonstration is divided into three parts, namely:

(i) $\Phi_i \circ \Psi_i = \mathrm{id}_{R^i(A/S)} = \pi_{i,A}$,
(ii) $\Psi_i \circ \Phi_i = \mathrm{id}_{\wedge^i R^1(A/S)} = \lambda_i \circ s_i \circ \pi_{i,A^i}$,
(iii) Φ is compatible with the multiplicative structures.

(i) We have

$$\begin{aligned}\Phi_i \circ \Psi_i &= \left(\frac{1}{i!}[{}^t\Gamma_{\Delta_i}] \circ \pi_i \circ \lambda_i \circ s_i\right) \circ (\pi_i \circ \lambda_i \circ s_i \circ [{}^t\Gamma_{\Sigma_i}]) \\ &= \frac{1}{i!}[{}^t\Gamma_{\Delta_i}] \circ \lambda_i \circ [{}^t\Gamma_{\Sigma_i}] \circ \pi_i \\ &= \frac{1}{i!2^i} \sum_{(a_1,\ldots,a_i)\in\mu_2^{\times i}} \chi(a_1, \ldots, a_i)[{}^t\Gamma_{\Sigma_i \mathrm{mult}(a_1,\ldots,a_i)\Delta_i}] \circ \pi_i \\ &= \frac{1}{i!2^i} \sum_{(a_1,\ldots,a_i)\in\mu_2^{\chi}} \chi(a_1, \ldots, a_i)[{}^t\Gamma_{\mathrm{mult}(\sigma(a_1,\ldots,a_i))}] \circ \pi_i\end{aligned}$$

where we have denoted $a_1 + \cdots + a_i$ by $\sigma(a_1, \ldots, a_i)$. Now by the defining property of the π_i our assertion is a direct consequence of

LEMMA (3.3.2). *For every nonnegative integer* i *we have*

$$\sum_{(a_1,\dots,a_i)\in\mu_2^{\times i}} \chi(a_1,\dots,a_i)\sigma(a_1,\dots,a_i)^k = \begin{cases} i!2^i & \text{for } k=i, \\ 0 & \text{for } 0\le k<i. \end{cases}$$

PROOF. This follows by induction on i. □

(ii) Actually we would have to show

$$\pi_i\circ s_i\circ\lambda_i=\Psi_i\circ\Phi_i=\frac{1}{i!}\pi_i\circ\lambda_i\circ[{}^{\mathrm t}\Gamma_{\Delta_i\circ\Sigma_i}]\circ\lambda_i.$$

Instead we show the transposed equality

$${}^{\mathrm t}\lambda_i\circ{}^{\mathrm t}s_i\circ{}^{\mathrm t}\pi_i=\frac{1}{i!}{}^{\mathrm t}\lambda_i\circ[\Gamma_{\Delta_i\circ\Sigma_i}]\circ{}^{\mathrm t}\lambda_i\circ{}^{\mathrm t}\pi_i,$$

which by (8) and (3.1.1iii) is equivalent to

$$\text{(10)}\qquad {}^{\mathrm t}\lambda_i\circ{}^{\mathrm t}s_i\circ\log([\Gamma_{\mathrm{id}}])^{*i}=\frac{1}{i!}{}^{\mathrm t}\lambda_i\circ[\Gamma_{\Delta_i\circ\Sigma_i}]\circ{}^{\mathrm t}\lambda_i\circ\log([\Gamma_{\mathrm{id}}])^{*i}.$$

To prove this we need some more notation. We define the projection on the nth factor and the nth inclusion map:

$$\mathrm{pr}_n=(\lambda\times_S\cdots\times_S\mathrm{id}\times_S\cdots\times_S\lambda)\colon A^i=A\times_S\cdots\times_S A\to A,$$
$$j_n=(e_A\times_S\cdots\times_S\mathrm{id}\times_S\cdots\times_S e_A)\colon A\to A^i.$$

We start transforming the right-hand side of (10) using (1.2.2), (1.3.4), (9), (1.4.2), and the formula

$$\Sigma_i\circ\mathrm{mult}(a_1,\dots,a_i)=\sum_{n=1}^{i}\mathrm{pr}_n\circ\mathrm{mult}(a_n)$$

in order to obtain

$$\frac{1}{i!2^i}\sum_{(a_1,\dots,a_i)\in\mu_2^{\times i}}\chi(a_1,\dots,a_i)^{\mathrm t}\lambda_i\circ[\Gamma_{\Delta_i}]\circ\log([\Gamma_{\Sigma_i\circ\mathrm{mult}(a_1,\dots,a_i)}])^{*i}$$
$$=\frac{1}{i!2^i}\sum_{(a_1,\dots,a_i)\in\mu_2^{\times i}}\chi(a_1,\dots,a_i)^{\mathrm t}\lambda_i\circ[\Gamma_{\Delta_i}]\circ\left(\sum_{n=1}^{i}\log([\Gamma_{\mathrm{pr}_n\circ\mathrm{mult}(a_n)}])\right)^{*i}.$$

Now we have

$$\begin{aligned}\log([\Gamma_{\mathrm{pr}_n\circ\mathrm{mult}(a_n)}])&=[\Gamma_{\mathrm{pr}_n}]\circ([\Gamma_{\mathrm{mult}(a_n)}]\circ{}^{\mathrm t}\pi_1)\\&=[\Gamma_{\mathrm{pr}_n}]\circ{}^{\mathrm t}(\pi_1\circ[{}^{\mathrm t}\Gamma_{\mathrm{mult}(a_n)}])\\&=a_n[\Gamma_{\mathrm{pr}_n}]^{\mathrm t}\pi_1\\&=a_n\log([\Gamma_{\mathrm{pr}_n}]),\end{aligned}$$

which gives

$$=\frac{1}{i!2^i}\sum_{\substack{n_1,\dots,n_i=1\\(a_1,\dots,a_i)\in\mu_2^{\times i}}}^{i}\chi(a_1,\dots,a_i)\chi(a_{n_1},\dots,a_{n_i})^{\mathrm t}\lambda_i\circ[\Gamma_{\Delta_i}]\circ G_{n_1,\dots,n_i}$$

where $G_{n_1,\dots,n_i}$ denotes the product $\log([\Gamma_{\mathrm{pr}_{n_1}}]) * \cdots * \log([\Gamma_{\mathrm{pr}_{n_i}}])$. If $(n_1, \dots, n_i)$ is not a permutation of $(1, \dots, i)$ there is a $j \in \{1, \dots, i\}\backslash\{n_1, \dots, n_i\}$, and the corresponding terms for $a_j = 1$ and $a_j = -1$ cancel each other. Hence we have to consider only permutations:

$$= \frac{1}{i!}\sum_{\sigma\in S_i} {}^t\lambda_i \circ [\Gamma_{\Delta_i}] \circ (\log([\Gamma_{\mathrm{pr}_{\sigma(1)}}]) * \cdots * \log([\Gamma_{\mathrm{pr}_{\sigma(i)}}]))$$

$$= {}^t\lambda_i \circ [\Gamma_{\Delta_i}] \circ (\log([\Gamma_{\mathrm{pr}_1}]) * \cdots * \log([\Gamma_{\mathrm{pr}_i}])).$$

Since ${}^ts_i \circ [\Gamma_{\Delta_i}] = [\Gamma_{\Delta_i}]$, we obtain

$$= {}^t\lambda_i \circ s_i^t \circ (\log([\Gamma_{\Delta_i\circ\mathrm{pr}_1}]) * \cdots * \log([\Gamma_{\Delta_i\circ\mathrm{pr}_i}]))$$

$$= {}^t\lambda_i \circ {}^ts_i \circ (\log([\Gamma_{\Sigma_{n_1=1}^i j_{n_1}\circ\mathrm{pr}_1}]) * \cdots * \log([\Gamma_{\Sigma_{n_i=1}^i j_{n_i}\circ\mathrm{pr}_i}]))$$

$$= \sum_{n_1,\dots,n_i=1}^{i} {}^t\lambda_i \circ {}^ts_i \circ (\log([\Gamma_{j_{n_1}\circ\mathrm{pr}_1}]) * \cdots * \log([\Gamma_{j_{n_i}\circ\mathrm{pr}_i}])).$$

Again if $(n_1, \dots, n_i)$ is not a permutation of $(1, \dots, i)$ it is not difficult to see that ${}^t\lambda_i$ forces the corresponding term to be zero. Hence

$$= \sum_{\sigma\in S_i} {}^t\lambda_i \circ {}^ts_i \circ (\log([\Gamma_{j_{\sigma(1)}\circ\mathrm{pr}_1}]) * \cdots * \log([\Gamma_{j_{\sigma(i)}\circ\mathrm{pr}_i}])).$$

Furthermore, we use ${}^ts_i \circ [\Gamma_\sigma] = {}^ts_i$ to get

$$= \sum_{\sigma\in S_i} {}^t\lambda_i \circ {}^ts_i \circ (\log([\Gamma_{j_1\circ\mathrm{pr}_1}]) * \cdots * \log([\Gamma_{j_i\circ\mathrm{pr}_i}])),$$

which by the same arguments as above is seen to be equal to

$$= \sum_{n_1,\dots,n_i=1}^{i} {}^t\lambda_i \circ {}^ts_i \circ (\log([\Gamma_{j_{n_1}\circ\mathrm{pr}_{n_1}}]) * \cdots * \log([\Gamma_{j_{n_i}\circ\mathrm{pr}_{n_i}}]))$$

$$= {}^t\lambda_i \circ {}^ts_i \circ \left(\sum_{n=1}^{i} \log([\Gamma_{j_n\circ\mathrm{pr}_n}])\right)^{*i}$$

$$= {}^t\lambda_i \circ {}^ts_i \circ \log([\Gamma_{\mathrm{id}}])^{*i}.$$

Thus we have shown (10), and hence, (ii).

(iii) This is a direct consequence of the definitions. □

References

[Beau1] A. Beauville, *Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne*, Algebraic Geometry (Tokyo/Kyoto 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, Heidelberg and New York, 1983, pp. 238–260.

[Beau2] ——, *Sur l'anneau de Chow d'une variété abélienne*, Math. Ann. **273** (1986), 647–651.

[Bloch] S. Bloch, *Some elementary theorems about algebraic cycles on abelian varieties*, Invent. Math. **37** (1976), 215–228.

[DeMi] P. Deligne and J. S. Milne, *Tannakian categories*, Hodge Cycles, Motives and Shimura Varieties (P. Deligne, et al., eds.), Lecture Notes in Math., vol. 900, Springer, Berlin, Heidelberg, and New York, 1982, pp. 101–228.

[DeMu] C. Deninger and J. Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.

[Fult] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, Heidelberg, and New York, 1984.

[Jann] U. Jannsen, *Deligne homology, Hodge-D-conjecture and motives*, Beilinson's Conjectures on Special Values on L-functions, (M. Rapoport, N. Schappacher, and P. Schneider, eds.) Academic Press, London, 1988.

[Klei] S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Cohomologie des Schémas (A. Grothendieck and N. H. Kuiper, eds.), North-Holland, Amsterdam, 1968, pp. 359–386.

[Man] Y. I. Manin, *Correspondences, motives and monoidal transformations*, Math. Sbor. **77** (1968), 475–507; English transl., Math. USSR Sb. **6** (1968), 439–470.

[Miln] J. S. Milne, *Abelian varieties*, Arithmetic Geometry (G. Cornell and J. H. Silverman, eds.), Springer, Berlin, Heidelberg, and New York, 1986, pp. 103–150.

[Muk] S. Mukai, *Duality Between $D(X)$ and $D(\widehat{X})$ with its applications to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[Mur] J. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.

[Sh] A. M. Shermenev, *The motive of an abelian variety*, Funct. Anal. **8** (1974), 47–53.

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Weight Filtrations in Algebraic K-Theory

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ABSTRACT. We survey briefly some of the K-theoretic background related to the theory of mixed motives and motivic cohomology.

1. Introduction

The recent search for a motivic cohomology theory for varieties, described elsewhere in this volume, has been largely guided by certain aspects of the higher algebraic K-theory developed by Quillen in 1972. It is the purpose of this article to explain the sense in which the previous statement is true, and to explain how it is thought that the motivic cohomology groups with rational coefficients arise from K-theory through the intervention of the Adams operations. We give a basic description of algebraic K-theory and explain how Quillen's idea **[42]** that the Atiyah-Hirzebruch spectral sequence of topology may have an algebraic analogue guides the search for motivic cohomology.

There are other useful survey articles about algebraic K-theory: **[50, 46, 23, 56, 40]**.

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2. Constructing topological spaces

In this section we explain the considerations from combinatorial topology that give rise to the higher algebraic K-groups. The first principle is simple enough to state but hard to implement: when given an interesting group (such as the Grothendieck group of a ring) arising from some algebraic situation, try to realize it as a low-dimensional homotopy group (especially π_0 or π_1)

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of a space X constructed in some combinatorial way from the same algebraic inputs and study the homotopy type of the resulting space.

The groups which can easily be described as π_0 of some space are those that are presented as the set of equivalence classes for some equivalence relation on a set S. We may then take for the space T the graph that has S as its set of vertices, and as its edges some subset of $S \times S$ that generates the given equivalence relation.

The groups that can easily be described as π_1 of some space are those that are presented by generators and relations. One may then take for T the connected cell-complex constructed with one vertex, one edge for each generator of the group, and one 2-cell for each relation (appropriately glued into edges to impose the desired relations).

Neither of the two spaces T mentioned above has particularly interesting homotopy groups, due to the absence of higher-dimensional cells. One hopes that in the algebraic situation at hand there is some particularly evident and natural way of adding cells of higher dimension to T. The most natural and fruitful framework for adding higher-dimensional cells to such spaces hinges on the notion of geometric realization of a simplicial set, as invented by John Milnor in **[36]**. The cells used are simplices (triangles, tetrahedra, etc.), and a *simplicial set* is a sort of combinatorial object which amounts to a convenient way of labeling the faces of the simplices in preparation for gluing. For each integer $n \geq 0$ we give ourselves a set X_n and regard it as the set of labels for the simplices of dimension n to be used in the gluing construction. Then for each labeled simplex of dimension n we assign labels of the appropriate dimension to each of the faces.

The faces of a simplex of dimension n may be accounted for as follows. Let $\underline{\underline{n}}$ denote the ordered set $\{0 < 1 < 2 < \cdots < n\}$, and write the points of the standard n-dimensional simplex Δ^n as formal linear combinations $\sum_{i=0}^{n} a_i\langle i\rangle$, where $\langle i\rangle$ is simply a symbol and where the coefficients a_i are nonnegative real numbers with $\sum_{i=0}^{n} a_i = 1$. The faces of Δ^n are the affine-linear spans of subsets of $\{\langle 0\rangle, \ldots, \langle n\rangle\}$. We can index the m-dimensional faces of Δ^n by the injective maps $s : \underline{\underline{m}} \to \underline{\underline{n}}$ that are *increasing* in the sense that $i \leq j \Rightarrow s(i) \leq s(j)$. We consider the unique affine-linear map $s_* : \Delta^m \to \Delta^n$ satisfying $s_*(\langle i\rangle) = \langle s(i)\rangle$, which embeds Δ^m as a face of Δ^n. If $x \in X_n$ is a label for an n-dimensional simplex, then the label we assign to the face given by the image of s_* should be an element of X_m which we will dub $s^*(x)$. We have to do this for each s and for each x. It turns out to be convenient to do this also for increasing maps s that are not necessarily injective.

The total system of compatibilities that this system of labels must satisfy is codified as follows. Let Ord denote the category of finite nonempty ordered sets $\underline{\underline{n}}$, where the arrows are the increasing maps. Then the collection of sets X_n together with the collection of maps $s^* : X_n \to X_m$ should constitute a contravariant functor from Ord to the category of sets. Such a functor X is called a *simplicial set*, and the corresponding space $|X|$ obtained by gluing simplices together is called the *geometric realization* of X.

A label $x \in X_n$ is called an *n-simplex* of X.

3. Nerves of categories

The primary example of a simplicial set arises from a category $\mathscr{C}$ in the following way. Interpret the ordered set $\underline{\underline{n}}$ as a category $0 \to 1 \to 2 \to \cdots \to n$, and let $\mathscr{C}_n = \mathscr{C}(\underline{\underline{n}})$ denote the set $\operatorname{Hom}(\underline{\underline{n}}, \mathscr{C})$ of functors from $\underline{\underline{n}}$ to $\mathscr{C}$. The resulting simplicial set, which we may also write as $\mathscr{C}$ without too much fear of confusion, is called the *nerve* of $\mathscr{C}$. The space $|\mathscr{C}|$ will have a vertex for each object of $\mathscr{C}$, an edge for each arrow of $\mathscr{C}$, and so on.

Geometric realization is a functor from the category of simplicial sets to the category of spaces, and taking the nerve is a functor from the category of small categories to the category of simplicial sets. There is a dictionary of corresponding notions in these three categories which are related by these two functors. Thus, the nerve of a functor $F : \mathscr{C} \to \mathscr{C}'$ is a natural transformation, i.e., a map of simplicial sets, and the geometric realization of that is a continuous map. The geometric realization of the category $0 \to 1$ can be identified with the unit interval $I = [0, 1]$, and investigating the extent to which compatibility with products holds shows that the nerve of a natural transformation $\mathscr{C} \times \underline{\underline{1}} \to \mathscr{C}'$ is a simplicial homotopy, and the geometric realization of that is a homotopy $|\mathscr{C}| \times I \to |\mathscr{C}'|$. In particular, if $\mathscr{C}$ has an initial object, then contraction along the cone of initial arrows amounts to a null-homotopy of $|\mathscr{C}|$, so that $|\mathscr{C}|$ is a contractible space.

4. Classifying spaces of groups

The simplest examples of categories for which it is useful to consider the geometric realization arise from groups. Let G be a group, and let $G[1]$ denote the category with one object $*$ and with G as its monoid of arrows. One finds that

$$\pi_i|G[1]| = \begin{cases} G & i = 1, \\ 0 & i \neq 1. \end{cases} \tag{4.1}$$

To prove this, one considers the category $\widetilde{G}$ whose set of objects is G and in which there is for each $g, h \in G$ a unique arrow $g \to h$ labeled hg^{-1}. The labels are there only for the purpose of describing the map $\widetilde{G} \to G[1]$ that sends an arrow of $\widetilde{G}$ labeled hg^{-1} to the arrow hg^{-1} of $G[1]$. One sees that G acts freely on $|\widetilde{G}|$ on the right and that the map $|\widetilde{G}| \to |G[1]|$ is the covering map corresponding to the quotient by this action. Moreover, the category $\widetilde{G}$ has an initial object, so $|\widetilde{G}|$ is contractible. Putting this information together yields the result.

Another name for the space $|G[1]|$ is BG, and it is called the *classifying space* of the group G.

An explicit calculation with simplicial chains shows that the singular homology group $H_n(BG, \mathbb{Z})$ is isomorphic to the group cohomology group $H^n(G, \mathbb{Z})$, as calculated using the bar resolution. In fact, the difference between the normalized bar resolution and the ordinary bar resolution amounts to the homeomorphism $|\widetilde{G}|/G \cong |G[1]|$.

5. Simplicial abelian groups

There is a third important class of examples of simplicial sets. Consider a simplicial abelian group A, which by definition is a contravariant functor from Ord to the category of abelian groups. If we forget the abelian group structure on each A_n, we are left with a simplicial set whose geometric realization $|A|$ we may consider. The striking fact here is that $\pi_i|A| = H_i NA$, where by NA we mean the normalized chain complex associated to A. The functor $A \mapsto NA$ is an equivalence of categories from the category of simplicial abelian groups to the category of homological chain complexes of abelian groups **[11, 28]**. (The appropriate definition of the inverse functor to N is easy to deduce from Yoneda's lemma.) This Dold-Kan equivalence allows us to embed the theory of homological algebra into homotopy theory.

We remark that if X is a simplicial set and we let $\mathbb{Z}[X]$ denote the simplicial abelian group whose group of n-simplices is the free abelian group $\mathbb{Z}[X_n]$ on the set X_n, then the homotopy groups of $|\mathbb{Z}[X]|$ are the homology groups of $|X|$. Thus $\mathbb{Z}[X]$ is a drastic form of abelianization for simplicial sets.

6. Eilenberg-Mac Lane spaces

Here is the fourth important class of examples of simplicial sets, obtained as a special case of the third. Let G be an abelian group, and let $n \geq 0$ be an integer. Then consider the homological chain complex that has G in dimension n and the group 0 in all the other dimensions. Let $G[n]$ denote the corresponding simplicial abelian group, obtained according to the Dold-Kan equivalence. We get a space $|G[n]|$ which has the abelian group $G = \pi_n|G[n]|$ as its only nonvanishing homotopy group; it is an *Eilenberg-Mac Lane space.*

Consider pointed spaces (CW-complexes) V and W, and let $[V, W]$ denote the set of homotopy classes of (base-point preserving) maps from V to W. There is a natural isomorphism $[V, |G[n]|] \cong H^n(V, G)$, which we will use later. To see the plausibility of the isomorphism, consider the case where V is a sphere and identify the two sides.

7. The lower K-groups

Grothendieck considered the group $K(R)$ generated by the isomorphism classes of finitely generated projective R-modules, modulo relations $[P] = [P'] + [P'']$ coming from short exact sequences $0 \to P' \to P \to P'' \to 0$. The group $\mathrm{Gl}(R)$ is the group of invertible matrices over R with a countable number of rows and columns, equal to the identity outside of some square. Bass defined $K_1(R)$ to be the abelianization $\mathrm{Gl}(R)^{\mathrm{ab}}$ of the infinite general linear group $\mathrm{Gl}(R)$ and renamed the Grothendieck group to $K_0(R)$ because of six-term exact sequences he was able to prove involving the Grothendieck group and his new group [2]. Milnor [37] found the correct definition for $K_2(R)$ and supported its correctness by extending the exact sequences of Bass.

8. The construction of the higher K-groups

We can now describe Quillen's first construction of algebraic K-theory for a ring R.

By adding a single two-cell and a single three-cell to the space $\mathrm{BGl}(R)$ Quillen was able to construct a space $\mathrm{BGl}(R)^+$ with the property that the map $\mathrm{BGl}(R) \to \mathrm{BGl}(R)^+$ induces an isomorphism on homology groups with integer coefficients and induces the map $\mathrm{Gl}(R) \twoheadrightarrow \mathrm{Gl}(R)^{\mathrm{ab}}$ on fundamental groups. (The construction provides a functor from rings to spaces, because the cells' attaching maps used in the case $R = \mathbb{Z}$ work for any ring R.) Quillen also proved that the space $\mathrm{BGl}(R)^+$ is an abelian group in the homotopy category of pointed spaces. This construction therefore serves as a modest form of abelianization for spaces such as this, whose commutator subgroup is perfect.

Quillen's first definition of the higher algebraic K-groups is given in terms of this plus-construction by setting $K_i(R) = \pi_i \mathrm{BGl}(R)^+$ for $i > 0$.

The Hurewicz mapping

$$K_i(R) \to H_i(\mathrm{BGl}(R)^+, \mathbb{Z}) \cong H_i(\mathrm{BGl}(R), \mathbb{Z}) \cong H_i(\mathrm{Gl}(R), \mathbb{Z}) \tag{8.1}$$

gives an initial stab at the relationship between the higher K-groups and the homology groups of the general linear group.

9. K-groups of exact categories

The plus-construction is a curious creature. It permits some explicit computations to be performed, by virtue of its close connection with the general linear group. For example, using it, Quillen proved that $K_1(R)$ agrees with the group of the same name defined by Bass and that $K_2(R)$ agrees with the group of the same name defined by Milnor in terms of the Steinberg group [37]. But it suffers from two drawbacks. First, one would prefer to have a space $K(R)$ which has all of the K-groups appearing as its homotopy groups, including $K_0(R)$. The naive construction of such a space, $K_0(R) \times \mathrm{BGl}(R)^+$, has such homotopy groups but, when regarded as an abelian group in the homotopy category, fails to mix π_0 with π_1 appropriately, unless it happens that $K_0(R) = \mathbb{Z}$. Second, the Grothendieck group is defined for almost any sort of category with a notion of exact sequence; the higher K-groups should be defined for such categories, too.

The definitions of K-theory that do not suffer from these two drawbacks deal with an *exact* category $\mathcal{M}$. An *exact* category $\mathcal{M}$ is an additive category equipped with a set of sequences $0 \to M' \to M \to M'' \to 0$ called *exact* sequences, which arises as a full subcategory $\mathcal{M} \subseteq \mathcal{A}$ closed under extensions in some (unspecified) abelian category $\mathcal{A}$, equipped with the collection of all short sequences of $\mathcal{M}$ that are exact in $\mathcal{A}$. As examples of exact categories, we mention the category $\mathcal{P}(R)$ of finitely generated projective left R-modules, the category $\mathcal{M}(R)$ of finitely generated left R-modules, the category $\mathcal{P}(Z)$ of locally free $\mathcal{O}_Z$-modules of finite type on a scheme Z, and the category $\mathcal{M}(Z)$ of quasi-coherent $\mathcal{O}_Z$-modules of finite type on a scheme Z. The corresponding higher K-groups are all of interest.

The first such definition of K-theory I intend to discuss is Quillen's Q-construction, **[43]**. The category $Q\mathcal{M}$ has the same objects as does $\mathcal{M}$, but an arrow $M' \to M$ of $Q\mathcal{M}$ is an isomorphism of M' with an admissible subobject of an admissible quotient object of M. Admissibility of a subobject refers to the requirement that the corresponding quotient object also lies in $\mathcal{M}$ or, more precisely, that the inclusion map for the subobject is part of a short exact sequence in $\mathcal{M}$. One may check eventually that the connected space $|Q\mathcal{M}|$ satisfies $\pi_1|Q\mathcal{M}| \cong K_0\mathcal{M}$, and one defines $K_i(\mathcal{M}) = \pi_{i+1}|Q\mathcal{M}|$ for all $i \geq 0$. It is then a theorem of Quillen **[22]** that $K_i(R) \cong K_i(\mathcal{P}(R))$. We define the K-theory space $K(\mathcal{M})$ to be the loop space $\Omega|Q\mathcal{M}|$ of $|Q\mathcal{M}|$, so that $K_i(\mathcal{M}) = \pi_i(K(\mathcal{M}))$.

If X is a scheme, we may define $K_i(X) = K_i(\mathcal{P}(X))$, where $\mathcal{P}(X)$ denotes the category of locally free $\mathcal{O}_X$-modules of finite type on X. Much of what is stated below for commutative rings R applies equally well to schemes X.

If R is a ring, we define $K_i'(R) := K_i(\mathcal{M}(R))$, where $\mathcal{M}(R)$ denotes the category of finitely generated R-modules. If X is a scheme, we define $K_i'(X) := K_i(\mathcal{M}(X))$, where $\mathcal{M}(X)$ denotes the category of quasi-coherent $\mathcal{O}_X$-modules locally of finite type.

A second definition of K-theory is provided by Waldhausen's S-construction **[57]**. The simplicial set $S\mathcal{M}$ is defined in such a way that its set of n-simplices consists of chains $0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_n$ of admissible monomorphisms of $\mathcal{M}$, together with objects of $\mathcal{M}$ representing all the quotient objects. One sees that there is exactly one vertex in the space $|S\mathcal{M}|$, one edge for each object M of $\mathcal{M}$, and one triangle for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathcal{M}$. If we let $[M]$ denote the class in $\pi_1\mathcal{M}$ arising from the edge labeled by M, then the triangles are glued to the edges so that the relation $[M] = [M'] + [M'']$ is imposed. It follows that $\pi_1 S\mathcal{M} \cong K_0\mathcal{M}$—this fits in well with the earlier remark about expressing groups given by generators and relations as π_1 of a space. It is a theorem of Waldhausen that $|S\mathcal{M}|$ is homotopy equivalent to $|Q\mathcal{M}|$, so that $\pi_{i+1}|S\mathcal{M}| \cong K_i\mathcal{M}$ for all $i \geq 0$.

The S-construction or the Q-construction can be used to show that the space $K(\mathcal{M})$ is naturally an infinite loop space; the deloopings obtained thereby are increasingly connected, so the homotopy groups in negative dimension that arise thereby are all zero. This makes the space $K(\mathcal{M})$ into an Ω-*spectrum* (or an infinite loop space), which is essentially a space Z, together with compatible choices of deloopings $\Omega^{-n}Z$ for every $n > 0$. The sort of spectrum just obtained, in which the deloopings are increasingly connected, with no new homotopy groups arising in low degrees, is called *connective*.

Another equivalent definition for K-theory is presented in **[20]** by Gillet and Grayson. It hinges on the elementary fact that every element of $K_0(\mathcal{M})$ can be expressed as a difference $[P] - [Q]$, where P and Q are objects of $\mathcal{M}$ and $[P]$ denotes the class in $K_0(\mathcal{M})$. Thus $K_0(\mathcal{M})$ is a quotient of the set of pairs (P, Q) of objects of $\mathcal{M}$ by the equivalence relation where $(P, Q) \sim (P', Q')$ if and only if $[P] - [Q] = [P'] - [Q']$. According to the earlier remark about groups whose elements are equivalence classes of a relation, we may try to express $K_0(\mathcal{M})$ as π_0 of some space. That space would have vertices that are pairs (P, Q) of objects, its edges would generate the equivalence relation, and its higher-dimensional simplices should be defined in a natural and simple way. It is an exercise to show that our equivalence relation is generated by the requirement that $(P, Q) \sim (P', Q')$ whenever there are admissible monomorphisms $P' \rightarrowtail P$ and $Q' \rightarrowtail Q$ whose cokernels are isomorphic. This suggests that an edge of our simplicial set should be a triple consisting of two such monomorphisms and an isomorphism of their cokernels. The simplicial set $G\mathcal{M}$ is defined by saying that an n-simplex is a pair of chains of admissible monomorphisms $P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n$ and $Q_0 \rightarrowtail Q_1 \rightarrowtail \cdots \rightarrowtail Q_n$, together with a commutative diagram of isomorphisms of quotients as illustrated here.

$$\begin{array}{ccccccc} P_1/P_0 & \longrightarrow & P_2/P_0 & \longrightarrow & \cdots & \longrightarrow & P_n/P_0 \\ \cong\downarrow & & \cong\downarrow & & & & \cong\downarrow \\ Q_1/Q_0 & \longrightarrow & Q_2/Q_0 & \longrightarrow & \cdots & \longrightarrow & Q_n/Q_0 \end{array}$$

It is then a theorem that $K_i(\mathcal{M}) = \pi_i|G\mathcal{M}|$ for all $i \geq 0$ and that $|G\mathcal{M}|$ is a loop space of $|S\mathcal{M}|$. It turns out that having $K_0(\mathcal{M})$ appear as π_0 rather than as π_1 offers a technical advantage when constructing operations on K-groups that are homomorphisms on K_i only for $i > 0$, for then the operations may arise from maps of spaces **[24]**.

10. Some theorems of algebraic K-theory

Using the Q-construction, Quillen **[43]** was able to prove the following four foundational theorems (among others).

The first theorem is the additivity theorem. It states that if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of exact functors $\mathcal{M} \to \mathcal{M}'$, then the map $F_* : K_n(\mathcal{M}) \to K_n(\mathcal{M}')$ satisfies the formula $F_* = F'_* + F''_*$.

The second theorem is the Jordan-Hölder theorem for higher K-theory. It states that if $\mathcal{M}$ is an Artinian abelian category (i.e., every object has a composition series), then $K_n(\mathcal{M}) = \bigoplus_V K_n(\mathrm{End}(V))$, where the direct sum runs over the isomorphism classes of simple objects V of $\mathcal{M}$. An important application of this theorem is the following. Let R be a Noetherian ring, and let $\mathcal{M}^p(R)$ denote the category of finitely generated R-modules whose support has codimension $\geq p$. Then the quotient abelian category $\mathcal{M}^p/\mathcal{M}^{p+1}$ is Artinian, and the theorem implies that $K_n(\mathcal{M}^p/\mathcal{M}^{p+1}) = \bigoplus_x K_n(k(x))$, where x runs over points $x \in \mathrm{Spec}(R)$ of height p and $k(x)$ denotes the residue field at x. If we take $n = 0$ then we see that $K_0(\mathcal{M}^p/\mathcal{M}^{p+1}) = \bigoplus_x \mathbb{Z}$ is the group of algebraic cycles of codimension p. This is the way that algebraic cycles are related to algebraic K-theory.

The third theorem is the resolution theorem, which implies that if X is a regular Noetherian scheme, then $K_n(\mathcal{P}(X)) = K_n(\mathcal{M}(X))$. It hinges on the fact that any $M \in \mathcal{M}(X)$ has a resolution of finite length by objects of $\mathcal{P}(X)$. For X affine, that resolution is simply a projective resolution.

The fourth theorem is the localization theorem for abelian categories. It says that if $\mathcal{B}$ is a Serre subcategory of an abelian category $\mathcal{A}$ (which means that it is closed under taking subobjects, taking quotient objects, and taking extensions), then there is a fibration sequence $K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B})$, where $\mathcal{A}/\mathcal{B}$ denotes the quotient abelian category. The main import of being a fibration sequence is that there results the following long exact sequence of K-groups:

$$\cdots \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to K_{n-1}(B) \to \cdots .$$

This theorem can be applied notably in the case above where $\mathcal{A} = \mathcal{M}^p(R)$ and $\mathcal{B} = \mathcal{M}^{p+1}(R)$ or in the case where X is a Noetherian scheme, Y is a closed subscheme, $A = \mathcal{M}(X)$, $\mathcal{B} = \mathcal{M}(Y)$, and $\mathcal{A}/\mathcal{B} \cong \mathcal{M}(X - Y)$.

11. Some computations

We now present some explicit computations of some algebraic K-groups.

There is a map $R^\times = \mathrm{Gl}_1(R) \subseteq \mathrm{Gl}(R) \to K_1(R)$; let us use $\{u\}$ to denote the image of a unit u under this map, so that we can write the group law in $K_1(R)$ additively, $\{u\} + \{v\} = \{uv\}$. When R is commutative this map is split by the determinant map, so that $K_1(R)$ has $R^\times$ as a direct summand. It is known that when R is a field, a local ring, or a euclidean domain, then $K_1(R) \cong R^\times$.

Concerning the K-groups of a field F, we know that

$$\begin{aligned} &K_0 F = \mathbb{Z}, \\ (11.1)\qquad &K_1 F = F^\times, \\ &K_2 F = (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle a \otimes (1-a) \mid a \in F - \{0, 1\}\rangle. \end{aligned}$$

The standard notation for the image of $a \otimes b$ in $K_2 F$ is $\{a, b\}$, and the relation $\{a, 1-a\} = 0$ is called the Steinberg relation. From the Steinberg relation one can deduce that $\{a, b\} = -\{b, a\}$ [37, p. 95].

For finite fields, the following K-groups are completely known [**45**]:

$$K_0\mathbb{F}_q = \mathbb{Z},$$
$$K_{2i+1}\mathbb{F}_q \cong \mathbb{Z}/(q^{i+1} - 1),$$
$$K_{2i+2}\mathbb{F}_q = 0.$$

In fact, what motivated Quillen's original definition of algebraic K-theory was the discovery of a space (the homotopy fixed-point set of the Adams operation ψ^q acting on topological K-theory) that has these homotopy groups, whose homology groups are isomorphic to the homology groups of the general linear group.

Suslin's important work on the K-groups of algebraically closed fields [**54**] shows that the torsion subgroup of $K_n\mathbb{C}$ is $\mathbb{Q}/\mathbb{Z}$ for n odd and is 0 for n even and nonzero. The quotient modulo the torsion is a uniquely divisible group. (For example, $K_1(\mathbb{C}) = \mathbb{C}^\times$, so the torsion subgroup is the group of roots of unity, thus isomorphic to $\mathbb{Q}/\mathbb{Z}$.) For a general exposition of these matters and others in the K-theory of fields, see [**23**].

As for the field $\mathbb{Q}$ of rational numbers, Tate proved [**37**, 11.6] that $K_2\mathbb{Q} \cong \mathbb{Z}/2 \oplus \bigoplus_{p \text{ prime}}(\mathbb{Z}/p)^\times$.

The higher K-groups of $\mathbb{Q}$ are not known exactly, but the ranks are. In fact, the ranks are known for any ring of algebraic integers; so consider now the case where F is a number field and $\mathscr{O}_F$ is the ring of integers in F. Write $F \otimes \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, so that r_1 is the number of real places of F, and r_2 is the number of complex places.

For $\mathscr{O}_F$ and, indeed, for any Dedekind domain, one knows that $K_0\mathscr{O}_F = \mathbb{Z} \oplus \mathrm{Pic}(\mathscr{O}_F)$, where $\mathrm{Pic}(\mathscr{O}_F)$ denotes the ideal class group of $\mathscr{O}_F$.

It is a theorem of Bass, Milnor, and Serre [**3**] that $K_1(\mathscr{O}_F) \cong \mathscr{O}_F^\times$. This is not a general fact about Dedekind domains.

Quillen [**44**] has shown that $K_n(\mathscr{O}_F)$ is a finitely generated abelian group for all $n \geq 0$. (In fact, it is conjectured by Bass that $K_n(R)$ is a finitely generated group whenever R is a finitely generated regular commutative ring.) We list the ranks as

$$\text{rank}\, K_n\mathscr{O}_F = \begin{cases} 1 & \text{if } n = 0, \\ r_1 + r_2 - 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2k \text{ and } k > 0, \\ r_1 + r_2 & \text{if } n = 4k+1 \text{ and } k > 0, \\ r_2 & \text{if } n = 4k+3 \text{ and } k \geq 0. \end{cases} \tag{11.2}$$

In this table, the case $n = 0$ amounts to the finiteness of the ideal class group; the case $n = 1$ is Dirichlet's unit theorem; Borel's theorem, [**10**], handles the case $n \geq 2$ through a detailed study of harmonic forms on symmetric spaces associated to arithmetic groups and depends on earlier work of Borel and Serre on compactifying these symmetric spaces. It is customary now to refer to a nontorsion element of $K_{2i-1}(\mathscr{O}_F)$ as a *Borel class*.

The interesting fact about these ranks is that in the odd cases, the rank of $K_{2i-1}(\mathscr{O}_F)$ is equal to the order of vanishing of the Dedekind zeta function $\zeta_F(s)$ at $s = 1 - i$. It was Lichtenbaum who spawned the current endeavor by predicting this coincidence, well before Borel computed the ranks of the K-groups.

Only the first four K-groups of $\mathbb{Z}$ are known. We know that $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \mathbb{Z}/2$. Milnor showed [**37**, 10.1] that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$. Lee and Szczarba [**32**] proved that $K_3\mathbb{Z} \cong \mathbb{Z}/48$.

12. Products in K-theory

Henceforth we shall deal with rings R that are commutative. In that case the tensor product operation $P \otimes_R Q$ on finitely generated projective R-modules leads to an operation

$$K_m R \otimes K_n R \to K_{m+n} R \tag{12.1}$$

which endows $\bigoplus_{n=0}^{\infty} K_n R$ with the structure of a skew-commutative graded ring, by virtue of the essential commutativity and associativity of the tensor product operation. The unit element of the ring is $1 = [R] \in K_0(R)$. In the case $m = n = 1$ the product agrees with the Steinberg symbol (at least up to a possible sign), in the sense that $\{u\} \cdot \{v\} = \{u, v\}$.

One defines the Milnor ring of a field F to be the quotient of the tensor algebra of $F^\times$ by the ideal generated by the Steinberg relations. This ring is a graded ring, and we let $K_n^M F$ denote its degree n part. It follows from what we have said that there is a natural map $K_n^M F \to K_n F$ which is an isomorphism for $0 \le n \le 2$ and which is known to be a nonisomorphism in general for $n > 2$.

13. Nonlinear operations on K-theory

If $F : \mathscr{M} \to \mathscr{M}'$ is an exact functor, then there is a natural map $K(\mathscr{M}) \to K(\mathscr{M}')$ induced by F. The exterior power operation $\Lambda^k P$ on finitely generated projective R-modules P gives rise to operations $\lambda^k : K_0(R) \to K_0(R)$, as shown by Grothendieck [**26**], even though Λ^k is not an exact functor. These *lambda* operations are defined by letting $\lambda^k([P] - [Q])$ be the coefficient of t^k in $\lambda_t([P])/\lambda_t([Q])$, where $\lambda_t([P]) := \sum_{k=0}^{\infty} [\Lambda^k P] t^k$ in the power series ring $K_0(R)[[t]]$. One uses the identity $\lambda_t([P \oplus Q]) = \lambda_t([P])\lambda_t([Q])$ to show that λ_t (and hence λ^k) is well defined. When R is a field, then $K_0(R) = \mathbb{Z}$, and $\lambda^k(n) = \binom{n}{k}$; the usual definition of the binomial coefficient for $n < 0$ is the one we are led to using in the artifice above.

We point out, for later use, that when $P \cong L_1 \oplus \cdots \oplus L_n$, where each L_i

is a projective module of rank 1, then

$$\lambda_t([P]) = \prod_{i=1}^{n} \lambda_t([L_i]) = \prod_{i=1}^{n}(1 + t[L_i]) = \sum_{k=0}^{n} t^k \sigma_k([L_1], \dots, [L_n]),$$

where σ_k is the elementary symmetric polynomial of degree k. We deduce that $\lambda^k([\mathscr{P}]) = \sigma_k([L_1], \dots, [L_n])$. We will see that the operations λ^k are useful for the same reason the elementary symmetric polynomials are: we can write other symmetric functions in terms of them.

In **[27]** is presented Quillen's method for defining the lambda operations on the higher K-groups. The resulting functions $\lambda^k : K_n(R) \to K_n(R)$ are group homomorphisms except when $n = 0$ and $k \neq 1$. Perhaps it is startling at first glance that functions that are decidedly not additive on K_0 are closely related to functions on the higher K-groups which are, but there is no other possibility. Any sort of operation on higher homotopy groups of a space will presumably have to arise from a map of spaces and so cannot avoid being a homomorphism.

It is possible to repair the nonadditivity of the lambda operations on K_0, thereby increasing their utility, by means of a natural sort of abelianization procedure which produces new *Adams operations* $\psi^k : K_n(R) \to K_n(R)$ which are group homomorphisms, even for $n = 0$. On K_0, the salient feature of the Adams operations, aside from being homomorphisms, is that if L is a projective module of rank 1, then $\psi^k([L]) = [L^{\otimes k}]$. Consequently, if $P \cong L_1 \oplus \cdots \oplus L_n$ is a direct sum of projective modules of rank 1, then $\psi^k([P]) = [L_1^{\otimes k}] + \cdots + [L_n^{\otimes k}]$. The symmetric polynomial $x_1^k + \cdots + x_n^k$ can be expressed as a polynomial with integer coefficients in the elementary symmetric polynomials, so there is a formula for $[L_1^{\otimes k}] + \cdots + [L_n^{\otimes k}]$ in terms of the exterior powers of P. It does not really matter what this formula is, but it may be compactly recorded in terms of generating functions as follows:

$$\sum_{k=0}^{\infty} \psi^k(x)(-t)^k = \operatorname{rank} x - t\frac{d}{dt}\log \lambda_t(x).$$

This formula serves as the definition of $\psi^k(x)$ for any $x \in K_0(R)$.

A unit u of the ring R arises in $K_1(R)$ by virtue of being an automorphism of the free module $L = R$ of rank 1. As such it gives rise to the automorphism $u^k = u^{\otimes k}$ of $L = L^{\otimes k}$, so we see that $\psi^k(u) = u^k$, or, writing it additively, $\psi^k u = ku$.

Heuristically speaking, ψ^k raises the functions (i.e., the elements of R) entering into a construction of an element of K-theory, to the kth power. To the extent that constructions of elements of K-theory from several functions of R involve those functions in a multi-multiplicative way, the effect of ψ^k

on an element of K-theory can be used to count the number of functions entering into its construction.

Here we summarize some properties of the Adams operations.

(1) If $x \in K_n(R)$ and $y \in K_n(R)$ then $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$.
(2) If x is the class $[L]$ of a line bundle (rank 1 projective R-module) L in $K_0(R)$, then $\psi^k(x) = [L^{\otimes k}]$.
(3) If x is the class in $K_0(R)$ of a free module, then $\psi^k(x) = x$.
(4) If $x \in K_p(R)$ and $y \in K_q(R)$ then $\psi^k(xy) = \psi^k(x)\psi^k(y) \in K_{p+q}(R)$.
(5) If $x \in R^\times$ then $\psi^k(\{x\}) = \{x^k\} = k\{x\}$.
(6) $\psi^k \circ \psi^\ell = \psi^{k\ell}$.
(7) If R is a regular Noetherian ring and M is a finitely generated R-module whose support has codimension $\geq p$, then $\psi^k([M]) = k^p[M]$ modulo torsion and classes of modules of codimension greater than p.

The last item above is related to the fact **[21]** that in K-theory with supports, if we have a regular sequence $t_1, \dots, t_p$ in R, and if $M = R/(t_1, \dots, t_p)$, then $\psi^k([M]) = [R/(t_1^k, \dots, t_p^k)]$.

In the first part of **[25]** the reader may find a concrete description of the kth Adams operation on K_0 as the secondary Euler characteristic of the Koszul complex; the Koszul complex is introduced there by taking the mapping cone of the identity map on $P \in \mathscr{P}(R)$ and considering the kth symmetric power of that complex of length 1.

14. Weight filtrations in the K-groups

The Adams operations were used by Grothendieck to provide an answer (up to torsion) to the following question. Suppose that R is a regular Noetherian ring and that M is a finitely generated R-module with the codimension of its support at least p. We get a class $[M] \in K_0(\mathscr{M}(R))$. By the resolution theorem, we have an isomorphism $K_0(R) \cong K_0(\mathscr{M}(R))$, so we get a class $[M] \in K_0(R)$. Let $F^p_{\text{top}}K_0(R)$ denote the subgroup of $K_0(R)$ generated by such classes; the collection of these subgroups is called the filtration by codimension of support. The question is, is there an algebraic construction of such a filtration which makes sense for any commutative ring R, not necessarily regular? The construction would have to be expressed in terms of projective modules alone, in the absence of the resolution theorem. In the next few paragraphs, we explore the construction of such a filtration.

Let us assume that R is a commutative domain. Then we have the *rank* homomorphism $K_0(R) \to \mathbb{Z}$ defined by $[P] \mapsto \operatorname{rank} P$. This map is an isomorphism when R has dimension 0, i.e., is a field. The kernel F^1_{alg} of the rank homomorphism consists of elements of the form $[P] - \operatorname{rank} P$, and we declare such elements to have *weight* ≥ 1.

The Picard group $\operatorname{Pic}(R)$ is the group of isomorphism classes $[L]$ of rank 1 projective R-modules L, with tensor product as the group operation. Given $P \in \mathscr{P}(R)$, we let $\det P$ denote the highest exterior power of P, and we let $[\det P]$ denote its isomorphism class in $\mathrm{Pic}(R)$. From the formula $\det(P \oplus P') \cong (\det P) \otimes (\det P')$ we find that there is a group homomorphism $K_0(R) \to \mathrm{Pic}(R)$. There is also a function $\mathrm{Pic}(R) \to K_0(R)$, defined by sending $[L] \in \mathrm{Pic}(R)$ to $[L] \in K_0(R)$, which is a group homomorphism from $\mathrm{Pic}(R)$ to the group of units $K_0(R)^\times$ of the ring $K_0(R)$. (We shall see shortly that $[L] - 1$ is a nilpotent element of the ring $K_0(R)$.)

Let us consider the simple case where R is a regular domain of dimension 1, i.e., is a Dedekind domain. In this case it is known that the homomorphism $K_0(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R)$ given by $[P] \mapsto (\operatorname{rank} P, [\det P])$ is an isomorphism. The proof depends on the algebraic fact that any $P \in \mathscr{P}(R)$ of rank n is the direct sum of a free module R^{n-1} and a projective module L of rank 1, from which it follows that $L \cong \det P$, allowing us to recover $[P]$ from $\operatorname{rank} P$ and $\det P$ by the formula $[P] = [\det P] - 1 + \operatorname{rank} P$. (The nilpotence of $[L]-1$ results from the isomorphism $L \oplus L \cong L^{\otimes 2} \oplus R$, which allows us to prove that $([L]-1)^2 = 0$.)

For an arbitrary commutative domain R the kernel F^2_{alg} of the surjective homomorphism $K_0(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R)$ is the subgroup generated by elements of the form $([\det P]-1) - ([P] - \operatorname{rank} P)$, for the vanishing of such elements is all that is required for the proof from the previous paragraph. We declare such elements to be of *weight* ≥ 2.

What ought we to use for the elements of weight ≥ 3? One problem confronting us is the lack of a function analogous to $\operatorname{rank} P$ and $\det P$ that would vanish on such elements. (Using the second Chern class associated with some cohomology theory seems unnatural and unprofitable.)

We try to get an idea by examining $([\det P]-1) - ([P] - \operatorname{rank} P)$ when P is decomposable. For example, when $P \cong L_1 \oplus L_2$, where L_1 and L_2 have rank 1, we see that

$$\begin{aligned}([\det P]-1) - ([P] - \operatorname{rank} P) &= [L_1 \otimes L_2] + 1 - [L_1 \oplus L_2] \\ &= ([L_1]-1)([L_2]-1),\end{aligned}$$

showing that our element of weight ≥ 2 is a product of two elements of weight ≥ 1. This suggests that the weight filtration we are constructing ought to be compatible with multiplication. That in turn fits in well with our original topological filtration, for if D_1 is a divisor whose defining ideal is isomorphic to L_1, then we have $[D_1] = 1 - [L_1]$, and if D_2 is a divisor whose ideal is isomorphic to L_2, and D_1 and D_2 intersect regularly, then $D_1 \cap D_2$ has codimension 2 and $[D_1 \cap D_2] = (1 - [L_1])(1 - [L_2])$.

A first attempt at a weight filtration might involve declaring a product such as

$$([L_1]-1) \cdot \cdots \cdot ([L_k]-1),$$

where each L_i is a rank 1 projective module, to have weight $\geq k$. This is not quite enough because it fails to assign weight ≥ 2 to the element

$([\det P]-1)-([P]-\operatorname{rank} P)$ when P is an arbitrary projective module, for P may not decompose as a direct sum of rank 1 modules; P may even fail to have a filtration whose successive quotients are rank 1 projectives.

Supposing $P \cong L_1 \oplus \cdots \oplus L_n$, where each L_i is a rank 1 projective, we let $x_i = [L_1]-1$ and compute

$$\begin{aligned}
&([\det P]-1)-([P]-\operatorname{rank} P)\\
&\quad = ([L_1 \otimes \cdots \otimes L_n]-1)-([L_1 \oplus \cdots \oplus L_n]-n)\\
&\quad = (x_1+1)\cdot \cdots \cdot (x_n+1)-(1+(x_1+\cdots+x_n))\\
&\quad = \sum_{k=2}^{n} \sigma_k(x_1, \ldots, x_n).
\end{aligned}$$

One can show that each term in the latter sum has trivial rank and determinant, and that suggests singling them out. Setting $x = x_1+\cdots+x_n = [P]-n$, we define $\gamma^k(x) := \sigma_k(x_1, \ldots, x_n)$. We justify the notation by showing that $\gamma^k(x)$ is a well-defined function of x as follows. Using the definition of the elementary symmetric polynomials, we compute

$$\begin{aligned}
\sum_{k=0}^{n} t^k \gamma^k(x) &= \sum_{k=0}^{n} t^k \sigma_k(x_1, \ldots, x_n)\\
&= \prod_{i=1}^{n}(1+tx_i)\\
&= \prod_{i=1}^{n}(1-t+t[L_i])\\
&= (1-t)^n \prod_{i=1}^{n}\left(1+\frac{t}{1-t}[L_i]\right).
\end{aligned}$$

The latter product is $\lambda_t([P])$ with t replaced by $u := \frac{t}{1-t}$, and we use $\lambda_u([P])$ to denote it. We find that

$$\begin{aligned}
\sum_{k=0}^{n} t^k \gamma^k(x) &= (1-t)^n \lambda_u([P]) = \left(1+\frac{t}{1-t}\right)^{-n} \lambda_u([P])\\
&= \lambda_u(1)^{-n}\lambda_u([P]) = \lambda_u([P]-n) = \lambda_u(x).
\end{aligned}$$

This formula shows that $\gamma^k(x)$ depends only on x and provides a definition of $\gamma^k(x)$ which is useable for any element $x \in K_0(R)$, without assuming that x has the form $\sum([L_i]-1)$.

We observe that $\gamma^0(x) = 1$ and $\gamma^1(x) = x$. When $\operatorname{rank} x = 0$, from the equation $\gamma_t(x)\gamma_t(-x) = 1$ and the fact that both $\gamma_t(x)$ and $\gamma_t(-x)$ are polynomials with constant term 1, we see that the coefficients $\gamma^k(x)$ must be nilpotent for $k \geq 1$.

One can show that $\det(\gamma^k(x)) = 0$ when $k \geq 2$ and $\operatorname{rank} x = 0$, using the splitting principle. Thus $F^2_{\mathrm{alg}}K_0(R)$ is generated by such elements $\gamma^k(x)$.

We now have sufficient motivation to define the gamma filtration on $K_0(R)$. For $k \geq 1$ we let $F_\gamma^k K_0(R)$ be the subgroup generated by all products of the form $\gamma^{k_1}(y_1) \cdot \cdots \cdot \gamma^{k_m}(y_m)$ where each $y_i \in K_0(R)$ has $\operatorname{rank} y_i = 0$ and $\sum_{i=1}^m k_m \geq k$. In particular, $F_\gamma^1 K_0(R) = \{x \in K_0(R) | \operatorname{rank} x = 0\}$. It is evident from the preceding discussion that for a domain R we have isomorphisms $F_\gamma^0 K_0(R)/F_\gamma^1 K_0(R) \cong \mathbb{Z}$ and $F_\gamma^1 K_0(R)/F_\gamma^2 K_0(R) \cong \operatorname{Pic}(R)$.

Now we take a look at the effect of the Adams operations on the gamma filtration. With the notation as above, we have

$$\begin{aligned}\psi^k(x_i) &= \psi^k([L_i] - 1) = [L_i^{\otimes k}] - 1 = [L_i]^k - 1 \\ &= (1 + x_i)^k - 1 = kx_i + \binom{k}{2} x_i^2 + \cdots + x_i^k,\end{aligned}$$

so that $\psi^k(x_i) \equiv kx_i$ modulo $F_\gamma^2 K_0(R)$. Then

$$\begin{aligned}\psi^k(\gamma^r(x)) &= \psi^k \sigma_r(x_1, \ldots, x_n) \\ &= \sigma_r([L_1]^k - 1, \ldots, [L_n]^k - 1) \\ &= \sigma_r(kx_1 + \cdots, \ldots, kx_n + \cdots) \\ &= \sigma_r(kx_1, \ldots, kx_n) + g(x_1, \ldots, x_n) \\ &= k^r \sigma_r(x_1, \ldots, x_n) + g(x_1, \ldots, x_n) = k^r \gamma_r(x) + g(x_1, \ldots, x_n),\end{aligned}$$

where $g(x_1, \ldots, x_n)$ is some symmetric polynomial in n variables of degree $\geq r + 1$. Writing $g(x_1, \ldots, x_n)$ as a polynomial $G(\gamma^{r+1}(x), \gamma^{r+2}(x), \ldots)$ yields a formula

$$\psi^k(\gamma^r(x)) = k^r \gamma^r(x) + G(\gamma^{r+1}(x), \gamma^{r+2}(x), \ldots)$$

which is true for every $x \in F_\gamma^1 K_0(R)$, by the splitting principle. It follows that $\psi^k(\gamma^r(x)) \equiv k^r \gamma^r(x)$ modulo F_γ^{r+1}, and from this it follows (by multiplication) that $\psi^k(z) \equiv k^r z$ modulo F_γ^{r+1} for any $z \in F_\gamma^r K_0(R)$.

If for any $x \in F_\gamma^1 K_0(R)$ we consider the expression $(\psi^k - k^r) \circ (\psi^k - k^{r-1}) \circ \cdots \circ (\psi^k - k)(x)$ for large r, we see that it is a sum of monomials in $\gamma^1(x), \ldots, \gamma^n(x)$ of high degree, so by the nilpotence of the elements $\gamma^i(x)$, the expression will be zero when r is sufficiently large. Combining this with the fact that for any $x \in K_0(R)$, the element $(\psi^k - 1)(x)$ lies in $F_\gamma^1 K_0(R)$, we see (from linear algebra) that an element of $K_0(R)_\mathbb{Q} := K_0(R) \otimes \mathbb{Q}$ is a sum of eigenvectors of ψ^k and that the eigenvalues occurring are nonnegative powers of k. We let $K_0(R)_\mathbb{Q}^{(i)}$ denote the eigenspace for ψ^k with the eigenvalue k^i, so that

$$K_0(R)_\mathbb{Q} = \bigoplus_{i=0}^{\infty} K_0(R)_\mathbb{Q}^{(i)}.$$

This eigenspace can be shown to be independent of the choice of k by considering an expression like $(\psi^k - k^r) \circ (\psi^k - k^{r-1}) \circ \cdots \circ (\psi^{k'} - (k')^i) \circ \cdots \circ (\psi^k - 1)(x)$, which must be zero for sufficiently large r, for the same reason as before.

Finally, because we know that ψ^k acts as k^i on $F_\gamma^i / F_\gamma^{i+1}$, we see that the natural map $K_0(R)_{\mathbb{Q}}^{(i)} \to F_\gamma^i K_0(R)_{\mathbb{Q}} / F_\gamma^{i+1} K_0(R)_{\mathbb{Q}}$ is an isomorphism.

Quillen's method for extending the facts about the gamma filtration and Adams operations to the higher K-groups is described in [**27**]. Let $K_n(R)_{\mathbb{Q}}$ denote $K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q}$, and fix a value $k > 1$. For each $i \geq 0$ define $K_n(R)_{\mathbb{Q}}^{(i)}$ to be the eigenspace of $\psi^k : K_n(R)_{\mathbb{Q}} \to K_n(R)_{\mathbb{Q}}$ for the eigenvalue k^i; we call this group the *weight i part* of $K_n(R)$. The *gamma filtration* on the higher K-groups is defined as follows. Let F_γ^1 denote the kernel of the rank homomorphism $K_*(R) \to K_0(R) \to H^0(\mathrm{Spec}(R), \mathbb{Z})$. Let F_γ^i denote the subgroup of $K_*(R)$ generated by elements $\gamma^{r_1}(x_1) \cdot \cdots \cdot \gamma^{r_m}(x_m)$ with $x_1, \ldots, x_m \in F_\gamma^1$, at most one x_i not in $K_0(R)$, and $\sum r_j \geq i$. (The values of the gamma operations are multiplied using the product operation (12.1).) The main hurdle Quillen had to overcome in showing that $K_n(R)_{\mathbb{Q}} = \bigoplus_{i=0}^{\infty} K_n(R)_{\mathbb{Q}}^{(i)}$ using the techniques analogous to those above was to show that for an element $x \in K_n(R)$ with $n > 0$, the expression $\gamma^k(x)$ vanishes for sufficiently large k. Any such x arises from a pointed map $S^n \to \mathrm{BGl}_N(R)^+$ for some N, and he showed that $\gamma^k(x) = 0$ for $k > N$.

There are two other plausible definitions of the gamma filtration on the K-groups $K_n(R)$ which agree up to torsion with the one above ([**51**, 1.5–1.6] and [**29**]). In one, we allow more than one x_i to lie outside K_0. In the other, we insist that $m = 1$. It is not important that there is no compelling reason to choose one of these filtrations over the other (integrally), because the goal is to define the gamma filtration on the space rather than on its homotopy groups.

Let X be a nonsingular quasi-projective variety over a field, and let $\mathrm{CH}^i(X)$ denote the Chow group codimension i algebraic cycles modulo rational equivalence. Grothendieck's theorem [**26**], answering the question mentioned earlier, asserts that the filtration of $K_0(X)$ by codimension of support agrees with the gamma filtration up to torsion and

$$K_0(X)_{\mathbb{Q}}^{(i)} \cong \mathrm{CH}^i(X)_{\mathbb{Q}}. \tag{14.1}$$

15. Computations and conjectures about weights

Let us return to the explicit computations of K-groups mentioned earlier and add the weights to the description of the K-groups. Classes of free modules in K_0 have weight 0, classes $\{u\}$ of units u in $K_1(R)$ have weight 1, and a product $\{u_1\} \cdot \cdots \cdot \{u_n\} \in K_n(R)$ of units has weight n. Thus, for a field F, the image of $K_n^M(F) \to K_n(F)$ has weight n. In particular, $K_n(F)$

has weight n for $n \leq 2$, but this is not true for $n > 2$. Kratzer has shown **[29]** that in $K_*(R)$ the weight 0 occurs only in $K_0(R)$ and the weight 1 occurs only in $K_0(R)$ and $K_1(R)$; moreover, the units $R^\times$ account for all of the weight 1 part of $K_1(R)$.

Soulé has shown **[51,** Corollary 1, p. 498] that

$$K_n(F)^{(i)}_{\mathbb{Q}} = 0 \qquad \text{for } i > n, \tag{15.1}$$

or in other words, $K_n(F)_{\mathbb{Q}}$ has all its weights $\leq n$. The result depends on stability results due to Suslin **[52]**; the proof has roughly the following flavor: elements in $K_n(F)$ come from n by n matrices, and the kth exterior power of a vector space of dimension n is zero if $k > n$. Soulé shows, more generally, that for Noetherian rings R of dimension d, the group $K_n(R)_{\mathbb{Q}}$ has all its weights $\leq n + d$.

Soulé **[51,** 2.9] and Beilinson **[5,** 2.2.2] have conjectured independently that $K_{2i}(R)$ and $K_{2i-1}(R)$ have all their weights $\geq i$. This conjecture is not known even for fields, except for finite fields, and fields closely related to those. Explicit computations involving cyclic cohomology **[17]** have shown recently that we must append the hypothesis of regularity of the ring R to the conjecture. It would be very useful to have a proof of this conjecture, for Marc Levine has shown that it implies the existence of an interesting construction for a category of mixed motives. In **[4]** the conjecture is strengthened to assert that $K_{2i}(F)$ has all of its weights $\geq i + 1$.

We would like to say that $K_{2i-1}(\mathbb{F}_q)$ has weight i, but this statement is vacuously true because the group is finite. Nevertheless, Hiller **[27]** and Kratzer **[29]** showed that the Adams operation ψ^k acts on $K_{2i-1}(\mathbb{F}_q)$ via multiplication by k^i, so it is plausible to think of i as the weight.

In the decomposition $K_0(\mathscr{O}_F) = \mathbb{Z} \oplus \operatorname{Pic}(\mathscr{O}_F)$, the factor $\mathbb{Z}$ accounts for the free modules and has weight 0, and the factor $\operatorname{Pic}(\mathscr{O}_F)$ accounts for the line bundles and has weight 1. The groups $K_{2i-1}(\mathscr{O}_F)$ are weight i; this follows from examining the proof of Borel's theorem and eventually boils down to the fact that the unit sphere in $\mathbb{C}^i$ is of dimension $2i - 1$.

16. The Atiyah-Hirzebruch spectral sequence

Now let X be a finite cell complex, and let $\mathbb{C}(X^{\text{top}})$ denote the topological ring of continuous functions $X \to \mathbb{C}$. It turns out that there is a way to take the topology of a ring into account when defining the algebraic K-groups, yielding the topological K-groups $K_n(X^{\text{top}}) := K_n(\mathbb{C}(X^{\text{top}}))$, **[41]**. One can do this with any of the definitions above, and let $K(X^{\text{top}})$ denote the space obtained, so that $K_n(X^{\text{top}}) = \pi_n K(X^{\text{top}})$. It turns out that the plus-construction no longer is needed, and one may define $K_n(X^{\text{top}}) = \pi_{n+1} \mathrm{BGl}(\mathbb{C}(X^{\text{top}}))$ for $n > 0$, and prove this is the same as $\pi_n \mathrm{Gl}(\mathbb{C}(X^{\text{top}}))$. Thus $\mathrm{Gl}(\mathbb{C}(X^{\text{top}}))$ is the connected component of the identity in $K(X^{\text{top}})$. As mentioned before (during the discussion of the S-construction), the space

$K(X^{\text{top}})$ is naturally an infinite loop space, with the deloopings getting more and more connected. The connective spectrum corresponding to this infinite loop space structure on $K(*^{\text{top}})$ is denoted by bu.

Let $*$ denote the one-point space. Bott computed the following homotopy groups of $K(*^{\text{top}})$:

$$K_{2i}(*^{\text{top}}) = \pi_{2i}K(*^{\text{top}}) = \mathbb{Z}(i),$$
$$K_{2i-1}(*^{\text{top}}) = \pi_{2i-1}K(*^{\text{top}}) = 0.$$

I write $\mathbb{Z}(i)$ above to mean simply the group $\mathbb{Z}$, together with an action of weight i by the Adams operations. The identification $\mathbb{Z}(1) = K_2(*^{\text{top}})$ can be decomposed as a sequence of isomorphisms

$$K_2(*^{\text{top}}) = \pi_2 \operatorname{BGl}(\mathbb{C}^{\text{top}}) \cong \pi_2 \operatorname{BGl}_1(\mathbb{C}^{\text{top}})) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}(1).$$

The group $\pi_1(\mathbb{C}^\times)$ is weight 1 for the Adams operations, and hence we write it as $\mathbb{Z}(1)$, essentially because the map $\psi^k : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $u \mapsto u^k$ multiplies homotopy classes by k.

Picking a generator for $\pi_1(\mathbb{C}^\times)$ gives us a generator β for $K_2(*^{\text{top}})$. Bott's theorem includes the additional statement that multiplication by β gives a homotopy equivalence of spaces $K(*^{\text{top}}) \to \Omega^2 K(*^{\text{top}})$. This homotopy equivalence gives us a nonconnected delooping $\Omega^{-2}K(*^{\text{top}})$ of $K(*^{\text{top}})$, which is $K(*^{\text{top}})$ itself. These deloopings can be composed to give deloopings of every order and, hence, yields an Ω-spectrum BU that has $K(*^{\text{top}})$ as its underlying infinite loop space and whose homotopy group in dimension $2i$ is $\mathbb{Z}$ for every integer i. The spectrum BU can also be thought of as arising from the connective spectrum bu by inverting the action of multiplication by β.

Let X_+ denote X with a disjoint base point adjoined. There is a homotopy equivalence of the mapping space $K(*^{\text{top}})^X$ with $K(X^{\text{top}})$, and from this it follows that $K_n(X^{\text{top}}) = [X_+, \Omega^n BU]$. When $n < 0$ there might be a bit of ambiguity about what we might mean when we write $K_n(X^{\text{top}})$; we let it always denote $[X_+, \Omega^n BU]$, so that $K_n(X^{\text{top}}) = K_{n+2}(X^{\text{top}})$ for all $n \in \mathbb{Z}$.

We introduce the notation $F_\gamma^0 K_n(X)$ for $[X_+, \Omega^n bu]$. We see that

$$F_\gamma^0 K_n(X) = \begin{cases} K_n(X^{\text{top}}), & n \geq 0, \\ 0, & n < -\dim X. \end{cases}$$

The topological K-groups behave nicely with respect to the Adams operations; it is a theorem of Atiyah and Hirzebruch that

$$K_n(X^{\text{top}})_{\mathbb{Q}}^{(i)} = H^{2i-n}(X, \mathbb{Q}).$$

For $n = 0$ this formula should be compared with (14.1). It was obtained in [1] from a spectral sequence known as the Atiyah-Hirzebruch spectral sequence. The construction of the spectral sequence uses the skeletal filtration $\operatorname{sk}_p X$ of X as follows.

A cofibration sequence $A \subseteq B \twoheadrightarrow B/A$ of pointed spaces and an infinite loop space F give rise to a long exact sequence $\cdots \to [A, \Omega^1 F] \to [B/A, F] \to [B, F] \to [A, F] \to [B/A, \Omega^{-1}F] \to \cdots$.

We construct an exact couple $D_1^{p-1,q} \to E_1^{pq} \to D_1^{pq} \to D_1^{p-1,q+1} \to \cdots$ by setting

$$E_1^{pq} := [\mathrm{sk}_p X/\mathrm{sk}_{p-1} X, \Omega^{-p-q} BU] \quad \text{and} \quad D_1^{pq} := [(\mathrm{sk}_p X)_+, \Omega^{-p-q} BU].$$

The explicit computation of the homotopy groups of BU presented above, together with the fact that the space $\mathrm{sk}_p X/\mathrm{sk}_{p-1} X$ is a bouquet of the p-cells from X, leads to the computation that

$$E_1^{pq} = \begin{cases} C^p(X, \mathbb{Z}(-q/2)) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd} \end{cases}$$

where C^p denotes the group of cellular cochains. (We abbreviate this conclusion by regarding $\mathbb{Z}(-q/2)$ as zero when q is odd.) The differential $d_1 : E_1^{pq} \to E_1^{p+1,q}$ is seen to be the usual differential for cochains, so that $E_2^{pq} = H^p(X, \mathbb{Z}(-q/2))$. The exact couple gives rise to a convergent spectral sequence because X is a finite-dimensional cell complex. The abutment is $[X_+, \Omega^{-p-q} BU] = K_{-p-q}(X)$, so the resulting spectral sequence may be displayed as

$$E_2^{pq} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow K_{-p-q}(X).$$

This spectral sequence is concentrated in quadrants I and IV, is nonzero only in the rows where q is even, and is periodic with respect to the transation $(p, q) \mapsto (p, q-2)$ (if you ignore the action of the Adams operations). Using the Chern character map, Atiyah and Hirzebruch show that the differentials in this spectral sequence vanish modulo torsion and obtain a canonical isomorphism $K_n(X^{\mathrm{top}})_{\mathbb{Q}} \cong \bigoplus_i H^{2i-n}(X, \mathbb{Q}(i))$. It follows that $K_n(X^{\mathrm{top}})_{\mathbb{Q}}^{(i)} \cong H^{2i-n}(X, \mathbb{Q}(i))$. Again, the notation $\mathbb{Q}(i)$ simply denotes $\mathbb{Q}$ together with the action of Adams operations defined by $\psi^k(x) = k^i x$.

17. An alternate approach

In **[12]** Dwyer and Friedlander faced the problem of constructing an Atiyah-Hirzebruch spectral sequence for a scheme X. In this rarified algebraic environment, the notions of triangulation and skeleton are no longer available. One of the ingredients of their solution to the problem was to use the Postnikov tower of BU. In this section we describe that idea, but we remain in the topological situation. We use bu instead of BU and produce an exact couple involving $[X_+, bu]$ that arises from fibration sequences involving bu rather than cofibration sequences involving X_+. This leads naturally to a weight filtration on the K-theory space $K(X^{\mathrm{top}})$.

In the construction of the Atiyah-Hirzebruch spectral sequence presented above, we could have used bu instead of BU. The resulting spectral sequence converges just as well but must be written as

$$(17.1) \qquad E_2^{pq} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow F_\gamma^0 K_{-p-q}(X^{\text{top}})$$

and be accompanied by the announcement that this is a fourth-quadrant spectral sequence, i.e., this time we interpret $\mathbb{Z}(-q/2)$ as zero if $q > 0$ or q is odd. This spectral sequence degenerates modulo torsion just as the other one does, and one sees that $F_\gamma^0 K_{-n}(X^{\text{top}})_{\mathbb{Q}} = H^n(X, \mathbb{Q}(0)) \oplus H^{n+2}(X, \mathbb{Q}(1)) \oplus \cdots$.

It turns out that the gamma filtration exists as a filtration on the space $K(*^{\text{top}})$, but we must interpret the notion of "filtration" homotopy theoretically: the notion of "inclusion" does not survive modulo homotopy. So for our purposes, a filtration on a space T is simply a diagram of spaces $T = T^0 \leftarrow T^1 \leftarrow \cdots$. We will interpret the symbol T^p/T^{p+1} to denote that space that fits into a fibration sequence $T^{p+1} \to T^p \to T^p/T^{p+1}$, if such a space exists. Such a space almost exists, in the sense that its loop space exists as the homotopy fiber of the map $T^{p+1} \to T^p$. If, moreover, the spaces T^p and T^{p+1} are infinite loop spaces and the map $T^{p+1} \to T^p$ is an infinite loop space map, then a delooping of the homotopy fiber will provide a representative for T^p/T^{p+1}. (The K-theory spaces we shall be dealing with are all infinite loop spaces.) This notation is suggestive and convenient; for example, we may rephrase the localization theorem for abelian categories as saying that $K(\mathscr{A}/\mathscr{B}) = K(\mathscr{A})/K(\mathscr{B})$.

As we saw above, the space $K(*^{\text{top}})$ is a simple sort of space, for it has the nice property that each homotopy group is all of one weight for the Adams operations, and each weight occurs in just one homotopy group. This helps us find a weight filtration for it, for there is a standard filtration of any space T, called the Postnikov tower, which peels off one homotopy group at a time.

The Postnikov tower is constructed inductively as follows. (There is also a combinatorial construction in [**35**].) Suppose that T^p is a space with π_p as its lowest nonvanishing homotopy group. For simplicity of exposition we suppose $p > 1$. By the Hurewicz theorem, there is an isomorphism $\pi_p(T^p) \cong H_p(T^p, \mathbb{Z})$; let G denote this group. The identity map $H_p(T^p, \mathbb{Z}) \to G$ corresponds to a cohomology class in $H^p(T^p, G)$ and thus to a map $\phi : T^p \to |G[p]|$ which induces an isomorphism on π_p. We let T^{p+1} be the homotopy fiber of the map and observe that

$$\pi_i(T^{p+1}) = \begin{cases} 0, & i \le p, \\ \pi_i(T^p), & i \ge p+1. \end{cases}$$

The "quotient" T^p/T^{p+1} is identified with $|G[p]|$ and has exactly one nonvanishing homotopy group, namely, $\pi_p(T^p)$.

In the case of the space $K(*^{\text{top}})$, alternate homotopy groups vanish, so we can strip off homotopy groups at double speed, getting a filtration

$$K(*^{\text{top}}) = W_0 \leftarrow W_{-1} \leftarrow W_{-2} \leftarrow \cdots$$

with $W_{-i}/W_{-i-1} = |\mathbb{Z}(i)[2i]|$. Since the pieces of the filtration are of distinct pure weights with respect to Adams operations, it is reasonable to call this the weight filtration of $K(*^{\mathrm{top}})$.

We remark that if $F \to E \to B$ is a fibration sequence of pointed spaces, then each adjacent triple of spaces in the long sequence $\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$ is a fibration sequence. (Here ΩE denotes the loop space of the pointed space E.) If we are dealing with infinite loop spaces, this sequence can be extended to the right with $F \to E \to B \to \Omega^{-1}F \to \Omega^{-1}E \to \Omega^{-1}B \cdots$, where $\Omega^{-1}E$ denotes the chosen delooping of the space E. We will do this sort of thing below to ensure that our long exact sequences continue onward to the right, thereby obtaining legitimate exact couples. This is a concern whenever the map $\pi_0 E \to \pi_0 B$ is not surjective.

If $F \to E \to B$ is a fibration sequence of infinite loop spaces and X is a space, then the mapping spaces fit into a fibration sequence $F^X \to E^X \to B^X$. Taking homotopy groups of this fibration gives a long exact sequence

$$\cdots \to \pi_n F^X \to \pi_n E^X \to \pi_n B^X \to \pi_{n-1} F^X \to \cdots$$

which can be rewritten as

$$\cdots \to [X_+, \Omega^n F] \to [X_+, \Omega^n E] \to [X_+, \Omega^n B] \to [X_+, \Omega^{n-1} F] \to \cdots.$$

As before, we regard this sequence as continuing onward through negative values of n.

The identity $K(*^{\mathrm{top}})^X \cong K(X^{\mathrm{top}})$ can be used to construct a weight filtration for the space $K(X^{\mathrm{top}})$ from the weight filtration for $K(*^{\mathrm{top}})$ considered above. We simply define

$$W_q K(X^{\mathrm{top}}) := (W_q K(*^{\mathrm{top}}))^X$$

and

$$\begin{aligned} W_q K(X^{\mathrm{top}})/W_{q-1} K(X^{\mathrm{top}}) &:= (W_q K(*^{\mathrm{top}})/W_{q-1} K(*^{\mathrm{top}}))^X \\ &= |\mathbb{Z}(-q)[-2q]|^X. \end{aligned}$$

Using this filtration of $K(X^{\mathrm{top}})$ we define $D_2^{pq} := \pi_{-p-q} W_q K(X^{\mathrm{top}})$ and

$$\begin{aligned} E_2^{pq} &:= \pi_{-p-q} W_q K(X^{\mathrm{top}})/W_{q-1} K(X^{\mathrm{top}}) \\ &= [X_+, \Omega^{-p-q}(W_q K(*^{\mathrm{top}})/W_{q-1} K(*^{\mathrm{top}}))] \\ &= [X_+, \Omega^{-p-q}|\mathbb{Z}(-q)[-2q]|] \\ &= [X_+, |\mathbb{Z}(-q)[p-q]|] \\ &= H^{p-q}(X, \mathbb{Z}(-q)) \end{aligned}$$

and produce an exact couple

$$\cdots \to D_2^{p+1,q-1} \to D_2^{pq} \to E_2^{pq} \to D_2^{p+2,q-1} \to \cdots.$$

The resulting spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow F_\gamma^0 K_{-p-q}(X^{\mathrm{top}})$$

is related to the spectral sequence (17.1). Since we were able to strip off the homotopy groups of bu at double speed, the rows in (17.1) with q odd (which are all zero) do not appear here. (In fact, there is a simple renumbering scheme that will accomplish the same thing for any spectral sequence whose odd numbered rows are zero.)

18. Motivic cohomology

Now return to the case where X is an algebraic variety. There is no suitable cohomology theory for X with integer coefficients, but taking the Atiyah-Hirzebruch theorem as a guide, we may guess a definition of a cohomology theory with rational coefficients. Following Beilinson [**5**, 2.2.1] we define *motivic* cohomology groups

$$H^{2i-n}_{\mathscr{M}}(X, \mathbb{Q}(i)) := K_n(X)^{(i)}_{\mathbb{Q}},$$

or, equivalently,

$$H^{m}_{\mathscr{M}}(X, \mathbb{Q}(i)) := K_{2i-m}(X)^{(i)}_{\mathbb{Q}}. \tag{18.1}$$

The rational motivic cohomology groups defined in (18.1) have the correct functorial properties and localization theorems expected for groups which we like to call "cohomology" groups, for in [**51**] Soulé defines motivic cohomology groups with supports and motivic homology groups and proves part of what is required from them to form a "Poincaré duality theory with supports" in the sense of Bloch and Ogus [**8**]. The proof uses the localization theorems of Quillen for the algebraic K-groups, as well as the Riemann-Roch theorem without denominators for algebraic K-theory proved by Gillet [**19**]. Soulé's definition of a motivic homology theory for a variety Z, possibly singular, involves embedding Z in a nonsingular variety X. Using the Adams operations for X and the complement $X - Z$, together with the localization theorem, allows one to define Adams operations on $K'_m(Z)_{\mathbb{Q}}$ which are ultimately independent of the choice of X. The corresponding eigenspaces $K'_m(Z)_{(j)}$ then give the proposed homology groups. When $Z \subseteq Y$ are varieties, possibly singular, there is a cap product operation $K^Z_m(Y) \otimes K'_n(Y) \xrightarrow{\cap} K'_{m+n}(Z)$ arising from the tensor product operation, but it is not yet known how to show this cap product is compatible with the Adams operations, in the sense that it induces pairings $K^Z_m(Y)^{(i)} \otimes K'_n(Y)_{(j)} \xrightarrow{\cap} K'_{m+n}(Z)_{(j-i)}$.

The theorem of Soulé mentioned before (15.1), when rephrased in terms of motivic cohomology groups, becomes a statement about cohomological dimension. It says, for a field F, that $H^j_{\mathscr{M}}(\mathrm{Spec}(F), \mathbb{Q}(i)) = 0$ for $j > i$. More generally, if X is an affine scheme of dimension d, then $H^j_{\mathscr{M}}(X, \mathbb{Q}(i)) = 0$ for $j > i + d$.

Let X be a (nonsingular) variety. The conjecture of Beilinson and Soulé about weights, when rephrased in terms of the motivic cohomology groups, says that if $j < 0$ then $H^j_{\mathscr{M}}(X, \mathbb{Q}(i)) = 0$. (The strengthened form includes the statement that $H^0_{\mathscr{M}}(X, \mathbb{Q}(i)) = 0$ for $i > 0$.) It is expected that there is an alternate construction of these motivic cohomology groups so that $H^j_{\mathcal{M}}(X, \mathbb{Q}(i))$ appears as a Yoneda-Ext group $\operatorname{Ext}^j(\mathbb{Q}, \mathbb{Q}(i)) = 0$ of certain objects (Tate motives) in an abelian category (yet to be defined) of "mixed motives". The vanishing conjecture would be an immediate consequence of such a construction or an obstacle to performing the construction.

One could ask the following question. Given a nonsingular algebraic variety X, consider the algebraic K-theory space $K(X)$; does it have a *weight* filtration $K(X) = W_0K(X) \leftarrow W_{-1}K(X) \leftarrow W_{-2}K(X) \leftarrow \cdots$ whose successive quotients are of pure weight for the Adams operations? (It might also be necessary to view all the spaces in sight as spectra with negative homotopy groups and to sheafify things with respect to the topology on X.) Were there such a filtration we could define motivic cohomology groups with integer coefficients $H^m_{\mathcal{M}}(X, \mathbb{Z}(i)) := \pi_{2i-m}(W_{-i}K(X)/W_{-i-1}K(X))$ and assemble these groups into an Atiyah-Hirzebruch-type spectral sequence, as we did above. In addition, in order for the motivic cohomology groups to be "cohomology" groups in the usual sense of the word, it would be necessary for the space $W_{-i}K(X)/W_{-i-1}K(X)$ to be homotopy equivalent to the geometric realization of a simplicial abelian group. The normalized chain complex associated to that simplicial abelian group would be the motivic complex sought by Beilinson in **[6,** 5.10D] and by Lichtenbaum in **[33]** and **[34]**. We reverse the numbering of the complex and shift its numbering scheme by $2i$ so that $H^m_{\mathcal{M}}(X, \mathbb{Z}(i))$ is the mth cohomology group of a cohomological complex; compatibility with the vanishing conjecture of Soulé and Beilinson would then require the complex to be exact except in degrees $1, \ldots, i$ when $i > 0$ and X is the spectrum of a field or a local ring.

19. Constructing motivic chain complexes

There has been a flurry of activity centering upon attempts to build these motivic chain complexes directly.

A promising direct construction of a chain complex aiming to fulfill this role is presented by Bloch in **[9]**; the construction involves algebraic cycles on the standard cosimplicial affine space over X. The resulting *higher Chow groups* bear the same relation to the usual Chow groups as the higher K-groups bear to the Grothendieck group.

Various other constructions of such chain complexes are presented in **[7]** by Beilinson, MacPherson, and Schechtmann; their *Grassmannian complex* involves "linear" algebraic cycles, i.e., formal linear combinations of linear subspaces of vector spaces satisfying a transversality condition. Gerdes **[18]** proves that the map from the homology of the Grassmannian complex to K-theory is a surjection rationally, but unfortunately is not an isomorphism rationally; thus the Grassmannian complex needs to be repaired in some as yet unknown way.

In **[30]** and **[31]** Landsburg modifies Bloch's approach by introducing the categories of modules of codimension $\geq p$ on the standard cosimplicial affine space over X.

In [**56**] Thomason presents an adelic construction of motivic complexes which has the virtue of being directly connected to K-theory, but is not based upon algebraic cycles. He shows how to combine the gamma filtration on rational algebraic K-theory with the canonical weight filtration on ℓ-adic étale topological K-theory to get a good weight filtration on the integral spectrum of algebraic K-theory localized (in the sense of Bousfield) at BU.

In [**16**] Friedlander and Gabber introduce a complex $A_r(Y, X)$, called the algebraic bivariant cycle complex, which is a possible candidate for a motivic complex. It is related to Lawson homology, but has the advantage that it incorporates rational equivalence in its construction rather than algebraic equivalence.

20. The motivic spectral sequence

In the context of topological K-theory and singular cohomology, the rational cohomology groups, were they to be defined as the appropriate weight spaces in rational topological K-theory, would immediately be seen to be computable in terms of singular cochains by considering the influence of the skeletal filtration of the finite cell complex X on the K-theory. For an algebraic variety X, the nearest thing we have to a skeletal filtration on X is the filtration by codimension of support of the category of coherent sheaves on X. As mentioned above, the group $K_0(\mathcal{M}^p/\mathcal{M}^{p+1})$ is the group of algebraic cycles of codimension p, so it is expected that algebraic cycles on the variety X will play the same role that singular cochains play on a finite cell complex.

In Figure 20.1 is a handy chart of the prospective motivic Atiyah–Hirzebruch spectral sequence

$$E_2^{pq} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$

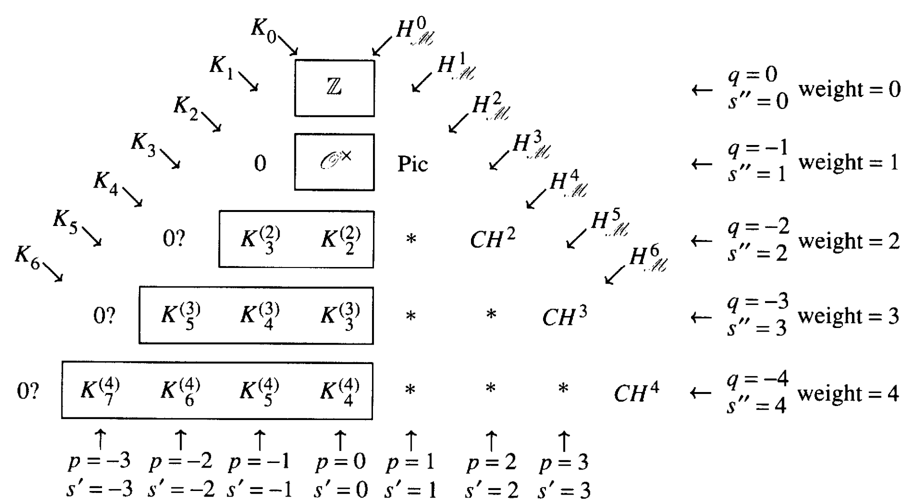

Figure 20.1

Beilinson conjectured its existence in **[6,** 5.10B]. The spectral sequence is drawn so that the degree of the d_2 differential is the customary $(2, -1)$, but we do not draw the arrows for these differentials, as they are expected to die modulo controllable torsion anyway. Let $n = -p - q$ and $i = -q$, so that we are concerned with the weight i part of K_n. If we define $s' = p$ and $s'' = -q$ and take X to be a scheme of finite type, then the rank of the group E_2^{pq} is conjecturally **[49]** associated with the order of zero of the L-function for $H_{\text{ét}}^{p-q-1}$ at the point s', which is to the left of the critical line. The corresponding regulator map to Deligne cohomology should involve a polylogarithm function of order $-q$. The point s'', to the right of the critical line, is the one related to s' by the conjectural functional equation for the L-series. The real point on the critical line is $(s'+s'')/2 = (p-q)/2$, and the distance from it to either of the other two points is $(s''-s')/2 = n/2$. When $n = 0$ we are at the center of the critical strip, and the Beilinson conjectures are like the Birch and Swinnerton-Dyer conjectures. When $n = 1$ we are at the edge of the critical strip, and the Beilinson conjectures are like the Tate conjecture. When $n \geq 2$ we are outside the critical strip, and s'' is in the region of absolute convergence. When $p = 0$, then we are concerned with the weight i part of K_i; this is the part corresponding to Milnor K-theory, so the line $p = 0$ is called the Milnor line. When $p - q = 1$, then we are concerned with the motivic cohomology groups $H^1_{\mathscr{M}}(X, \mathbb{Z}(i))$, which is where the Borel classes in the higher algebraic K-theory of algebraic number rings appear; thus, the line $p - q = 1$ is called the Borel line.

The boxes in the diagram contain the groups that are expected to be nonzero when X is the spectrum of a field, according to the conjecture of Beilinson and Soulé. In addition, one can imagine the E_1 term of the spectral sequence underlying the diagram, so that the groups in the boxes would be cohomology groups of the motivic complexes lying in each row, with the differential maps going to the right.

The stars in the right-hand portion of the diagram, together with the indicated Picard and Chow groups, represent the groups that are known to be zero when $\dim X < p$, according to the vanishing theorem of Soulé. We may say that a group in column p reflects the structure of X in codimension p. (Will this phrase eventually have a meaning for negative values of p?)

The question marks on the left side of the diagram together with the cells off the chart to the left are the locations where the (strengthened) vanishing conjecture is not yet known.

21. Related spectral sequences

Let A be a finitely generated regular ring. In **[42]**, motivated by conjectures of Lichtenbaum relating K-theory to étale cohomology, Quillen hopes that there is an Atiyah-Hirzebruch spectral sequence of the form

$$(21.1) \qquad E_2^{pq} = H^p_{\text{ét}}(\operatorname{Spec}(A[\ell^{-1}]), \mathbb{Z}_\ell(-q/2)) \Rightarrow K_{-p-q}(A) \otimes \mathbb{Z}_\ell$$

converging at least in degrees $-p - q > \dim(A) + 1$; it would degenerate in case A is the ring of integers in a number field, and either ℓ is odd or A is totally imaginary. This conjecture is now known as the Quillen-Lichtenbaum conjecture; it is not known yet, but inspired by work of Soulé, three approximations to this conjecture have been proved. We describe them now.

In **[15]** Friedlander defined étale K-theory in terms of mapping spaces from the étale homotopy type of X to a BU and used the approach of §17 to produce an Atiyah-Hirzebruch spectral sequence connecting étale cohomology to étale K-theory. But he was limited to considering the case of varieties over algebraically closed fields. In order to treat more general schemes X, it is necessary to produce a suitable algebraic model for bu and to account for the way it changes from point to point in the space X. This involves using the simplicial scheme $\mathrm{BGl}(\mathscr{O}_X)$ as the algebraic model for bu, and defining the étale ℓ-adic K-theory space as a space of sections for the fibrewise ℓ-adic completion of the map $\mathrm{BGl}(\mathscr{O}_X)_{\text{ét}} \to X_{\text{ét}}$. The resulting étale K-groups depend on the choice of a prime ℓ which is invertible on X and are denoted by $K_n^{\text{ét}}(X;\mathbb{Z}_\ell)$. Under the assumption that X is a scheme of finite $\mathbb{Z}/\ell$-cohomological dimension, Dwyer and Friedlander **[12,** 5.1] construct a fourth-quadrant spectral sequence

$$E_2^{pq} = H_{\text{ét}}^p(X, \mathbb{Z}_\ell(-q/2)) \Rightarrow K_{-p-q}^{\text{ét}}(X;\mathbb{Z}_\ell) \tag{21.2}$$

converging in positive degrees. (Presumably, it is possible to ensure convergence in all degrees by replacing the abutment by the zero-th stage of its gamma filtration, as we did with (17.1).) Dwyer and Friedlander rephrase the Quillen-Lichtenbaum conjecture as the assertion that the natural map $K_n(X;\mathbb{Z}_\ell) \to K_n^{\text{ét}}(X;\mathbb{Z}_\ell)$ from algebraic K-theory to étale K-theory (with $\mathbb{Z}_\ell$ coefficients) is an isomorphism for all $n \geq 0$, for the case where $X = \mathrm{Spec}(\mathscr{O})$, with $\mathscr{O}$ the ring of integers in a number field, and X has finite cohomological dimension. When X has infinite cohomological dimension, but has a finite étale covering which has finite cohomological dimension (as is the case for $\mathrm{Spec}(\mathbb{Z})$ at $\ell = 2$), they provide **[13]** a modification to the definition of $K_n^{\text{ét}}(X;\mathbb{Z}_\ell)$, which makes use of this covering and provides an analogous rephrasing of the Quillen-Lichtenbaum conjecture. The homotopy type of the corresponding space $K^{\text{ét}}(\mathrm{Spec}(\mathbb{Z});\mathbb{Z}_\ell)$ is obtained for the case $\ell = 2$ in **[13]**, and for ℓ a regular prime this is done in **[14]**. Building on this work, S. Mitchell obtained the following completely explicit conjectural calculation of the K-groups of $\mathbb{Z}$, for $n \geq 2$; its validity depends on the truth of the Quillen-Lichtenbaum conjecture, together with the conjecture of Vandiver which states that for every prime ℓ, if $\zeta_\ell = e^{2\pi i/\ell}$, the class number of the totally real number ring $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$ is not divisible by ℓ.

$$K_n(\mathbb{Z}) \cong \begin{cases} 0, & n \equiv 0 \pmod 8, \\ \mathbb{Z} \oplus \mathbb{Z}/2, & n \equiv 1 \pmod 8, \\ \mathbb{Z}/c_k \oplus \mathbb{Z}/2, & n \equiv 2 \pmod 8, \\ \mathbb{Z}/8d_k, & n \equiv 3 \pmod 8, \\ 0, & n \equiv 4 \pmod 8, \\ \mathbb{Z}, & n \equiv 5 \pmod 8, \\ \mathbb{Z}/c_k, & n \equiv 6 \pmod 8, \\ \mathbb{Z}/4d_k, & n \equiv 7 \pmod 8. \end{cases} \tag{21.3}$$

Here c_k and d_k are defined so that c_k/d_k is the fraction in lowest terms representing the number B_k/k, with B_k being the kth Bernoulli number, and with k being chosen so that n is $4k-1$ or $4k-2$. For example, from $B_6 = -691/2730$ we may read off the conjectural equation $K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691$. (Warning: in certain other numbering schemes for the Bernoulli numbers, it is B_{12} that is $-691/2730$.)

Let X be a nice scheme on which the prime ℓ is invertible. If $\ell = 2$, assume further that $\sqrt{-1}$ exists on X. In favorable cases an element $\beta \in K_2(X; \mathbb{Z}/\ell^\nu)$ arises via the long exact sequence

$$\cdots \to K_2(X; \mathbb{Z}/\ell^\nu) \to K_1(X) \xrightarrow{\ell^\nu} K_1(X) \to \cdots$$

from a primitive ℓ^νth root of unity. This element serves as a replacement for the Bott element of topological K-theory, which is missing in this algebraic context. Thomason (**[55]** and **[56]**) constructs a spectral sequence

$$E_2^{pq} = H^p_{\text{ét}}(X, \mathbb{Z}/\ell^\nu(-q/2)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/\ell^\nu)[\beta^{-1}] \tag{21.4}$$

in the first and fourth quadrants which succeeds admirably in linking algebra and topology. The abutment is something obtained from K-theory itself by inverting the action of β on the level of topological spectra.

The latest exciting result in this direction is due to Mitchell **[38]**. (See also **[39]**.) He strengthens Thomason's theorem by identifying the abutment with something closer to algebraic K-theory. Thomason's spectral sequence then becomes

$$E_2^{pq} = H^p_{\text{ét}}(X, \mathbb{Z}/\ell^\nu(-q/2)) \Rightarrow (L_\infty K)_{-p-q}(X; \mathbb{Z}/\ell^\nu) \tag{21.5}$$

converging for $-p-q \gg 0$. The abutment $L_\infty K$ is the *harmonic* localization of algebraic K-theory, which is a much weaker localization of K-theory than the one obtained by inverting the Bott element. I refer the reader to the highly readable survey paper **[40]** for further details.

22. Maps from K-theory to cohomology

According to the motivic philosophy, the motivic cohomology groups ought to be universal in some sense, so there ought to be maps from the motivic cohomology groups to any other reasonable cohomology theory of algebraic varieties. For the rational cohomology groups defined above, such maps were constructed by Soulé in **[47]** for étale cohomology and by Gillet in **[19]** for any cohomology theory $H^*_\alpha(—, \mathbb{Z}(i))$ satisfying a certain general set of axioms. These maps hinge on the construction of Chern class maps

$$c_i : K_n(X) \to H^{2i-n}_\alpha(X, \mathbb{Z}(i)). \tag{22.1}$$

Starting with an element $x \in K_n(X)$ apply the Hurewicz map to get an element in homology, $x' \in H_n(\mathrm{BGl}(R), \mathbb{Z})$. Since $\mathrm{Gl}(R) = \bigcup_m \mathrm{Gl}_m(R)$, we may choose m large enough so that x' comes from an element $x'' \in H_n(\mathrm{BGl}_m(R), \mathbb{Z})$. The functor $R \mapsto \mathrm{Gl}_m(R)$ from rings to groups is represented by the algebraic group Gl_m, and the functor $R \mapsto \mathrm{BGl}_m(R)$ is represented by the simplicial scheme BGl_m. For each R, the free module R^m is a $\mathrm{Gl}_m(R)$-module; these modules assemble into a simplicial vector bundle $\mathscr{O}^m$ on BGl_m. Grothendieck's theory of Chern classes gives elements $c_i(\mathscr{O}^m) \in H^{2i}_\alpha(\mathrm{BGl}_m, \mathbb{Z}(i))$. For any scheme Z there is a natural evaluation map $Z(R) \times \mathrm{Spec}(R) \to Z$. Let $f : \mathrm{BGl}_m(R) \times \mathrm{Spec}(R) \to \mathrm{BGl}_m$ denote the corresponding evaluation map of simplicial schemes. There is a cap product operation **[53,** 5.1.6]

$$H_n(\mathrm{BGl}_m(R), \mathbb{Z}) \otimes H^{2i}_\alpha(\mathrm{BGl}_m(R) \times X, \mathbb{Z}(i)) \xrightarrow{\cap} H^{2i-n}_\alpha(X, \mathbb{Z}(i)).$$

Using it, we obtain an element $c_i(x) := x'' \cap f^* c_i(\mathscr{O}^m) \in H^{2i-n}_\alpha(X, \mathbb{Z}(i))$. One checks that this procedure is independent of the choices involved, yielding the map c_i of (22.1).

As an example, if we take $\alpha = \mathscr{D}$, where $H^*_\mathscr{D}$ is the Deligne cohomology theory, one may show that the map

$$c_1 : K_1(X) \to H^1_\mathscr{D}(X, \mathbb{Z}(i)) \cong H^0(X, \mathscr{O}_X^\times)$$

is the usual determinant map.

The usual methods yield Chern *character* maps

$$ch_i : K_n(X)_\mathbb{Q} \to H^{2i-n}_\alpha(X, \mathbb{Q}(i)),$$

whose direct product is a graded ring homomorphism. By compatibility of ch_i with the Adams operations, it turns out that ch_i vanishes on the part of $K_n(X)_\mathbb{Q}$ of weight unequal to i, so it is natural to consider the induced map

$$ch_i : K_n(X)^{(i)}_\mathbb{Q} \to H^{2i-n}_\alpha(X, \mathbb{Q}(i))$$

as the object of primary importance. Letting $m = 2i - n$ and rewriting the source of this map in terms of the motivic cohomology groups defined earlier allows us to present this map as

$$ch_i : H^m_\mathscr{M}(X, \mathbb{Q}(i)) \to H^m_\alpha(X, \mathbb{Q}(i)).$$

Gillet has proved **[19]** an important Grothendieck-Riemann-Roch theorem for these Chern class maps and Chern character maps.

If X is a regular scheme of finite type over $\mathbb{Z}$, then the Chern character map

$$ch_i : H^m_{\mathscr{M}}(X, \mathbb{Q}(i)) \to H^m_{\mathscr{D}}(X \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{R}(i))^+$$

is known as a *regulator* map. The target group (in which the superscript + denotes taking the invariants under complex conjugation) is a finite-dimensional real vector space. These regulator maps are central to the theorem of Borel (11.2) about ranks of K-groups of rings of algebraic integers and to the conjecture of Beilinson.

REFERENCES

1. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Differential Geometry, Proceedings of a conference held at University of Arizona in 1960, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, RI, 1961, pp. 7–38.
2. H. Bass, *Algebraic K-Theory*, W. A. Benjamin, Inc., New York and Amsterdam, 1968.
3. H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. **33** (1967), 59–137.
4. A. A. Beilinson, Letter to C. Soulé.
5. ——, *Higher regulators and values of L-functions*, Current Problems in Mathematics **24** (1984), 181–238, Itogi Nauki i Tekhniki, Moscow, 1984.
6. ——, *Height pairing between algebraic cycles*, K-Theory, Arithmetic and Geometry; Seminar, Moscow, 1984–1986 (Yu. I. Manin, ed.), Lecture Notes in Math., vol. 1289, Springer-Verlag, Berlin, Heidelberg, and New York, 1987, pp. 1–26.
7. A. Beilinson, R. MacPherson, and V. Schechtman, *Notes on motivic cohomology*, Duke Math. J. **54** (1987), 679–710.
8. S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201.
9. S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.
10. A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. **7** (1974), 235–272.
11. A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. of Math. (2) **68** (1958), 54–80.
12. W. G. Dwyer and E. M. Friedlander, *Algebraic and étale K-theory*, Trans. Amer. Math. Soc. **292** (1985), 247–280.
13. ——, *Conjectural calculations of general linear group homology*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part I, Proceedings of a Summer Research Conference held June 12–18, 1983, Boulder, CO, Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, pp. 135–147.
14. ——, *Topological models for arithmetic*, Topology (to appear).
15. E. M. Friedlander, *Étale K-theory,* I. *Connections with étale cohomology and algebraic vector bundles*, Invent. Math. **60** (1980), 105–134.
16. E. M. Friedlander and O. Gabber, *Cycle spaces and intersection theory*, in Topological Methods in Modern Mathematics, Publish or Perish, 1993, pp. 325–370.
17. S. Geller and C. Weibel, *Hodge decompositions of Loday symbols in K-theory and cyclic homology*, preprint.
18. W. Gerdes, *The linearization of higher Chow cycles of dimension one*, Duke Math. J. **62** (1991), 105–129.
19. H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Adv. Math. **40** (1981), 203–289.
20. H. Gillet and D. R. Grayson, *The loop space of the Q-construction*, Illinois J. Math. **31** (1987), 574–597.
21. H. Gillet and C. Soulé, *Intersection theory using Adams operations*, Invent. Math. **90** (1987), 243–278.
22. D. R. Grayson, *Higher algebraic K-theory.* II [*after Daniel Quillen*], Algebraic K-Theory, Evanston 1976, Lecture Notes in Math, vol. 551, Springer-Verlag, Berlin, Heidelberg, and New York, 1976, pp. 217–240.

23. D. R. Grayson, *On the K-theory of fields*, Algebraic K-Theory and Algebraic Number Theory, Proceedings of a Seminar held January 12–16, 1987, Honolulu, HI, Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 31–55.
24. ——, *Exterior power operations on algebraic K-Theory*, K-Theory **3** (1989), 247–260.
25. ——, *Adams operations on higher K-Theory*, K-theory **6** (1992), 97–111.
26. A. Grothendieck, P. Berthelot, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch,* Séminaire de Géométrie Algébrique du Bois-Marie 1966/67, SGA 6, Lecture Notes in Math., vol. 225, Springer-Verlag, Berlin, Heidelberg, and New York, 1971.
27. H. L. Hiller, *λ-Rings and algebraic K-theory*, J. Pure Appl. Algebra **20** (1981), 241–266.
28. D. Kan, *Functors involving c.s.s. complexes*, Trans. Amer. Math. Soc. **87** (1958), 330–346.
29. C. Kratzer, *λ-Structure en K-théorie algébrique*, Comment. Math. Helv. **55** (1980), 233–254.
30. S. Landsburg, *Relative cycles and algebraic K-theory*, Amer. J. Math. **111** (1989), 599–632.
31. ——, *Some filtrations on higher K-Theory and related invariants*, K-Theory **6** (1992), 431–455.
32. R. Lee and R. H. Szczarba, *The group $K_3\mathbb{Z}$ is cyclic of order* 48, Ann. of Math. (2) **104** (1976), 31–60.
33. S. Lichtenbaum, *Values of zeta-functions at non-negative integers*, Number Theory (Journées Arithmetique, Noordwijkerhout, 1983) (H. Jager, ed.), Lecture Notes in Math, vol. 1068, Springer-Verlag, Berlin, Heidelberg, and New York, 1984, pp. 127–138.
34. ——, *The construction of weight-two arithmetic cohomology*, Invent. Math. **88** (1987), 183–215.
35. J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Math. Stud., vol. 11, Van Nostrand, Princeton, NJ, 1967.
36. J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. of Math. (2) **65** (1957), 357–362.
37. ——, *Introduction to algebraic K-Theory*, Ann. of Math. Stud., vol. 72, Princeton University Press, Princeton, NJ, 1971.
38. S. A. Mitchell, *The Morava K-theory of algebraic K-theory spectra*, K-Theory **3** (1990), 607–626.
39. ——, *Harmonic localization of algebraic K-theory spectra*, Trans. Amer. Math. Soc. **332** (1992), 823–837.
40. ——, *On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic viewpoint*, Proceedings of the Topology and K-theory year, 1989–90, Math. Sci. Res. Inst. Publ., Springer-Verlag, Berlin, Heidelberg, and New York (to appear).
41. M. Paluch, *Algebraic and topological K-theory*, thesis, Univ. of Illinois, Chicago.
42. D. Quillen, *Higher Algebraic K-theory*, Proceedings of the International Congress of Mathematicians, Vancouver 1974, **I**, Canad. Math. Congr., 1975, pp. 171–176.
43. ——, *Higher algebraic K-Theory.* I, Algebraic K-Theory. I. Higher K-Theories, Lecture Notes in Math. **341**, Springer-Verlag, Berlin, Heidelberg, and New York, 1973, pp. 85–147.
44. ——, *Finite generation of the groups K_i of rings of algebraic integers, Algebraic K-Theory* I. Higher K-Theories, Lecture Notes in Math. **341**, Springer-Verlag, Berlin, Heidelberg, and New York, 1973, pp. 179–198.
45. ——, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), 552–586.
46. P. Schneider, *Introduction to the Beilinson conjectures*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, P. Schneider, and N. Schappacher, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 1–36.
47. C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, C. R. Acad. Sci., Paris Sér. I Math. **55** (1979), 251–295.
48. ——, *Régulateurs*, numéro 644, Séminaire Bourbaki (1984/85), Secrétariat mathématique, Paris.
49. ——, *K-théorie et zéros aux points entiers de fonctions zêta*, Proceedings of the International Congress of Mathematicians, Warszawa, August 16–24, 1983, **1**, Elsevier Science Publishing Company, Inc., New York, pp. 437–445.

50. ——, *On K-theory and values of zeta functions*, Current Trends in Algebraic Topology, Part 2 (London, Ontario, 1981), Contemp. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1982, pp. 49–58.
51. ——, *Opérations en K-théorie algébrique*, Canad. J. Math. **37** (1985), 488–550.
52. A. A. Suslin, *Stability in algebraic K-theory*, Lecture Notes in Math., vol. 966, Springer-Verlag, Berlin, Heidelberg, and New York, 1982, pp. 304–333.
53. ——, *Homology of GL_n, characteristic classes and Milnor K-theory*, Algebraic K-Theory, Number Theory, Geometry, and Analysis, Proceedings, Bielfeld 1982 (A. Bak, ed.), Lecture Notes in Math., vol. 1046, Springer-Verlag, Berlin, Heidelberg, and New York, 1984, pp. 347–375.
54. ——, *On the K-theory of local fields*, J. Pure Appl. Algebra **34** (1984), 301–318.
55. R. W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), 437–552.
56. ——, *Survey of algebraic vs. étale topological K-theory*, Algebraic K-Theory and Algebraic Number Theory, Proceedings of a Seminar held January 12–16, 1987, Honolulu, HI, Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 393–444.
57. F. Waldhausen, *Algebraic K-theory of spaces, Algebraic and Geometric Topology*, Lecture Notes in Math., vol. 1126, Springer-Verlag, Berlin, Heidelberg, and New York, 1985, pp. 318–419.

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An Elementary Presentation for K-Groups and Motivic Cohomology

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One daunting aspect of the Beilinson conjectures relating K-theory to values of L-functions is the difficult technical machinery necessary to develop higher K-theory with Adams operations, Chern classes, and regulator maps. My purpose in what follows is to give an elementary approach to these, using algebraic cycles on "spheres". The basic idea is to get a presentation for K_n of a regular ring R in terms of the Grothendieck group of certain modules on affine n-space (Theorem 3). This leads to an interpetation of $\mathrm{gr}_\gamma^m K_n(R)$ as a sort of Chow group of codimension m cycles on $\mathbb{A}_R^n$ (or more intuitively as a Chow group of the "sphere" S_R^n obtained by gluing two copies of $\mathbb{A}_R^n$ along the simplex defined by $t_1 \cdot t_2 \cdot \dots \cdot t_n \cdot (1 - t_1 - \dots - t_n) = 0)$. The regulator map in this context is identified with the cycle class map. The cycles in question have support in the smooth locus of S_R^n, so there are no new technical problems involved in defining their classes.

The cycle-theoretic point of view espoused here has a number of advantages. In addition to being simpler than the abstract K-theory, cycles are quite evidently motivic. Realizations automatically exist in any cohomology theory with reasonable algebraic cycle classes. It also throws an interesting light on a number of deep conjectures in the subject, including the Soulé-Beilinson conjecture that $\mathrm{gr}_\gamma^m K_n(R) \otimes \mathbb{Q} = (0)$ for R regular and $m \le n/2$ (unless $m = n = 0$) and the Kato conjecture for Galois cohomology, which states that $H^*(F, \mu_n^{\otimes *})$ is generated as a ring by $H^1(F, \mu_n)$.

Let R be a regular ring. Write $R_.$ for the simplicial ring

$$R_n = R[t_0, \dots, t_n] \Big/ \left(\sum t_i - 1\right)$$

with face (resp. degeneracy) operators $t_i \mapsto 0$ (resp. $t_i \mapsto t_i + t_{i+1}$). To my knowledge, this ring was first introduced in [A]. Let $\Sigma^n \subset \mathrm{Spec}(R_{n+1})$ be the

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union of the faces. I am indebted to A. Suslin for suggesting how to prove

PROPOSITION 1. $\pi_n(\mathrm{GL}(R.)) \cong K_1(\Sigma^n)/K_1(R)$ *for* $n \geq 0$.

PROOF. An element in $\pi_n(\mathrm{GL}(R.))$ is represented by a map $\Sigma^n \to \mathrm{GL}$, i.e., by an element in $\mathrm{GL}(R_{n+1}/I_{n+1})$ where $I_{n+1} \subset R_{n+1}$ is the ideal of the closed subset Σ^n. Indeed, $I_{n+1} = t_0 \cdot \cdots \cdot t_{n+1} R_{n+1}$, so an element in $\mathrm{GL}(R_{n+1}/I_{n+1})$ is a collection of elements $\alpha_i \in \mathrm{GL}(R_n)$, $i = 0, \dots, n+1$, agreeing on the overlap of these faces in $\mathrm{Spec}(R_{n+1})$. It is easy to see that such a collection $\{\alpha_0, \dots, \alpha_0\}$ represents an element α in $\pi_n(\mathrm{GL}(R.))$ and that α is trivial if and only if there exists a $\beta \in \mathrm{GL}(R_{n+1})$ such that $\beta|\Sigma^n = \alpha$. (The point is that a simplicial group necessarily satisfies the Kan extension property. For a proof of this, see [**May**, Theorem 17.1].)

On the other hand, thinking of α as an element in $\mathrm{GL}(R_{n+1}/I_{n+1})$, we see that the image of α in $K_1(\Sigma^n)/K_1(R)$ is trivial if and only if $\alpha \in \mathrm{GL}(R) \cdot E(R_{n+1}/I_{n+1})$, where E denotes the group of elementary matrices. Since $\mathrm{GL}(R_{n+1}) = \mathrm{GL}(R) \cdot E(R_{n+1})$ and $E(R_{n+1}) \twoheadrightarrow E(R_{n+1}/I_{n+1})$, it follows that α is trivial in $K_1(\Sigma^n)/K_1(R)$ if and only if there exists $\beta \in \mathrm{GL}(R_{n+1})$ such that $\beta|\Sigma^n = \alpha$. This completes the proof. □

COROLLARY 2. *Let* R *be regular as above. Then* $K_n(R) \cong K_0(R_n, I_n)$.

PROOF. We have by [**A**]

$$\begin{aligned} K_n(R) &\cong \pi_n(\mathrm{BGL}(R.)) \cong \pi_{n-1}(\mathrm{GL}(R.)) \\ &\cong K_1(R_n/I_n)/K_1(R) \cong K_0(R_n, I_n). \end{aligned}$$

The isomorphism on the right follows from the relative sequence

$$K_1(R_n) \to K_1(R_n/I_n) \to K_0(R_n, I_n) \to K_0(R_n) \to K_0(R_n/I_n),$$

the isomorphism $K_*(R_n) \cong K_*(R)$, and the existence of an augmentation $R_n/I_n \to R$. □

We continue to assume R is a regular ring. Let $\mathscr{S}_n$ denote the category of finitely generated R_n-modules whose support does not meet Σ^{n-1}. Let $\widetilde{R}_n = R_n[1 + I_n]^{-1}$ and $\widetilde{I}_n = I_n \cdot \widetilde{R}_n$. Note $\widetilde{R}_n/\widetilde{I}_n \cong R_n/I_n$.

THEOREM 3. *There is a presentation*

$$(1 + \widetilde{I}_n)^\times \to K_0(\mathscr{S}_n) \to K_n(R) \to 0.$$

PROOF. Define $\widetilde{D}_n$ (resp. D_n) by the Cartesian square of rings

$$\begin{array}{ccc} \widetilde{D}_n & \longrightarrow & \widetilde{R}_n \\ \downarrow & & \downarrow \\ R_n & \longrightarrow & R_n/I_n \end{array}$$

(resp. the corresponding square with $\widetilde{R}_n$ replaced by R_n). Since $\widetilde{D}_n$ is an augmented R_n-algebra, the Mayer-Vietoris sequence [**M**, Theorem 6.4] yields, for $i = 0, 1$,

$$K_{i+1}(\widetilde{R}_n) \to K_{i+1}(R_n/I_n) \to K_i(\widetilde{D}_n)/K_i(R_n) \to K_i(\widetilde{R}_n) \to K_i(R_n/I_n).$$

Comparing this with the long exact sequence of relative K-theory for the ideal $\widetilde{I}_n \subset \widetilde{R}_n$ [**M**, Theorem 6.2], we conclude

$$(1) \qquad K_i(\widetilde{R}_n, \widetilde{I}_n) \cong K_i(\widetilde{D}_n)/K_i(R_n), \qquad i = 0, 1.$$

The localization sequence for K-theory gives

$$(2) \qquad K_1(\widetilde{D}_n)/K_1(R_n) \to K_0(\mathcal{S}_n) \to K_0(D_n)/K_0(R_n) \to K_0(\widetilde{D}_n)/K_0(R_n).$$

One has [**M**, §4]

$$(3) \qquad K_n(R) \cong K_0(R_n, I_n) \overset{\text{def}}{=} K_0(D_n)/K_0(R_n).$$

Substituting (1) and (3) into (2), we get

$$(4) \qquad K_1(\widetilde{R}_n, \widetilde{I}_n) \to K_0(\mathcal{S}_n) \to K_n(R) \to K_0(\widetilde{R}_n, \widetilde{I}_n).$$

Note $\widetilde{I}_n \subset \text{radical}(\widetilde{R}_n)$. By [**B**, V.9.1] we have

$$(5) \qquad K_1(\widetilde{R}_n, \widetilde{I}_n) \cong (1 + \widetilde{I}_n)^\times.$$

Also $\mathrm{GL}(\widetilde{R}_n) \twoheadrightarrow \mathrm{GL}(\widetilde{R}_n/\widetilde{I}_n)$, so

$$(6) \qquad K_0(\widetilde{R}_n, \widetilde{I}_n) \cong \mathrm{Ker}[K_0(\widetilde{R}_n) \to K_0(\widetilde{R}_n/\widetilde{I}_n)].$$

But one has maps (choosing an R-point of $\Sigma^{n-1} = \mathrm{Spec}(\widetilde{R}_n/\widetilde{I}_n)$)

$$(7) \qquad K_0(R) \cong K_0(R_n) \twoheadrightarrow K_0(\widetilde{R}_n) \to K_0(\widetilde{R}_n/\widetilde{I}_n) \to K_0(R)$$

composing to the identity. It follows from (6) and (7) that

$$(8) \qquad K_0(\widetilde{R}_n, \widetilde{I}_n) = (0).$$

Substituting (5) and (8) into (4) yields the theorem. □

We assume R regular Noetherian, and Spec(R) connected. Our intention is to compare the γ-filtration on $K_n(R)$ defined by the γ-filtration on $K_0(D_n)$ with the filtration by codimension of supports on $K_0(\mathcal{S}_n)$. The arguments are all simple adaptations either of the original ideas of Grothendieck or of the arguments in [**FL**].

Recall that the "Lambda operations" $\lambda^i[E] = [\Lambda^i E] \in K_0(D_n)$ for E projective over D_n are defined and are functorial for change of ring, as are the γ-operations defined by [**FL**, III, §1]

$$\sum \gamma^i(x)t^i = \gamma_t(x) = \lambda_{t/(1-t)}(x) = \sum \lambda^i(x)(t/(1-t))^i.$$

The γ-filtration $F^{\cdot}K_0$ is the decreasing filtration defined by

$$F^1 = \mathrm{Ker}(\text{rank}: K_0 \to \mathbb{Z});$$

F^n = $\mathbb{Z}$-module generated by all $\gamma^{r_1}(x_1)\cdot\cdots\cdot\gamma^{r_p}(x_p)$ for $x_i \in F^1$ and $\sum r_i \geq n$. We have as above

$$K_n(R) \cong K_0(D_n)/K_0(R) \cong \mathrm{Ker}(K_0(D_n) \to K_0(R_n)).$$

Since the γ-operations are compatible with change of ring, there is an induced γ-filtration on $K_n(R)$.

Another way to think about these things is via the Adams operations [**FL**, I, §6]

$$\sum \psi^i(x)t^i = \psi_t(x) = \mathrm{rank}(x) - t(d/dt)\log(\lambda_{-t}(x)).$$

The arguments in [**FL**, III, §3] carry over immediately to show

$$K_n(R) \otimes \mathbb{Q} = \bigoplus V(m),$$

where $V(m) \subset K_n(R) \otimes \mathbb{Q}$ is the eigenspace for ψ^i with eigenvalue i^m. ($V(m)$ is independent of i.) One has also

$$V(m) \cong \mathrm{gr}_\gamma^m K_n(R) \otimes \mathbb{Q}.$$

Define another filtration $\mathrm{supp}^\cdot K_n(R)$ by

$$\mathrm{supp}^i K_n(R) = \mathrm{Image}(K_0(\mathrm{supp}^i \mathcal{S}_n) \to K_0(\mathcal{S}_n)),$$

where $\mathrm{supp}^i \mathcal{S}_n \subset \mathcal{S}_n$ is the Serre subcategory of modules whose support has codimension $\geq i$ in $\mathrm{Spec}(R_n)$. Note this is *not* the same as the filtration by codimension of support in $\mathrm{Spec}(R)$ on $K_n(R)$.

THEOREM 4. *The γ-filtration and the support filtration on $K_n(R) \otimes \mathbb{Q}$ coincide. In particular,*

$$\mathrm{gr}_\gamma^m K_n(R) \otimes \mathbb{Q} \cong \mathrm{gr}_{\mathrm{supp}}^m K_n(R) \otimes \mathbb{Q}.$$

PROOF. The inclusion $\mathbb{Q} \otimes \mathrm{supp}^m \subset \mathbb{Q} \otimes F^m$ is proved as in [**FL**, VI.5.5]. Let $V \subset \mathrm{Spec}(R_n)$ be closed, not meeting Σ^{n-1}. We embed $\mathrm{Spec}(R_n) \subset \mathrm{Spec}(D_n)$ as one of the hemispheres. It will suffice to show $[\mathcal{O}_V] \in \mathbb{Q} \otimes F^m K_0(D_n)$. For this, one can by Noetherian induction replace V by $V - S$ and $\mathrm{Spec}(D_n)$ by $\mathrm{Spec}(D_n) - S$ for a closed $S \subset V$ with $\dim(S) < \dim(V)$. Indeed, since S lies in the regular points of $\mathrm{Spec}(D_n)$, the map $F^1 K_0(D_n) \to F^1 K_0(\mathrm{Spec}(D_n) - S)$ is onto, as is therefore the corresponding map on F^m. By Noetherian induction, the kernel of this map lies in F^{m+1}, so

$$[\mathcal{O}_{V-S}] \in F^m K_0(\mathrm{Spec}(D_n) - S) \otimes \mathbb{Q} \Leftrightarrow [\mathcal{O}_V] \in F^m K_0(\mathrm{Spec}(D_n)) \otimes \mathbb{Q}.$$

Since $V - S \hookrightarrow \mathrm{Spec}(D_n) - S$ can be taken to be a regular embedding, the desired inclusion follows from [**FL**, V.6.4(a)].

To prove the theorem, it will suffice to show the filtration $\mathrm{supp}^\cdot K_n(R) \otimes \mathbb{Q}$ is stable under the Adams operator ψ^i, and ψ^i acts by i^m on $\mathrm{gr}_{\mathrm{supp}}^m K_n(R)$. The following lemma is an immediate consequence of [**FL**, V.6.3]. We omit the proof.

LEMMA 5. *Let $f\colon X \hookrightarrow Y$ be a regular embedding of codimension m. Assume the normal bundle of X in Y is trivial. Then the following diagram commutes.*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{i^m\psi^i} & K_0(X) \\ f_*\big\downarrow & & f_*\big\downarrow \\ K_0(Y) & \xrightarrow{\psi^i} & K_0(Y). \end{array}$$

The theorem now follows as before by applying the lemma to inclusions $V - S \hookrightarrow \mathrm{Spec}(D_n) - S$ and to $[\mathscr{O}_{V-S}] \in K_0(V - S)$. □

Let $H^*(\mathrm{Spec}(R), n)$ for $n \in \mathbb{Z}$ denote either (continuous) étale cohomology with $\mathbb{Q}_\ell(n)$ coefficients or (if R is a k-algebra for $k \subset \mathbb{C}$) Deligne cohomology $H^*_{\mathscr{D}}(\mathrm{Spec}(R), \mathbb{Q}(n))$. We define a Chern character map

$$\mathrm{ch}\colon \mathrm{gr}^m_\gamma K_n(R) \otimes \mathbb{Q} \to H^{2m-n}(\mathrm{Spec}(R), m)$$

as follows. It is straightforward to verify that

$$\begin{aligned} H^*(S^n_R, m)/H^*(\mathrm{Spec}(R), m) &= H^*(\mathrm{Spec}(D_n), m)/H^*(\mathrm{Spec}(R), m) \\ &\cong H^{*-n}(\mathrm{Spec}(R), m); \end{aligned}$$

i.e., from the point of view of these cohomology theories, S^n really is the n-sphere. We can therefore employ the usual Chern character on $K_0(D_n)$:

$$(*) \quad \begin{aligned} \mathrm{ch}\colon \mathrm{gr}^m_\gamma K_n(R) \otimes \mathbb{Q} &\cong \mathrm{gr}^m_\gamma [K_0(D_n)/K_0(R)] \otimes \mathbb{Q} \\ &\to H^*(\mathrm{Spec}(D_n), m)/H^*(\mathrm{Spec}(R), m) \cong H^{*-n}(\mathrm{Spec}(R), m). \end{aligned}$$

With a bit more work, it is possible using cohomology with support to define integral classes

$$\mathrm{cyc}\colon \mathrm{gr}^m_{\mathrm{supp}} K_n(R) \to H^{2m-n}(\mathrm{Spec}(R), \mathbb{Z}(m) \text{ or } \mathbb{Z}_l(m)).$$

Let X be a smooth quasi-projective variety defined over a field k. The Beilinson regulator arises from a Chern character map

$$\mathrm{ch}_{\mathrm{B}}\colon \mathrm{gr}^m_\gamma K_n(X) \otimes \mathbb{Q} \to H^{2m-n}(X, m).$$

Suppose for a moment $X = \mathrm{Spec}(R)$ is affine. In order to check that the Beilinson-Chern character coincides with the one defined in $(*)$, one can, for example, use Karoubi-Villamayor techniques to reduce to a question on K_0, where it is straightforward. For X arbitrary quasi-projective, a lemma of Jouanolou [J, 1.5] says there exists a torseur $W \to X$ for a vector bundle on X such that $W = \mathrm{Spec}(A)$ is affine. Both the K-theory and the cohomology of W coincide with that of X, so $(*)$ can be used to define a Chern character in this case as well.

We broaden our notation a bit to include the case $H^*(\cdot, m) = H^*_{\text{ét}}(\cdot, \mu_N^{\otimes m})$. The isomorphism

$$H^*(S^n_R, m)/H^*(\mathrm{Spec}(R), m) \cong H^{*-n}(\mathrm{Spec}(R), m)$$

suggests a more elaborate version of the Grothendieck coniveau filtration. Fix integers n and p, and let $H_p^*(S_R^n, m) \subset H^*(S_R^n, m)$ be the classes supported on some codimension p closed subset $V \subset \operatorname{Spec}(R_n)$ such that $V \cap \Sigma^{n-1} = \varnothing$. Define

$$
\begin{aligned}
&{}_n\mathrm{fil}^p H^q(\operatorname{Spec}(R), m) \\
&\quad = \operatorname{Image}[H_p^{q+n}(S_R^n, m) \to H^{q+n}(S_R^n, m)/H^{q+n}(\operatorname{Spec}(R), m)].
\end{aligned}
$$

As an example, let us suppose $n = p = q = m$ and R is a field. Then ${}_n\mathrm{fil}^n H^n(\operatorname{Spec}(R), \mu_N^{\otimes n})$ coincides with the image of classes in $H^{2n}(S_R^n, \mu_N^{\otimes n})$ with punctual support. It is not hard to show that this is the image under cup product of $H^1(\operatorname{Spec}(R), \mu_N)^{\otimes n} \to H^n(\operatorname{Spec}(R), \mu_N^{\otimes n})$. Thus, the Kato conjecture that this map is onto amounts to the conjecture

$$
{}_n\mathrm{fil}^n H^n(\operatorname{Spec}(R), \mu_N^{\otimes n}) = H^n(\operatorname{Spec}(R), \mu_N^{\otimes n}).
$$

It would be of interest to formulate a generalization of the Kato conjecture, describing the filtration ${}_n\mathrm{fil}^p H^q$ for arbitrary n, p, and q.

Bibliography

[A] D. W. Anderson, *Relationship among K-theories,* Algebraic K-theory I, Lecture Notes in Math., vol. 341, Springer-Verlag, New York, 1973, pp. 57–73.

[B] H. Bass, *Algebraic K-theory,* W. A. Benjamin, New York, 1968.

[FL] W. Fulton and S. Lang, *Riemann–Roch Algebra,* Springer-Verlag, New York, 1985.

[J] J.-P. Jouanolou, *Une suite exacte de Mayer-Vietoris en K-théorie algébrique,* Algebraic K-theory I, Lecture Notes in Math., vol. 341, Springer-Verlag, New York, 1973, pp. 293–317.

[May] P. May, *Simplicial Objects in Algebraic Topology,* Van Nostrand Math. Stud., vol. 11, Van Nostrand, Princeton, NJ, 1967.

[M] J. Milnor, *Introduction to Algebraic K-Theory,* Princeton Univ. Press, Princeton, NJ, 1971.

[R] M. Rapoport et al., *Beilinson's Conjectures on Special Values of L-functions,* Perspect. Math., vol. 4, Academic Press, New York, 1988.

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Motivic Sheaves and Filtrations on Chow Groups

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Grothendieck's motives, as described in **[Dem, Kl2, Ma]** are designed as a tool to understand the cohomology of smooth projective varieties and the algebraic cycles modulo homological and numerical equivalence on them. According to Beilinson and Deligne, Grothendieck's category of pure motives should embed in a bigger category of mixed motives that allows the treatment of arbitrary varieties and an understanding of the whole Chow group of cycles modulo rational equivalence, in fact, even of all algebraic K-groups of the varieties.

In this paper we review some of these ideas and discuss some consequences. In particular, we show how the vast conjectural framework set up by Beilinson leads to very explicit conjectures on the existence of certain filtrations on Chow groups of smooth projective varieties. These filtrations would offer an understanding of several phenomena and counterexamples that for some time have led people to believe that the behaviour of the algebraic cycles is absolute chaos for codimension bigger than one.

In §1 we review some basic facts on Chow groups, correspondences, and cycle maps into cohomology theories. We recall a counterexample of Mumford implying that in general Chow groups are not representable and the Abel-Jacobi map has a huge kernel and some investigations of Bloch on this topic.

In §§2 and 4 we state altogether four versions of Beilinson's conjectures on mixed motives and filtrations on Chow groups, increasing in generality and sophistication. The first one does not even mention mixed motives and proposes finite filtrations $F^0 \supset F^1 \supset \cdots$ on rational Chow groups $CH^j(X)_{\mathbb{Q}}$ that are uniquely determined by their behaviour under algebraic correspondences. The first step is homological equivalence, but the following steps differ very much from those considered classically. For example, algebraic equivalence does not appear, and the second step is something like the kernel

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of (all) Abel-Jacobi maps. The crucial property is that the actions of correspondences on the graded pieces $\mathrm{Gr}_F^\nu CH^j(X)_\mathbb{Q}$ factor through homological equivalence. This would very nicely bridge the gap between homological and rational equivalence, in finitely many steps.

The described "filtration conjecture" can thus be regarded as a counterpart of the standard conjectures, being responsible for the part not covered by the latter. It is amazing to realize that the filtration conjecture would follow from the injectivity of suitable cycle maps, while (parts of) the standard conjectures would follow from the surjectivity of cycle maps, viz., the conjectures of Hodge and Tate.

While by definition $\mathrm{Gr}_F^0 CH^j(X)_\mathbb{Q}$ is the space of algebraic cycles in the cohomology $H^{2j}(X)$ of degree $2j$, the higher graded pieces are related to $H^{2j-1}(X)$, $H^{2j-2}(X)$, and so on. In fact, version 2 of the conjecture proposes the description (called "Beilinson's formula" in the following)

$$(0) \qquad \mathrm{Gr}_F^\nu CH^j(X)_\mathbb{Q} = \mathrm{Ext}^\nu_{\mathcal{MM}_k}(1, h^{2j-\nu}(X)(j)),$$

in terms of Yoneda extensions in the conjectural category $\mathcal{MM}_k$ of mixed motives, where 1 is the trivial motive and $h^i(X)$ is the pure motive corresponding to the ith cohomology of X. This is closely related to higher cycle maps, and indeed Mumford's counterexample and the above isomorphisms suggest studying higher than secondary (= Abel-Jacobi) maps to understand the whole Chow group.

Version 3 of the conjecture reveals more of the framework of mixed motives. Recall that $X \mapsto \bigoplus h^i(X)$ is the universal Weil cohomology theory for smooth projective varieties X and that by definition an element $\alpha \in CH^j(X)_\mathbb{Q}$ corresponds to a morphism $\varphi_\alpha \colon 1 \to h^{2j}(X)(j)$ in Grothendieck's category $\mathcal{M}_k$ of pure motives. Roughly speaking, Beilinson's formula should come from a derived version: If one believes that $h^\cdot(X)$ arises as the cohomology of a complex $R(X)$ in $D^b(\mathcal{MM}_k)$, the bounded derived category of $\mathcal{MM}_k$, and the morphism φ_α from a morphism $\eta_\alpha \colon 1 \to R(X)(j)[2j]$ (such that φ_α is obtained by passing to the 0th cohomology), then this leads to a cycle map

$$(1) \qquad CH^j(X)_\mathbb{Q} \to \mathrm{Hom}_{D^b(\mathcal{MM}_k)}(1, R(X)(j)[2j])$$

and to an induced filtration via the spectral sequence

$$(2) \qquad \mathrm{Ext}^p_{\mathcal{MM}_k}(1, h^q(X)(j)) \Rightarrow \mathrm{Hom}_{D^b(\mathcal{MM}_k)}(1, R(X)(j)[p+q]).$$

The optimistic conjecture says that (1) is an isomorphism. This together with the degeneration of (2) would give Beilinson's formula (0).

All this only reflects the situation encountered in the ℓ-adic cohomology, and in fact, (2) is the motivic analogue of the Hochschild-Serre spectral sequence and (1) is the analogue of the cycle map into ℓ-adic cohomology over k. The most general version 4 of Beilinson's conjecture expresses the hope that there is a theory of mixed motives that resembles and parallels closely the general theory of ℓ-adic cohomology and ℓ-adic sheaves. In particular, one hopes for relative and local versions of mixed motives, the so-called mixed motivic sheaves.

In §3 we state some remarkable consequences of Beilinson's conjectures which already follow from the down-to-earth version 1, but which we derive via Beilinson's formula, to demonstate its use and usefulness. In particular, we show that the conjectures would lead to a good understanding of the representability of Chow groups. Guided by the conjectural theory, we prove some results on the nonrepresentability of Chow groups. This extends work of Mumford, Bloch, Roitman, and others and may be interesting in its own right.

Finally, in §5 we discuss the relationship with a conjectural filtration proposed by Murre. This proposal has the advantage of being quite explicit, in the formulation close to the standard conjectures, and more amenable to being proved in part. Whereas Murre arrived at this conjecture by the consideration of decompositions of motives, it was quickly clear to several experts that it is implied by Beilinson's conjectures. At the Seattle "motives" conference I discussed a partial converse, and soon after it occurred to me that Murre's conjecture is in fact equivalent to version 1 of Beilinson's conjecture.

This paper would not be complete without mentioning that Grothendieck certainly envisioned a much more general theory of motives than just for smooth projective varieties over a field. In particular, he already thought about motives over arbitrary bases and a general motivic duality formalism. This becomes quite clear in a letter written by Grothendieck to Illusie in 1973 which is reproduced in an appendix. This letter also addresses several interesting questions on motives that I have not seen discussed elsewhere.

I am indebted to L. Illusie and W. Messing for providing me with the letter in the appendix and other "historical" material on motives, and to S. Kleiman for numerous helpful comments on this paper. The main result of §5 was obtained during a visit of Leiden University, which I thank for its hospitality. It is a pleasure to thank J. P. Murre and S. Saito for stimulating discussions. My stay at the Seattle conference was partially supported by the DFG, whose support is gratefully acknowledged.

1. Higher cycle maps and Bloch's conjecture

1.1. Let X be a smooth projective variety over a field k. For any integer $j \geq 0$, the set $X^{(j)} = \{x \in X | \operatorname{codim} x = j\}$ of points of codimension j can be identified with the set of closed irreducible subvarieties Z of codimension j in X (by mapping x to its closure $Z = \overline{\{x\}}$ and Z to its generic point). Recall that the group of cycles of codimension j on X is the free abelian group $Z^j(X) = \bigoplus_{x \in X^{(j)}} \mathbb{Z}$ on $X^{(j)}$ and that the jth Chow group $CH^j(X)$ is the quotient of $Z^j(X)$ modulo cycles that are rationally equivalent to zero

[Kl2, §2]. By Quillen [Qui, §7, proof of 5.14], we can write this as

$$(1.1)\qquad CH^j(X) = \operatorname{Coker}\left(\bigoplus_{x\in X^{(j-1)}} k(x)^\times \overset{\text{div}}{\to} \bigoplus_{x\in X^{(j)}} \mathbb{Z}\right),$$

where $k(x)$ is the residue field of x (= the function field of $Z = \overline{\{x\}}$) and div is the divisor map (cf. also [Fu, 1.3 and 1.6] for Chow groups that are graded by dimension).

The following operations on cycles are (only) well defined modulo rational equivalence. One has an intersection product

$$CH^i(X) \times CH^j(X) \to CH^{i+j}(X),$$
$$(\alpha, \beta) \mapsto \alpha \cdot \beta,$$

making $CH^*(X)$ into a commutative ring, and for a morphism $f\colon X \to Y$ of smooth projective varieties one has pull-back maps

$$f^*\colon CH^j(Y) \to CH^j(X), \qquad j \geq 0,$$

inducing a ring morphism $CH^*(Y) \to CH^*(X)$, and a push-forward map

$$f_*\colon CH^*(X) \to CH^*(Y)$$

mapping $CH^{d+i}(X)$ to $CH^{e+i}(Y)$, if X and Y are of pure dimensions d and e, respectively. These operations enjoy the following compatibilities:

(a) (functoriality) $(gf)^* = f^*g^*$ and $(gf)_* = g_*f_*$ for a second morphism $g\colon Y \to Z$.

(b) (projection formula) $f_*\alpha \cdot \beta = f_*(\alpha \cdot f^*\beta)$.

(c) (base change) For a Cartesian diagram of projections

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{p_{XY}} & X \times Y \\ \downarrow{\scriptstyle p_{YZ}} & & \downarrow{\scriptstyle q} \\ Y \times Z & \xrightarrow{p} & Y \end{array}$$

one has $p^*q_* = (p_{YZ})_*p_{XY}^*$.

These properties ensure that one has a bilinear, associative composition law of correspondences (= algebraic cycles on products)

(1.2)

$$\begin{array}{ccccc} CH^{e+s}(X \times Y \times Z) & \times & CH^{d+r}(X \times Y \times Z) & \xrightarrow{\cdot} & CH^{d+r+e+s}(X \times Y \times Z) \\ \uparrow{\scriptstyle p_{YZ}^*} & & \uparrow{\scriptstyle p_{XY}^*} & & \downarrow{\scriptstyle (p_{XZ})_*} \\ CH^{e+s}(Y \times Z) & \times & CH^{d+r}(X \times Y) & \to & CH^{d+r+s}(X \times Z) \end{array}$$

$$(g, f) \mapsto g \circ f = (p_{XZ})_*(p_{YZ}^*g \cdot p_{XY}^*f)$$

for X and Y of pure dimensions d and e, respectively, making the diagram commutative. In particular, $CH^d(X \times X)$ is an associative ring with unit (consider $X = Y = Z$; the unit is the class of the diagonal $\Delta\colon X \hookrightarrow X \times X$), and one has an action of correspondences on Chow groups (consider $X = \operatorname{Spec} k$):

$$CH^{e+s}(Y \times Z) \times CH^r(Y) \to CH^{r+s}(Z),$$
$$(\alpha, y) \mapsto \alpha y = (p_Z)_*(\alpha \cdot p_Y^* y).$$

1.2. More generally, algebraic correspondences act on generalized cohomology theories. These are contravariant functors

$$H\colon X \mapsto H^*(X, *)$$

from the category $\mathscr{V}_k$ of smooth projective varieties over k to the category of bi-graded R-algebras, for a ring R, equipped with the following additional structure:

(I) For every morphism $f\colon X \to Y$ in $\mathscr{V}_k$ there is a map of R-modules $f_*\colon H^*(X, *) \to H^*(Y, *)$, mapping $H^{2d+i}(X, d+j)$ to $H^{2e+i}(Y, e+j)$ if X and Y are of pure dimensions d and e, respectively, such that the properties (a)–(c) hold correspondingly, where $f^* = H(f)$.

(II) There are cycle maps

$$cl\colon CH^j(X) \to H^{2j}(X, j)$$

compatible with pull-back, push-forward, and products.

In fact, by (I) one has analogues of the diagram (1.2) and, hence, composition and action of "cohomological" correspondences, i.e., elements in $H^*(X \times Y, *)$. By (II), algebraic correspondences act via their cycle classes. Explicitly, we have the following diagram, for X of pure dimension d,

(1.3)

$$\begin{array}{ccccc}
H^{2d+2r}(X \times Y, d+r) & \times & H^i(X \times Y, m) & \to & H^{2d+i+2r}(X \times Y, d+m+r) \\
\uparrow{\scriptstyle \mathrm{cl}} & & \uparrow{\scriptstyle p_X^*} & & \downarrow{\scriptstyle (p_Y)_*} \\
CH^{d+r}(X \times Y) & \times & H^i(X, m) & \to & H^{i+2r}(Y, m+r).
\end{array}$$

1.3. Examples of generalized cohomology theories are:

(1) The Chow theory:

$$H^i(X, j) = \begin{cases} CH^j(X), & i = 2j, \\ 0 & \text{otherwise}, \end{cases}$$

where $R = \mathbb{Z}$ and cl is the identity.

(2) Every Weil cohomology theory $X \mapsto H^*(X)$ (cf. [**Kl3**, Chapter 3])

$$H^i(X, j) = H^i(X) \quad \text{for all } j.$$

(3) Singular cohomology with Hodge-Tate twists

$$H^i(X, j) = H^i(X(\mathbb{C}), \mathbb{Z}(j)), \qquad \mathbb{Z}(j) = \mathbb{Z}(2\pi\sqrt{-1})^j,$$

for $k = \mathbb{C}$, where $R = \mathbb{Z}$ and the cycle class is renormalized (cf. the discussion in [**Ja3**, §5]);

(4) ℓ-adic cohomology with Tate twists

$$H^i(X, j) = H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(j)), \qquad \overline{X} = X \times_k \overline{k},$$

for $\ell \neq \operatorname{char}(k)$, where $R = \mathbb{Z}_\ell$. Here $\overline{k}$ is a separable closure of k and $\mathbb{Z}_\ell(j) = \mathbb{Z}_\ell(1)^{\otimes j}$, where $\mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}$ for the sheaves μ_{ℓ^n} of ℓ^nth roots of unity.

(5) Deligne cohomology **[Be1; EV**, §1**]**

$$H^i(X, j) = H^i_{\mathscr{D}}(X, \mathbb{Z}(j))$$

for $k = \mathbb{C}$.

(6) ℓ-adic cohomology over k

$$H^i(X, j) = H^i_{\text{ét}}(X, \mathbb{Z}_\ell(j))$$

(naive or continuous **[Ja1]**) for $\ell \neq \operatorname{char}(k)$.

After tensoring with $\mathbb{Q}$, examples (3) and (4) are also examples (the principal ones) of Weil cohomology theories, but we have some additional structure: In example (3) the cohomology group $H^i(X, j)$ carries a pure Hodge structure of weight $i - 2j$, by identifying it with the tensor product of $H^i(X(\mathbb{C}), \mathbb{Z})$, which has the usual Hodge structure of weight i, and the Hodge structure $\mathbb{Z}(j)$, which has weight $-2j$ (cf. **[De2**, 2.1 and 2.2**]**). Then cup-product, pull-backs, and push-forwards are compatible with the Hodge structures, and the fact that the cycle map has image in

$$H^{2j}(X(\mathbb{C}), \mathbb{Z}(j)) \cap H^{j,j}(X)$$

implies that the action of correspondences respects the Hodge structures. The last statement becomes quite obvious, if one identifies the above group with

$$\operatorname{Hom}_{\text{HS}}(\mathbb{Z}, H^{2j}(X(\mathbb{C}), \mathbb{Z}(j))),$$

the group of homomorphisms of Hodge structures, where $\mathbb{Z}$ is the trivial Hodge structure. In example (4), the absolute Galois group $G_k = \operatorname{Gal}(\overline{k}/k)$ of k acts continuously on $H^i(X, j)$ by functoriality of étale cohomology. Since the cycle map has image in the fixed module

$$H^{2j}(\overline{X}, \mathbb{Z}_\ell(j))^{G_k} \cong \operatorname{Hom}_{G_k}(\mathbb{Z}_\ell, H^{2j}(\overline{X}, \mathbb{Z}_\ell(j))),$$

the action of correspondences respects the Galois actions. We remark that $H^i(X, j)$ is again pure of weight $i - 2j$ as a Galois module, i.e., $\mathbb{Z}_\ell$-sheaf on $(\operatorname{Spec} k)_{\text{ét}}$ (cf. **[De3**, 3.4.11**]**).

1.4. Although the cycle maps in (3) and (4) and, more generally, the cycle maps into Weil cohomology theories are already objects of interesting study and deep conjectures, namely, the Hodge conjecture, the Tate conjecture **[Ta]**, and the standard conjectures **[Kl3]**, this only covers a small part of the Chow groups, namely, cycles modulo homological equivalence. Recall that a cycle $\alpha \in Z^j(X)$ is called:

(i) homologically equivalent to zero ($\alpha \sim_{\text{hom}} 0$), if $cl(\alpha) = 0$ for a cycle map cl into a Weil cohomology, and

(ii) numerically equivalent to zero ($\alpha \sim_{\text{num}} 0$), if the intersection number $(\alpha \cdot \beta) = \deg \alpha \cdot \beta = 0$ for all $\beta \in CH^{d-j}(X)$, if X is of pure dimension d, say.

From these definitions it is only clear that $\alpha \sim_{\text{hom}} 0$ implies $\alpha \sim_{\text{num}} 0$, and a priori $\sim_{\text{hom}}$ depends on the chosen Weil cohomology, but according to the standard conjectures $\sim_{\text{hom}}$ and $\sim_{\text{num}}$ should coincide, which would imply the independence as well. Now $CH^j(X)/\sim_{\text{num}}$ is a finitely generated abelian group (cf. [**Kl3**, 5-2]), while

$$CH^j(X)_{\text{hom}} = \{\alpha \in CH^j(X) | cl(\alpha) = 0\}$$

can be huge. To a certain extent this group can be studied by secondary cycle maps, namely, the Abel-Jacobi maps.

For $k = \mathbb{C}$ this is a map

$$cl' \colon CH^j(X)_0 \to \frac{H^{2j-1}(X(\mathbb{C}), \mathbb{C})}{H^{2j-1}(X(\mathbb{C}), \mathbb{Z}(j)) + F^j}$$

into the Weil-Griffiths intermediate Jacobian (cf. [**Lie**, p. 131]). Here $F^j = \bigoplus_{p+q=2j-1, p \geq j} H^{p,q}(X)$ is the jth step of the Hodge filtration, $CH^j(X)_0$ is the kernel of the cycle map cl in example 1.3(3), and we have normalized the lattice to be $H^{2j-1}(X, \mathbb{Z}(j))$ instead of the more classical $H^{2j-1}(X, \mathbb{Z})$. Note that $CH^j(X)_0$ is of finite index in $CH^j(X)_{\text{hom}}$. By the classical Abel-Jacobi theorem, cl' is an isomorphism for $j = 1$.

The ℓ-adic analogue, for $\ell \neq \text{char}(k)$, is a map

$$cl' \colon CH^j(X)_0 \to H^1(G_k, H^{2j-1}(\overline{X}, \mathbb{Z}_\ell(j))),$$

which can be defined by the cycle map into $H^{2j}(X, \mathbb{Z}_\ell(j))$ and the Hochschild-Serre spectral sequence

$$(1.4) \qquad E_2^{p,q} = H^p(G_k, H^q(\overline{X}, \mathbb{Z}_\ell(j))) \Rightarrow H^{p+q}(X, \mathbb{Z}_\ell(j))$$

(cf. [**Ja1**, 6.15c)]). Here $CH^j(X)_0$ is the kernel of the cycle map into $H^{2j}(\overline{X}, \mathbb{Z}_\ell(j))$; it is of finite index in

$$CH^j(X)_{\text{hom}} = \text{Ker}(CH^j(X) \to H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))).$$

For $j = 1$ and a finitely generated field k, cl' is injective up to torsion prime to ℓ by Kummer theory and the Mordell-Weil theorem (cf. [**Ja1**, 6.15a], where "up to torsion prime to ℓ" should be added).

In the ℓ-adic setting we have in addition higher than secondary cycle classes. In fact, let $F^0 \supset F^1 \supset \cdots$ be the filtration on $H^{2j}(X, \mathbb{Q}_\ell(j))$ (continuous étale cohomology over k) induced by the Hochschild-Serre spectral sequence

$$(1.5) \qquad E_2^{p,q} = H^p(G_k, H^q(\overline{X}, \mathbb{Q}_\ell(j))) \Rightarrow H^{p+q}(X, \mathbb{Q}_\ell(j)).$$

Since this spectral sequence degenerates (cf. the remark in **[Ja1**, 6.15b]; this fact also follows from the considerations in **[De5]**), we have isomorphisms

$$\mathrm{Gr}^{\nu}_F H^{2j}(X, \mathbb{Q}_\ell(j)) \cong H^{\nu}(G_k, H^{2j-\nu}(\overline{X}, \mathbb{Q}_\ell(j)).$$

Now let F_ℓ be the descending filtration on $CH^j(X)$ obtained by pull-back via the cycle map

$$cl\colon CH^j(X) \to H^{2j}(X, \mathbb{Q}_\ell(j)).$$

Hence $F^0_\ell = CH^j(X)$, $F^1_\ell = CH^j(X)_{\mathrm{hom}}$, and $F^2_\ell =$ kernel of the Abel-Jacobi map cl', up to torsion. Then we obtain higher cycle maps

$$cl^{(\nu)}\colon F^{\nu}_\ell CH^j(X) \to H^{\nu}(G_k, H^{2j-\nu}(\overline{X}, \mathbb{Q}_\ell(j))) \tag{1.6}$$

with kernel $F^{\nu+1}_\ell CH^j(X)$ extending the cases $\nu = 0$ and 1 discussed before.

As we remarked, $CH^1(X)_{\mathrm{hom}}$ can effectively be recovered by Abel-Jacobi maps. But in any case, the structure of $CH^j(X)$ is well understood for $j = 1$: One has a canonical isomorphism $CH^1(X) \cong \mathrm{Pic}(X)$ and an exact sequence

$$0 \to \mathrm{Pic}^{\circ}(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X/k) \to 0,$$

where the " k-rational Néron-Severi group" $\mathrm{NS}(X/k)$ is a finitely generated abelian group and $\mathrm{Pic}^{\circ}(X)$ is the group of points of an abelian variety—namely, the Jacobian variety $\mathrm{Jac}(X)$—if X has a rational point. In fact, $\mathrm{Jac}(X) = (\mathrm{Pic}^{\circ}_{X/k})_{\mathrm{red}}$ for the Picard group scheme $\mathrm{Pic}_{X/k}$ (cf. **[Gr2**, 3.2]), there is an exact sequence $0 \to \mathrm{Pic}(X) \xrightarrow{j} \mathrm{Pic}_{X/k}(k) \to \mathrm{Br}(k)$ in which j is an isomorphism for $X(k) \neq \varnothing$ (cf. **[Gr1**, 2.1]), and we obtain the result by putting $\mathrm{Pic}^{\circ}(X) = j^{-1}\mathrm{Pic}^{\circ}_{X/k}(k)$, since $\mathrm{NS}(X/k)$ is finitely generated for algebraically closed k (cf. **[SGA6**, XIII, 5.1]). It is known that $\mathrm{Pic}^{\circ}(X)$ has finite index in $CH^1(X)_{\mathrm{num}}$.

The naive hope that similar structure results would hold also for $j \geq 2$ was destroyed by the following counterexample of Mumford. For X of dimension d over k let $CH^d(X)_0$ be the Chow group of zero cycles of degree zero on X. If $k = \mathbb{C}$ (or a universal domain), then, following Mumford, call $CH^d(X)_0$ finite dimensional, if there is an $n \in \mathbb{N}$ such that the natural map

$$\begin{aligned} S^n X(k) \times S^n X(k) &\to CH^d(X)_0, \\ (a, b) &\to \text{class of } a - b \end{aligned} \tag{1.7}$$

is surjective, where $S^n X$ is the nth symmetric power of X.

THEOREM 1.5 **[Mum1**, p. 203]. *Let X be a smooth projective surface over $\mathbb{C}$. If $H^2(X, \mathcal{O}_X) \neq 0$, then $CH^2(X)_0$ is not finite dimensional.*

For the discussion of this result it is useful to investigate various possible characterizations of a "nice behaviour" of $CH^d(X)_0$.

PROPOSITION 1.6. *Let X be a smooth, projective, geometrically irreducible variety of dimension d over a field k, and let $\Omega \supseteq k$ be an algebraically closed field. Consider the following statements.*

(i) (*resp.* (i′)) *There is an $n \in \mathbb{N}$ such that the natural map*

$$S^n X(\Omega) \times S^n X(\Omega) \to CH^d(X_\Omega)_0$$

(*resp.* $S^n X(\Omega) \times S^n X(\Omega) \to CH^d(X_\Omega)_0 \otimes \mathbb{Q}$) *is surjective, where $X_\Omega = X \times_k \Omega$.*

(ii) (*resp.* (ii′)) *There exists a smooth projective curve C over Ω and a morphism $f\colon C \to X_\Omega$ such that*

$$f_*\colon \mathrm{Pic}^0(C) = CH^1(C)_0 \to CH^d(X_\Omega)_0$$

(*resp.* $f_*\colon CH^1(C)_0 \otimes \mathbb{Q} \to CH^d(X_\Omega)_0 \otimes \mathbb{Q}$) *is surjective.*

(iii) (*resp.* (iii′)) *There exists a closed subscheme $Y \subseteq X$ of dimension* 1 *such that*

$$CH^d((X - Y)_\Omega) = 0$$

(*resp.* $CH^d((X - Y)_\Omega)_\mathbb{Q} = 0$), *where $CH^j(Z)$ is defined for $j \geq 0$ and any algebraic Ω-scheme Z by formula* (1.1), *applied to Z over Ω in place of X over k.*

(iv) (*resp.* (iv′)) *If $B \subseteq X_\Omega$ is a smooth linear space section of dimension* 1, *then*

$$CH^d(X_\Omega - B) = 0$$

(*resp.* $CH^d(X_\Omega - B)_\mathbb{Q} = 0$).

(v) (*resp.* (v′)) *The canonical map*

$$a_{X_\Omega}\colon CH^d(X_\Omega)_0 \to \mathrm{Alb}(X)(\Omega)$$

is an isomorphism (*resp. an isomorphism after tensoring with* $\mathbb{Q}$), *where* $\mathrm{Alb}(X)$ *is the Albanese variety of X.*

Then (v) ⇔ (v′) ⇒ (iv) ⇔ (iv′) ⇒ (iii) ⇔ (iii′) ⇒ (ii) ⇔ (ii′) ⇒ (i) ⇔ (i′). *If Ω is uncountable, then* (i) ⇒ (v), *so all statements are equivalent. Property* (v) *holds over an algebraically closed field $\Omega \supseteq k$ if and only if it holds for all algebraically closed fields Ω', $\Omega \supseteq \Omega' \supseteq k$, of finite transcendence degree over k. In particular,* (v) *holds for all algebraically closed fields $\Omega \supseteq k$, if it holds for one which is uncountable.*

PROOF. (v) ⇒ (iv) Let $B \overset{i}{\hookrightarrow} X_\Omega$ be as in (iv), and consider the following commutative diagram.

$$\begin{array}{ccc} a_B\colon CH^1(B)_0 & \xrightarrow{\sim} & \mathrm{Alb}(B)(\Omega) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ a_{X_\Omega}\colon CH^d(X_\Omega)_0 & \longrightarrow & \mathrm{Alb}(X_\Omega)(\Omega) = \mathrm{Alb}(X)(\Omega) \end{array}$$

The right-hand map i_* is surjective by the choice of B (cf. [**Wei**, Corollary 1 to Theorem 7]). The canonical map a_B is an isomorphism, since $\mathrm{Jac}(C) = \mathrm{Alb}(C)$ for every smooth projective curve C. Hence (v) implies that the left-hand map i_* is surjective as well, and then the same is true for $i_*\colon CH^1(B) \to CH^d(X_\Omega)$, since B has a point of degree 1. Now (iv) follows from the exact sequence

$$CH^1(B) \xrightarrow{i_*} CH^d(X_\Omega) \to CH^d(X_\Omega - B) \to 0$$

(cf. 3.10(2)).

(iv) $\Rightarrow$ (iii) There always exists a linear section B as in (iv) that is defined over a finite separable extension of k, and we may take Y = image of B in X (which is not necessarily smooth). Since B is a closed subvariety of Y_Ω, the natural restriction $CH^d(X_\Omega - B) \to CH^d(X_\Omega - Y_\Omega)$ (cf. 3.10(1)) is surjective; hence the implication.

(iii) $\Rightarrow$ (ii) We may assume that Y is of pure dimension 1 and then take C = normalization of $(Y_\Omega)_{\mathrm{red}}$. In fact, (iii) implies that the first map in the exact sequence

$$CH^1(Y_\Omega) \to CH^d(X_\Omega) \to CH^d(X_\Omega - Y_\Omega) \to 0$$

(cf. 3.10(2)) is surjective, but it is easy to see that the map f_* in (ii) induced by $f\colon C \to (Y_\Omega)_{\mathrm{red}} \to Y_\Omega \to X_\Omega$ factors as

$$f_*\colon CH^1(C) \to CH^1((Y_\Omega)_{\mathrm{red}}) = CH^1(Y_\Omega) \to CH^1(X_\Omega),$$

where the first map is induced by $C^{(1)} \to Y_\Omega^{(1)}$, $x \mapsto f(x)$ and, hence, is surjective.

(ii) $\Rightarrow$ (i) If C is an irreducible smooth projective curve of genus g over Ω, then it follows from the Riemann-Roch theorem that $S^gC(\Omega) \times S^gC(\Omega) \to CH^1(C)_0$ is surjective (note that $S^nC(\Omega)$ is identified with the set of effective divisors of degree n). Hence the claim follows from the commutative diagram

$$\begin{array}{ccc} S^nX(\Omega) \times S^nX(\Omega) & \to & CH^d(X_\Omega)_0 \\ \uparrow f_* & & \uparrow f_* \\ S^nC(\Omega) \times S^nC(\Omega) & \twoheadrightarrow & CH^1(C)_0 \end{array}$$

That (i) implies (v), for an uncountable field Ω, is a theorem of Roitman ([**Roi1**, Theorem 4, p. 585]; in Roitman's paper it is generally assumed that $\mathrm{char}(\Omega) = 0$, but the proof of this result does not involve this assumption).

By another theorem of Roitman [**Roi2**, 3.1], a_{X_Ω} always induces an isomorphism

(1.8) $$\mathrm{Tor}(CH^d(X_\Omega)_0) \xrightarrow{\sim} \mathrm{Tor}(\mathrm{Alb}(X)(\Omega))$$

on the torsion subgroups (for all smooth projective varieties X of dimension d over k and $\Omega \supseteq k$ algebraically closed). This implies the equivalence of

(v) and (v′), since a_{X_Ω} is always surjective. Another consequence is that

$$i_*\colon \mathrm{Tor}(CH^1(B)_0) \twoheadrightarrow \mathrm{Tor}(CH^d(X_\Omega)_0)$$

is surjective for any smooth linear space section of dimension 1, $i\colon B \hookrightarrow X_\Omega$, by the surjectivity of $i_*\colon \mathrm{Alb}(B) \to \mathrm{Alb}(X_\Omega)$. Since $CH^d(X_\Omega)_0$ is divisible, this implies the equivalence of (iv) and (iv′), and the equivalence of (iii) and (iii′), (ii) and (ii′), or (i) and (i′) follows as well, using the ingredients of the proofs of the various implications.

Finally, if $\Omega \supseteq \Omega'$ are algebraically closed extensions of k and if $a_{X_\Omega} \otimes \mathbb{Q}$ is injective, then so is $a_{X_{\Omega'}} \otimes \mathbb{Q}$, since the restriction map $CH^d(X_{\Omega'})_{\mathbb{Q}} \to CH^d(X_\Omega)_{\mathbb{Q}}$ is injective (cf. 3.10(4)). Conversely, since

$$CH^d(X_\Omega) = \varinjlim CH^d(X_{\Omega'}),$$

where the limit is over all algebraically closed fields Ω', $\Omega \supset \Omega' \supset k$, of finite transcendence degree over k (cf. 3.10(3)), a_{X_Ω} is injective if $a_{X_{\Omega'}}$ is injective for all such Ω'. This shows the remaining claims. For the last claim note that X can be defined over a field k_0 which is finitely generated over the prime field and that an uncountable algebraically closed field $\Omega \supseteq k$ contains all fields $\Omega' \supseteq k_0$ of finite transcendence degree over k_0.

REMARK 1.7. Condition 1.6(iii) and its generalization to $\dim Y \geq 1$ appears in work of Bloch and Srinivas **[BS]** and is further investigated in **[MS]** and **[SaS]** (cf. also §3). Condition 1.6(ii) appears in [**Bl1**, lecture 1, Appendix], and the equivalence of (iii) and (v) is stated without proof in [**MS**, 1.2(b)]. Following Bloch (cf. [**Bl1**, Definition (1.1); **BS**, p. 1238]), we shall call $CH^d(X)_0$ representable, if a_{X_Ω} is an isomorphism over a universal domain $\Omega \supset k$.

If $k = \mathbb{C}$, then there is an isomorphism

$$\mathrm{Alb}(X)(\mathbb{C}) \xrightarrow{\sim} \frac{H^{2d-1}(X(\mathbb{C}), \mathbb{C})}{H^{2d-1}(X(\mathbb{C}), \mathbb{Z}(d)) + F^d}$$

such that the complex Abel-Jacobi map cl' on $CH^d(X)_0$ can be identified with a_X. Hence Mumford's theorem implies the following: If X is a complex surface with $H^2(X, \mathcal{O}_X) \neq 0$, then the Abel-Jacobi map

$$cl'\colon CH^2(X)_0 \xrightarrow{a_X} \mathrm{Alb}(X)(\mathbb{C}) \xrightarrow{\sim} \frac{H^3(X(\mathbb{C}), \mathbb{C})}{H^3(X(\mathbb{C}), \mathbb{Z}(2)) + F^2}$$

has a huge kernel. (In fact, by Roitman's isomorphism (1.8), $\mathrm{Ker}\, cl'$ is torsion-free and divisble, and hence, a $\mathbb{Q}$-vector space. If it had a finite basis, it would lie in the image of $g_*\colon \mathrm{Pic}^0(C') \to CH^2(X)_0$ for some morphism $g\colon C' \to X$ of a smooth projective curve C' into X, and we would obtain 1.6(ii), a contradiction). Moreover, $CH^2(X)_0$ cannot be given the structure of (the group of points of) an abelian variety, in a reasonable way. In fact, correctly interpreted this would mean that a_X is an isomorphism (since a_X is universal for regular maps (cf. [**BS**, p. 1238]) into abelian varieties).

Bloch proposed the following converse of Mumford's theorem.

CONJECTURE 1.8 [**Bl1**, Lecture 1]. *If* $H^2(X, \mathscr{O}_X) = 0$, *then* $a_X: CH^2(X)_0 \to \mathrm{Alb}(X)(\mathbb{C})$ *is an isomorphism.*

Moreover, he extended both Mumford's theorem and this conjecture to arbitrary base fields, by using ℓ-adic cohomology. Namely, he observed the following equivalence for the surface $X/\mathbb{C}$:

$$H^2(X, \mathscr{O}_X) = 0,$$

$$\Leftrightarrow H^2(X(\mathbb{C}), \mathbb{C}) = H^{1,1} \quad \text{[by Hodge theory]},$$

$\Leftrightarrow H^2(X(\mathbb{C}), \mathbb{C})$ is algebraic, i.e., generated by cycle classes of divisors [by Lefschetz's theorem].

The last statement makes sense for any Weil cohomology, and Bloch proves Theorem 1.9 and makes Conjecture 1.10 [**Bl1**, Lecture 1].

THEOREM 1.9. *Let* X *be a smooth projective surface over a field* k, *and let* $\Omega \supset k$ *be a universal domain. If* $a_X: CH^2(X_\Omega)_0 \to \mathrm{Alb}(X)(\Omega)$ *is an isomorphism, then* $H^2(\overline{X}, \mathbb{Q}_\ell(1))$ *is algebraic for* $\ell \neq \mathrm{char}(k)$.

CONJECTURE 1.10. *The converse holds; i.e., if* $H^2(\overline{X}, \mathbb{Q}_\ell(1))$ *is algebraic for some* $\ell \neq \mathrm{char}(k)$, *then* $CH^2(X)$ *is representable.*

REMARKS 1.11. (a) The consideration of a universal domain or at least of a "field containing many parameters" is essential here. In fact, the homomorphism $a_{\overline{X}}: CH^{\dim(X)}(\overline{X})_0 \to \mathrm{Alb}(X)(\overline{k})$ is known to be an isomorphism for finite fields [**KS**, §9] and is conjectured to be an isomorphism for number fields k by Bloch and Beilinson (cf. [**Be3**, 5.2]).

(b) If $k = \mathbb{C} = \Omega$, then Conjectures 1.8 and 1.10 are equivalent, by the canonical comparison isomorphism between singular and étale cohomology.

(c) We have seen that $\mathrm{Ker}\, a_X = \mathrm{Ker}\, cl'$ for the complex Abel-Jacobi map cl' on $CH^d(X)_0$. If k is finitely generated and $\ell \neq \mathrm{char}(k)$, then one can show as well (cf. [**Ja3**, 9.14]) that $(\mathrm{Ker}\, a_X) \otimes \mathbb{Z}_\ell = (\mathrm{Ker}\, cl') \otimes \mathbb{Z}_\ell$ for the ℓ-adic Abel-Jacobi map cl' on $CH^d(X)_0$.

Conjecture 1.10 is known to be true for an abelian surface X over a field k of characteristic $p > 0$. In fact, since Tate's conjecture for $H^2(\overline{X}, \mathbb{Q}_\ell(1))$ is known by work of Zarhin [**Za**] and Mori [**Mo**], one easily sees that $H^2(\overline{X}, \mathbb{Q}_\ell(1))$ is algebraic if and only if X is isogeneous to a product of two supersingular elliptic curves. The representability of $CH^2(X)_0$ for such surfaces was proved by Maruyama and Suwa [**MS**, Theorem 3.2].

In general, Bloch proposes the following strategy. He introduces the following three-step filtration

$$
\begin{array}{ll}
 & CH^2(X) \\
 & \quad \cup| \\
(1.9) & CH^2(X)_0 = \mathrm{Ker}(\deg\colon CH^2(X) \to \mathbb{Z}) \\
 & \quad \cup| \\
 & T(X) = \mathrm{Ker}(a_X\colon CH^2(X)_0 \to \mathrm{Alb}(X)(k))
\end{array}
$$

The action of correspondences, i.e., of $CH^2(X \times X)$, respects this filtration, and one can show

THEOREM 1.12 (cf. **[Bl1**, 1.11]). *If the action of correspondences on $T(X_\Omega)$ factors through homological equivalence, then Conjecture* 1.10 *is true.*

Finally, Bloch discusses various aspects of the mysterious relation

$$T(X) \leftrightarrow H^2.$$

In the next section we review how Beilinson "explains" this relation on the basis of mixed motives. His approach heads toward an understanding of arbitrary Chow groups (in fact, arbitrary motivic cohomology), even in the nonrepresentable case.

2. Beilinson's formula

In his paper on height pairings **[Be3**, 5.10], Beilinson stated a conjecture on mixed motivic sheaves that leads to the following explicit conjecture in which no mixed motives are mentioned.

CONJECTURE 2.1 (Version 1 of Beilinson's conjecture). *Let k be a field. For every smooth projective variety X over k there exists a descending filtration F on $CH^j(X)_{\mathbb{Q}}$, for all $j \geq 0$, such that*

(a) $F^0CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathbb{Q}}$, $F^1CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathrm{hom},\mathbb{Q}}$, *for some fixed Weil cohomology $H^{\cdot}(X)$;*

(b) $F^rCH^i(X)_{\mathbb{Q}} \cdot F^sCH^j(X)_{\mathbb{Q}} \subseteq F^{r+s}CH^{i+j}(X)_{\mathbb{Q}}$ *under the intersection product;*

(c) $F^{\cdot}$ *is respected by f^* and f_* for morphisms $f\colon X \to Y$;*

(d) (*assuming the algebraicity of the Künneth components of the diagonal*) $\mathrm{Gr}_F^\nu CH^j(X)_{\mathbb{Q}}$ *depends only on the motive modulo homological equivalence $h^{2j-\nu}(X)$; and*

(e) $F^\nu CH^j(X)_{\mathbb{Q}} = 0$ *for $\nu \gg 0$.*

The meaning of (d) is as follows. By (b) and (c) the action of correspondences respects the filtration $F^{\cdot}$, and by (a) (applied to $X \times X$) the induced action on $\mathrm{Gr}_F^\nu CH^j(X)_{\mathbb{Q}} = F^\nu CH^j(X)_{\mathbb{Q}}/F^{\nu+1}CH^j(X)_{\mathbb{Q}}$ factors through homological equivalence (i.e., $CH^d(X \times X)_{\mathrm{hom}}$ acts as zero). For defining the motive $h^i(X)$ we have to assume that the Künneth components

$$\pi_i \in H^{2d-i}(X) \otimes H^i(X), \qquad d = \dim X,$$

of the diagonal Δ are algebraic ([**Kl3**, conjecture $C(X)$]; this is a fairly weak consequence of the standard conjectures or the Tate conjecture (cf. [**Ta**, §3]) and holds for varieties over finite fields [**KM**, Theorem 2]). This means that the idempotent

$$H^{\cdot}(X) \to H^i(X) \to H^{\cdot}(X)$$

is given by an algebraic correspondence (again denoted by π_i) which is unique and an idempotent in $CH^d(X \times X)_{\mathbb{Q}}/\sim_{\mathrm{hom}}$. The motive $h^i(X)$ can then be defined as the triple $(X, \pi_i, 0)$ (cf. [**Scho**]), and (d) means that

$$\text{(2.1)} \qquad \pi_i | \mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} = \delta_{i, 2j-\nu} \cdot \mathrm{id},$$

where $\delta_{a,b}$ is the Kronecker symbol.

Let us write this out in more detail, for further reference. Recall [**Scho**] that the category $\mathscr{M}_k^{\sim}$ of motives over k modulo some adequate equivalence relation $\sim$ can be defined as follows. Objects are triples $(X, p, m) = (X, p, m)_{\sim}$, where X is smooth projective over k, $p^2 = p \in \mathrm{Corr}^0(X, X)$ is an idempotent, and $m \in \mathbb{Z}$, and we have

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q\,\mathrm{Corr}^{n-m}(X, Y)p.$$

Here $\mathrm{Corr}^r(X, Y) = \bigoplus_i CH^{d_i+r}(X_i \times Y)_{\mathbb{Q}}/\sim$, for $X = \coprod X_i$, X_i of pure dimension d_i, is the group of correspondences of degree r, and composition is the one of correspondences recalled in §1. If $h(X) = (X, \mathrm{id}, 0)_{\mathrm{hom}}$ is the motive modulo homological equivalence associated to X, then

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X),$$

since the π_i are pairwise orthogonal idempotents in $CH^d(X \times X)_{\mathbb{Q}}/\sim_{\mathrm{hom}} = \mathrm{End}\, h(X)$ and

$$\text{(2.2)} \qquad \mathrm{End}\, h(X) = \bigoplus_{i=0}^{2d} \mathrm{End}\, h^i(X) = \bigoplus_{i=0}^{2d} \pi_i CH^d(X \times X)_{\mathbb{Q}}/\sim_{\mathrm{hom}} \pi_i,$$

since the π_i are central. Now (2.1) and (d) both mean that the action of $\mathrm{End}\, h(x)$ on $\mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}}$ factors through the quotient $\mathrm{End}\, h^i(X)$.

One would in fact expect the following stronger form of 2.1 to be true:

STRONG CONJECTURE 2.1. *This is the same as Conjecture* 2.1, *except that* (e) *is replaced by*

(*strong e*) $\quad F^{j+1} CH^j(X)_{\mathbb{Q}} = 0$.

LEMMA 2.2. *If the standard conjecture of Lefschetz type* $B(X)$ *is true, then Conjecture* 2.1 *implies its strong form.*

PROOF (Compare [Ja3, 11.2]). By $B(X)$ there is a hard Lefschetz isomorphism of motives

$$L^{d-s}: h^s(X) \xrightarrow{\sim} h^{2d-s}(X)(d-s) \qquad (s \le d),$$

where $M(n)$ is the n-fold Tate twist of a motive (in the notation of [Scho] we have $M(n) = M \otimes \mathbb{L}^{\otimes -n}$ and $h^{2d-s}(X)(d-s) = (X, \pi_{2d-s}, d-s))$. Hence $\pi_s: h(X) \to h^s(X) \to h(X)$ factors as a composition $\pi_s = \alpha' \circ \alpha$, where $\alpha \in \pi_{2d-s} CH^{2d-s}(X \times X)_{\mathbb{Q}}/\sim_{\text{hom}} = \operatorname{Hom}(h(X), h^{2d-s}(X)(d-s))$ and $\alpha' \in CH^s(X \times X)_{\mathbb{Q}}/\sim_{\text{hom}} = \operatorname{Hom}(h^{2d-s}(X)(d-s), h(X))$. If α is represented by an element $\Gamma \in CH^{2d-s}(X \times X)_{\mathbb{Q}}$, then Γ maps $CH^j(X)_{\mathbb{Q}}$ to $CH^{d-s+j}(X)_{\mathbb{Q}}$:

$$CH^j(X)_{\mathbb{Q}} \xrightarrow{\text{pr}_1^*} CH^j(X \times X)_{\mathbb{Q}} \xrightarrow{\Gamma} CH^{2d-s+j}(X \times X)_{\mathbb{Q}} \xrightarrow{(\text{pr}_2)_*} CH^{d-s+j}(X)_{\mathbb{Q}},$$

and this group vanishes for $s < j$. By (2.1), we conclude that $\operatorname{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} = 0$ for $\nu > j$, hence the result. We note that Lemma 2.2 makes sense and stays true, if we restrict the consideration of Conjecture 2.1 to X and $X \times X$.

A relation with mixed motives is given by the following version of Beilinson's conjecture. From now on, let $\mathcal{M}_k = \mathcal{M}_k^{\text{hom}}$ be the category of motives over k modulo homological equivalence. We sometimes call these Grothendieck motives, since they are the object of his standard conjectures, but, of course, Grothendieck also defined and considered other variants.

CONJECTURE 2.3 (Version 2 of Beilinson's conjecture). *This is the same as Conjecture* 2.1, *except that* (d) *is replaced by*

(d′) (*assumptions as in Conjecture* 2.1(d)) *There is an abelian category* $\mathcal{MM}_k$ (*of "mixed motives over* k") *containing the category* $\mathcal{M}_k$ *of Grothendieck motives as a full subcategory, and a functorial isomorphism*

$$\text{(2.3)} \qquad \operatorname{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} \cong \operatorname{Ext}^{\nu}_{\mathcal{MM}_k}(1, h^{2j-\nu}(X)(j)),$$

where $1 = h(\operatorname{Spec} k)$ *is the trivial motive.*

In the following we call (2.3) "Beilinson's formula". It makes more evident and precise (granting the existence of $\mathcal{MM}_k$!) how $\operatorname{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}}$ depends on the motive $h^{2j-\nu}(X)$. Note that the category $\mathcal{M}_k$ is expected to be semisimple, so there are no nontrivial (Yoneda) extension groups $\operatorname{Ext}^i_{\mathcal{M}_k}(-,-)$ for $i \ge 1$. The existence of nontrivial extensions is a specific feature of the "mixed" situation.

2.4. Let us consider Beilinson's formula for $\nu = 0$. Since 1 and $h^{2j}(X)(j)$ are in $\mathcal{M}_k$, the assumption that $\mathcal{M}_k$ is a full subcategory of $\mathcal{MM}_k$ means that we have an isomorphism

$$\operatorname{Hom}_{\mathcal{M}_k}(1, h^{2j}(X)(j)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{MM}_k}(1, h^{2j}(X)(j)).$$

On the other hand, by (a) we want

$$\mathrm{Gr}_F^0 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathbb{Q}}/\sim_{\mathrm{hom}}.$$

But the equality

$$CH^j(X)_{\mathbb{Q}}/\sim_{\mathrm{hom}} = \mathrm{Hom}_{\mathcal{M}_k}(1, h^{2j}(X)(j))$$

holds by construction of the category $\mathcal{M}_k$; by definition, the left-hand side is $\mathrm{Hom}_{\mathcal{M}_k}(1, h(X)(j))$, and this is equal to $\mathrm{Hom}_{\mathcal{M}_k}(1, h^{2j}(X)(j))$, since $\pi_i CH^j(X)_{\mathbb{Q}}/\sim_{\mathrm{hom}} \subseteq \pi_i H^{2j}(X) = 0$ for $i \neq 2j$.

For $\nu = 1$ we want a map

$$(2.4) \qquad F^1 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathrm{hom},\mathbb{Q}} \to \mathrm{Ext}^1_{\mathcal{MM}_k}(1, h^{2j-1}(X)(j))$$

whose kernel is $F^2CH^j(X)_{\mathbb{Q}}$. While a category $\mathcal{MM}_k$ as in (d′) has not yet been identified (but cf. **[Li]** for a discussion of several proposals), we have already encountered the reflection of (2.4) in cohomology. For example, consider the ℓ-adic cohomology, $\ell \neq \mathrm{char}(k)$. By the well-known isomorphisms

$$(2.5) \qquad H^\nu(G_k, V) \cong \mathrm{Ext}^\nu_{G_k}(\mathbb{Q}_\ell, V)$$

for any (finite-dimensional) $\mathbb{Q}_\ell$-representation V of G_k, we can in fact reinterprete the ℓ-adic Abel-Jacobi map (after tensoring with $\mathbb{Q}$ and $\mathbb{Q}_\ell$) as a morphism

$$(2.6) \qquad cl' : CH^j(X)_{\mathrm{hom},\mathbb{Q}} \to \mathrm{Ext}^1_{G_k}(\mathbb{Q}_\ell, H^{2j-1}(\overline{X}, \mathbb{Q}_\ell)(j)),$$

the ℓ-adic version of (2.4). If $k = \mathbb{C}$, then we obtain something similar in the Hodge theory; particularly, if MHS is the category of mixed Hodge structures **[De2**, 2.3; **St]**, then Carlson **[Ca**, Proposition 2] and Beilinson **[Be2**, 1.7] have constructed an isomorphism

$$(2.7) \qquad \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H) = W_0 H_{\mathbb{C}}/(W_0 H + F^0 W_0 H_{\mathbb{C}})$$

for a mixed Hodge structure H, where $W_\cdot$ is the ascending weight filtration. The same formula holds for $\mathrm{Ext}^1_{A\text{-MHS}}(A, H)$, where A is $\mathbb{Q}$ or $\mathbb{R}$, A-MHS is the category of A-Hodge structures, and H is in A-MHS (loc. cit.). Since $H^{2j-1}(X(\mathbb{C}), \mathbb{Q}(j))$ is of weight $-1 < 0$, we can thus regard the complex Abel-Jacobi map (after tensoring with $\mathbb{Q}$) as a morphism

$$(2.8) \qquad cl' : CH^j(X)_{\mathrm{hom},\mathbb{Q}} \to \mathrm{Ext}^1_{\mathbb{Q}\text{-MHS}}(\mathbb{Q}, H^{2j-1}(X(\mathbb{C}), \mathbb{Q}(j))).$$

The map (2.4) should be compatible with the Abel-Jacobi maps into cohomology; i.e., these should factor as

$$\begin{array}{ccc}
 & H^1(G_k, H^{2j-1}(\overline{X}, \mathbb{Q}_\ell(j))) = & \mathrm{Ext}^1_{G_k}(\mathbb{Q}_\ell, H^{2j-1}(\overline{X}, \mathbb{Q}_\ell(j))) \\
 \nearrow & & \uparrow H_\ell \\
CH^j(X)_{\mathrm{hom},\mathbb{Q}} & \overset{(2.4)}{\rightarrow} & \mathrm{Ext}^1_{\mathcal{MM}_k}(1, h^{2j-1}(X)(j)) \\
 \searrow & & \downarrow H_B \\
 & \dfrac{H^{2j-1}(X(\mathbb{C}), \mathbb{C})}{H^{2j-1}(X(\mathbb{C}), \mathbb{Q}(j)) + F^j} = & \mathrm{Ext}^1_{\mathbb{Q}\text{-MHS}}(\mathbb{Q}, H^{2j-1}(X, \mathbb{Q}(j))).
\end{array}$$

Here the vertical maps should be induced by exact, faithful "realization" functors

$$\begin{aligned} &H_\ell : \mathcal{MM}_k \to \operatorname{Rep}(G_k, \mathbb{Q}_\ell) := \text{category of finite-dimensional} \\ (2.10) \qquad &\qquad\qquad\qquad\qquad \mathbb{Q}_\ell\text{-representations of } G_k, \\ &H_B : \mathcal{MM}_k \to \mathbb{Q}\text{-MHS}, \end{aligned}$$

which extend the existing functors H_ℓ and H_B on $\mathcal{M}_k$ induced by ℓ-adic and singular cohomology, respectively. Note that by definition we have

$$H_\ell(1) = \mathbb{Q}_\ell, \qquad H_\ell(h^i(X)(j)) = \pi_i H^\cdot(\overline{X}, \mathbb{Q}_\ell(j)) = H^i(\overline{X}, \mathbb{Q}_\ell(j)),$$
$$H_B(1) = \mathbb{Q}, \qquad H_B(h^i(X)(j)) = H^i(X(\mathbb{C}), \mathbb{Q}(j)),$$

where $\mathbb{Q}_\ell$ and $\mathbb{Q}$ are the trivial $\mathbb{Q}_\ell$-representation and trivial $\mathbb{Q}$-Hodge structure, respectively. Note also that the corresponding diagram for $\nu = 0$,

(2.11)

$$\begin{array}{ccccc} & & H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))^{G_k} & = & \operatorname{Hom}_{G_k}(\mathbb{Q}_\ell, H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))) \\ & \nearrow & & & \uparrow H_\ell \\ CH^j(X)_\mathbb{Q} & \to & CH^j(X)_\mathbb{Q}/\sim_{\text{hom}} & = & \operatorname{Hom}_{\mathcal{M}_k}(1, h^{2j}(X)(j)) \\ & \searrow & & & \downarrow H_B \\ & & H^{2j}(X(\mathbb{C}), \mathbb{Q}(j)) \cap H^{j,j} & = & \operatorname{Hom}_{\mathbb{Q}\text{-HS}}(\mathbb{Q}, H^{2j}(X(\mathbb{C}), \mathbb{Q}(j))), \end{array}$$

exists and commutes by definition.

2.5. The extensions given by Abel-Jacobi maps can in fact be constructed in a universal and geometric way: If $z \in Z^j(X)$ is a cycle that is homologically equivalent to zero, let Z be the support of z and put $U = X - Z$. For the ℓ-adic cohomology ($\ell \neq \operatorname{char} k$) we obtain a commutative diagram in the category of $\mathbb{Q}_\ell$-representations of G_k

(2.12)

$$\begin{array}{ccccccccc} 0 & \to & H^{2j-1}(\overline{X}, \mathbb{Q}_\ell(j)) & \to & H^{2j-1}(\overline{U}, \mathbb{Q}_\ell(j)) & \to & H^{2j}_{\overline{Z}}(\overline{X}, \mathbb{Q}_\ell(j)) & \to & H^{2j}(\overline{X}, \mathbb{Q}_\ell(j)) \\ & & \| & & \cup| & & \uparrow z & \nearrow & cl(z)=0 \\ 0 & \to & H^{2j-1}(\overline{X}, \mathbb{Q}_\ell(j)) & \to & E_\ell & \to & \mathbb{Q}_\ell & \to & 0 \end{array}$$

Here the exact top row is part of the localization sequence [**Mi**, III, 1.25], where by purity $H^{2j-1}_{\overline{Z}}(\overline{X}, \mathbb{Q}_\ell(j)) = 0$ and $H^{2j}_{\overline{Z}}(\overline{X}, \mathbb{Q}_\ell(j)) \cong \mathbb{Q}_\ell^A$ as a $\mathbb{Q}_\ell$-representation of G_k, where A is the set of irreducible components of $\overline{Z}$. This gives the map denoted by z (mapping $1 \in \mathbb{Q}_\ell$ to the local cycle class of z; cf. [**Mi**, VI, §9]), and the bottom exact sequence is obtained by pull-back via z. It is shown in [**Ja3**, 9.4] that the extension class of this sequence is the image of z under the map cl' in (2.6).

An analogous result holds for the complex version (2.8), by using the corresponding diagram of mixed $\mathbb{Q}$-Hodge structures given by singular cohomology (loc. cit., 9.2 and 9.7c)). This suggests that there should exist a similar diagram in $\mathcal{MM}_k$

$$\begin{array}{ccccccccc} 0 & \to & h^{2j-1}(X)(j) & \to & h^{2j-1}(U)(j) & \to & h^{2j}_Z(X)(j) & \to & h^{2j}(X)(j) \\ & & \| & & \cup| & & \uparrow & & \\ 0 & \to & h^{2j-1}(X)(j) & \to & E & \to & 1 & \to & 0 \end{array}$$

mapping to (2.12) and its Hodge analogue via H_ℓ and H_B. This consideration sheds some more light on the expected nature of $\mathcal{MM}_k$ and the isomorphism in Beilinson's formula. Like the target categories in (2.10), $\mathcal{MM}_k$ should contain objects corresponding to the cohomology of arbitrary varieties over k. More precisely, $\mathcal{MM}_k$ should be an abelian tensor category, and there should be a twisted Poincaré duality theory $((X, Z) \mapsto H_Z^{\cdot}(X, \cdot), X \mapsto H_{\cdot}(X, \cdot))$ in the sense of Bloch-Ogus **[BO]** with values in $\mathcal{MM}_k$ (cf. **[Ja3**, §6]), mapping to ℓ-adic and singular cohomology via H_ℓ and H_B, respectively. The isomorphisms (2.3) should be compatible with the cycle maps via the realization functors (2.10). This leads to versions 3 and 4 of Beilinson's conjecture, which will be discussed in §4.

2.6. Consider the ℓ-adic cohomology ($\ell \neq \operatorname{char} k$). The compatibility between Beilinson's formula and the higher cycle maps for all $\nu \geq 0$ means the following. Let $F^0 \supset F^1 \supset \cdots$ be the filtration on $H^{2j}(X, \mathbb{Q}_\ell(j))$ introduced in 1.4. Since the action of correspondences respects the Hochschild-Serre spectral sequence (1.5), it respects $F^{\cdot}$, and the induced action on $\mathrm{Gr}_F^\nu H^{2j}(X, \mathbb{Q}_\ell(j)) \cong H^\nu(G_k, H^{2j-\nu}(\overline{X}.\mathbb{Q}_\ell(j)))$ factors through homological equivalence (for the Weil cohomology $H^{\cdot}(\overline{X}, \mathbb{Q}_\ell(\cdot))$). Moreover, from the above isomorphism it is clear that π_i acts as $\delta_{i,2j-\nu} \cdot \mathrm{id}$ on this space; i.e., it depends only on $h^{2j-\nu}(X)$. It follows that, under the cycle map

$$cl\colon CH^j(X)_{\mathbb{Q}} \to H^{2j}(X, \mathbb{Q}_\ell(j)),$$

the conjectured filtration $F^{\cdot}$ on $CH^j(X)_{\mathbb{Q}}$ would map into the filtration $F^{\cdot}$ on $H^{2j}(X, \mathbb{Q}_\ell(j))$ considered here. The mentioned compatibility means the commutativity of the diagram

$$\text{(2.13)}\qquad \begin{array}{ccc} \mathrm{Gr}_F^\nu CH^j(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{Gr}_F^\nu cl} & \mathrm{Gr}_F^\nu H^{2j}(X, \mathbb{Q}_\ell(j)) \\ \| & & \| \\ \mathrm{Ext}^\nu_{\mathcal{MM}_k}(1, h^{2j-\nu}(X)(j)) & \xrightarrow{H_\ell} & \mathrm{Ext}^\nu_{G_k}(\mathbb{Q}_\ell, H^{2j-\nu}(\overline{X}, \mathbb{Q}_\ell(j))) \end{array}$$

where H_ℓ associates to a Yoneda-ν-extension

$$0 \to h^{2j-\nu}(X)(j) \to E_\nu \to \cdots \to E_2 \to E_1 \to 1 \to 0$$

in $\mathcal{MM}_k$ the ν-extension of $\mathbb{Q}_\ell - G_k$-representations

$$0 \to H^{2j-\nu}(\overline{X}, \mathbb{Q}_\ell(j)) \to H_\ell(E_\nu) \to \cdots \to H_\ell(E_1) \to \mathbb{Q}_\ell \to 0.$$

It would be interesting to construct the ν-extensions in the image of

$$cl^{(\nu)}\colon F_\ell^\nu CH^j(X)_{\mathbb{Q}} \to \mathrm{Ext}^\nu_{G_k}(\mathbb{Q}_\ell, H^{2j-\nu}(\overline{X}, \mathbb{Q}_\ell(j)))$$

(cf. 1.4) in a geometric way also for $\nu > 1$. By proceeding along the lines of 2.5 one can relate these ν-extensions to certain canonical extensions related to the complex computing $H^{\cdot}(\overline{U}, \mathbb{Q}_\ell)$ (notation as in 2.5), but we do not have a geometric interpretation for these.

The analogy with ℓ-adic cohomology certainly provides a major motivation for the conjectures on mixed motives (cf. discussion in §4). The following observation provides a direct link.

LEMMA 2.7. *If the cycle map*

$$cl\colon CH^j(X)_{\mathbb{Q}} \to H^{2j}(X, \mathbb{Q}_\ell(j))$$

is injective for all X *in* $\mathscr{V}_k$ *and all* $j \geq 0$, *then Beilinson's conjecture* (*version* 1) *is true and Beilinson's filtration* $F^\cdot$ *agrees with the filtration* $F_\ell^\cdot$ *of* 1.4.

PROOF. Since the cycle map is compatible with the action of correspondences, the considerations in 2.6 imply that $F_\ell^\cdot$ satisfies properties (a)–(d). The assumed injectivity implies that $\mathrm{Gr}^\nu_{F_\ell} CH^j(X)_{\mathbb{Q}} = 0$ for $\nu \gg 0$, and the equality $F^\cdot = F_\ell^\cdot$ follows from the uniqueness result 5.7.

We remark that as in 2.2 it makes sense to restrict to certain subcategories of $\mathscr{V}_k$. Also, similar statements would hold for other suitable cycle maps into absolute cohomology theories (cf. [**Ja3**, 11.5]).

Of course, cl will not in general be injective, for example, not over an algebraically closed field. But it seems natural to ask

QUESTION 2.8. Is $cl\colon CH^j(X)_{\mathbb{Q}} \to H^{2j}(X, \mathbb{Q}_\ell(j))$ injective for a finitely generated field k?

2.9. For the discussion of the Hodge realization, we again consider the case of zero cycles on a complex surface X. For the conjectural filtration we have

$$F^1 CH^2(X)_{\mathbb{Q}} = CH^2(X)_0 \otimes \mathbb{Q}$$

by definition, and since the standard conjecture $B(X)$ holds for surfaces and singular cohomology (cf. [**Kl3**, 4-3], Lemma 2.2 implies $F^3 CH^2(X)_{\mathbb{Q}} = 0$ (this is the argument of [**Bl1**, (1.9)]. The following lemma now implies that F coincides with Bloch's filtration (1.9) after tensoring with $\mathbb{Q}$.

LEMMA 2.10. *Let* X *be a smooth, projective, irreducible variety of dimension* d *over a field* k. *If* $\dim H^1(X) = 2\dim \mathrm{Jac}(X)$ *for the considered Weil cohomology theory* (*e.g., if we consider* ℓ-*adic or singular cohomology, or if* $B(X)$ *holds*), *then for Beilinson's filtration* $F^\cdot$ *we must have*

$$F^2 CH^d(X)_{\mathbb{Q}} = T(X)_{\mathbb{Q}},$$

where $T(X)$ *is the kernel of* $a_X\colon CH^d(X)_0 \to \mathrm{Alb}(X)(k)$.

PROOF. Murre constructed a certain idempotent $\tilde{\pi}_{2d-1} \in CH^d(X \times X)_{\mathbb{Q}}$ (cf. [**Mur1**, Theorem 2]) that lifts the Künneth component π_{2d-1} of the diagonal under the stated assumptions on the Weil cohomology theory (loc. cit. for ℓ-adic cohomology, hence for the singular cohomology by the comparison isomorphism; in the other cases the claim easily follows from the arguments in [**Kl1**, Appendix 2]). By property (2.1) of Beilinson's filtration we then must have $F^2CH^d(X)_{\mathbb{Q}} = $ kernel of $\tilde{\pi}_{2d-1}$ on $CH^d(X)_0 \otimes \mathbb{Q}$ (this is a special case of 5.5). But according to [**Mur1**, Theorem 2ii)], this kernel is $T(X)_{\mathbb{Q}}$.

Returning to our complex surface X, we now have

$$CH^2(X)/CH^2(X)_0 \overset{cl}{\underset{\sim}{\to}} \mathrm{Hom}_{\mathrm{HS}}(\mathbb{Z}, H^4(X(\mathbb{C}), \mathbb{Z}(2))),$$
(2.14)
$$CH^2(X)_0/T(X) \overset{cl'}{\underset{\sim}{\to}} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^3(X(\mathbb{C}), \mathbb{Z}(2))),$$

which fits well with Beilinson's formula (2.3) for $\nu = 0, 1$. For $\nu = 2$, (2.3) gives

$$T(X) \overset{?}{=} \mathrm{Ext}^2_{\mathcal{MM}_{\mathbb{C}}}(1, h^2(X)(2)), \tag{2.15}$$

which in a way makes more precise the relation between $T(X)$ and H^2 envisioned by Bloch. Unfortunately, there is no analogue in the Hodge realization, since

$$\mathrm{Ext}^{\nu}_{A\text{-MHS}}(A, -) = 0 \quad \text{for } \nu \geq 2, \ A = \mathbb{Z}, \mathbb{Q}, \text{ or } \mathbb{R}, \tag{2.16}$$

as easily follows from the right-exactness of $\mathrm{Ext}^1_{A\text{-MHS}}(A, -)$. This together with Mumford's counterexample shows that the category of mixed Hodge structures is too coarse to detect all cycles or all mixed motives.

QUESTION 2.11. Is there another abelian category $(A\text{-MHS})_{\mathrm{fine}}$ of "refined mixed Hodge structures" and a forgetful functor F to A-MHS such that the singular cohomology H_B factors through F and there are nontrivial 2-exensions in $(A\text{-MHS})_{\mathrm{fine}}$?

$(A\text{-MHS})_{\mathrm{fine}}$ should be a rigid abelian tensor category with a weight filtration (cf. [**Ja3**, 6.3]), and F should embed the subcategory of pure objects fully faithfully into the category A-HS of (pure) A-Hodge structures, if one believes in the classical Hodge conjecture. A possible approach could be to consider the *subcategory* of A-MHS formed by the mixed Hodge structures of "geometric origin" (cf. the considerations in [**SaM2**]), but this definition seems hard to handle for computational purposes.

3. "Applications" and theorems

In this section we want to show how nicely Beilinson's formula could be applied to determine Chow groups. In fact, these "applications" can also be deduced from the less fancy version 1 of Beilinson's conjecture, but the "proofs" based on Beilinson's formula are more instructive and shorter. The first two examples are only written down to illustrate this; these consequences have already been deduced from version 1. In the following, let X be smooth, projective, irreducible of dimension d over k.

LEMMA 3.1 (cf. 2.2). *If the standard conjecture of Lefschetz type $B(X)$ is true, then necessarily $F^{j+1}CH^j(X)_{\mathbb{Q}} = 0$.*

PROOF. By $B(X)$, we have a hard Lefschetz isomorphism

$$h^{2j-\nu}(X)(j) \cong h^{2d-2j+\nu}(X)(d-2j+\nu+j).$$

We compute

$$\begin{aligned}\mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} &= \mathrm{Ext}^{\nu}_{\mathcal{MM}_k}(1, h^{2j-\nu}(X)(j)) \\ &= \mathrm{Ext}^{\nu}_{\mathcal{MM}_k}(1, h^{2d-2j+2\nu-\nu}(X)(d-j+\nu)) \\ &= 0 \quad \text{for } \nu > j,\end{aligned}$$

as a subquotient of $CH^{d-j+\nu}(X)_{\mathbb{Q}}$.

LEMMA 3.2 (cf. 1.12). *Beilinson's formula implies Bloch's conjecture.*

PROOF. We may assume that k is algebraically closed and show that the group

$$T(X) = \mathrm{Ext}^2_{\mathcal{MM}_k}(1, h^2(X)(2))$$

vanishes, if $H^2(X)$ is generated by algebraic cycles. Indeed, this implies that $h^2(X)(1)$ is a sum of trivial objects 1, but

$$\mathrm{Ext}^2_{\mathcal{MM}_k}(1, 1(1)) = 0,$$

as a subquotient of $CH^1(\mathrm{Spec}\, k)_{\mathbb{Q}} = 0$.

The next two "applications" we state as separate conjectures—which are much in the spirit of Bloch's conjecture—giving "proofs" based on Beilinson's formula. Then we shall prove actual converse theorems, in the spirit of Mumford's theorem and Bloch's generalization.

Recall Grothendieck's filtration by coniveau on the ℓ-adic cohomology, $\ell \neq \mathrm{char}\, k$ **[Gr3**, 10.1]:

$$\begin{aligned} N^{\nu} H^i(\overline{X}, \mathbb{Q}_{\ell}) &= \bigcup_{Z \subseteq X} \mathrm{Im}(H^i_{\overline{Z}}(\overline{X}, \mathbb{Q}_{\ell}) \to H^i(\overline{X}, \mathbb{Q}_{\ell})) \\ &= \bigcup_{Z \subset X} \mathrm{Ker}(H^i(\overline{X}, \mathbb{Q}_{\ell}) \to H^i(\overline{X - Z}, \mathbb{Q}_{\ell})), \end{aligned} \tag{3.1}$$

where Z runs over all closed subschemes of X that are of codimension $\geq \nu$. The equivalence of the definitions comes from the long exact localization sequence. A variant is

$$\widetilde{N}^{\nu} H^i(\overline{X}, \mathbb{Q}_{\ell}) = \bigcup_{f:\ Y \to X} \mathrm{Im}(f_*: H^{i-2\nu}(\overline{Y}, \mathbb{Q}_{\ell}(-\nu)) \to H^i(\overline{X}, \mathbb{Q}_{\ell})), \tag{3.2}$$

where $f: Y \to X$ runs over all morphisms from smooth projective varieties Y of pure dimension $d - \nu$ into X. Obviously $\widetilde{N}^{\nu} \subseteq N^{\nu}$, and equality holds if we have resolution of singularities, e.g., if $\mathrm{char}\, k = 0$. In fact, if $\pi: Y \to Z$ is proper and surjective, with Y smooth and projective, of pure dimension $d - \nu$, then the Gysin morphism for $f: Y \xrightarrow{\pi} Z \xrightarrow{i} X$ factors as

$$\begin{array}{ccccc} f_*: H^{i-2\nu}(\overline{Y}, \mathbb{Q}_{\ell}(-\nu)) & \to & H^i_{\overline{Z}}(\overline{X}, \mathbb{Q}_{\ell}) & \to & H^i(\overline{X}, \mathbb{Q}_{\ell}) \\ \downarrow & & \downarrow & & \downarrow \\ H_{2d-i}(\overline{Y}, \mathbb{Q}_{\ell}(d)) & \xrightarrow{\pi_*} & H_{2d-i}(\overline{Z}, \mathbb{Q}_{\ell}(d)) & \xrightarrow{i_*} & H_{2d-i}(\overline{X}, \mathbb{Q}_{\ell}(d)) \end{array}$$

so that $\operatorname{Im} f_* \subseteq \operatorname{Im}(H^i_{\overline{Z}}(\overline{X}, \mathbb{Q}_\ell))$, and we have $\operatorname{Im} f_* = \operatorname{Im} i_*$ by weight arguments: Since $H^i(\overline{X}, \mathbb{Q}_\ell)$ is pure of weight i, the image of i_* equals the image of $W_i H_{2d-i}(\overline{Z}, \mathbb{Q}_\ell(d))$, but $H_{2d-i}(\overline{Y}, \mathbb{Q}_\ell(d))$ surjects onto this space via π_* (cf. **[Ja3**, 7.7], where $H_0(X, b)$ should read $H_a(X, b)$). For ètale homology and the discussion of weights we refer to **[Ja3**, 6.7, 6.8.2, and 6.11.1] (for our application, the technical 6.11.1 loc. cit. for $\operatorname{char} k = 0$ can be avoided by using comparison isomorphisms with singular cohomology and Deligne's result quoted in loc. cit. 7.7).

For $k = \mathbb{C}$, we define filtrations $N^\cdot$ and $\widetilde{N}^\cdot$ on $H^i(X(\mathbb{C}), \mathbb{Q})$ by the same formulae as above, and by resolution of singularities we have $N^\cdot = \widetilde{N}^\cdot$. In the following we put

$$H^i(X, j) = \begin{cases} H^i(\overline{X}, \mathbb{Q}_\ell(j)) & \text{if } \operatorname{char} k \neq \ell, \text{ or} \\ H^i(X(\mathbb{C}), \mathbb{Q}(j)) & \text{if } k = \mathbb{C}, \end{cases}$$

to treat both cases in a parallel way. Similarly we use the notation $H^i_Z(X, j)$ for $H^i_{\overline{Z}}(\overline{X}, \mathbb{Q}_\ell(j))$ or $H^i_Z(\mathbb{C})(X(\mathbb{C}), \mathbb{Q}(j))$, and $H_a(X, b)$ for $H_a(\overline{X}, \mathbb{Q}_\ell(b))$ or $H_a(X(\mathbb{C}), \mathbb{Q}(b))$ (Borel-Moore homology). In fact, everything below could be extended to any suitable Poincaré duality theory with weights satisfying the axioms a)–m) of **[Ja3**, §6, 7] and such that $H^i(X, j)$ defines a Weil cohomology on V_k after applying a forgetful functor to vector spaces. Moreover, we sometimes omit Tate twists, by writing $H^i(X)$ for $H^i(X, j)$, etc.

Note that for a surface X we have

$$H^2(X) \text{ is algebraic} \Leftrightarrow N^1 H^2(X) = H^2(X).$$

Hence the following is a generalization of Bloch's conjecture.

CONJECTURE 3.3. *If $H^i(X)$ is supported in codimension* 1 *(i.e., $H^i(X) = N^1 H^i(X)$), for $i = 2, \ldots, d$, then $CH^d(X)_0$ is representable.*

"PROOF". *Assuming Beilinson's formula, the standard conjectures, and resolution of singularities over k*—The assumption implies that there is a smooth projective variety Y of dimension $d-1$ and an epimorphism

$$h^{i-2}(Y)(-1) \twoheadrightarrow h^i(X), \qquad i = 2, \ldots, d. \tag{3.3}$$

By Poincaré duality this gives a monomorphism

$$h^{2d-i}(X)(d) \hookrightarrow h^{2d-i}(Y)(d), \qquad i = 2, \ldots, d.$$

Since $\mathcal{M}_k$ is semisimple, we obtain an induced injection

$$\begin{array}{ccc} \operatorname{Ext}^\nu_{\mathcal{MM}_k}(1, h^{2d-\nu}(X)(d)) & \hookrightarrow & \operatorname{Ext}^\nu_{\mathcal{MM}_k}(1, h^{2d-\nu}(Y)(d)) \\ \| & & \| \\ \operatorname{Gr}^\nu_F CH^d(X)_{\mathbb{Q}} & & \operatorname{Gr}^\nu_F CH^d(Y)_{\mathbb{Q}} = 0 \end{array}$$

for $\nu \geq 2$ and obtain

$$0 = F^2CH^d(X)_{\mathbb{Q}} = T(X)_{\mathbb{Q}},$$

where $T(X) = \mathrm{Ker}(a_X\colon CH^d(X)_0 \to \mathrm{Alb}(X)(k))$ (cf. 2.10). The same reasoning shows that $T(X_E) = 0$ over any algebraically closed extension field E of k (since the surjection (3.3) carries over to the base extensions Y_E, X_E), hence the result. Note that we need Beilinson's formula for X and Y over all such E or over a universal domain $\Omega \supset k$.

CONJECTURE 3.4. *If $H^{\cdot}(X)$ is algebraic, then $CH^{\cdot}(X_E)_{\mathrm{num}} \otimes \mathbb{Q} = 0$ over any extension field E of k.*

"PROOF". *Assuming Beilinson's formula*—We may assume that k is algebraically closed. Then the assumption means that

$$H^i(X) = \begin{cases} 0 & \text{for } i \text{ odd}, \\ \text{generated by cycle classes} & \text{for } i \text{ even.} \end{cases}$$

This implies that $\sim_{\mathrm{hom}} = \sim_{\mathrm{num}}$ and that the standard conjectures of Lefschetz type are true for X. From this we easily deduce that

$$h^{2j-\nu}(X)(j) = \begin{cases} 0 & \text{if } \nu \text{ is odd}, \\ \text{sum of copies of } 1(\mu) & \text{if } \nu = 2\mu \text{ is even}, \end{cases}$$

for all j and ν. But

$$\mathrm{Ext}^{2\mu}_{\mathcal{MM}_k}(1, 1(\mu)) = CH^{\mu}(\mathrm{Spec}\, k)_{\mathbb{Q}} = 0 \quad \text{for } \mu > 0,$$

showing that $F^1CH^j(X)_{\mathbb{Q}} = 0$ for all $j \geq 0$, and the same is true over all extension fields.

The cohomology is known to be algebraic for the standard cellular varieties, like projective spaces, Grassmannians, Schubert varieties, etc., but for these it is also known that $CH^{\cdot}(X)_{\mathrm{num}} \otimes \mathbb{Q} = 0$! We now come to the converse theorems, showing that the conditions in 3.3 and 3.4 are in fact necessary.

THEOREM 3.5. (a) *If $CH^d(X)_0$ is representable, then $H^i(X)$ is supported in codimension 1, for $i = 2, \ldots, d$.*

(b) *More generally, say that $CH^d(X)_0$ has* rank $\leq \mu$ $(0 \leq \mu \leq d)$, *if there is a closed subscheme $Z \subset X$ of dimension μ such that the restriction*

$$CH^d(X_\Omega)_{\mathbb{Q}} \to CH^d((X - Z)_\Omega)_{\mathbb{Q}}$$

is zero, where $\Omega \supset k$ is a universal domain (e.g., $\Omega = k$, if $k = \mathbb{C}$). If $CH^d(X)_0$ has rank $\leq \mu$, then $H^i(X)$ is supported in codimension 1 for $i = \mu + 1, \ldots, d$.

THEOREM 3.6. (a) *If $CH^{\cdot}(X_\Omega)_{\mathrm{hom}} \otimes \mathbb{Q} = 0$ for a universal domain $\Omega \supset k$, then $H^{\cdot}(X)$ is algebraic.*

(b) *If* $CH^{\nu}(X_{\Omega})_{\hom} \otimes \mathbb{Q} = 0$ *for* $\nu = i, \dots, d$, *then* $H^{\mu}(X)$ *is algebraic for* $\mu \leq 2(d-i+1)$ *and* $H^{\mu}(X) = N^{d-i+1}H^{\mu}(X)$ *for* $\mu \geq 2(d-i+1)$.

The two theorems will be proved later in 3.10. Similar results have been obtained independently by Lewis **[Le]**, Schoen **[Schoe]**, Colliot-Thélène, Raskind, and Saito **[CTRS]**, and S. Saito **[SaS]**. In particular, we refer to the last paper for many generalizations and a thorough discussion of the relation between filtrations on Chow groups, the coniveau filtration, and the ranks of Chow groups. All these papers consider only fields of characteristic zero (to use resolution of singularities), and Theorem 3.6 is only obtained conditionally (assuming the Hodge conjecture or some standard conjecture). Theorem 3.6(a) improves a result of Bloch (cf. **[Kl2**, 3.12]).

We note that in Theorems 3.5 and 3.6 we may replace the coniveau filtration by the following, more concrete filtrations: Following Grothendieck **[Gr3**, 10.2; **Gr4]**, define the filtrations

$$N_H^{\nu}H^i(X(\mathbb{C}), \mathbb{Q}) = \text{the union of sub-Hodge structures } H \text{ of } H^i(X(\mathbb{C}), \mathbb{Q}) \text{ for which } H(\nu) \text{ is effective}$$

for $k = \mathbb{C}$, and

$$N_{\ell}^{\nu}H^i(\overline{X}, \mathbb{Q}_{\ell}) = \text{the union of sub-}G_k\text{-representations } V \text{ of } H^i(\overline{X}, \mathbb{Q}_{\ell}) \text{ for which } V(\nu) \text{ is effective}$$

if k is finitely generated and $\ell \neq k$. A Hodge structure is effective, if $p \geq 0$ and $q \geq 0$ for the occurring Hodge types (p, q), and a pure G_k-representation V is effective if the same is true for the "ℓ-adic Hodge numbers" (as defined by Deligne **[De3**, 3.3.7]) of the Frobenius eigenvalues at all good places. Equivalently, V is effective, if V and $\mathrm{Hom}(V, \mathbb{Q}_{\ell}(-w))$ are both entire, for $w =$ weight of V, where a G_k-representation W is called entire, if the Frobenius eigenvalues at good places are algebraic integers. Then one has

$$N^{\nu}H^i(X(\mathbb{C}), \mathbb{Q}) \subseteq N_H^{\nu}H^i(X(\mathbb{C}), \mathbb{Q}),$$

and the generalized Hodge conjecture **[Gr4]** predicts that equality holds. Similarly

$$N^{\nu}H^i(\overline{X}, \mathbb{Q}_{\ell}) \subseteq N_{\ell}^{\nu}H^i(\overline{X}, \mathbb{Q}_{\ell}),$$

and equality holds by the generalized Tate conjecture (cf. **[Gr3**, 10.3; **Ja3**, 10.2b]). Note

$$H^i(X(\mathbb{C}), \mathbb{Q}) = N_H^{\nu}H^i(X(\mathbb{C}), \mathbb{Q})$$
$$\Leftrightarrow p \geq \nu,\ q \geq \nu \text{ for the occurring Hodge types } (p, q),$$

similarly for the ℓ-adic cohomology. Hence

$$H^i(X(\mathbb{C}), \mathbb{Q}) = N^1H^i(X(\mathbb{C}), \mathbb{Q}) \Rightarrow H^0(X, \Omega^i) = 0,$$

and from Theorem 3.5 we, in particular, obtain Roitman's theorem **[Roi1**, Theorem 3]:

Corollary 3.7. *Let $k = \mathbb{C}$. If $H^0(X, \Omega^i) \neq 0$ for some $i = 2, \dots, d$, then $CH^d(X)_0$ is not representable.*

This was proved by Roitman by completely different methods. For arbitrary fields we obtain

Corollary 3.8. *Let $\ell \neq \operatorname{char} k$. If $H^i(\overline{X}, \mathbb{Q}_\ell)(1)$ is not entire, for some $i = 2, \dots, d$, then $CH^d(X)_0$ is not representable.*

We note that $\operatorname{Hom}(H^i(\overline{X}, \mathbb{Q}_\ell), \mathbb{Q}_\ell(-i)) \cong H^i(\overline{X}, \mathbb{Q}_\ell)$ by Poincaré duality and hard Lefschetz. Hence $H^i(\overline{X}, \mathbb{Q}_\ell)(\nu)$ is effective if and only if it is entire. We now prove Theorems 3.5 and 3.6, by refining Bloch's method for surfaces.

Proposition 3.9. *Let z be a cycle of codimension d on $X \times X$.*

(a) *If z is supported on $X \times Y$, for $Y \subset X$ of codimension μ, then z as a correspondence maps $H^i(X)$ to $N^\mu H^i(X)$ for all i.*

(b) *If z is supported on $Y' \times X$, for $Y' \subset X$ of dimension ν, then z as a correspondence maps $H^i(X)$ to 0 for $i > 2\nu$ and to $N^1 H^i(X)$ for $\nu < i \leq 2\nu$.*

Proof. Without loss of generality we may assume that $z = Z$ is a prime cycle, i.e., $i: Z \hookrightarrow X \times X$ closed, integral, of dimension d. Define the maps

$$X \xleftarrow{f_1} Z \xrightarrow{f_2} X$$

by $f_j = \mathrm{pr}_j i$, where pr_1 and $\mathrm{pr}_2 : X \times X \to X$ are the first and second projection, respectively. Then the correspondence Z is the composition

$$H^i(X) \xrightarrow{f_1^*} H^i(Z) \xrightarrow{\alpha_Z} H_{2d-i}(Z) \xrightarrow{(f_2)_*} H_{2d-i}(X) \underset{\sim}{\xrightarrow{\alpha_X^{-1}}} H^i(X)$$

where α_Z and α_X are the canonical maps, obtained by cap-product with the fundamental classes η_Z and η_X respectively (cf. [**Ja3**, 6.1i), j)]). This follows from the projection formula and the compatibility of cup- and cap-product, making the following diagram commute:

$$\begin{array}{ccccccc}
H^i(Z) & \times & H_{2d}(Z) & \xrightarrow{\cap} & H_{2d-i}(Z) & & \\
{\scriptstyle i^*}\uparrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \\
H^i(X \times X) & \times & H_{2d}(X \times X) & \xrightarrow{\cap} & H_{2d-i}(X \times X) & \xrightarrow{(\mathrm{pr}_2)_*} & H_{2d-i}(X) \\
\| & & \uparrow{\scriptstyle \alpha_{X\times X}} & & \uparrow{\scriptstyle \alpha_{X\times Y}} & & \uparrow{\scriptstyle \alpha_X} \\
H^i(X \times X) & \times & H^{2d}(X \times X) & \xrightarrow{\cup} & H^{2d+i}(X \times X) & \xrightarrow{(\mathrm{pr}_2)_*} & H^i(X) \\
{\scriptstyle \mathrm{pr}_1^*}\uparrow & & & & & & \\
H^i(X) & & & & & &
\end{array}$$

where $(\alpha_{X\times X})^{-1} i_*$ maps η_Z to the class of Z in $H^{2d}(X \times X)$.

Then (a) follows from the fact that f_2 factors as $Z \to Y \to X$, so that the correspondence Z factors through $H_{2d-i}(Y) \xrightarrow{\sim} H^i_Y(X)$. Now consider

(b) (cf. [**Ja3**, proof of Theorem 10.1]): One shows the existence of a diagram

$$\begin{array}{ccccccc} & & U' & \leftarrow & V & \xrightarrow{g_1} & U \\ & & \cap| & & \cap| & & \cap| \\ X & \leftarrow & Y' & \leftarrow & Z & \xrightarrow{f_1} & X \end{array}$$

with vertical open immersions, in which U' is affine, the composition $Z \to Y' \to X$ is f_1, $V = f_1^{-1}(U)$ so that g_1 is proper, and U is nonempty. The commutative diagram

$$\begin{array}{ccccccccccccc} & & H^i(U') & \to & H^i(V) & \xrightarrow{\alpha_V} & H_{2d-i}(V) & \xrightarrow{(g_1)_*} & H_{2d-i}(U) & \xrightarrow{\alpha_U^{-1}} & H^i(U) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^i(X) & \to & H^i(Y') & \to & H^i(Z) & \xrightarrow{\alpha_Z} & H_{2d-i}(Z) & \xrightarrow{(f_1)_*} & H_{2d-i}(X) & \xrightarrow{\alpha_X^{-1}} & H^i(X) \end{array}$$

shows that the image of the correspondence Z is zero in $H^i(X)$ if $i > 2\dim Y'$ (since then $H^i(Y') = 0$) and maps to zero in $H^i(U)$ if $i >$ *dim* $Y' =$ *dim* U' *(since then* $H^i(U') = 0$ *by weak Lefschetz).*

3.10. For the proofs of Theorems 3.5 and 3.6 we use the following four well-known facts about Chow groups.

(1) Chow groups $CH^j(X)$ can be defined for arbitrary Noetherian schemes X, as the cokernel of Quillen's divisor map

$$\bigoplus_{y \in X^{(j-1)}} \kappa(y)^\times \xrightarrow{\text{div}} \bigoplus_{x \in X^{(j)}} \mathbb{Z}, \tag{3.4}$$

$\text{div} = d_1^{j-1,-j}$ in Quillen's spectral sequence for algebraic K-theory [**Qui**, §7, Theorem 5.4], compare formula (1.1). They are contravariant for flat morphisms, since (3.4) is (loc. cit.). Explicitly, $\text{div}(f) = \sum \text{ord}_x(f)$ for $f \in \kappa(y)^\times$, where the sum is over all $x \in X^{(j)} \cap Y$, $Y = \overline{\{y\}}$, and ord_x is the order function of $\mathcal{O}_{Y,x}$: $\text{ord}_x(f) = \text{length}(\mathcal{O}_{Y,x}/(f))$ for $f \in \mathcal{O}_{Y,x}$. Moreover, for a flat morphism $f: X' \to X$ and $x \in X^{(j)}$ one has $f^*(x) = \sum e(x'|x)x'$, where the sum is over all $x' \in (X')^{(j)}$ with $f(x') = x$, and where $e(x'|x) = \text{length}(\mathcal{O}_{X',x'} \otimes \kappa(x))$ is the ramification index of $x'|x$. (For nonequidimensional schemes, these definitions differ from those in [**Fu**], since we are grading by codimension instead of dimension as in [**Fu**]. In particular, there is no contradiction to loc. cit., Example 1.7.1.)

(2) If X is algebraic over a field and $Y \subseteq X$ is a closed subscheme of pure codimension c, then there is an exact sequence

$$CH^{j-c}(Y) \to CH^j(X) \to CH^j(X - Y) \to 0,$$

induced by the obvious exact sequence

$$0 \to \bigoplus_{x \in Y^{(j-c)}} \mathbb{Z} \to \bigoplus_{x \in X^{(j)}} \mathbb{Z} \to \bigoplus_{x \in U^{(j)}} \mathbb{Z} \to 0.$$

(3) If $X = \varprojlim_i X_i$ is the projective limit of Noetherian schemes, with affine flat transition morphisms, then

$$CH^j(X) = \varinjlim_i CH^j(X_i),$$

since (3.4) has this limit property [**Qui**, §7, Theorem 5.4].

(4) If X is smooth over a field k, and if L is any field extension of k, then

$$CH^j(X) \to CH^j(X_{L]})$$

is injective up to torsion (cf. [**Bl1**, lecture 1, appendix]).

PROOF OF THEOREM 3.5. (a) If $CH^d(X)_0$ is representable, then by Proposition 1.6 and 3.10(4) there exists a finitely generated field $k_0 \subset k$ such that X has a model X_0 over k_0 and a curve $C \subset X_0$ such that

$$CH^d((X_0 - C)_L)_{\mathbb{Q}} = 0$$

for every field extension L that is finitely generated over k_0. Looking at the restriction

$$CH^d(X_0 \times X_0)_{\mathbb{Q}} \to CH^d(X_0 \times_{k_0} k_0(X_0))_{\mathbb{Q}},$$

where $k_0(X_0)$ is the function field of X_0 over k_0, we see from 3.10(2), (3) that we have a decomposition $\Delta = \Gamma_1 + \Gamma^1$ of the diagonal in $CH^d(X_0 \times X_0)_{\mathbb{Q}}$ such that Γ_1 is supported on $C \times X_0$ and Γ^1 is supported on $X_0 \times D$ for some divisor D. Since Δ acts as the identity on $H^i(X)$, Theorem 3.5(a) immediately follows from Proposition 3.9.

(b) If $\mathrm{rk}(CH^d(X)_0) = \mu$, then the above holds with C a subscheme of dimension μ instead of a curve, and the result follows from the proposition as well.

PROOF OF THEOREM 3.6. Obviously (a) follows from (b). We may assume that k is the algebraic closure of a finitely generated field. If $CH^d(X)_{\mathrm{hom}} \otimes \mathbb{Q} = 0$, then taking any closed point $x \in X$ we have

$$CH^d((X - \{x\})_{k(X)})_{\mathbb{Q}} = 0.$$

As above we conclude that $\Delta = \Gamma_0 + \Gamma^1$, where Γ_0 is supported on $\{x\} \times X$ and Γ^1 is supported on $X \times D$ for a divisor D. By the proposition, Γ_0 is zero on $H^i(X)$ for $i > 0$ and Γ^1 maps $H^i(X)$ to $N^1H^i(X)$ for all i. Since Δ is the identity, we get

$$H^1(X) = 0,$$
$$H^i(X) = N^1H^i(X) \quad \text{for all } i \geq 2.$$

Now look at $\Gamma^1 \in CH_d(X \times D)_{\mathbb{Q}} = CH^{d-1}(X \times D)_{\mathbb{Q}}$. If D is irreducible, let $k(D)$ be the function field of D and let $p\colon \mathrm{Spec}\, k(d) \to \mathrm{Spec}\, k$ be the structural map. By choosing a chain $\mathrm{Spec}\, k = Z_0 \subset Z_1 \subset \cdots \subset Z_{d-1} = D$ of integral closed subvarieties Z_i such that Z_{i-1} has codimension 1 in Z_i, we get a specialization map sp

$$CH^{d-1}(X) \underset{p^*}{\overset{\text{sp}}{\leftrightarrows}} CH^{d-1}(X \times_k k(D))$$

with $\text{sp}\, p^* = \text{id}$ (cf. [**DV**, exp III]). This specialization map is compatible with the corresponding one in the ℓ-adic cohomology via the cycle map (cf. [**Ja3**, pp. 201–202]), as indicated in the following commutative diagram.

$$\begin{array}{ccc} CH^{d-1}(X)_{\mathbb{Q}} & \underset{p^*}{\overset{\text{sp}}{\leftrightarrows}} & CH^{d-1}(X \times_k k(D))_{\mathbb{Q}} \\ cl \downarrow & & \downarrow cl \\ H^{2d-2}(\overline{X}, \mathbb{Q}_\ell(d-1)) & \underset{p^*}{\overset{\text{sp}}{\leftrightarrows}} & H^{2d-2}(X \times_k \overline{k(D)}, \mathbb{Q}_\ell(d-1)) \end{array}$$

Since p^* and sp are isomorphisms for the ℓ-adic cohomology [**Mi**, VI, 4.2 and 4.3], we have

$$cl(\alpha) - cl\, p^*\, \text{sp}(\alpha) = cl(\alpha) - p^*\, \text{sp}\, cl(\alpha) = 0$$

for every $\alpha \in CH^{d-1}(X \times k(D))_{\mathbb{Q}}$. If now cl is injective on this group, we deduce that $p^*\,\text{sp} = \text{id}$ on $CH^{d-1}(X \times k(D))_{\mathbb{Q}}$. Hence there exists a cycle Γ_1' in $CH^{d-1}(X)_{\mathbb{Q}}$ such that $\Gamma^1 - \Gamma_1' \times D$ maps to zero in $CH^{d-1}(X \times k(D))_{\mathbb{Q}}$, i.e.,

$$\Gamma^1 = \Gamma_1 + \Gamma^2 \quad \text{in } CH^d(X \times X)_{\mathbb{Q}},$$

where $\Gamma_1 = \Gamma_1' \times D$ is supported on $Y_1 \times X$ with $\dim Y_1 = 1$ and Γ^2 is supported on $X \times Y^2$ with $\operatorname{codim} Y^2 = 2$. This is also true if D is not irreducible: if $D_1, \dots, D_r$ are the irreducible components of D, then we can write

$$\Gamma^1 = \sum_{i=1}^{r} \Gamma_i^1 + \widetilde{\Gamma}^2,$$

with Γ_i^1 supported on $X \times D_i$ and $\widetilde{\Gamma}^2$ supported on $X \times \widetilde{Y}^2$ with $\operatorname{codim} \widetilde{Y}^2 = 2$, and apply the above to the Γ_i^1.

If $k = \mathbb{C}$, we can apply a similar argument to the singular cohomology, by specializing from generic to special points, or one can still work with ℓ-adic cohomology, using the fact that by the comparison isomorphism with singular cohomology, homological equivalence is the same for them. Hence the above is established in both cases, and we proceed with the proof.

By the proposition, Γ_1 maps $H^i(X)$ to zero for $i > 2$ and Γ^2 maps $H^i(X)$ to $N^2H^i(X)$. Since Γ^1 is the identity on $H^i(X)$ for $i > 0$, we see that

$$\begin{aligned} &H^3(X) = 0, \\ &H^\mu(X) = N^2 H^\mu(X) \quad \text{for } \mu \geq 4. \end{aligned}$$

Inductively, we now show: If $CH^{\nu}(X_{\Omega})_{\hom,\mathbb{Q}} = 0$ for $\nu = d - i, \dots, d$, then there are cycles of codimension d on $X \times X$

$$\Gamma_j \text{ supported on } Y_j \times X, \ \dim Y_j = j,$$
$$\Gamma^j \text{ supported on } X \times Y^j, \ \operatorname{codim} Y^j = j,$$

for $j = 0, \dots, i$ (Γ^j also for $j = i+1$) such that $\Gamma^0 = \Delta$, $\Gamma^{d+1} = 0$,

$$\Gamma^j = \Gamma_j + \Gamma^{j+1},$$

and Γ^j acts as the identity on $H^i(X)$ for $i > 2j - 2$. By looking at Γ^{j+1} on $H^{2j+1}(X)$ and $H^{2j+2}(X)$, we obtain that

$$H^{2j+1}(X) = N^{j+1}H^{2j+1}(X) = 0,$$
$$H^{2j+2}(X) = N^{j+1}H^{2j+2}(X)$$

for $j = 0, \dots, i$, and by looking at Γ^{i+1} we see that

$$H^{\mu}(X) = N^{i+1}H^{\mu}(X), \qquad \mu \geq 2i + 2;$$

hence the result.

REMARKS 3.11. (a) The motivation for Theorem 3.6 came from the following potential application, in connection with results of Faber **[Fa]**. It should be possible to extend Theorem 3.6 to Q-varieties (i.e., varieties that are locally the quotient of a smooth variety by a finite group) and, in particular, to the compactifications $\overline{\mathscr{M}}_g$ of the moduli spaces $\mathscr{M}_g$ of curves of genus g. Then one could try to use some knowledge on $CH^{\cdot}(\overline{\mathscr{M}}_g)_{\mathbb{Q}}$ to prove that $H^i(\mathscr{M}_g)$ is generated by algebraic classes for $i \ll g$ as conjectured by Mumford [**Mum3**, p. 272].

(b) A variant of Theorem 3.6 was recently proved by Esnault and Levine **[EL]**, with similar methods. They give necessary conditions for the injectivity of the cycle map into Deligne cohomology, for a smooth projective variety over $\mathbb{C}$. Also, we refer the reader to a recent paper by Paranjape **[Pa]**, in which methods similar to those of this section have been used to check other predictions of the conjectures of Bloch and Beilinson.

We conclude this section with

CONJECTURE 3.12. *The functor* $X \mapsto CH^j(X)$ *is of order* $2j$ *on* $\mathscr{V}_k$.

"PROOF". *For* $CH^j(X)_{\mathbb{Q}}$, *assuming Beilinson's formula*—Recall (cf. [**Mum2**, p. 55]) that a contravariant functor F on varieties with values in an abelian category is called of order n, if for any irreducible varieties $X_0, \dots, X_n$ the morphism

$$\sum_{i=0}^{n} p_i^*: \bigoplus_{i=0}^{n} F(X_0 \times \cdots \times \widehat{X}_i \times \cdots \times X_n) \to F(X_0 \times \cdots \times X_n)$$

is surjective, where

$$p_i\colon X_0 \times \cdots \times X_n \to X_0 \times \cdots \times \widehat{X}_i \times \cdots \times X_n$$

is the projection ($\widehat{X}_i$ indicating the omission of the factor X_i in the product). If the Künneth components of the diagonal are algebraic, then the Künneth formula for the chosen Weil cohomology implies that, with the notation just introduced, $h^n(X_0 \times \cdots \times X_n)$ is direct factor of $\bigoplus_{i=0}^n h^n(X_0 \times \cdots \times \widehat{X}_i \times \cdots \times X_n)$ via $\sum p_i^*$. Consequently, Beilinson's formula implies that

$$\mathrm{Gr}_F^\nu CH^j(X)_{\mathbb{Q}} \cong \mathrm{Ext}^\nu_{\mathcal{MM}_k}(1, h^{2j-\nu}(X)(j))$$

is of order $2j-\nu$; hence the result.

We note that trivially $CH^0(X)$ is of order 0, while $CH^1(X)$ is of order 2 by the theorem of the cube (loc. cit.). Therefore Conjecture 3.12 is sometimes called the (conjectural) theorem of the hypercube. For abelian varieties over finite fields it has been proved by Soulé [**Sou1**, 3.5] (cf. also [**Kü2**]).

4. Mixed motivic sheaves

Before we state Beilinson's conjecture in its most general form, we formulate an absolute version that already reveals a lot of the basic philosophy about mixed motives and their relation with Chow groups.

CONJECTURE 4.1 (Version 3 of Beilinson's conjecture). *There exists a rigid abelian $\mathbb{Q}$-linear tensor category $\mathcal{MM}_k$ (of mixed motives over k) and a contravariant functor*

$$R\colon \mathcal{V}_k \to D^b(\mathcal{MM}_k),$$

where $D^b(\mathcal{MM}_k)$ is the derived category formed by bounded complexes in $\mathcal{MM}_k$, such that the following hold.

(i) (*Künneth formula*) *There are functorial quasi-isomorphisms*

$$R(X) \otimes^L R(Y) \to R(X \times Y),$$

where $\otimes^L$ denotes the left derivative of $\otimes$, satisfying an obvious compatibility with the associativity and commutativity constraints of $\mathcal{MM}_k$.

(ii) (*trace map*) *If X is pure of dimension d, then there is a canonical morphism*

$$\eta_X\colon R(X) \to 1(-d)[-2d],$$

where $1(-d) = H^2(R(\mathbb{P}_k^1))^{\otimes d} \in Ob(\mathcal{MM}_k)$. If Y is pure of dimension e, and if we let pt = Spec k, *then the following diagrams commute:*

$$\begin{array}{ccc} R(X) \otimes^L R(Y) & \to & R(X \times Y) \\ {\scriptstyle \eta_X \otimes \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_{X \times Y}} \\ 1\{-d\} \otimes 1\{-e\} & \to & 1\{-d-e\} \end{array} \qquad \begin{array}{ccc} R(\mathrm{pt}) \otimes^L R(X) & \to & R(X) \\ {\scriptstyle \eta_{\mathrm{pt}} \otimes \mathrm{id}}\downarrow & & \| \\ 1 \otimes R(X) & \xrightarrow{\sim} & R(X), \end{array}$$

where $1(0) = 1$ *is the neutral object of* $\mathcal{MM}_k$, *and where we put* $1\{-d\} = 1(-d)[-2d]$.

(iii) (*Poincaré duality*) *If* X *is pure of dimension* d, *then the pairing*

$$R(X) \otimes^L R(X) \to R(X \times X) \xrightarrow{\Delta^*} R(X) \xrightarrow{\eta_X} 1(-d)[-2d]$$

induces a quasi-isomorphism

$$R(X)(d)[2d] \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R(X), 1). \tag{4.1}$$

Here $R\underline{\mathrm{Hom}}$ *is the right derivative of* $\underline{\mathrm{Hom}}$, *the internal hom in* $\mathcal{MM}_k$, *and for an object* C *in* $D^b(\mathcal{MM}_k)$ *and* $n \in \mathbb{Z}$ *we put*

$$C(n) = C \otimes 1(n),$$

where for $n \geq 0$ *we let* $1(n) = \underline{\mathrm{Hom}}(1(-n), 1)$, *the dual of* $1(-n)$.

(iv) (*cycle map*) *There are functorial isomorphisms*

$$cl = cl_X^j : CH^j(X)_{\mathbb{Q}} \to \mathrm{Hom}_{\mathcal{D}}(1, R(X)(j)[2j]),$$

where we denote $\mathcal{D} = \mathcal{D}^b(\mathcal{MM}_k)$ *for short, such that the following compatiblities are satisfied*:

(a) *via* cl, *the intersection pairing*

$$CH^i(X) \otimes CH^j(X) \to CH^{i+j}(X)$$

is compatible with the pairing induced by

$$R(X) \otimes^L R(X) \to R(X \times X) \xrightarrow{\Delta^*} R(X);$$

(b) *if* X *and* Y *are of pure dimensions* d *and* e, *respectively, and* $f: X \to Y$ *is a morphism, then via* cl, *the push-forward*

$$f_*: CH^{d+i}(X) \to CH^{e+i}(Y)$$

is compatible with the morphism

$$f_*: R(X)(d)[2d] \to R(Y)(e)[2e]$$

induced by f *via* (4.1) *and the functorality of* $R(X)$.

(v) (*relation with motivic cohomology*) *More generally, there are functorial isomorphisms for* $i, j \in \mathbb{Z}$

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(1, R(X)(j)[i]),$$

where $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = K_{2j-i}(X)^{(j)}$ *is motivic cohomology as defined via algebraic* K*-theory* [**Be1**, 2.2.1; **Sou2**; **Gray**], *such that compatibilities analogous to* (iv)(a), (b) *hold.*

(vi) (*relation with Grothendieck motives*) *The contravariant functor*

$$X \mapsto \bigoplus_i H^i(R(X)) \in Ob(\mathcal{MM}_k)$$

identifies the category $\mathcal{M}_k$ of Grothendieck motives with a full abelian tensor subcategory of $\mathcal{MM}_k$ such that

$$H^i(R(X)) = h^i(X) \in Ob(\mathcal{M}_k)$$

under this identification.

REMARKS 4.2. (a) Since $\otimes$ and $\underline{\mathrm{Hom}}$ are exact functors for a rigid abelian tensor category (cf. [**DM**, 1.16]), they derive trivially. Hence for complexes $A^\cdot, B^\cdot, C^\cdot$ in $\mathcal{MM}_k$, $A^\cdot \otimes^L B^\cdot$ is represented by $A^\cdot \otimes B^\cdot$, $R\underline{\mathrm{Hom}}(A^\cdot, B^\cdot)$ is represented by $\underline{\mathrm{Hom}}^\cdot(A^\cdot, B^\cdot)$, and we have formulae like

$$\mathrm{Hom}_{\mathscr{D}}(A^\cdot, R\underline{\mathrm{Hom}}(B^\cdot, C^\cdot)) \cong \mathrm{Hom}_{\mathscr{D}}(A^\cdot \otimes^L B^\cdot, C^\cdot).$$

In fact, $\otimes^L$ and $R\underline{\mathrm{Hom}}$ give $D^b(\mathcal{MM}_k)$ the structure of a rigid $\mathbb{Q}$-linear tensor category.

(b) By (a), every pairing $A \otimes^L B \to C$ induces a morphism $B \to R\underline{\mathrm{Hom}}(A, C)$. This explains the morphism (4.1).

(c) It follows from the axioms that $\eta_Y f_* = \eta_X$ for $f: X \to Y$. One gets an equivalent set of axioms, if instead of (iv)(b) one postulates: if X is of pure dimension d, then via cl, the homomorphism

$$(p_Y)_*: CH^j(X \times Y) \to CH^{j-d}(Y)$$

induced by the projection $p_Y: X \times Y \to Y$ is compatible with the composition

$$R(X \times Y) \overset{(i)}{\xleftarrow{\sim}} R(X) \otimes^L R(Y) \overset{\eta_X \otimes \mathrm{id}}{\to} 1(-d)[-2d] \otimes R(Y).$$

(d) The meaning of Conjcture 4.1(vi) is as follows. Let X be of pure dimension d. Then we have canonical isomorphisms

$$CH^d(X \times Y)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(R(X), R(Y)) \tag{4.2}$$

by composiing $cl^d_{X\times Y}$ with the isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{D}}(1, R(X \times Y)(d)[2d]) \\ &\quad \cong \mathrm{Hom}_{\mathscr{D}}(1, R(X)(d)[2d] \otimes^L R(Y)) \quad \text{[from (i)]} \\ &\quad \cong \mathrm{Hom}_{\mathscr{D}}(1, R\underline{\mathrm{Hom}}(R(X), 1) \otimes^L R(Y)) \quad \text{[from (iii)]} \\ &\quad \cong \mathrm{Hom}_{\mathscr{D}}(1, R\underline{\mathrm{Hom}}(R(X), R(Y))) \quad \text{[cf. [DM, (1.6.10)]]} \\ &\quad \cong \mathrm{Hom}_{\mathscr{D}}(R(X), R(Y)) \quad \text{[cf. (a)].} \end{aligned} \tag{4.3}$$

Moreover, one checks that via these isomorphisms composition of correspondences corresponds to composition of morphisms in $\mathscr{D}$. Now Conjecture 4.1(vi) means that by passing to the cohomology, (4.2) induces an *isomorphism*

$$\begin{array}{c} CH^d(X \times Y)_{\mathbb{Q}}/\sim_{\mathrm{hom}} \xrightarrow{\sim} \bigoplus_i \mathrm{Hom}_{\mathcal{MM}_k}(H^i(R(X)), H^i(R(Y))) \\ \| \\ \mathrm{Hom}_{\mathcal{M}_k}(h(X), h(Y)) \end{array} \tag{4.4}$$

such that for $Y = X$ the central idempotent projecting to $\mathrm{End}_{\mathcal{MM}_k}(H^i(R(X)))$ corresponds to π_i, the ith Künneth component of the diagonal (cf. (2.2)). It is easy to see that then the association

$$h^i(X) \mapsto H^i(R(X)) \in Ob(\mathcal{MM}_k)$$

induces a fully faithful embedding $\mathcal{M}_k \subset \mathcal{MM}_k$ as wanted. In particular, this implies that the objects $1(n)$ are the usual Tate objects considered in $\mathcal{M}_k$.

(e) The complex $R(X)$ should be thought of as the motivic analogue of the complex $R\Gamma(\overline{X}, \mathbb{Q}_\ell)$ in the derived category $D^b(G_k, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-representations of G_k computing $H^{\cdot}(\overline{X}, \mathbb{Q}_\ell)$ or of the complex $R\Gamma(X(\mathbb{C}), \mathbb{Q})$ in $D^b(\mathbb{Q}\text{-MHS})$ computing $H^{\cdot}(X(\mathbb{C}), \mathbb{Q})$ with its Hodge structure (cf. **[Be2]**). In fact, a natural complement to Conjecture 4.1 is the assumption

(f) There exist exact faithful tensor functors H_ℓ and H_B on $\mathcal{MM}_k$ as in (2.10) such that there are functorial quasi-isomorphisms

$$H_\ell R(X) \simeq R\Gamma(\overline{X}, \mathbb{Q}_\ell) \qquad (\ell \neq \operatorname{char} k),$$
$$H_B R(X) \simeq R\Gamma(X(\mathbb{C}), \mathbb{Q}) \qquad (k = \mathbb{C}),$$

compatible with the structures in Conjecture 4.1(i)–(v).

LEMMA 4.3. *If $X \mapsto R(X)$ is a functor satisfying Conjecture* 4.1(i)–(iv), (vi), *then one has a* (*noncanonical*) *quasi-isomorphism*

$$R(X) \xrightarrow{\sim} \bigoplus_{i \geq 0} h^i(X)[-i].$$

PROOF (cf. **[De1**, 1.11**]**). Let $\pi_j \in CH^d(X \times X)_{\mathbb{Q}}$ $(d = \dim X)$ also denote some lifts of the Künneth components of the diagonal $(j = 0, \dots, 2d)$. Via (4.2), each π_j can be regarded as an endomorphism of $R(X)$, and by (4.3) and the identification $H^i(R(X)) = h^i(X)$, the induced endomorphism on $H^i(R(X))$ is $\delta_{i,j} \cdot \mathrm{id}$. Since the Ext spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{MM}_k}(H^i(R(X)), H^q(R(X))) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(H^i(R(X)), R(X)[p+q])$$

is functorial with respect to the endomorphisms π_j of $R(X)$, we immediately conclude that it degenerates. Hence the edge morphism

$$\mathrm{Hom}_{\mathcal{D}}(H^i(R(X)), R(X)[i]) \to \mathrm{Hom}_{\mathcal{MM}_k}(H^i(R(X)), H^i(R(X))),$$
$$f \mapsto H^i(f)$$

is surjective. Thus, for all i there are $a_i\colon H^i(R(X))[-i] \to R(X)$ inducing the identity on the ith cohomology. Then

$$\sum a_i\colon \bigoplus_i H^i(R(X))[-i] \to R(X)$$

is a quasi-isomorphism.

Proposition 4.4. *Version 3 implies version 2 of Beilinson's conjecture.*

Proof. Let $F^{\cdot}$ with $F^{\nu}/F^{\nu+1} = E_{\infty}^{\nu, 2j-\nu}$ be the descending filtration on

$$CH^j(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[2j]) \tag{4.5}$$

given by the Ext spectral sequence

$$E_2^{p,q} = \mathrm{Hom}_{\mathscr{D}}(1, H^q(R(X)(j))[p]) \Rightarrow \mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[p+q]). \tag{4.6}$$

By Lemma 4.3 (or by the argument used in its proof) this spectral sequence degenerates, and we get

$$\mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} \cong E_2^{\nu, 2j-\nu} = \mathrm{Ext}^{\nu}_{\mathscr{MM}_k}(1, h^{2j-\nu}(X)(j)), \tag{4.7}$$

i.e., Beilinson's formula. In fact, for an abelian category $\mathscr{A}$ and objects A, B in $\mathscr{A}$ one has

$$\mathrm{Hom}_{D^b(\mathscr{A})}(A, B[m]) = \begin{cases} 0, & m < 0, \\ \mathrm{Ext}^m_{\mathscr{A}}(A, B), & m \geq 0, \end{cases} \tag{4.8}$$

where A and B are regarded as complexes concentrated in degree zero and $\mathrm{Ext}^m_{\mathscr{A}}$ denotes the Yoneda-Ext groups **[Ver**, 2.3]. Since $R(X)$ is a bounded complex, the spectral sequence converges, and we have $F^0 = CH^j(X)_{\mathbb{Q}}$ and $F^{\nu} = 0$ for $\nu \gg 0$, i.e., property (e) of version 2. The rest of the properties (a)–(c) and (d$'$) easily follows from (i)–(iv) and (vi) of version 3. Indeed, the equality $F^1 = CH^j(X)_{\mathrm{hom},\mathbb{Q}}$ is a special case of (vi): we have

$$\begin{aligned} \mathrm{Gr}_F^0 CH^j(X)_{\mathbb{Q}} = E_2^{0,2j} &= \mathrm{Hom}_{\mathscr{MM}_k}(1, h^{2j}(X)(j)) \\ &= \mathrm{Hom}_{\mathscr{M}_k}(1, h^{2j}(X)(j)) = CH^j(X)_{\mathbb{Q}}/\sim_{\mathrm{hom}}; \end{aligned}$$

cf. (4.4). For the functoriality in (b), (c), and (d$'$) note that by definition the Ext spectral sequence (4.6) is induced by the canonical filtration $\tau_{\leq}$, which is respected by all morphisms in the derived cateogry. Hence (4.6) and consequently $F^{\cdot}$ is functorial for morphisms in $\mathscr{D}$ (this was already used in the proof of 4.3). Since (4.5) is functorial, we obtain that $F^{\cdot}$ is respected by f^* for morphisms $f\colon X \to Y$. More generally, via (4.5) the action of correspondences α on $CH^{\cdot}(\ )$ corresponds to the map obtained on $\mathrm{Hom}_{\mathscr{D}}(1, R(-)(-)[-])$ by functoriality from the morphism $cl(\alpha)$ in $\mathscr{D}$ associated to α via (4.3)—this is a special case of the compatibility of (4.3) with composition. In particular, $F^{\cdot}$ is also respected by f_*, showing (c), and (4.7) is functorial for correspondences, completing (d$'$).

For 2.1(b) we note that by our definition we have

$$\begin{aligned} F^{\nu} CH^j(X)_{\mathbb{Q}} = \mathrm{Im}(&\mathrm{Hom}_{\mathscr{D}}(1, \tau_{\leq -\nu}(R(X)(j)[2j])) \\ &\to \mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[2j])). \end{aligned} \tag{4.9}$$

But

$$\tau_{\leq -r}(R(X)(i)[2i]) \otimes^L \tau_{\leq -s}(R(Y)(j)[2j])$$

maps to $\tau_{\leq -r-s}(R(X) \otimes^L R(Y)(i+j)[2i+2j])$ and hence to

$$\tau_{\leq -r-s}(R(X \times Y)(i+j)[2(i+j)])$$

under the quasi-isomorphism (i) (quasi-isomorphically, since $\otimes$ is exact). By the compatibility between intersection product and the pairing on $R(X)$ stated in (iv)(a), we get (b) of version 2.

REMARKS 4.5. (a) A more concrete description is obtained by the decomposition in 4.3. Via this isomorphism,

$$\tau_{\leq m} R(X) \simeq \bigoplus_{i=0}^{m} h^i(X)[-i].$$

Note that the decomposition is not unique whereas the filtration is. But the decomposition implies that the map in (4.9) is injective and that we have noncanonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[2j]) &\cong \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathscr{D}}(1, h^i(X)(j)[2j-i]) \\ &= \bigoplus_{i=0}^{2j} \mathrm{Ext}^{2j-i}_{\mathscr{M}\mathscr{M}_k}(1, h^i(X)(j))\end{aligned}$$

under which

$$F^\nu \mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[2j]) \cong \bigoplus_{i=0}^{2j-\nu} \mathrm{Ext}^{2j-i}_{\mathscr{M}\mathscr{M}_k}(1, h^i(X)(j)).$$

(b) Part (v) of version 3 would imply similar filtrations $F^\cdot$ on motivic cohomology $H^i_{\mathscr{M}}(X, \mathbb{Q}(j))$. We would get

($\tilde{\mathrm{a}}$) $F^0 H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = H^i_{\mathscr{M}}(X, \mathbb{Q}(j))$, $F^1 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathrm{hom}, \mathbb{Q}}$.

($\tilde{\mathrm{b}}$) $F^r H^{i_1}_{\mathscr{M}}(X, \mathbb{Q}(j_1)) \cdot F^s H^{i_2}_{\mathscr{M}}(X, \mathbb{Q}(j_2)) \subset F^{r+s} H^{i_1+i_2}_{\mathscr{M}}(X, \mathbb{Q}(j_1+j_2))$ for the product in motivic cohomology.

($\tilde{\mathrm{c}}$) F is respected by f^* and f_* for morphisms $f: X \to Y$.

($\tilde{\mathrm{d}}$) There are functorial isomorphisms

$$\mathrm{Gr}^\nu_F H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = \mathrm{Ext}^\nu_{\mathscr{M}\mathscr{M}_k}(1, h^{i-\nu}(X)(j)), \tag{4.10}$$

in particular, $\mathrm{Gr}^\nu_F H^i_{\mathscr{M}}(X, \mathbb{Q}(j))$ only depends on $h^{i-\nu}(X)$, as a functor on $\mathscr{M}_k$.

($\tilde{\mathrm{e}}$) $F^\nu H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = 0$ for $\nu \gg 0$.

Denote by $\mathscr{CHM}_k = \mathscr{M}_k^{\mathrm{rat}}$ the category of motives modulo rational equivalence; i.e., we take the whole Chow groups for the correspondences. We sometimes call the objects in $\mathscr{CHM}_k$ Chow motives.

LEMMA 4.6. *If $X \mapsto R(X)$ is a functor satisfying* 4.1(i)–(iv), (vi), *it induces a fully faithful embedding of $\mathbb{Q}$-linear tensor categories*

$$i: \mathscr{CHM}_k \hookrightarrow D^b(\mathscr{M}\mathscr{M}_k).$$

PROOF. Let $ch(X) = (X, \mathrm{id}, 0)_{\mathrm{rat}}$ be the Chow motive associated to $X \in Ob(\mathscr{V}_k)$. Then the formula (4.2) means that we have the embedding i on the full subcategory of $\mathscr{CHM}_k$ formed by all $ch(X)$, by putting $i(ch(X)) = R(X)$. By the definitions, this embedding is compatible with tensor products, and the canonical decomposition $ch(\mathbb{P}^1_k) = 1 \oplus 1(-1)$ carries over to the decomposition $R(\mathbb{P}^1_k) = 1 \oplus 1(-1)[-2]$. Hence we have to show that for a projector $p = p^2 \in \mathrm{End}(R(X))$ the kernels of p and $1-p$ exist and that the canonical morphisms induce an isomorphism

$$\mathrm{Ker}\, p \oplus \mathrm{Ker}(1-p) \xrightarrow{\sim} R(X). \tag{4.11}$$

Consider the decomposition

$$R(X) \cong \bigoplus_i h^i(X)[-i] \tag{4.12}$$

of Lemma 4.3. Recall (cf. (4.8)) that we have

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{D}}(h^i(X)[-i], h^j(X)[-j]) \\ &\quad \cong \mathrm{Hom}_{\mathscr{D}}(h^i(X), h^j(X)[i-j]) = 0 \quad \text{for } i < j.\end{aligned}$$

This shows that the kernel J of the epimorphism

$$\mathrm{End}_{\mathscr{D}}(R(X)) \twoheadrightarrow \bigoplus_i \mathrm{End}_{\mathscr{M}_k}(h^i(X))$$

is a nilpotent ideal. Hence every projector $p \in \mathrm{End}(R(X))$ can be written as

$$p = (1-\eta)p'(1-\eta)^{-1}, \tag{4.13}$$

where p' is "diagonal" with respect to the decomposition (4.12) and $\eta \in J$ (cf. Lemma 5.4). Obviously, $\mathrm{Ker}\, p'$ and $\mathrm{Ker}(1-p')$ exist, and the decomposition (4.11) holds for p', since this is true for its components in $\mathrm{End}_{\mathscr{D}}(h^i(X)[-i]) \cong \mathrm{End}_{\mathscr{M}_k}(h^i(X))$. But then (4.13) implies the claim for p.

4.7. As Deligne points out (cf. **[De4**, §3]), it might be too optimistic to hope for a description inside a category $D^b(\mathscr{MM}_k)$ as above. A weaker and more cautious conjecture would result after the following changes:

(1) Replace $D^b(\mathscr{MM}_k)$ by a general triangulated category $\mathscr{D}$ with a t-structure $(\mathscr{D}^{\geq 0}, \mathscr{D}^{\leq 0})$ in the sense of **[BBD**, 1.3].

(2) Replace $H^i(-)$ by the cohomological functors ${}^tH^i(-) = {}^t\tau_{\leq i}{}^t\tau_{\geq i}$ associated to the t-structure **[BBD**, 1.3.6].

(3) Replace $\otimes^L$ by some tensor law $\otimes$ on $\mathscr{D}$ that is compatible with the triangulation and t-structure and $R\underline{\mathrm{Hom}}$ by an internal hom associated to $\otimes$ ($\mathscr{D}$ should be rigid).

(4) $\mathscr{MM}_k$ would then be the heart $\mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$ of the t-structure, which is an abelian category (loc. cit.).

The point here is that $\mathscr{D}$ is not necessarily equivalent to $D^b(\mathscr{MM}_k)$. The analogues of Lemmas 4.3 and 4.6 would hold, and the theory of exact couples would still give a degenerate spectral sequence

$$E_2^{p,q} = \mathrm{Hom}_{\mathscr{D}}(1, {}^tH^q(R(X)(j)[p])) \Rightarrow \mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[p+q]).$$

This would still give a filtration on the limit terms $\mathrm{Hom}_{\mathscr{D}}(1, R(X)(j)[i]) \cong H^i_{\mathscr{M}}(X, \mathbb{Q}(j))$ with all the properties in version 1 of Beilinson's conjecture, but for the successive quotients we would get

$$\mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} = \mathrm{Hom}_{\mathscr{D}}(1, h^{2j-\nu}(X)(j)[\nu]),$$

which is not necessarily isomorphic to

$$\mathrm{Ext}^{\nu}_{\mathscr{MM}_k}(1, h^{2j-\nu}(X)(j)) = \mathrm{Hom}_{D^b(\mathscr{MM}_k)}(1, h^{2j-\nu}(X)(j)[\nu])$$

(cf. [**BBD**, 3.1.6]).

This caveat is incorporated in the following broadest version of Beilinson's conjecture. Since it mainly serves as a philosophical background for more precise conjectures, and it would be somewhat difficult to express all data and compatibilities precisely, we only state it in a vague form and then add some discussion.

CONJECTURE 4.8 (Version 4 of Beilinson's conjecture; cf. [**Be3**, 5.10]). *For every algebraic k-scheme X there is a triangulated $\mathbb{Q}$-linear tensor category $\mathscr{DM}(X)$ with a t-structure such that the following hold.*

(1) *There are the usual six functors*

$$f_*, f_!, f^*, f^!, \otimes, \underline{\mathrm{Hom}}$$

between these categories, i.e.,

(a) *On each $\mathscr{DM}(X)$ there is a tensor law $\otimes$ and an associated internal hom $\underline{\mathrm{Hom}}$;*

(b) *If $f: X \to Y$ is a morphism, there are functor*

$$\mathscr{DM}(X) \begin{array}{c} \xrightarrow{f_*} \\ \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xleftarrow{f^!} \end{array} \mathscr{DM}(Y)$$

satisfying the usual properties of Grothendieck-Verdier duality theory; e.g., f^* is left adjoint to f_*, $f_!$ is left adjoint to $f^!$, there is a projection formula for f_* and f^*, and one for $f^!$ and $f_!$, the operation $f \mapsto f_*$ is functorial, and so forth.

(2) *Let $\mathscr{M}(X)$ (the category of mixed motivic sheaves on X) be the abelian $\mathbb{Q}$-linear category which is the heart of the t-structure on $\mathscr{DM}(X)$. Then there is a weight filtration on $\mathscr{M}(X)$; i.e., the objects M in $\mathscr{M}(X)$ have functorially associated increasing "weight" filtrations $W.M$ such that $M \mapsto$*

W_mM is an exact functor for every $m \in \mathbb{Z}$. In addition, the objects $\mathrm{Gr}_m^W M$ are semisimple.

(3) *There are canonical ("Tate") objects $\mathbb{Q}_{\mathcal{M}}(i) = \mathbb{Q}_{\mathcal{M},X}(i)$ in $\mathcal{DM}(X)$, for $i \in \mathbb{Z}$, together with compatible isomorphisms $\mathbb{Q}_{\mathcal{M}}(i) \otimes \mathbb{Q}_{\mathcal{M}}(j) \xrightarrow{\sim} \mathbb{Q}_{\mathcal{M}}(i+j)$.*

(4) *The subcategory of semisimple objects in $\mathcal{M}(\operatorname{Spec} k)$ is equivalent to the category $\mathcal{M}_k$ of Grothendieck motives over k. Under this equivalence, $\mathbb{Q}_{\mathcal{M},\operatorname{Spec} k}(j)$ corresponds to the Tate object $1(j)$, and for a smooth projective variety $a: X \to \operatorname{Spec} k$ the object $H^i(a_*\mathbb{Q}_{\mathcal{M},X})$ corresponds to $h^i(X) \in Ob(\mathcal{M}_k)$ (here $H^i = {}^tH^i$ is the* ith *cohomological functor associated to the t-structure on $\mathcal{DM}(\operatorname{Spec} k)$).*

(5) *There are canonical homomorphisms*

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to \operatorname{Hom}_{\mathcal{DM}(X)}(\mathbb{Q}_{\mathcal{M}}, \mathbb{Q}_{\mathcal{M}}(j)[i])$$

that are isomorphisms for regular X.

(6) *There are exact ("realization") functors*

$$r_\ell : \mathcal{DM}(X) \to D^b_m(X_{\text{ét}}, \mathbb{Q}_\ell), \qquad \ell \neq \operatorname{char}(k),$$
$$r_B : \mathcal{DM}(X) \to D^b(\mathcal{MH}(X)), \qquad k = \mathbb{C},$$

where $D^b_m(X_{\text{ét}}, \mathbb{Q}_\ell)$ is the derived category of bounded complexes of étale $\mathbb{Q}_\ell$-sheaves on X with mixed constructible cohomology [**BBD**, 5.15] *and $\mathcal{MH}(X)$ is the category of mixed Hodge modules on X* [**SaM1**, 2.17]. *These functors are compatible with the above structures and map $\mathcal{M}(X)$ to the categories $\mathcal{MP}(X_{\text{ét}}, \mathbb{Q}_\ell)$ of mixed perverse étale $\mathbb{Q}_\ell$-sheaves on X* [**BBD**, 5.1.7] *and to $\mathcal{MH}(X)$, respectively. Furthermore, they are exact and faithful on $\mathcal{M}(X)$.*

4.9. Let us add some explanation. The existence of the homomorphisms in (5) should follow from the data in (1) and (3), perhaps together with some extra assumptions. Namely, as for the ℓ-adic cohomology, the "duality theory" of (1) should imply that the associations

$$(Z \overset{i}{\hookrightarrow} X) \mapsto H^i_Z(X, j) = \operatorname{Hom}_{\mathcal{DM}(X)}(\mathbb{Q}_{\mathcal{M}}, i_* i^! \mathbb{Q}_{\mathcal{M}}(j)[i]),$$
$$(X \xrightarrow{f} \operatorname{Spec} k) \mapsto H_a(X, b) = \operatorname{Hom}_{\mathcal{DM}(X)}(\mathbb{Q}_{\mathcal{M}}, f^! \mathbb{Q}_{\mathcal{M}}(-a)[-b])$$

define a twisted Poincaré duality theory in the sense of Bloch and Ogus [**BO**, §1]. Some additional properties (as in [**Beil**, 2.3] or in [**Gi**, 1.1 and 1.2]) should give canonical Chern character maps

$$K_{2j-i}(X) \to H^i(X, j) \tag{4.14}$$

inducing the maps in (5).

As for (6), the categories $D^b_m(X, \mathbb{Q}_\ell)$ and $\mathcal{MP}(X, \mathbb{Q}_\ell)$ have only been defined under some finiteness conditions in [**BBD**], but a more sophisticated theory of $\mathbb{Q}_\ell$-sheaves (cf., e.g., [**Ek**]) should allow defining them in general. Concerning the analogy between r_ℓ and r_B, we note that Beilinson has proved remarkable equivalences of categories (cf. [**Be4**, 1.3]),

$$(4.15) \qquad D^b(\mathscr{MP}(X_{\text{ét}}, \mathbb{Q}_\ell)) \overset{\text{real}}{\underset{\sim}{\to}} D^b_m(X_{\text{ét}}, \mathbb{Q}_\ell), \qquad \ell \neq \operatorname{char} k,$$

$$(4.16) \qquad D^b(\mathscr{P}(X, \mathbb{Q})) \overset{\text{real}}{\underset{\sim}{\to}} D^b_c(X(\mathbb{Q}), \mathbb{C}), \qquad k = \mathbb{C},$$

where $D^b_c(X(\mathbb{C}), \mathbb{Q})$ is the derived category of bounded complexes of $\mathbb{Q}$-sheaves on $X(\mathbb{C})$ (with the strong topology) with algebraically constructible cohomology and $\mathscr{P}(X, \mathbb{Q}) \subset D^b_c(X(\mathbb{C}), \mathbb{Q})$ is the category of algebraically constructible perverse $\mathbb{Q}$-sheaves on $X(\mathbb{C})$. On the other hand, one has a forgetful functor [**SaM1**]

$$\mathscr{MH}(X) \to \mathscr{P}(X, \mathbb{Q}).$$

The compatibility of r_ℓ or r_B with the structures in (1)–(5) means, e.g., the following: r_ℓ "commutes" with the six functors which also exist on $D^b_c(-, \mathbb{Q}_\ell)$. For example, one should have $r_\ell f_* = f_* r_\ell$, where we also write f_* for what is often denoted $Rf_*: D^b_m(X_{\text{ét}}, \mathbb{Q}_\ell) \to D^b_m(Y_{\text{ét}}, \mathbb{Q}_\ell)$. Furthermore, $\mathbb{Q}_{\mathscr{M}}(i)$ should map to the sheaf $\mathbb{Q}_\ell(i)$ on X, and the weight filtration should map to the weight filtration in $\mathscr{MP}(X_{\text{ét}}, \mathbb{Q}_\ell)$ via r_ℓ. Similar facts should hold for r_B. Note that r_ℓ and r_B would induce maps

$$\operatorname{Hom}_{D^b_m(X_{\text{ét}}, \mathbb{Q}_\ell)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(j)[i]) = H^i(X_{\text{ét}}, \mathbb{Q}_\ell(j))$$

$$\nearrow$$

$$\operatorname{Hom}_{\mathscr{DM}(X)}(\mathbb{Q}_{\mathscr{M}}, \mathbb{Q}_{\mathscr{M}}(j)[i])$$

$$\searrow$$

$$\operatorname{Hom}_{D^b(\mathscr{MH}(X))}(\mathbb{Q}, \mathbb{Q}(j)[i]) = H^i_{\mathscr{H}}(X, \mathbb{Q}(j))$$

into continuous étale and absolute Hodge cohomology (cf. [**Be2**]; $H^i_{\mathscr{H}}(X, \mathbb{Q}(j))$ coincides with Deligne cohomology $H^i_{\mathscr{D}}(X, \mathbb{Q}(j))$ if $i \leq 2j$), respectively. The compositions with the maps in (5) should be the usual Chern character maps.

4.10. According to Beilinson's philosophy, the category $\mathscr{M}(X)$ of mixed motivic sheaves should correspond to the categories of mixed perverse sheaves in the realizations. For $X = \operatorname{Spec} k$, one has

(4.18)

$$\begin{aligned} \mathscr{MP}((\operatorname{Spec} k)_{\text{ét}}, \mathbb{Q}_\ell) &= \text{mixed constructible } \mathbb{Q}_\ell\text{-sheaves on } (\operatorname{Spec} k)_{\text{ét}} \\ &= \text{mixed finite-dimensional } \mathbb{Q}_\ell\text{-representations of } G_k \\ \mathscr{MH}(\operatorname{Spec} \mathbb{C}) &= \text{MHS}. \end{aligned}$$

In general, $\mathscr{MP}(X_{\text{ét}}, \mathbb{Q}_\ell)$ is different from the category of mixed constructible $\mathbb{Q}_\ell$-sheaves on $X_{\text{ét}}$—and is better behaved than the latter; for example, it is an artinian category, and one has the decomposition theorem [**BBD**, 4.3.1 and 6.2.5].

4.11. One would deduce version 3 of Beilinson's conjecture from this version 4 as follows. Put

$$\mathcal{MM}_k = \mathcal{M}(\operatorname{Spec} k) \tag{4.19}$$

(cf. the analogues (4.18)) and

$$R(X) = a_*\mathbb{Q}_{\mathcal{M},X} \in Ob(\mathcal{DM}(\operatorname{Spec} k)) \tag{4.20}$$

for $a\colon X \to \operatorname{Spec} k$ smooth and projective (recall that this corresponds to $Ra_*\mathbb{Q}_{\mathcal{M},X}$ in more traditional notation). The analogy with the realizations suggests that we should have

$$\mathbb{Q}_{\mathcal{M},X} = a^*\mathbb{Q}_{\mathcal{M},\operatorname{Spec} k}. \tag{4.21}$$

This would imply isomorphisms

$$\begin{aligned} H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) &\cong \operatorname{Hom}_{\mathcal{DM}(X)}(a^*\mathbb{Q}_{\mathcal{M}}, a^*\mathbb{Q}_{\mathcal{M}}(j)[i]) \\ &\cong \operatorname{Hom}_{\mathcal{DM}(\operatorname{Spec} k)}(\mathbb{Q}_{\mathcal{M}}, a_*a^*\mathbb{Q}_{\mathcal{M}}(j)[i]) \\ &= \operatorname{Hom}_{\mathcal{DM}(\operatorname{Spec} k)}(1, R(X)(j)[i]), \end{aligned} \tag{4.22}$$

by (5), adjunction, and (4), hence the isomorphisms in (iv) and (v) of version 3, as modified in 4.7. Their compatibility with product and push-forward (as in 4.7(iv)(a), (b)) would be a consequence of a Riemann-Roch theorem for the maps (4.14) as in [**Bei1**, 2.3.3] or [**Gi**, 4.1], if these are constructed as indicated above.

As for properties (i)–(iii) of version 3, we note that the corresponding statements hold for the ℓ-adic analogues $Ra_*\mathbb{Q}_\ell$ of the $R(X)$. Hence the properties for $R(X)$ can be seen as a consequence of the principle that the motivic theory "behaves the same way" as the theory of mixed ℓ-adic sheaves [**Bei3**, 5.10, A]. We can deduce Conjecture 4.1(i)–(iii) in a more precise way from (6), viz., the assumption that r_ℓ is faithful on $\mathcal{M}(X)$. This assumption has the following two consequences. First, an object C in $\mathcal{M}(X)$ is zero, if $r_\ell(C)$ is (look at the identity of C). Second, a morphism φ in $\mathcal{DM}(X)$ is an isomorphism, if $r_\ell(\varphi)$ is (apply the first property to the cohomology of C in an exact triangle $A \xrightarrow{\varphi} B \to C \to A[1]$ in $\mathcal{DM}(X)$).

Conjecture 4.1(i) can now be obtained as follows. The Cartesian diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_1 \downarrow & & \downarrow b \\ X & \xrightarrow{a} & \operatorname{Spec} k \end{array}$$

induces a morphism in $\mathcal{DM}(\operatorname{Spec} k)$

$$\begin{aligned} a_*\mathbb{Q}_{\mathcal{M}} \otimes b_*\mathbb{Q}_{\mathcal{M}} &\xrightarrow{\sim} a_*(\mathbb{Q}_{\mathcal{M}} \otimes a^*b_*\mathbb{Q}_{\mathcal{M}}) && \text{[projection formula]} \\ &\to a_*(\mathbb{Q}_{\mathcal{M}} \otimes (p_1)_*p_2^*\mathbb{Q}_{\mathcal{M}}) && \text{[base change]} \\ &\xrightarrow{\sim} (ap_1)_*(p_1^*\mathbb{Q}_{\mathcal{M}} \otimes p_2^*\mathbb{Q}_{\mathcal{M}}) && \text{[projection formula]} \\ &\xrightarrow{\sim} (a_{X\times Y})_*\mathbb{Q}_{\mathcal{M}} && [a_{X\times Y} = ap_1], \end{aligned}$$

since $p_1^*\mathbb{Q}_{\mathscr{M},X} \otimes p_2^*\mathbb{Q}_{\mathscr{M},Y} \cong \mathbb{Q}_{\mathscr{M},X\times Y} \otimes \mathbb{Q}_{\mathscr{M},X\times Y} \cong \mathbb{Q}_{\mathscr{M},X\times Y}$ by (4.21) and (3). Hence we have a morphism

$$R(X) \otimes R(Y) \to R(X \times Y),$$

which is a quasi isomorphism for smooth and proper X and Y, since it is so after applying r_ℓ, by the ℓ-adic Künneth formula (cf. [**Mi**, VI, p. 261]). Similarly, we could have noted that the base change morphism

$$a^* b_* \mathbb{Q}_{\mathscr{M}} \to (p_1)_*(p_2)^* \mathbb{Q}_{\mathscr{M}},$$

obtained from the usual duality formalism, is a quasi-isomorphism, since its ℓ-adic image is.

Concerning Conjecture 4.1(ii), the trace map $\eta_X \colon R(X) \to 1(-d)[-2d]$ for X smooth projective of dimension d is obtained from the fact that

$$H^i(a_*\mathbb{Q}_{\mathscr{M}}) \cong h^i(X) = \begin{cases} 0, & i > 2d, \\ 1(-d), & i = 2d. \end{cases}$$

The quasi-isomorphy in Conjecture 4.1(iii) follows as before, since it holds for the image under r_ℓ by the ℓ-adic Poincaré duality. Finally, property (vi) of version 3 is a consequence of property (4) of version 4.

Remarks 4.12. (a) Let X be smooth and projective. The assumption on r_ℓ does not imply that r_ℓ is faithful on the triangulated category $\mathscr{D}\mathscr{M}(X)$, and this cannot be expected. In fact, the related map (cf. (4.17))

$$H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) \to H^i(X_{\text{ét}}, \mathbb{Q}_\ell(j))$$

will not in general be injective, as remarked after Lemma 2.7. The same is true for r_B, for example, by Mumford's counterexample recalled in §1. However, one may hope that r_ℓ is faithful on $\mathscr{D}\mathscr{M}(X)$ if k is a finitely generated field; cf. Question 2.8.

(b) Beilinson hopes that the motivic theory of Conjecture 4.8 exists for arbitrary schemes (instead of just schemes of finite type over a field) and more general coefficients (instead of $\mathbb{Q}$-coefficients as above); cf. [**Bei3**, 5.10].

(c) For a finitely generated field k of Kronecker dimension r (= Krull dimension of an integral $\mathbb{Z}$-algebra of finite type with field of fractions k) one expects that the cohomological dimension of $\mathscr{M}\mathscr{M}_k = \mathscr{M}(\operatorname{Spec} k)$ is r, i.e., that $\operatorname{Ext}^\nu_{\mathscr{M}\mathscr{M}_k} = 0$ for $\nu > r$. We discuss two examples.

(i) $k = \mathbb{F}_q$, *a finite field* $(r = 0)$. Here one expects that $\mathscr{M}\mathscr{M}_k = \mathscr{M}_k$, i.e., that every mixed motive is a direct sum of pure motives, and for a smooth variety X/k one expects that the map

(4.23) $$H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) \to H^i(\overline{X}, \mathbb{Q}_\ell(j)) \qquad (\ell \neq \operatorname{char} k)$$

is injective. In fact, if $\operatorname{Ext}^\nu_{\mathscr{M}\mathscr{M}_k} = 0$ for $\nu > 0$, then

$$\begin{aligned} &\operatorname{Hom}_{D^b(\mathscr{M}\mathscr{M}_k)}(1, R(X)(j)[i]) \\ &\quad \cong \operatorname{Hom}_{\mathscr{M}\mathscr{M}_k}(1, H^i(R(X)(j))) \\ &\quad \overset{(6)}{\hookrightarrow} \operatorname{Hom}_{G_k}(\mathbb{Q}_\ell, H^i(\overline{X}, \mathbb{Q}_\ell(j))). \end{aligned}$$

For further discussion of the map (4.23) in this case, cf. [**Ja3**, §12].

(ii) k = *a number field or a global function field* $(r = 1)$. Here one expects $D^b(\mathcal{MM}_k) \xrightarrow{\sim} \operatorname{grad}\mathcal{MM}_k$, where $\operatorname{grad}\mathcal{MM}_k$ is the category of $\mathbb{Z}$-graded objects $\bigoplus A^j$ in $\mathcal{MM}_k$ for which $A^j = 0$ for almost all $j \in \mathbb{Z}$. For a smooth projective variety X/k one expects a short exact sequence

$$0 \to \operatorname{Ext}^1_{\mathcal{MM}_k}(1, h^{i-1}(X)(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to \operatorname{Hom}_{\mathcal{M}_k}(1, h^i(X)(j)) \to 0.$$

This is related to the conjecture of Beilinson and Bloch that the Abel-Jacobi maps are injective on $CH^{\cdot}(X)_{\mathrm{hom}} \otimes \mathbb{Q}$. For further discussion of the expected properties of the realization maps (4.17) in this case we refer the readers to [**Nek**; **BK**, **FPR**, **Ja3**]. Here we only point out the following two consequences. If $H^i_{\mathcal{M}}(X, \mathbb{Q}(j))_0$ denotes the kernel of the realization maps into $H^i(\overline{X}, \mathbb{Q}_\ell(j))$ $(\ell \neq \operatorname{char} k)$ or $H^i(X \times_\sigma \mathbb{C}, \mathbb{Q}(j))$ $(\operatorname{char} k = 0,\ \sigma\colon k \hookrightarrow \mathbb{C}$ some embedding), then

$$J = H^{\cdot}_{\mathcal{M}}(X, \mathbb{Q}(\cdot))_0$$

should be an ideal of square zero (cf. [**Bei2**, 8.5.1; **Bei3**, 5.7]). Also the hard Lefschetz for $h^{\cdot}(X)$ should imply a hard Lefschetz for $H^{\cdot}_{\mathcal{M}}(X, \mathbb{Q}(\cdot))_0$, i.e., if $L \in CH^1(X)$ is the class of a hyperplane section, then the map

$$L^{d-i+1}\colon H^i_{\mathcal{M}}(X, \mathbb{Q}(j))_0 \to H^{2d-i+2}_{\mathcal{M}}(X, \mathbb{Q}(j+d-i+1))_0$$

should be an isomorphism for $i - 1 \leq d$ (cf. [**Bei3**, Conjecture 5.3]).

(d) One may formulate still another version of Beilinson's conjecture (more general than version 3, but still absolute and not "relative" as version 4), by postulating the existence of a suitable category $\mathcal{MM}_k$ of mixed motives over k, and of complexes

$$\underline{R\Gamma}_Z(X, j),\ \underline{R\Gamma}'(X, b) \qquad (j, b \in \mathbb{Z})$$

in $D^b(\mathcal{MM}_k)$ for arbitrary algebraic k-schemes X and closed subschemes $Z \subset X$ forming something like a "derived" twisted Poincaré duality theory (cf. [**J3**, 11.3]). One would deduce this from version 4 by putting

$$\underline{R\Gamma}_Z(X, j) = (a_Z)_* i^! \mathbb{Q}_{\mathcal{M}, X}(j),$$
$$\underline{R\Gamma}'(X, b) = (a_X)_* a_X^! \mathbb{Q}_{\mathcal{M}, \operatorname{Spec} k}(-b),$$

where $a_X\colon X \to \operatorname{Spec} k$ is the structural morphism and $i\colon Z \hookrightarrow X$ is the closed immersion. The functors

$$(Z \subset X) \mapsto H^i_{M,Z}(X, j) = H^i(\underline{R\Gamma}_Z(X, j)),$$
$$X \mapsto H^M_a(X, b) = H^{-a}(\underline{R\Gamma}'(X, b))$$

would from a twisted Poincaré duality theory with values in $\mathcal{MM}_k$ is indicated in 2.5.

Depending on the point of view, the following lemma could be regarded as an "application" or as a test for the philosophy of motivic sheaves (this property of motivic cohomology was independently conjectured by Beilinson and Soulé [**Bel**, 2.2.2; **Sou2**, 2.9].

LEMMA 4.13. *It is a consequence of the formalism of (mixed) motivic sheaves that*

$$H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = 0 \quad \text{for } i < 0,$$

if X is smooth over a field k.

PROOF. If $a: X \to \operatorname{Spec} k$ is the structural morphism, we would have

$$H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = \operatorname{Hom}_{D^b(\mathscr{M}\mathscr{M}_k)}(1, a_*\mathbb{Q}_{\mathscr{M}}(j)[i])$$

as in (4.22). Now $a_*\mathbb{Q}_{\mathscr{M}}$ is a complex that is concentrated in degrees ≥ 0; this follows, for example, from the property that

$$r_\ell H^\nu(a_*\mathbb{Q}_{\mathscr{M}}(j)) = H^\nu(r_\ell a_*\mathbb{Q}_{\mathscr{M}}(j)) = H^\nu(\overline{X}, \mathbb{Q}_\ell(j)) = 0 \quad \text{for } \nu > 0,$$

and the faithfulness of r_ℓ on $\mathscr{M}\mathscr{M}_k$. Hence the above group vanishes for $i < 0$ (1 is placed in degree zero), by standard properties of the derived category.

If X is a smooth variety, then $K_m(X) = 0$ for $m < 0$, and hence trivially $H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = K_{2j-i}(X)^{(j)} = 0$ for $i > 2j$. It is reassuring to know that this is also a consequence of the motivic picture.

LEMMA 4.14. *The formalism of motivic sheaves implies that*

$$\operatorname{Hom}_{D^b(\mathscr{M}\mathscr{M}_k)}(1, a_*\mathbb{Q}_{\mathscr{M}}(j)[i]) = 0 \quad \text{for } i > 2j,$$

if $a: X \to \operatorname{Spec} k$ is smooth.

The proof uses the following general fact.

LEMMA 4.15. *Let $\mathscr{C}$ be an abelian category with a weight filtration such that the pure objects are simisimple. Then*

$$\operatorname{Ext}^\nu_{\mathscr{C}}(N, M) = 0$$

for objects M, N in $\mathscr{C}$, provided

$$\min\{\textit{weights of } M\} > \max\{\textit{weights of } N\} - \nu.$$

PROOF. By induction on ν. For every ν, an obvious devissage via the weight filtrations of M and N reduces the question to the case that M and N are pure of weights m and n, respectively. Then $\operatorname{Hom}_{\mathscr{M}}(N, M) = 0$ if $m \neq n$. For $\nu = 1$ consider an extension

$$0 \to M \to E \to N \to 0$$

and the subextension

$$0 \to W_n M \to W_n E \to N \to 0.$$

Since $m > n - 1$, we have either $m > n$ or $m = n$. In the first case $W_n M = 0$, and $W_n E$ provides a splitting of the given extension. In the second case E is pure of weight n, and the sequence splits by the assumed semisimplicity. For $\nu > 1$, every $\chi \in \operatorname{Ext}^\nu_{\mathscr{C}}(N, M)$ is the Yoneda product of $\chi_1 \in \operatorname{Ext}^{\nu-1}_{\mathscr{C}}(N, X)$ and $\chi_2 \in \operatorname{Ext}^1_{\mathscr{C}}(X, M)$ for some object X in $\mathscr{C}$. If χ_1 and χ_2 are nonzero, then

$$m + 1 \leq \text{weights of } X \leq n - \nu + 1$$

by induction, and hence the claim for ν.

PROOF OF 4.14. By the Ext spectral sequence, it suffices to show that

$$\operatorname{Ext}^p_{\mathscr{MM}_k}(1, H^q(a_*\mathbb{Q}_{\mathscr{M}}(j))) = 0 \quad \text{for } p + q > 2j.$$

However, the weights of $H^q(a_*\mathbb{Q}_{\mathscr{M}}(j))$ are $\geq q - 2j$, since this holds for $h^q(\overline{X}, \mathbb{Q}_\ell(j)) = r_\ell H^q(a_*\mathbb{Q}_{\mathscr{M}}(j))$ by results of Deligne **[De3**, 3.3.5]. (In the case of smooth projective X we even know that $H^q(a_*\mathscr{Q}_{\mathscr{M}}(j)) = h^q(X)(j)$ is pure of weight $q - 2j$.) Hence the vanishing follows from Lemma 4.15.

5. Murre's filtration

The following concrete proposal for a filtration on Chow groups was made by Murre **[Mur2**, 1.4], based on his investigations on decompositions of Chow motives (cf. **[Mur1]**). Let X be an irreducible smooth projective variety over a field k, and assume that the Künneth components π_j^{hom} of the diagonal are algebraic (for some fixed Weil cohomology theory), for $j = 0, \dots, 2d$, where $d = \dim(X)$. Then $\{\pi_j^{\text{hom}}\}$ forms an orthogonal set of idempotents in

$$CH^d(X \times X)_{\mathbb{Q}}/\sim_{\text{hom}},$$

with $\sum_{j=0}^{2d} \pi_j^{\text{hom}} =$ class of the diagonal $\Delta = \text{id}$ as a correspondence.

CONJECTURE 5.1 (Murre). (A) *The* π_j^{hom} *lift to an orthogonal set of idempotents* $\{\pi_j\}$ *in* $CH^d(X \times X)_{\mathbb{Q}}$, *with* $\sum_{j=0}^{2d} \pi_j = \Delta = \text{id}$.

(B) *The correspondences* $\pi_{2j+1}, \dots, \pi_{2d}$ *act as zero on* $CH^j(X)_{\mathbb{Q}}$.

(C) *Let* $F^\nu CH^j(X)_{\mathbb{Q}} = \operatorname{Ker}\pi_{2j} \cap \operatorname{Ker}\pi_{2j-1} \cap \cdots \cap \operatorname{Ker}\pi_{2j-\nu+1}$. *Then* $F^\cdot$ *is independent of the choice of the* π_j.

(D) $F^1 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\text{hom},\mathbb{Q}}$.

In fact, Murre stated the following stronger form:

STRONG CONJECTURE 5.1. *This is the same as Conjecture* 5.1, *except that* (B) *is replaced by*

(*strong* B) *The correspondences* $\pi_0, \dots, \pi_{j-1}$ *and* $\pi_{2j+1}, \dots, \pi_{2d}$ *act as zero on* $CH^j(X)_{\mathbb{Q}}$.

The main result of this section is

THEOREM 5.2. *Murre's conjecture is equivalent to version* 1 *of Beilinson's conjecture, and the filtrations coincide; i.e., if one filtration exists then the other does and they agree. The same is true for the strong forms.*

Before we start with the proof, we explain the motivic background. If we start from version 3 of Beilinson's conjecture, we have a ring isomorphism

$$CH^d(X \times X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{End}_{D^b(\mathcal{MM}_k)}(R(X))$$

(cf. (4.2)) and a decomposition in $D^b(\mathcal{MM}_k)$

$$R(X) = \bigoplus_{i \geq 0} h^i(X)[-i]$$

(cf. Lemma 4.3). Any such decomposition gives an orthogonal set of idempotents $\{\pi_j\}$ as in (A), by letting $\pi_j \colon R(X) \to h^j(X)[-j] \to R(X)$ be the projection on the jth factor. Note that the image of π_j in

$$CH^d(X \times X)_{\mathbb{Q}}/\sim_{\mathrm{hom}} \xrightarrow{\sim} \bigoplus_{i \geq 0} \mathrm{End}_{\mathcal{M}_k}(h^i(X))$$

is in fact π_j^{hom}. The explicit description of Beilinson's filtration in Remark 4.5(a), namely,

$$F^{\nu} CH^j(X)_{\mathbb{Q}} = \bigoplus_{i=0}^{2j-\nu} \mathrm{Hom}_{D^b(\mathcal{MM}_k)}(1, h^i(X)(j)[2j-i]),$$

immediately implies the properties (B)–(D). As a motivation for the following considerations, we recall that under the full embedding $\mathcal{CHM}_k \hookrightarrow D^b(\mathcal{MM}_k)$ we have a correspondence

$$\text{Chow motive } M_i = (X, \pi_i, 0)_{\mathrm{rat}} \leftrightarrow h^i(X)[-i] \in Ob(D^b(\mathcal{MM}_k))$$

and that

$$(5.1) \quad \mathrm{Hom}_{D^b(\mathcal{MM}_k)}(h^i(X)[-i], h^{i'}(X)[-i']) = \begin{cases} 0 & \text{for } i < i', \\ \mathrm{End}_{\mathcal{M}_k}(h^i(X)) & \text{for } i = i'. \end{cases}$$

We now give a proof of Theorem 5.2 that does not mention mixed motives. Let X be as above.

PROPOSITION 5.3. *Assume*

(N) $\quad CH^d(X \times X)_{\mathrm{hom},\mathbb{Q}}$ is a nilpotent ideal of $CH^d(X \times X)_{\mathbb{Q}}$.

Then Conjecture 5.1(A) *holds.*

This follows from

LEMMA 5.4. *Let A be a ring with unit (not necessarily commutative), and let $I \subset A$ be a nilpotent two-sided ideal. Then every set $\{e_1, \dots, e_m\}$ of pairwise orthogonal idempotents (i.e., $e_i e_j = \delta_{i,j} e_i$) in A/I can be lifted to a set $\{\pi_1, \dots, \pi_m\}$ of pairwise orthogonal idempotents in A. If $\{\pi'_1, \dots, \pi'_m\}$ is another such lifting, then there is an element $\eta \in I$ such that $\pi'_i =$*

$(1-\eta)\pi_i(1-\eta)^{-1}$ *for* $i = 1, \ldots, m$. *If* $\sum_{i=1}^m e_i = 1$, *then necessarily* $\sum_{i=1}^m \pi_i = 1$.

PROOF. By induction on the index of nilpotency we may assume $I^2 = 0$. The proof proceeds by induction on m. The case $m = 1$ is probably well known; we found it as Beilinson's lemma in [**Mur1**, 7.3]. For $m > 1$ assume that we have already an orthogonal lifting $\{\pi_1, \ldots, \pi_m\}$ of $\{e_1, \ldots, e_m\}$ and a lifting $\pi'_{m+1} = (\pi'_{m+1})^2$ of e_{m+1} which is orthogonal to $\pi_1, \ldots, \pi_n$ for some $0 \leq n < m$ (new induction on n starting with the empty case $n = 0$). Let

$$\pi_{n+1}\pi'_{m+1} = \varepsilon \in I.$$

Then $\varepsilon = \pi_{n+1}\varepsilon\pi'_{m+1}$ and

$$\begin{aligned}\pi_{n+1}(1-\varepsilon)\pi'_{m+1}(1+\varepsilon) &= \pi_{n+1}(\pi'_{m+1} + \pi'_{m+1}\varepsilon - \varepsilon\pi'_{m+1}) \\ &= \varepsilon + \varepsilon^2 - \pi_{n+1}\varepsilon\pi'_{m+1} = 0.\end{aligned}$$

Moreover, $\pi_i\varepsilon = \pi_i\pi_{n+1}\pi'_{m+1} = 0 = \pi_{n+1}\pi'_{m+1}\pi_i = \varepsilon\pi_i$ for $i \leq n$, so that

$$\pi''_{m+1} = (1-\varepsilon)\pi'_{m+1}(1+\varepsilon)$$

is an idempotent lifting e_{m+1}, which is orthogonal to $\{\pi_1, \ldots, \pi_n\}$ and in addition satisfies $\pi_{n+1}\pi''_{m+1} = 0$. Now let

$$\pi''_{m+1}\pi_{n+1} = \varepsilon' \in I.$$

Then $\varepsilon' = \pi''_{m+1}\varepsilon'\pi_{n+1}$, and

$$\begin{aligned}\pi_i\varepsilon' &= 0 = \varepsilon'\pi_i \quad \text{for } i \leq n, \\ \pi_{n+1}\pi''_{m+1} &= 0, \\ \pi_{n+1}\varepsilon' &= 0.\end{aligned}$$

As above we deduce that $\pi_{m+1} = (1+\varepsilon')\pi''_{m+1}(1-\varepsilon')$ is an idempotent lifting e_{m+1}, which is now orthogonal to $\{\pi_1, \ldots, \pi_{n+1}\}$, since

$$\begin{aligned}\pi_{m+1}\pi_{n+1} &= (\pi''_{m+1} + \varepsilon'\pi''_{m+1} - \pi''_{m+1}\varepsilon')\pi_{n+1} \\ &= \varepsilon' + (\varepsilon')^2 - \pi''_{m+1}\varepsilon'\pi_{n+1} = 0.\end{aligned}$$

By induction on n we get an orthogonal set of idempotents $\{\pi_1, \ldots, \pi_{m+1}\}$ lifting $\{e_1, \ldots, e_{m+1}\}$.

For the uniqueness assume that we have already conjugated a second orthogonal lifting $\{\pi'_1, \ldots, \pi'_{m+1}\}$ such that $\pi'_i = \pi_i$ for $i \leq m$. By the Beilinson lemma we have

$$\pi'_{m+1} = (1-\varepsilon)\pi_{m+1}(1+\varepsilon) = \pi_{m+1} - \varepsilon\pi_{m+1} + \pi_{m+1}\varepsilon$$

for some $\varepsilon \in I$. Since by assumption $\pi_i\pi'_{m+1} = 0 = \pi'_{m+1}\pi_i$ for $i \leq m$, we must have

$$\pi_i\varepsilon\pi_{m+1} = 0 = \pi_{m+1}\varepsilon\pi_i \quad \text{for } i \leq m.$$

Putting $\eta = \varepsilon\pi_{m+1} + \pi_{m+1}\varepsilon \in I$, one easily computes

$$\pi_i = (1-\eta)\pi_i(1+\eta) \quad \text{for } i \le m,$$
$$\pi'_{m+1} = (1-\eta)\pi_{m+1}(1+\eta),$$

which proves the claim.

Finally, if $\sum_{i=1}^{m} \pi_i = 1 + \varepsilon$ for $\varepsilon \in I$, then $1 + \varepsilon = (1+\varepsilon)^2 = 1 + 2\varepsilon$, and hence $\varepsilon = 0$.

PROPOSITION 5.5. *Assume one has a descending filtration* $F^{\cdot}$ *on* $CH^j(X)_{\mathbb{Q}}$ *with the following properties*:

(α) $F^0 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathbb{Q}}$.

(β) *The action of* $CH^d(X \times X)_{\mathbb{Q}}$ *on* $CH^j(X)_{\mathbb{Q}}$ *respects* $F^{\cdot}$, *factors through homological equivalence on* $\mathrm{Gr}_F^{\cdot} CH^j(X)_{\mathbb{Q}}$, *and one has*

$$\pi_i^{\mathrm{hom}} = \delta_{i,2j-\nu} \cdot \mathrm{id} \quad \text{on } \mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}}.$$

(γ) $F^N CH^j(X)_{\mathbb{Q}} = 0$ *for some* $N \in \mathbb{N}$.

Then for any liftings $\pi_0, \dots, \pi_{2d} \in CH^d(X \times X)_{\mathbb{Q}}$ *of* $\pi_0^{\mathrm{hom}}, \dots, \pi_{2d}^{\mathrm{hom}}$ (*not necessarily idempotent or orthogonal*) *and for every* $\mu \ge N$, *the following holds.*

(B$'$) *The correspondences* $\pi_{2j+1}^{\mu}, \dots, \pi_{2d}^{\mu}$ *and* $\pi_{2j-N}^{\mu}, \pi_{2j-N-1}^{\mu}, \dots, \pi_0^{\mu}$ *act as zero on* $CH^j(X)_{\mathbb{Q}}$.

$$(\mathrm{C}') \qquad \begin{aligned} F^{\nu} CH^j(X)_{\mathbb{Q}} &= \mathrm{Ker}\,\pi_{2j}^{\mu} \cap \mathrm{Ker}\,\pi_{2j-1}^{\mu} \cap \dots \cap \mathrm{Ker}\,\pi_{2j-\nu+1}^{\mu} \\ &= \mathrm{Im}\,\pi_{2j-\nu}^{\mu} + \mathrm{Im}\,\pi_{2j-\nu-1}^{\mu} + \dots + \mathrm{Im}\,\pi_{2j-\mu+1}^{\mu} \end{aligned}$$

for $\nu \ge 0$.

PROOF. By assumptions (β) and (γ), $\pi_{2d}, \dots, \pi_{2j+1}$ and $\pi_{2j-N}, \dots, \pi_0$ act as zero on $\mathrm{Gr}_F^{\cdot} CH^j(X)_{\mathbb{Q}}$. This obviously implies (B$'$), since the filtration has length N by (γ). Exactly the same argument shows that $\pi_{2d}^{N-\nu}, \pi_{2d-1}^{N-\nu}, \dots, \pi_{2j-\nu+1}^{N-\nu}$ act as zero on $F^{\nu} CH^j(X)_{\mathbb{Q}}$ for $\nu \ge 0$. Hence

$$F^{\nu} CH^j(X)_{\mathbb{Q}} \subseteq \mathrm{Ker}\,\pi_{2j}^{\mu} \cap \mathrm{Ker}\,\pi_{2j-1}^{\mu} \cap \dots \cap \mathrm{Ker}_{2j-\nu+1}^{\mu}.$$

The converse inclusion follows by induction on ν. Starting with the trivial case $\nu = 0$, we have to show

$$F^{\nu} \cap \mathrm{Ker}\,\pi_{2j-\nu}^{\mu} \subseteq F^{\nu+1}.$$

But since $\pi_{2j-\nu}^{\mu}$ is the identity on $\mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}}$, we have $x - \pi_{2j-\nu}^{\mu}(x) \in F^{\nu+1}$ for every $x \in F^{\nu}$. The proof of the second equality in (C$'$) is dual: Since $\pi_{2j-\nu}^{\nu}, \pi_{2j-\nu-1}^{\nu}, \dots, \pi_0^{\nu}$ act as zero on $CH^j(X)_{\mathbb{Q}}/F^{\nu}$, we have the inclusion

$$F^{\nu} CH^j(X)_{\mathbb{Q}} \supseteq \mathrm{Im}\,\pi_{2j-\nu}^{\mu} + \mathrm{Im}\,\pi_{2j-\nu-1}^{\mu} + \dots + \mathrm{Im}\,\pi_{2j-N+1}^{\mu}.$$

The equality follows by descending induction on ν, starting with the trivial case $\nu = N$. The induction step

$$F^{\nu}CH^{j}(X)_{\mathbb{Q}} \subseteq F^{\nu+1}CH^{j}(X)_{\mathbb{Q}} + \operatorname{Im}\pi^{\mu}_{2j-\nu}$$

is clear from the fact that $\pi_{2j-\nu}$ is the identity on $\operatorname{Gr}^{\nu}_{F}CH^{j}(X)_{\mathbb{Q}}$.

REMARKS 5.6. (a) If the π_i are idempotents, then $\pi_i^{\mu} = \pi_i$ for all i. If the π_i are pairwise orthogonal, then

$$F^{\nu}CH^{j}(X)_{\mathbb{Q}} = \operatorname{Im}(\pi^{\mu}_{2j-\nu} + \pi^{\mu}_{2j-\nu-1} + \cdots + \pi^{\mu}_{2j-N+1}) \tag{5.2}$$

for all $\mu \geq N$. In fact, by Proposition 5.5

$$F^{\nu}CH^{j}(X)_{\mathbb{Q}} = \operatorname{Im}\pi^{\mu+1}_{2j-\nu} + \operatorname{Im}\pi^{\mu+1}_{2j-\nu-1} + \cdots + \operatorname{Im}\pi^{\mu+1}_{2j-N+1},$$

and by orthogonality this is contained in the right-hand side of (5.2), by the relation $\sum \pi_i^{\mu+1}(x_i) = (\sum \pi_i^{\mu})(\sum \pi_r(x_r))$. The converse inclusion in (5.2) follows trivially from Proposition 5.5.

(b) A filtration as in Beilinson's conjecture (any version) satisfies (α) and (γ) (by definition) and (β) (as was observed after the formulation of version 1). The above assumptions are much weaker, since one only needs a filtration on $CH^{j}(X)_{\mathbb{Q}}$ and properties of self-correspondences acting on it. In fact, even if one deduces (α), (β), and (γ) from Beilinson's conjecture (version 1), one only needs to consider X, $X \times X$, X^3, and X^4.

COROLLARY 5.7. *A filtration $F^{\cdot}$ on $CH^{j}(X)_{\mathbb{Q}}$ satisfying Proposition 5.5(α), (β), and (γ) is unique. In particular, Beilinson's conjectured filtration is unique.*

We can now prove that *Beilinson's conjecture implies Murre's conjecture*: Indeed, let $F^{\cdot}$ be Beilinson's filtration as in Conjecture 2.1. Then it follows from its properties (b) and (c) that

$$F^{r}CH^{d}(X \times X)_{\mathbb{Q}} \circ F^{s}CH^{d}(X \times X)_{\mathbb{Q}} \subseteq F^{r+s}CH^{d}(X \times X)_{\mathbb{Q}}$$

under composition of correspondences. Hence

$$F^{1}CH^{d}(X \times X)_{\mathbb{Q}} \overset{(a)}{=} CH^{d}(X \times X)_{\mathrm{hom},\mathbb{Q}}$$

is a nilpotent ideal by (e), and we get part (A) of Murre's conjecture by Proposition 5.3. By Proposition 5.5 and Remark 5.6(a), every set $\{\pi_i\}$ of idempotents (not necessarily orthogonal) lifting $\{\pi_i^{\mathrm{hom}}\}$ satisfies part (B), if $F^{\cdot}$ satisfies (e), and (strong B), if $F^{\cdot}$ satifies (strong e). Moreover, Murre's filtration defined via these π_i agrees with Beilinson's, and in particular, we obtain (C), with the additional information that the π_i do not have to be orthogonal. Finally, Murre's (D) follows from property (a) of Beilinson's filtration $F^{\cdot}$.

For the converse implication in Theorem 5.2 the following fact (cf. (5.1)) is crucial.

Proposition 5.8. *Let $\{\pi_i\} = \{\pi_i^X\}$ be an orthogonal set of idempotents lifting $\{\pi_i^{\mathrm{hom}}\} = \{\pi_i^{X,\mathrm{hom}}\}$. Let $M_i = (X, \pi_i, 0)_{\mathrm{rat}}$ be the Chow motive associated to π_i $(i = 0, \dots, 2d)$. If Murre's conjecture holds for $X \times X$, then*

$$\mathrm{Hom}_{\mathscr{CHM}_k}(M_i, M_j) = \begin{cases} 0 & \text{for } i < j, \\ \mathrm{End}_{\mathscr{M}_k}(h^i(X)) & \text{for } i = j. \end{cases}$$

More generally, let Y be a smooth projective variety of pure dimension e over k, let $\{\pi_0^Y, \dots, \pi_{2e}^Y\} \subseteq CH^{2e}(Y \times Y)_{\mathbb{Q}}$ be an orthogonal set of idempotents lifting the set $\{\pi_0^{Y,\mathrm{hom}}, \dots, \pi_{2e}^{Y,\mathrm{hom}}\}$ of Künneth components of the diagonal Δ_Y for Y, and denote by $N_j = (Y, \pi_j, 0)_{\mathrm{rat}}$ the Chow motive associated to π_j^Y $(j = 0, \dots, 2e)$. If Murre's conjecture holds for $X \times Y$, then

$$\mathrm{Hom}_{\mathscr{CHM}_k}(M_i, N_j) = \begin{cases} 0 & \text{for } i < j, \\ \mathrm{Hom}_{\mathscr{M}_k}(h^i(X), h^i(Y)) & \text{for } i = j. \end{cases}$$

Proof. Obviously it suffices to show the second claim. Denote by $\alpha \mapsto {}^t\alpha$ the transposition on $CH^d(X \times X)_{\mathbb{Q}}$, i.e., the anti-involution induced by interchanging the factors of $X \times X$. Since clearly

$${}^t\pi_i^{\mathrm{hom}} = \pi_{2d-i}^{\mathrm{hom}},$$

$\{\tilde{\pi}_0, \dots, \tilde{\pi}_{2d}\}$, with $\tilde{\pi}_i = {}^t\pi_{2d-i}$, is an orthogonal set of idempotents lifting $\{\pi_0^{\mathrm{hom}}, \dots, \pi_{2d}^{\mathrm{hom}}\}$. For $\alpha \in CH^d(X \times X)_{\mathbb{Q}}$ and $\beta \in CH^e(Y \times Y)_{\mathbb{Q}}$ denote by $\alpha \times \beta \in CH^{d+e}(X \times X \times Y \times Y)_{\mathbb{Q}}$ the external product, regarded as a correspondence on $X \times Y$ via the isomorphism

$$CH^{d+e}(X \times X \times Y \times Y)_{\mathbb{Q}} \xrightarrow[\sim]{\varphi^*} CH^{d+e}(X \times Y \times X \times Y)$$

induced by $\varphi\colon X \times Y \times X \times Y \xrightarrow{\sim} X \times X \times Y \times Y$, $(x_1, y_1, x_2, y_2) \mapsto (x_1, x_2, y_1, y_2)$. Then the external products

$$\tilde{\pi}_i^X \times \pi_j^Y \in CH^{d+e}((X \times Y) \times (X \times Y)), \qquad i = 0, \dots, 2d, \ j = 0, \dots, 2e,$$

form an orthogonal set of idempotents, by the formula

$$(\alpha \times \beta) \circ (\alpha' \times \beta') = \alpha \circ \alpha' \times \beta \circ \beta'$$

for $\alpha, \alpha' \in CH^d(X \times X)_{\mathbb{Q}}$ and $\beta, \beta' \in CH^e(Y \times Y)_{\mathbb{Q}}$, which is easily established (cf. [**Ma**, p. 448, lemma]; use the formulae $(f_1 \times f_2)^*(\alpha \times \beta) = f_1^*\alpha \times f_2^*\beta$, $(f_1 \times f_2)_*(\alpha \times \beta) = (f_1)_*\alpha \times (f_2)_*\beta$, and $(\alpha \times \beta) \cdot (\alpha' \times \beta') = \alpha \cdot \alpha' \times \beta \cdot \beta'$). Furthermore, the Künneth formula shows that

$$\Pi_r = \sum_{i+j=r} \tilde{\pi}_i^X \times \pi_j^Y$$

is a lifting of the rth Künneth component of the diagonal $\Delta_{X \times Y}$ for $X \times Y$.

Part (B) of Murre's conjecture for $X \times Y$ asserts that $\Pi_{2d+1}, \Pi_{2d+2}, \dots, \Pi_{2d+2e}$, regarded as correspondences on $X \times Y$, act as zero on $CH^d(X \times Y)_{\mathbb{Q}}$. Together with the mentioned orthogonality this implies that $\tilde{\pi}_i^X \times \pi_j^Y$ acts as zero on $CH^d(X \times Y)_{\mathbb{Q}}$ for all pairs (i, j) with $i + j > 2d$. Hence

$$\begin{aligned} 0 &= (\tilde{\pi}_i^X \times \pi_j^Y) CH^d(X \times Y)_{\mathbb{Q}} \quad [\text{application of } \tilde{\pi}_i \times \pi_j] \\ &= \pi_j^Y CH^d(X \times Y)_{\mathbb{Q}}\, {}^t\tilde{\pi}_i^X \quad [\text{composition with } {}^t\tilde{\pi}_i \text{ and } \pi_j] \\ &= \pi_j^Y CH^d(X \times Y)_{\mathbb{Q}} \pi_{2d-i}^X \quad [\text{by definition}] \\ &= \mathrm{Hom}_{\mathcal{CHM}_k}(M_{2d-i}, N_j) \quad [\text{by definition, cf. §2}], \end{aligned}$$

provided $i + j > 2d$, i.e., $2d - i < j$. Here the second equality is an easy consequence of the definitions; cf. Lieberman's lemma in **[Kl2**, p. 73].

Part (D) of Murre's conjecture for $X \times Y$ implies that $CH^d(X \times Y)_{\mathrm{hom}, \mathbb{Q}} = F^1 CH^d(X \times Y)_{\mathbb{Q}} = \mathrm{Ker}\, \Pi_{2d}$. As above we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{CHM}_k}(M_i, N_i) &= (\tilde{\pi}_{2d-i}^X \times \pi_i^Y) CH^d(X \times Y) \\ &= (\tilde{\pi}_{2d-i}^X \times \pi_i^Y) CH^d(X \times Y)_{\mathbb{Q}}/\sim_{\mathrm{hom}} \\ &= \pi_i^{Y, \mathrm{hom}} (CH^d(X \times Y)_{\mathbb{Q}}/\sim_{\mathrm{hom}}) \pi_i^{X, \mathrm{hom}} \\ &= \mathrm{Hom}_{\mathcal{M}_k}(h^i(X), h^i(Y)). \end{aligned}$$

Hence 5.8 is proved.

We now prove that *Murre's conjecture implies Beilinson's conjecture*: Let X, Y, and the notation be as in Proposition 5.8, and assume Murre's conjecture for X, Y, and $X \times Y$. Define the filtration $F^\cdot$ on $CH^\cdot(X)_{\mathbb{Q}}$ as in part (C) of Murre's conjecture from $\{\pi_i^X\}$, similarly for Y. Then (a) and (e) (resp. (strong e)) of Beilinson's conjecture follow, for this filtration, by definition, by (D) and by (B) and (strong B), respectively.

For the functoriality in part (c) of Beilinson's conjecture we use the following reinterpretation of $F^\cdot$. Recall that one can define rational Chow groups of Chow motives; for a Chow motive $(X, p, 0)_{\mathrm{rat}}$ we have

$$CH^j((X, p, 0)_{\mathrm{rat}})_{\mathbb{Q}} = p CH^j(X)_{\mathbb{Q}} = \mathrm{Im}\, p = \mathrm{Ker}(1 - p)$$

where $p \colon CH^j(X)_{\mathbb{Q}} \to CH^j(X)_{\mathbb{Q}}$ also denotes the endomorphism induced by the correspondence p. If M is the Chow motive associated to X, then

$$M = \bigoplus_{i=0}^{2d} M_i$$

by definition, and we have

$$F^{\nu}CH^{j}(X)_{\mathbb{Q}} = \bigcap_{k=2j-\nu+1}^{2d} \operatorname{Ker}\pi_k^X$$

(5.3)
$$= \bigoplus_{i=0}^{2j-\nu} \pi_i^X CH^j(X)_{\mathbb{Q}}$$

$$= \bigoplus_{i=0}^{2j-\nu} CH^j(M_i)_{\mathbb{Q}}$$

(this should be compared with the motivic considerations after Theorem 5.2). We have a similar description for $F^{\cdot}$ on $CH^j(Y)_{\mathbb{Q}}$, and now Proposition 5.8 implies that every correspondence $\alpha \in CH^d(X \times Y)_{\mathbb{Q}}$ maps $F^{\nu}CH^j(X)_{\mathbb{Q}}$ to $F^{\nu}CH^j(Y)_{\mathbb{Q}}$. In fact, α maps $\bigoplus_{i=0}^{2j-\nu} M_i$ to $\bigoplus_{i=0}^{2j-\nu} N_i$ by Proposition 5.8 (in more down-to-earth terms, Proposition 5.8 says that $\pi_k^Y \alpha \pi_i^X = 0$ for $i < k$, so that $\pi_k^Y \alpha F^{\nu} CH^j(X)_{\mathbb{Q}} = 0$ for $k > 2j - \nu$).

In particular, $F^{\cdot}$ is respected by f_* for a morphism $f: X \to Y$, and the same follows for f^* by interchanging the roles of X and Y (and considering Murre's conjecture for $CH^e(Y \times X)_{\mathbb{Q}}$).

For property (b) of Beilinson's conjecture we note that the intersection product coincides with the composition

$$CH^i(X)_{\mathbb{Q}} \otimes CH^j(X)_{\mathbb{Q}} \to CH^{i+j}(X \times X)_{\mathbb{Q}} \xrightarrow{\Delta^*} CH^{i+j}(X)_{\mathbb{Q}}$$

where the first arrow is the external product and the second one is the pull-back for the diagonal $\Delta: X \to X \times X$. If we define a filtration $F^{\cdot}$ on $CH^{i+j}(X \times X)_{\mathbb{Q}}$ by Murre's formula and the orthogonal idempotents

$$\Pi_r = \sum_{i+j=r} \pi_i \times \pi_j \qquad (r = 0, \dots, 4d)$$

lifting the Künneth components of the diagonal for $X \times X$, then the external product maps

$$F^{\mu}CH^i(X)_{\mathbb{Q}} \otimes F^{\nu}CH^j(X)_{\mathbb{Q}} = \left(\bigoplus_{k=0}^{2i-\mu} \pi_k CH^i(X)_{\mathbb{Q}} \right) \otimes \left(\bigoplus_{\ell=0}^{2j-\nu} \pi_\ell CH^j(X)_{\mathbb{Q}} \right)$$

to

$$\bigoplus_{\substack{k,\ell \\ k+\ell \leq 2(i+j)-(\mu+\nu)}} (\pi_k \times \pi_\ell) CH^{i+j}(X \times X)_{\mathbb{Q}} = F^{\mu+\nu}CH^{i+j}(X \times X)_{\mathbb{Q}},$$

and Δ^* maps $F^{\mu+\nu}CH^{i+j}(X \times X)_{\mathbb{Q}}$ to $F^{\mu+\nu}CH^{i+j}(X)_{\mathbb{Q}}$ by the previous step, if we assume Murre's conjecture for $X \times X \times X$.

Finally, property (d) of Beilinson's conjecture is clear from the formula

$$\mathrm{Gr}_F^{\nu} CH^j(X)_{\mathbb{Q}} \cong CH^j(M_{2j-\nu})_{\mathbb{Q}} = \pi_{2j-\nu} CH^j(X)_{\mathbb{Q}}.$$

Thus, Theorem 5.2 is completely proved, in view of the uniqueness result proved in Corollary 5.7.

The status of Murre's conjecture is as follows **[Mur2]**: It is trivially true for curves. For surfaces and for threefolds of type $S \times C$, where S is a surface and C is a curve, Murre has proved (A), as well as (B) and (D) for a natural choice of idempotents. For surfaces he shows that his filtration is the natural one (for zero cycles the one considered by Bloch; cf. (1.9)), in support of (C). For abelian varieites, (A) follows from work of Shermenev, Deninger-Murre, and Künnemann (cf. **[Kü1]**) and for a natural choice of idempotents part of (B) follows from work of Beauville **[Beau]**, and the rest of (B) is equivalent to a conjecture of Beauville.

REMARKS 5.9. (a) Murre's work and the considerations in §3 suggest finding π_i supported on $Y_i \times X$, where Y_i is of dimension i, for $0 \leq i \leq d$, and to take $\pi_{2d-i} = {}^t\pi_i$. (Cf. also **[Mur2**, Question 3.3].)

(b) S. Saito defined another filtration on Chow groups without assuming any special conditions, and he showed that it agrees with Beilinson's filtration, given the formalism of mixed motives as in version 3 of Beilinson's conjecture. We refer the reader to **[SaS]** for this and for further interesting discussion.

(c) As we have seen, one may easily define filtrations on Chow groups that are exhaustive (as in Murre's definition) or that satisfy Proposition 5.5(β) (this is the case for the filtration $F_\ell^\cdot$ in the proof of Lemma 2.7 and for Saito's definition). It seems to be very hard to define filtrations that satisfy both 5.5(β) and (γ).

Appendix: A letter from Grothendieck to Illusie

Buffalo le 3.5.1973

Cher Illusie,

Je t'envoie quelques afterthoughts de notre conversation mathématique sur les motifs. J'avais dit à tort que les isomotifs n'ont pas de "modules infinitésimaux", c'est-à-dire que si $i \colon S_0 \to S$ est une immersion nilpotente, le foncteur image inverse de motifs est une équivalence de catégories. Cela doit être vrai en car. $p > 0$ (plus généralement, si $\mathscr{O}_S$ est annulé par une puissance de p), pour la raison heuristique (qu'on peut expliciter entièrement lorsq'on travaille dans le contexte bien assis des schemas abéliens, ou des groupes de Barsotti-Tate) que lorsqu'on se ramène par dévissage au cas d'une nilimmersion d'ordre 1 ($J^2 = 0$), on peut définir une obstruction à la déformation sur S d'un homomorphisme (ou isomorphisme) de (pas iso) motif sur S_0, qui sera tuée par p^i si p^i tue J, donc qui sera tuée lorsqu'on passe aux *iso*motifs. Par contre, en caractéristique nulle, les schémas abéliens à isogénie près ont la même théorie des modules infinitésimaux que les schémas abéliens tout courts, et il faut s'attendre à la même chose pour les motifs et isomotifs. En termes des théories de systèmes de coefficients de de Rham ou de Hodge, l'élément de structure "filtration de DR" introduit bel et bien un élément de continuité, qui a pour effet de rendre faux le fait que pour ces coefficients, le foncteur image inverse par une nilimmersion soit une équivalence. Il semble donc qu'il faille bannir cette propriété (hors du cas des schémas de torsion) du yoga des "coefficients discrets". A moins qu'il se trouve que les besoins du formalisme (construction de foncteurs adjoints du type Rf_* etc.) nous impose de modifier la notion de faisceau de Hodge ou de DR sur un schéma X, en partant du genre de notion que nous avions regardée ensemble, et en passant ensuite aux catégories $\varinjlim$ des catégories correspondantes associées à X', où X' est réduit et $X' \to X$ est fini radiciel surjectif. Mais j'espère qu'il ne sera pas nécessaire de canuler ces notions ainsi. Une question liée est celle-ci: si X est de car. 0, un isomotif serein sur X qui est "effectif de poids 1" définit-il bien un schéma abélien à isogénie près, ou seulement un schéma abélien à isogénie près au dessus d'un X' comme ci-dessus? Ce dernier devrait être le cas en tous cas en car. $p > 0$, si on veut qu'un morphisme fini surjectif soit un morphisme de descente effective pour les isomotifs (et cela à son tour doit être vrai, étant vrai pour les $\mathbb{Q}_\ell$-faisceaux, si on veut que le foncteur isomotifs $\to \mathbb{Q}_\ell$-faisceaux commute aux opérations habituelles et est fidèle—et on le veut à tout prix!). Ainsi, en car. $p > 0$, si k est un corps, un isomotif effectif de poids 1 sur k devrait être, non un schéma abélien à isogénie près sur k, mais sur la clôture parfaite de k!

Je n'ai pas le coeur net non plus sur la nécessité de mettre du "iso" partout dans la théorie des motifs. Je ne serais pas tellement étonné qu'il y a en caractéristique nulle une théorie des motifs (*pas* iso), qui s'envoie dans les théories ℓ-adiques (sur $\mathbb{Z}_\ell$, pas $\mathbb{Q}_\ell$) pour tout ℓ. Pour ce qui est des coefficients de Hodge, il devrait être assez trivial de les définir "pas iso", de telle façon que les $\mathbb{Z}$-faisceaux de torsion algébriquement constructibles (sur X de type fini sur $\mathbb{C}$) en forment une sous-catégorie pleine, et avec un foncteur vers les $\mathbb{Z}$-faisceaux algébriquement constructibles ("foncteur de Betti"). En caractéristique $p > 0$, j'ai des doutes très sérieux pour l'existence d'une théorie des motifs pas iso du tout, à cause des phénomènes de p-torsion (surtout pour les schémas qui ne sont pas projectifs et lisses). Ainsi, si on admet la description de Deligne des "motifs mixtes" de niveau 1 comme le genre de chose permettant de définir un H^1 motivique d'un schéma pas projectif ou pas lisse, on voit que déjà pour une courbe algébrique sur un corps imparfait k, la construction ne peut fournir en général qu'un objet du type voulu sur la clôture parfaite de k. Par contre, il pourrait être vrai que seul la p-torsion canule, et qu'il suffise de localiser par tuage de p-torsion, c'est-à dire moralement de travailler avec des catégories $\mathbb{Z}[1/p]$-linéaires. On aurait alors encore des foncteurs allant des "motifs" (pas iso) vers les $\mathbb{Z}_\ell$-faisceaux (quel que soit $\ell \neq p$) mais pas vers les F-cristaux, mais seulement vers les

F-isocristaux. Dans cette théorie, on renoncerait donc simplement à regarder en car. p des phénomènes de p-torsion. Pourtant il est "clair" que ceux-ci existent et sont fort intéressants, tout au moins pour les morphismes propres et lisses, et on a bien l'impression que la cohomologie cristalline (plus fine que DR) pas iso en donne la clef. (Au fait, Berthelot est-il parvenu à des conjectures plausibles à cet sujet?) On peut donc espérer que pour les motifs sereins et semi-simples fibre par fibre, on a des catégories sur $\mathbb{Z}$, pas seulement sur $\mathbb{Z}[1/p]$, les Hom étant des $\mathbb{Z}$-modules de type finit. Cette impression peut être fondée par exemple sur le joli comportement des schémas abéliens sur le corps des fractions d'un anneau de val. discrète: dans la théorie de spécialisation, il se trouve qu'à un aucun moment la p-torsion ne canule.

Bien sûr, alors même qu'on arriverait à travailler avec des catégories des motifs pas iso, dans "l'état actuel de la science", pour en déduire une théorie de groupes de Galois motiviques, étant obligé de s'appuyer sur ce que Saavedra a rédigé, on est obligé à tensoriser tout par $\mathbb{Q}$, et on ne trouve que des groupes algébriques sur $\mathbb{Q}$ ou des extensions de $\mathbb{Q}$. Néanmoins, on a certainement dans l'idée que les "vrais" groupes de Galois motiviques (associés à des foncteurs-fibres comme la cohomologie ℓ-adique, ou la cohomologie de Betti) sont des schémas en groupes sur $\mathbb{Z}_\ell$ et sur $\mathbb{Z}$ plutôt que sur $\mathbb{Q}_\ell$ et sur $\mathbb{Q}$, et par là on devrait rejoindre le point de vue des groupes de type arithmétique de gens comme Borel, Griffiths, etc.

Encore une remarque: alors même qu'on travaille avec des isomotifs, on peut associer à un tel M quelque chose de mieux qu'une suite infinie de $\mathbb{Q}_\ell$-faisceaux (lorsqu'il y a une infinité de ℓ premiers aux car. résiduelles). En fait, on a ce qu'on pourrait appeler un faisceau "adélique", i.e. un faisceau de modules (moralement) sur l'anneau des adèles finis de $\mathbb{Q}$. De façon précise, on peut considérer tous les $T_\ell(M)$ sauf un nombre fini comme étant des $\mathbb{Z}_\ell$-faisceaux (pas seulement des $\mathbb{Q}_\ell$-faisceaux). Eliminant tout métaphysique motivique, on peut dire que la théorie de Jouanolou écrite en fixant un ℓ, pourrait être développée avec des modifications techniques mineures pour avoir une théorie des "A-faisceaux", où A est l'anneau des adèles, où un facteur direct A' de celui-ci obtenu en ne prenant qu'un paquet de nombres premiers (pas nécessairement tous). On obtient ainsi une théorie de coefficients (au sens technique dont nous avions discuté) ayant comme anneau de coefficients la $\mathbb{Q}$-algèbre A resp. A'. Comme A et A' sont "absolutement plats", il n'y a pas introduction de $\underline{\mathrm{Tor}}_i$ gênants et de canulars de degrés infinis dans cette théorie.

Pour en revenir au yoga des coefficients "discrets", où j'avais énoncé une propriété de trop apparemment, par contre il y en a une autre que nous n'avions pas explicitée. Il s'agit de la définition de l'objet de Tate sur S comme l'inverse de l'objet (inversible pour $\otimes$)

$$T(-1) = R^2 f_*(1_P) = R^2 g_!(1_E),$$

où $f\colon P \to S$ resp. $g\colon E \to S$ sont les projections de la droite projective resp. la droite affine sur S. D'autre part, ces objets (définis en fait sur le schéma de base S_0 de la théorie de coefficients) interviennent également dans la formulation des théorèmes de pureté relative ou absolus et la définition des classes fondamentales locales (qui, j'espère, doit être possible en termes des données initiales de la théorie de coefficients envisagée, sans constituer une donnée supplémentaire), et dans le calcul de $f^!$ pour f lisse (donc aussi pour f lissifiable), pour ne parler que du démarrage du formalisme cohomologique. En fait, on les retrouve ensuite à chaque pas.

Une dernière remarque. Je crois qu'il vaudrait la peine de formaliser, dans le cadre d'une théorie de coefficients plus ou moins arbitraire, les arguments de dévissage qui ont conduit, dans le cas des coefficients étales, aux théorèmes de finitude pour Rf_* pour f propre, puis pour f séparé de type fini seulement (moyennant résolution des singularités). Ces dévissages apparaitraient maintenant comme des pas destinés à prouver *l'existence* de Rf_* (en même temps, s'il y a lieu, que sa commutation aux changements de base). A vrai dire, il n'est pas clair pour moi qu'on arrivera à des formulations qui s'appliqueraient directement aux $\mathbb{Z}_\ell$-faisceaux, disons; en fait, ce n'est pas ainsi que procède Jouanolou dans ce cas, qui au contraire se ramène aux énoncés déjà connus dans le cas des coefficients de torision (procédé qui n'a guère de chance de s'axiomatiser dans le contexte qui nous intéresse). Par contre, pensant directement au cas des motifs, on peut songer à utiliser un dévissage qui s'appuie entre autres sur les propriétés suivantes (quitte à se tirer par les lacets de souliers pour les établir chemin faisant): (a) un (iso)motif se dévisse en motifs sereins sur des schémas irréd. normaux (NB on suppose qu'on travaille sur des schémas excellents); (b) un motif serein sur un schéma normal irréductible se dévisse en motifs sereins "simples"—en fait, il suffit de faire le dévissage en le point générique; (c) un motif simple (pourvu qu'on remplace la base S par un voisinage ouvert assez petit du point générique) est un facteur direct d'un $R^i f_*(1_X)$, où $f\colon X \to S$ est propre et lisse, tout du moins modulo tensorisation par un objet de Tate $T(j)$ convenable. Ainsi, moyennant au moins deux gros grains de sel qu'il faudrait essayer d'expliciter un jour, les motifs généraux (toujours iso, bien sûr) se ramènent aux motifs plus ou moins naïfs tels qu'ils sont décrits notamment dans Manin et Demazure. Cela s'applique tout au moins aux objets—quant aux morphismes, c'est une autre paire de manches—et encore pire pour les Ext^i . . . .

A ce propos, on peut se convaincre que l'application qui va des classes d'extension de deux motifs (dans la catégorie abélienne des motifs) vers le Ext^1 défini comme $\mathrm{Hom}(M, N[1])$ (Hom dans la catégorie triangulée) ne devrait pas être bijective (mais sans doute injective). Plaçons-nous en effet sur une base S spectre d'un corps fini, prenons pour M et N le motif unité $1_S = T_S(0) = T(0)$, de sorte que le Ext^1 n'est autre que $H^1(S, T(0))$. Les calculs ℓ-adiques du H^1 nous suggèrent fortement que le H^1 absolu motivique est canoniquement isomorphe à $\mathbb{Q}$. Mais d'autre part les classes d'extension de $T(0)$ par $T(0)$ doivent être nulles (plus généralement, les extensions de M par N doivent être nulles (sur tout corps K) si M et N sont des motifs de poids r et s avec $r \leq s$, si on admet le yoga de la filtration d'un motif par poids croissants, avec gradué associé semi-simple. (NB En fait, sur un corps fini, la catégorie des motifs devrait être toute entière semi-simple, i.e. toute extension devrait être triviale, i.e. la filtration croissante précédente devrait splitter canoniquement: cela résulte du fait que l'endomorphisme de frobenius du motif opère avec des "poids" différents sur les composants des différents poids—plus un petit exercice de catégories tannakiennes.)

Bien cordialement

Alexandre

REFERENCES

[Beau] A. Beauville, *Sur l'anneau de Chow d'une variété abélienne*, Math. Ann. **273** (1986), 647–651.

[Be1] A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet Math. **30** (1985), 2036–2070.

[Be2] ——, *Notes on absolute Hodge cohomology*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory. I, Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 35–68.

[Be3] ——, *Height pairing between algebraic cycles*, K-Theory, Arithmetic and Geometry (Y. Manin, ed.), Lecture Notes in Math., vol. 1289, Springer-Verlag, New York, 1987, pp. 1–26.

[Be4] ——, *On the derived category of perverse sheaves*, K-Theory, Arithmetic and Geometry (Y. Manin ed.), Lecture Notes in Math., vol. 1289, Springer-Verlag, New York, 1987, pp. 27–41.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analyse et Topologie sur les Espaces Singulières I, Astérisque **100** (1982).

[Bl1] S. Bloch, *An example in the theory of algebraic cycles*, Algebraic K-Theory, Evanston, 1976 (M. Stein, ed.), Lecture Notes in Math., vol. 551, Springer-Verlag, New York, 1976, pp. 1–29.

[Bl2] ——, *Lectures on Algebraic Cycles*, Duke Univ. Math. Ser., vol. IV, Duke Univ., Durham, NC, 1980.

[BK] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, Grothendieck Festschrift, vol. 1, Progr. Math., vol. 86, Birkhäuser, Boston, MA, 1990, pp. 333–400.

[BO] S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1979), 181–202.

[BS] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), 1235–1253.

[Ca] J. Carlson, *Extension of mixed Hodge structures*, Journées de Géométrie Algébrique d'Angers 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 107–127.

[CTRS] J.-L. Colliot-Thélène, W. Raskind, and S. Saito, *Nonrepresentability of Chow groups of 0-cycles and other K-cohomology groups*, preprint.

[CTSS] J.-L. Colliot-Thélène, J.-J. Sansuc, and C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. **50** (1983), 763–801.

[De1] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 107–126.

[De2] ——, *Théorie de Hodge*. II, Inst. Hautes Études Sci. Publ. Math. **40** (1972), 5–57.

[De3] ——, *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 313–428.

[De4] ——, *A quoi servent les motifs?*, these Proceedings, vol. 1, pp. 143–161.

[De5] ——, *Décompositions dans la catégorie dérivée*, these Proceedings, vol. 1, pp. 115–128.

[DM] P. Deligne and J. Milne, *Tannakian categories*, Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math., vol. 900, Springer-Verlag, New York, 1982, pp. 101–208.

[Dem] M. Demazure, *Motifs des variétés algébriques*, Sém. Bourbaki 1969–1970, exposé 365, Lecture Notes in Math., vol. 180, Springer-Verlag, New York, 1971.

[DV] A. Douady, J.-L. Verdier, et al., *Séminaire de géométrie analytique*, Astérisque 36–37, 1976.

[Ek] T. Ekedahl, *On the adic formalism*, Grothendieck Festschrift, vol. 2, Progr. Math., vol. 86, Birkhäuser, Boston, MA, 1990, pp. 197–218.

[EL] H. Esnault and M. Levine, *On the injectivity and surjectivity of cycle maps*, preprint 1992.

[EV] H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology*, Beilinson's Conjectures on Special Values of L-Functions (Rapoport, Schappacher, and Schneider, eds.), Perspect. Math., vol. 4, Academic Press, New York, 1988, pp. 43–91.

[Fa] C. Faber, *Chow rings of moduli spaces of curves.* I: *The Chow ring of* $\overline{\mathscr{M}}_3$, Ann. of Math. (2) **132** (1990), 331–419; II: *Some results on the Chow ring of* $\overline{\mathscr{M}}_4$, Ann. of Math. (2) **132** (1990), 421–449.

[FPR] J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: Cohomologie galoisienne et valeurs de fonctions* L, these Proceedings, vol. 1, pp. 599–706.

[Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, New York, 1984.

[Gi] H. Gillet, *Riemann-Roch theorems for higher algebraic* K*-theory*, Adv. Math. **40** (1981), 203–289.

[Gray] D. Grayson, *Weight filtrations in algebraic* K*-theory*, these Proceedings, vol. 1, pp. 207–237.

[Gr1] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique* (*TDTE*). V. *Les schémas de Picard. Théorèmes d'existence*, Sém. Bourbaki **14** (1961/62), exp. 232.

[Gr2] ——, *TDTE*. VI. *Les schémas de Picard. Propriétés générales*, Sém. Bourbaki **14** (1961/62), exp. 236.

[Gr3] ——, *Le groupe de Brauer.* III: *Exemples et complements*, Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, pp. 88–108.

[Gr4] ——, *Hodge's general conjecture is false for trivial reasons*, Topology **5** (1969), 299–303.

[Ja1] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.

[Ja2] ——, *Deligne homology, Hodge-D-conjecture, and motives*, Beilinson's Conjectures on Special Values of L-Functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, New York, 1988, pp. 305–372.

[Ja3] ——, *Mixed Motives and Algebraic* K*-Theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, New York, 1990.

[KS] K. Kato and S. Saito, *Unramified class field theory for arithmetical surfaces*, Ann. of Math. (2) **118** (1983), 241–275.

[KM] N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

[Kl1] S. Kleiman, *Algebraic cyles and the Weil conjectures*, Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, pp. 359–386.

[Kl2] ——, *Motives*, Algebraic Geometry, Oslo 1970 (F. Oort, ed.), Wolters-Noordhoff, Groningen, 1972, pp. 53–82.

[Kl3] S. Kleiman, *The standard conjectures*, these Proceedings, vol. 1, pp. 3–20.

[Kü1] K. Künnemann, *On the Chow motive of an abelian scheme*, these Proceedings, vol. 1, pp. 189–205.

[Kü2] ——, *A Lefschetz decomposition for Chow motives of abelian schemes*, preprint (1992).

[Le] J. Lewis, *Towards a generalization of Mumford's theorem*, J. Math. Kyoto Univ. **29** (1989), 195–204.

[Li] S. Lichtenbaum, *Motivic complexes*, these Proceedings, vol. 1, pp. 303–313.

[Lie] D. Lieberman, *Intermediate Jacobians*, Algebraic Geometry, Oslo 1970 (F. Oort, ed.), Wolters-Noordhoff, Groningen, 1972, pp. 125–139.

[Ma] Y. Manin, *Correspondences, motives, and monoidal transformations*, Mat. Sb. **77** (1968) = Math. USSR-Sb. **6** (1968), 439–470.

[MS] N. Maruyama and N. Suwa, *Remarques sur un article de Bloch et Srinivas sur les cycles algébriques*, J. Fac. Sci. Univ. Tokyo **35** (1988), 371–380.

[Mi] J. Milne, *Étale Cohomology*, Princeton Math. Ser., vol. 33, Princeton Univ. Press, Princeton, NJ, 1980.

[Mo] S. Mori, *On Tate conjecture concerning endomorphisms of abelian varieties*, Proceedings of the Internat. Sympos. of Alg. Geometry, Kyoto 1977 (M. Nagata, ed.), Kinokuniya Book-Store Co., Tokyo, pp. 219–230.

[Mum1] D. Mumford, *Rational equivalence of* 0*-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1969), 195–204.

[Mum2] ——, *Abelian Varieties*, Tata Inst. Fund. Res. Stud. Math., vol. 5, Oxford Univ. Press, London, 1970/1974/1985.

[Mum3] ——, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry (M. Artin and J. Tate, eds.), Birkhäuser, Boston, Basel, and Stuttgart, 1983, pp. 271–328.

[Mur1] J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.

[Mur2] ——, *On a conjectural filtration on the Chow groups of an algebraic variety* (parts I and II), Indag. Math. New Series Vol. 4 (1993), 177–201.

[Nek] J. Nekovář, *Beilinson's conjectures*, these Proceedings, vol. 1, pp. 537–570.

[Pa] K. Paranjape, *Cohomological and cycle theoretic connectivity*, preprint (1992).

[Roi1] A. Roitman, *Rational equivalence of zero-cycles*, Mat. Sb. **89** (1972) = Math. USSR-Sb. **18** (1974), 571–588.

[Roi2] ——, *The torsion in the group of zero-cycles modulo rational equivalence*, Ann. of Math. (2) **111** (1980), 553–569.

[SaM1] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.

[SaM2] ——, *On the formalism of mixed sheaves*, preprint RIMS Kyoto Univ. (1991).

[SaS] S. Saito, *Motives and filtrations on Chow groups*, preprint (1992).

[Schoe] C. Schoen, *On Hodge structures and non-representability of Chow groups*, preprint (1991).

[Scho] A. Scholl, *Classical motives*, these Proceedings, vol. 1, pp. 163–187.

[Sou1] C. Soulé, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), 317–345.

[Sou2] ——, *Opérations en K-théorie algébrique*, Canad. J. Math. **37** (1985), 488–550.

[St] J. Steenbrink, *A summary of mixed Hodge theory*, these Proceedings, vol. 1, pp. 31–41.

[Ta] J. Tate, *Conjectures on algebraic cycles in* ℓ*-adic cohomology*, these Proceedings, vol. 1, pp. 71–83.

[Ver] J.-L. Verdier, *Catégories dérivées (Etat* 0*)*, SGA $4\frac{1}{2}$ (Cohomologie Étale), Lecture Notes in Math., vol. 569, Springer-Verlag, New York, 1977, pp. 262–311.

[We] A. Weil, *Sur les critères d'équivalence en géométrie algébrique*, Math. Ann. **128** (1954), 95–127.

[Za] J. Zarhin, *Abelian varieties in characteristic* p, Mat. Zametki **19** (1976) = Math. Notes **19** (1976), 240–244.

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Motivic Complexes

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1. The original problems

One of the classic problems in the recent history of mathematics has been the search for an algebraically defined cohomology theory of algebraic varieties. Originally, at least two motivating questions were asked.

(1) Is it possible to give a definition of cohomology groups of an arbitrary algebraic variety such that these are finitely generated abelian groups and such that we recover the standard topological cohomology groups with integral coefficients when the variety is defined over the complex numbers? (Of course, these groups should satisfy a standard list of "expected properties".)

(2) Define a cohomology theory for all varieties with coefficients in a field of characteristic zero satisfying Poincaré duality. This, in particular, should give a cohomological description of the zeta-function for varieties over finite fields.

These two questions were eventually answered. Serre showed, by looking at supersingular elliptic curves, that the answer to the first question was no, and Grothendieck together with M. Artin and Berthelot showed that the answer to the second was yes, by constructing the ℓ-adic étale and crystalline cohomology theories. The ℓ-adic étale theory also answers the first question affirmatively, if "$\mathbf{Z}/\ell^n$ coefficients" replaces "integral coefficients".

2. Further questions

These two answers might seem to end the discussion, but it was soon realized that they did not. Perhaps the most important question left unanswered was this.

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QUESTION 1. Define a cohomology theory for varieties over number fields, or schemes of finite type over $\operatorname{Spec}\mathbf{Z}$, and thereby give a cohomological description of Hasse-Weil and scheme zeta-functions, including of course the Riemann zeta-function. Alternatively, we could ask for (getting progressively weaker):

(1a) Give such a description of the values at integral points of these zeta-functions (where "value" should always be interpreted as the first nonvanishing coefficient of the Laurent or power series defining the zeta-function in a neighborhood of $s = n$).

(1b) The same as (1a), but describe the values only up to sign (for L-functions, only up to units).

(1c) The same as (1a), but describe the values only up to a rational number (for L-functions up to an algebraic number).

(1d) Give such a description of the order of the zero or pole of the zeta-function at integral points.

How close are these questions to being answered? Question 1 is answered as above for varieties over finite fields, but there is not even a conjectural answer to this in the number field or number ring case.

Question 1(a) also does not have even a conjectural answer, even this time for varieties over finite fields.

Question 1(b) is conjecturally answered for varieties over finite fields by the conjectured theory of motivic cohomology (see **[L1]** for details). For varieties over number fields, Bloch and Kato **[BK]** give something like a conjectural answer in terms of Tamagawa measures. There should also be an answer more directly corresponding to the finite field answer in terms of an extension of definition of motivic cohomology. (Note that even though motivic cohomology is conjecturally defined for all schemes, answering Question 1(b) certainly involves a serious consideration of the infinite places as well. It is likely that Arakelov theory should intervene here in a nontrivial way.)

Questions 1(c), (d) are conjecturally answered by Beilinson in terms of algebraic K-theory. Beilinson defines motivic cohomology with rational coefficients to be the Adams weight r piece $K_i(X)^{(r)}$ of $K_i(X) \otimes \mathbf{Q}$, i.e., the piece where the Adams operations Ψ_k act as multiplication by k^r for all k. (It is known **[So]** that $K_i(X) \otimes \mathbf{Q}$ is the direct sum of its weight r pieces.)

Summing up, one first wants motivic cohomology to give descriptions of special values of zeta-functions, both in the finite field and in global settings. Related to this, one would also like an answer to

QUESTION 2. Find a "universal" cohomology theory for algebraic varieties, again satisfying some list of plausible axioms and mapping to all other such theories. Grothendieck suggested that this theory should take its values not in the category of abelian groups, but in a more general abelian category: the conjectural category of "motives".

QUESTION 3. Find a cohomology theory for algebraic varieties bearing the same relation to algebraic K-theory as topological cohomology does to topological K-theory.

It is by no means clear that there exists a single cohomology theory answering these three questions simultaneously.

3. What should motivic cohomology be?

The following picture, which is a variant of one originally proposed by Beilinson [**Be**], seems plausible:

For every scheme S, there should be an exact category called "mixed motives on S" and denoted $\mathbf{MMot}(S)$. If $\pi: T \to S$, $\pi^*: \mathbf{MMot}(S) \to \mathbf{MMot}(T)$ makes **MMot** a contravariant functor. $\mathbf{MMot}(S)$ should admit a tensor product operation $\tilde{\otimes}$, compatible with π^*. There are two "basic objects" $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$ in $\mathbf{MMot}(\mathrm{Spec}\,\mathbf{Z})$, and we define $\mathbf{Z}(r)$ to be $\mathbf{Z}(1)^{\tilde{\otimes} r}$ for $r \geq 0$. $\mathbf{Z}(0)$ should be the identity object for $\tilde{\otimes}$. We define $\mathbf{Z}_S(r)$ to be $\pi^*\mathbf{Z}(r)$ for $\pi: S \to \mathrm{Spec}\,\mathbf{Z}$. (Presumably $\tilde{\otimes}\mathbf{Z}_S(1)$ is invertible in $\mathbf{MMot}(S)$.)

If M is a mixed motive on S and T is a scheme over S, $M(T)$ should be a contravariant functor from schemes to complexes of abelian groups, so M determines a presheaf in the big étale or big Zariski site, and hence a corresponding complex of sheaves $\widetilde{M}$. We will also denote by $\mathbf{Z}_S(r)$ the complex of étale or Zariski sheaves defined by the motive $\mathbf{Z}_S(r)$.

EXAMPLE. The cohomology $\mathbf{MMot}(S)$ should include all complexes of commutative group schemes over S. $\mathbf{Z}(0)$ is the constant group scheme $\mathbf{Z}$, and the "Tate motive" $\mathbf{Z}(1)$ is $G_m[-1]$, and in particular, $\mathbf{MMot}(S)$ contains all Deligne 1-motives [D] over S.

Warnings. (1) Not every mixed motive (e.g., $(\mathbf{Z}(2))$ is a complex of commutative group-schemes.

(2) There should be a map from $\widetilde{M} \overset{L}{\otimes} \widetilde{N}$ to $(\widetilde{M \tilde{\otimes} N})$, but this map should *not* in general be an isomorphism.

(3) These definitions have the effect of making $\mathbf{Z}_S(r)$ actual complexes of sheaves on S, but the conjectured properties of $\mathbf{Z}_S(r)$ involve only its class in the derived category.

There should also be objects $\mathbf{Z}'_S(r)$ in the category $\mathbf{MMot}(S)$, with the property that $\mathbf{Z}'_S(r) = \mathbf{Z}_S(r)$ for all r if S is regular and quasi-projective over an affine scheme. We also denote by $\mathbf{Z}'_S(r)$ the associated complexes of sheaves on S.

The complexes of sheaves $\mathbf{Z}_S(r)$ and $\mathbf{Z}'_S(r)$ should satisfy the following "axioms".

(0) $\mathbf{Z}(0) =$ the constant sheaf $\mathbf{Z}$. $\mathbf{Z}(1) = G_m[-1]$. $\mathbf{Z}'_X(0) = \coprod_x (i_x)_*\mathbf{Z}$ where x runs through all points of X of codimension zero. $\mathbf{Z}'_X(1)$ is the complex $(\coprod_x (i_x)_* G_m \to \coprod_y (i_y)_*\mathbf{Z})[-1] = G'_m[-1]$ where x is as above and y runs through all points of X of codimension one. G'_m is the complex defined by Deninger [**Den**].

(1) $\mathbf{Z}(i)$ is acyclic outside of $[1, i]$ for $i \geq 1$ and $\mathbf{Z}'(i)$ is acyclic outside of $[0, i]$, both in the étale and Zariski sites.

(2a) (Kummer) Let n be a positive integer invertible on X. Then for the complex $\mathbf{Z}(i)$ of étale sheaves, there exists a triangle in the derived category $D(X)$

$$\mathbf{Z}(i) \xrightarrow{n} \mathbf{Z}(i) \to \mathbf{Z}/n\mathbf{Z}(i) \to \mathbf{Z}(i)[1]$$

where $\mathbf{Z}/n\mathbf{Z}(i)$ denotes the i-fold Tate twist $\mu_n^{\otimes i}$ of $\mathbf{Z}/n\mathbf{Z}$.

(2b) (Milne) Let X be a smooth scheme over a field k of characteristic p. Then there exists a triangle in the derived category $D(X)$

$$\mathbf{Z}(i) \xrightarrow{p^r} \mathbf{Z}(i) \to \mathbf{Z}/p^r\mathbf{Z}(i) \to \mathbf{Z}(i)[1]$$

where $\mathbf{Z}/p^r(i)$ denotes $v_r(i)[-i]$. We recall from **[Mi]** that $v_r(i)$ is the additive subsheaf of the piece $W_r\Omega^i_{X/K}$ of the deRham-Witt complex locally generated (for the étale topology) by sections of the form $d\log f_1 \wedge \cdots \wedge d\log f_i$. If r is equal to 1, this is equal to the additive subsheaf of $\Omega^i_{X/k}$ locally generated by exterior products of logarithmic differentials.

(3) There are maps in the derived category from $\mathbf{Z}(r)\overset{L}{\otimes}\mathbf{Z}(s)$ to $\mathbf{Z}(r+s)$ and $\mathbf{Z}(r)\overset{L}{\otimes}\mathbf{Z}'(s) \to \mathbf{Z}'(r+s)$ satisfying the same commutativity and associativity properties possessed by the K-theory and K'-theory groups and the maps $K_i(X) \otimes K_j(X) \to K_{i+j}(X)$ and $K_i(X) \otimes K'_j(X) \to K'_{i+j}(X)$. This should be true in either site.

(4) Let α_* be the functor which assigns to every étale sheaf its associated Zariski sheaf, and let α^* be the functor which assigns to every big site Zariski sheaf its étale sheafification. Then $R\alpha^*\mathbf{Z}(i)_{\text{zar}}$ is naturally isomorphic to $\mathbf{Z}(i)$ ($R\alpha^*$ may be identified with α^*, since α^* is exact). Also if X is regular, $t_{\leq i}R\alpha_*\mathbf{Z}(i) \simeq t_{\leq i+1}R\alpha_*\mathbf{Z}(i) \simeq \mathbf{Z}(i)_{\text{zar}}$. (This is false already if $i = 0$ if X is not assumed regular. Is it true for $\mathbf{Z}'(i)$ in the general case?) Note, in particular, if $X = \operatorname{Spec} F$ then the isomorphism $t_{\leq i}R\alpha_*\mathbf{Z}(i) \simeq t_{\leq i+1}R\alpha_*\mathbf{Z}(i)$ says that $H^{i+1}(F, \mathbf{Z}(i)) = 0$, which is the generalized Hilbert Theorem 90.

Evidently some truncation is necessary: the current axiom says

$$H^j_{\text{ét}}(X, \mathbf{Z}(i)) = H^j_{\text{zar}}(X, \mathbf{Z}(i)) \quad \text{for } j \leq i+1.$$

Without truncation, it would imply the same equality for all j, which is absurd if $X = \operatorname{Spec} F$ and $i = 0$ or 1, since $H^2(F, \mathbf{Z})$ and $H^2(F, G_m)$ are in general nonzero. It *is* possible that the isomorphism $R\alpha_*\mathbf{Z}(i) \simeq \mathbf{Z}(i)_{\text{zar}}$ may hold up to torsion.

(5) The rth cohomology sheaf $\mathscr{H}^r(X, \mathbf{Z}(r))$ of the complex of Zariski sheaves on X is isomorphic to the Zariski sheaf $\mathbf{K}^M_r(X)$ of Milnor K-groups on X. (This is the same as $R^r\alpha_*\mathbf{Z}(r)$, by Axiom 4).

(6) Let X be regular. Then there is an Atiyah-Hirzebruch spectral sequence starting from the Zariski motivic cohomology groups and converging to algebraic K-theory. More precisely, $H^p(X, \mathbf{Z}(-q/2)_{\text{zar}}) \Rightarrow K_{-p-q}(X)$ where $\mathbf{Z}(-q/2) = 0$ if q is odd. Furthermore, this spectral sequence should

degenerate up to torsion, in such a way that $H^p(X, \mathbf{Z}(-q/2)_{\mathrm{zar}}) \otimes \mathbf{Q} \simeq K^{(r)}_{-p-q} \otimes \mathbf{Q}$, where $r = -q/2$, and $K_i(X)^{(r)}$, as in §2, is the Adams weight r piece of $K_i(X)$.

(This agrees well with low-degree evidence, but does not in my mind have a satisfactory motivic explanation. Grayson has suggested that a motivic cohomology theory with such a spectral sequence might exist for any exact category. But why should the motivic cohomology of the category of coherent sheaves on X be the Zariski motivic cohomology rather than the étale one? Also, if X is not regular, are there two such spectral sequences, one with $\mathbf{Z}$ and K, the other with $\mathbf{Z}'$ and K'?)

If F is a field and $X = \operatorname{Spec} F$, the supposed fact (Axiom 1) that $H^p(X, \mathbf{Z}(r)) = 0$ for $p \leq 0$ would imply (Axiom 6) that $K_q^{(r)}(F)$ is torsion for $q \geq 2r$, which is the strong form of the Soulé-Beilinson conjecture ("strong" because $q \geq 2r$ rather than $q > 2r$). Axioms 4 and 5 imply that $H^r_{\text{ét}}(F, \mathbf{Z}(r)) = K_r^M(F)$, and Axiom 4 implies $H^{r+1}_{\text{ét}}(F, \mathbf{Z}(r)) = 0$. So Axiom (2a) implies that if $\operatorname{char}(F) \nmid r$, $K_r^M(F)/nK_r^M(F) \simeq H^r(F, \mu_n^{\otimes r})$; i.e., the axioms collectively imply the Kato conjecture.

Also the axioms imply that $H^p(F, \mathbf{Z}(r))$ is, up to torsion, the weight r piece of $K_{2r-p}(F)$ in either the Zariski or étale sites.

Returning to our original questions, the $H^p(X, \mathbf{Z}(r)_{\text{ét}})$ conjecturally give the value of the zeta-function of X up to sign and p-torsion, if X is a projective nonsingular variety over a finite field of characteristic p. If $\dim X = 2$, $H^3(X, \mathbf{Z}(1)_{\text{ét}})$ is the Brauer group $\operatorname{Br}(X)$ of X, whose order intervenes in the standard formulas for $\zeta(X, 2)$, which are true if $\operatorname{Br}(X)$ is finite **[T, M]**. The Zariski topology holds no hope of locating $\operatorname{Br}(X)$. So the stronger forms of Question 1 need the étale rather than the Zariski site.

For Question 2 the Kummer and Milne sequences directly relate motivic étale cohomology to ℓ-adic étale cohomology and crystalline cohomology theories, so again the étale site seems preferable.

For Question 3, the conjectured Atiyah–Hirzebruch spectral sequence involves the Zariski site, which as a result seems to be closer to algebraic K-theory.

4. Candidates for motivic cohomology complexes

(a) The first candidate beyond the obvious ones is the scissors congruence complex, essentially introduced by Sah and Dupont **[DS]** to study a geometric problem far removed from anything discussed in this article. The connection remains mysterious. Let F be a field, and let $Z(F - \{0, 1\})$ denote the free abelian group on the elements of F different from 0 and 1. Let $C(F)$, the "scissors congruence group", denote the quotient of $Z(F - \{0, 1\})$ by all relations of the form

$$[x] - [y] + [y/x] - [(1-y)/(1-x)] + [x(1-y)/y(1-x)]$$

for $x \neq y$ in $F - 0, 1$. Let $D(F)$ denote the quotient of $F^* \otimes_{\mathbf{Z}} F^*$ by all elements of the form $x \otimes y + y \otimes x$. Then the map that sends $[x]$ to $x^{-1} \otimes 1 - x^{-1}$ induces a map from $C(F)$ to $D(F)$, and the complex $B(F)$: $C(F) \to D(F)$ is an approximation to the complex $\mathbf{Z}(2)$ on F.

The desired properties for $\mathbf{Z}(2)$ as a complex of Zariski sheaves on $\operatorname{Spec} F$ (i.e., abelian groups) imply that $H^i(F, \mathbf{Z}(2))$ should be zero for $i \neq 1, 2$, equal to $K_2(F)$ for $i = 2$, and equal to the indecomposable part $K_3(F)[\text{ind}]$ for $i = 1$.

The complex $B(F)$ satisfies these conditions except that $H^1(F, B(F))$ is only isomorphic to $K_3(F)[\text{ind}]$ up to torsion. It does not seem possible to find a complex given explicitly by formulas that does better. Goncharov has found higher weight generalizations **[Go1, Go2]** which conjecturally should be $\mathbf{Z}(r)$ up to torsion, but relatively little is known.

(b) The best candidates so far for the complexes $\mathbf{Z}'(r)$ are Bloch's higher Chow groups **[Bl1]**. We recall the definition here:

Let Y be a scheme of finite type over a field k. The group of codimension d cycles $z^d(Y)$ is defined to be the free abelian group with generators the reduced and irreducible (integral) subschemes of Y of codimension d. If $W \subseteq Y$ is a closed subscheme which is a local complete intersection, there is a pull-back map $i^*: z^d(y)' \to z^d(W)$ defined, where $z^d(Y)' \subseteq z^d(Y)$ is the subgroup generated by integral subschemes of Y meeting W properly, i.e., in a subscheme of the correct dimension or in the empty set. Let

$$\Delta^n = \operatorname{Spec} k[t_0, \dots, t_n]/(t_0 + \cdots + t_n - 1).$$

Given an increasing map $\rho: \{0 \cdots m\} \to \{0 \cdots n\}$, we define $\tilde{\rho}: \Delta^n \to \Delta^m$ by $\tilde{\rho}(t_i) = \sum t_j$ over those j such that $\rho(j) = i$, and zero if there are none. If ρ is injective, we say ρ is a *face*. If ρ is surjective, ρ is a *degeneracy*. Define $z^d(X, n)$ to be the subgroup of $z^d(X \times \Delta^n)$ generated by integral subschemes meeting all faces $X \times \Delta^m \subseteq X \times \Delta^n$ properly. The $z^d(X, n)$ are stable under pullback by the degeneracy maps. We obtain in this way a simplicial abelian group $z^d(X, \cdot)$, whose faces and degeneracy maps are induced from the above face and degeneracy maps.

We define the higher Chow group complex $ZB^\bullet(X, r)$ by setting $ZB^n(X, r) = z^r(X, 2r - n)$ and taking the complex associated with the resulting simplicial abelian group. Note that we have changed from homological to cohomological indexing and shifted degrees by $2r$.

Since $ZB^\bullet(X, r)$ is contravariant for flat morphisms, we can sheafify $ZB^\bullet(X, r)$ in the étale and Zariski sites (for U over X, take the presheaf of complexes $ZB^\bullet(U, r)$ and take the associated sheaf), obtaining Bloch's candidates for the étale and Zariski motivic cohomology complexes which we denote by $\widetilde{ZB}^\bullet(X, r)$.

These complexes satisfy some of the "axioms" for motivic cohomology complexes. It is easy to see that $\widetilde{ZB}^\bullet(X, 0) = \mathbf{Z}'_X(0)$, and Nart has shown [N] that $\widetilde{ZB}^\bullet(X, 1) = \mathbf{Z}'(X, 1)$.

It has been shown independently by Suslin **[Su2]** and Totaro **[T]** that the complexes $ZB^{\bullet}(X, r)$ **[Zar]** satisfy the Milnor K-theory axiom when X is a field.

Unfortunately, Suslin found a gap in Bloch's proof of the localization sequence for the higher Chow group complexes, thus casting doubt on many of the results of **[Bl1]**. (Bloch has recently announced that he now has a correct proof.)

(c) We recall the construction in **[L2]** of a candidate $\mathbf{Z}L(2)$ for $\mathbf{Z}(2)$:

Let A be a regular commutative ring, let $W = \operatorname{Spec} A(T)$, and let $Z = \operatorname{Spec} A[T]/T(T-1)$. Let $B = b_1, b_2, \dots, b_n$ be a finite sequence of "exceptional units" of A, i.e., b_i and $1 - b_i$ are both units for all i. Let $Y_B = \operatorname{Spec} A[T]/\prod(T - b_i)$. Then there is an exact sequence coming from the localization sequence in relative K'-theory **[Bl2]**:

$$K_3(A) \to K_2(W - Y_B, Z) \to K_1'(Y_B) \to K_2(A).$$

Let $C(A)$ and $D(A)$ denote the direct limits of the two middle terms of this exact sequence, where the direct limit is taken over finite sets B of exceptional units ordered by inclusion, so we have a two-term complex $C(A) \to D(A)$.

If we sheafify this complex as above in (b) we obtain the complex $\mathbf{Z}L(2)$ on nonsingular schemes X. This complex satisfies **[L2, L3]** properties (0)–(5) and (7) above, except for possible 2-torsion problems, and for the Beilinson-Soulé conjecture. (We do not know that the weight 2 piece of $K_j(F)$ is zero for $j \geq 4$.)

In the case where X is not regular, the natural extension of this definition seems more closely related to $\mathbf{Z}'(2)$ rather than $\mathbf{Z}(2)$. It is not clear in any case what the analogous definition of $\mathbf{Z}(r)$ should be for $r > 2$.

(d) In **[BMS]** five definitions of motivic cohomology complexes were proposed. The first three were essentially variants of a linearized version of Bloch's higher Chow group complexes. These, however, were shown by Gerdes **[Ge]** not to satisfy property (7) for weight 2. Nonetheless they are very interesting objects of study, and their relationships to the true motivic cohomology complexes should continue to be explored. The last two definitions also look interesting, but I am not competent to discuss them further.

(e) Beilinson has emphasized the point of view that the motivic cohomology groups $H^i(X, \mathbf{Z}(r))$ should be $\operatorname{Ext}^i(\mathbf{Z}(0), \mathbf{Z}(r))$, where the Ext groups are taken in the hypothetical category of mixed motives on X. This is probably the natural way to view the situation, and we will return to this idea in the next section. Voevodsky **[V]** has embarked on a very ambitious program leading toward the construction of mixed motives satisfying many striking theorems. Levine **[Le]** also has an impressive construction of motivic cohomology from a very formal point of view.

5. Toward a theory of mixed motives

A sophisticated analysis of the stable homotopy of topological spaces leads to the beautiful theory of spectra **[A]**, and this theory should have an analogue in algebraic geometry. This we do not propose to discuss in this paper, but I would like to thank Tom Goodwillie for a long and still continuing series of conversations on this and the other topics discussed in the final two sections of this paper.

However, if one is only interested in the homology of topological spaces, it suffices to look at the "abelian objects" in the category of spectra. These "abelian objects" turn out to be bounded below chain complexes of abelian groups, up to quasi-isomorphism, i.e., the derived category of abelian groups. In order to discuss the homology of algebraic varieties, we should construct an algebraic analogue of this category.

To fix our ideas, we work in the category of algebraic varieties over a fixed algebraically closed field K. The most obvious candidate for an analogue is the category of complexes of commutative group schemes of finite type over K, perhaps up to quasi-isomorphism with some suitable definition. This shows some promise, since this category contains $\mathbf{Z}(0) = \mathbf{Z}$, $\mathbf{Z}(1) = G_m[-1]$, and Deligne's 1-motives **[D]**. (Recall that these consist of complexes $X \to G$ where X is a finitely generated free abelian group and G is an extension of an abelian variety by a torus.)

However, this category does not admit a reasonable notion of tensor product. In addition, the nonrepresentability of the group of zero-cycles modulo rational equivalence on a surface would force us to leave this category.

Before defining an enlargement of this category, recall that in the first part of this paper we observe that the category of mixed motives over K should map to the derived category of sheaves of abelian groups in the big étale or Zariski site on K. In particular, if M is a mixed motive, $M(K)$ should be a complex of abelian groups up to quasi-isomorphism.

Suslin **[Su1]** constructed for any variety Y over K, a very interesting candidate for such a complex. Let $\Delta^\bullet$ be the cosimplicial scheme over K which was used in the definition of the higher Chow group complex. Let $\mathrm{Sus}^n(Y)$ be the free abelian group generated by irreducible subvarieties of $\Delta^n \times Y$, finite and surjective over Δ^n. $\mathrm{Sus}^\bullet(Y)$ is clearly a simplicial abelian group, so we may regard it as a complex of abelian groups. Question: Is $\mathrm{Sus}^\bullet(Y)$ in some sense the points with values in K of the homological motive of Y?

If Y is a curve, then this seems very reasonable. In **[L4]** we show that $\mathrm{Sus}^\bullet(Y)$ has a filtration in the derived category sense whose associated graded pieces are $h_0(Y)$, $h_1(Y)$, and $h_2(Y)$. (Here $h_0(Y) = \mathbf{Z}^e$, where e is the number of connected components of Y. $\mathbf{Z}^e$ should be thought of as a 0-motive over K, i.e., a finitely generated abelian group. $h_2(Y) = (G_m[-1]^f)(K)$ where f is the number of complete irreducible components of Y. $h_1(Y)$ is in a very plausible sense the *homological* Deligne 1-motive attached to Y. Since $h_0(Y)$ is a 0-motive and $h_1(Y)$ is a 1-motive, $h_2(Y)$ should be interpreted as a 2-motive, which we tentatively think of as a three-term complex, in homological degrees 2, 1, 0 of the form $X \to G \to ?$, where X is a finitely generated abelian-group, G is an extension of an abelian variety by a torus, and ? is an object of unknown type. In the case of $h_2(Y)$, we have X and $? = 0$, $G = G_m^f$.)

Sulsin, however, did *more* than just describe this complex of abelian groups; in some sense he described it as the points with values in K of a mixed motive, without using that language. For $\mathrm{Sus}^\bullet(Y)$ may be viewed as the complex obtained by taking $\mathrm{Map}(\Delta^\bullet, \mathrm{Sp}(Y))$ and then taking the naive group completion. (Here $\mathrm{Sp}(Y)$ is the infinite symmetric product of Y, and the naive group completion is just the familiar process which constructs the integers from the natural numbers.) Suslin goes on to view $\mathrm{Map}(\Delta^n, \mathrm{Sp}(Y))$ as the K-valued points of an ind-scheme $\mathbf{Map}(\Delta^n, \mathrm{Sp}(Y))$ (inductive limit of schemes). In fact, the obvious addition in $\mathrm{Sp}(Y)$ makes it and, hence, $\mathbf{Map}(\Delta^n, \mathrm{Sp}(Y))$ into a commutative monoid ind-scheme. (*Not* an ind-monoid scheme; the monoid structure only exists after passing to the inductive limit.) So $\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(Y))$ becomes a simplicial commutative monoid ind-scheme (SCMI).

Now, returning to our original problem, we see that we have reached the category of SCMIs. But although a simplicial abelian group may be thought of as an Ω-spectrum by delooping it using the bar construction (as in the appendix), and although the category of such Ω-spectra is equivalent to the derived category of abelian groups, the Ω-spectrum determined by a simplicial monoid does not determine the monoid.

So the more natural construction is "spectra associated with SCMIs", i.e., a sequence of SCMIs, $\{E_n\}$ where $E_n = B^n A_0^\bullet$, and $A_0^\bullet$ is a fixed SCMI. This then is a possible candidate for the category of mixed motives. It is a rather large category, and it may be useful to identify those objects in it that are in some sense of finite type, but it will have to do for now.

6. Motive cohomology

Returning to our discussion in §2, we now have a category of mixed motives, with a $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$ but without a tensor product. We can, however, make some tentative steps toward a definition of this. Namely, we can try to define the tensor product of two *Suslin motives*, i.e., motives of the form $\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(Y))$.

We in fact would like the formula

$$\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(\widetilde{Y}_1)) \widetilde{\otimes} \mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(\widetilde{Y}_2)) = \mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(\widetilde{Y}_1 \wedge \widetilde{Y}_2)).$$

Here $\widetilde{Y}_1$, $\widetilde{Y}_2$ are pointed schemes. (If $\widetilde{Y} = (Y, *)$ is a pointed scheme, we define $\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}\,\widetilde{Y})$ to be $\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}\,\widetilde{Y})/\mathbf{Map}(\Delta^\bullet, *))$. Similarly, $\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(\widetilde{Y}_1 \wedge \widetilde{Y}_2)) = \mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(\widetilde{Y}_1 \times \widetilde{Y}_2))/\mathbf{Map}(\Delta^\bullet, \mathrm{Sp}(\widetilde{Y}_1 \vee \widetilde{Y}_2))$. (It is not totally clear in what sense we can take quotients here, but we would like it to go over to quotient complexes (cones ?) when we sheafify.)

EXAMPLE. $\mathbf{Map}(\Delta^\bullet, \mathbf{P}^1 - \{0, \infty\}, \{1\}) \simeq G_m = \mathbf{Z}(1)[1]$. Let $W = (\mathbf{P}^1 - \{0, \infty\}, \{1\})$. Then $\mathbf{Z}(r)$ should be given as $\mathbf{Map}(\Delta^\bullet, W^{\wedge r})[r]$. This definition makes sense in the category of complexes of sheaves to give $\mathbf{Z}(r)$ as an étale or Zariski complex of sheaves, and we can now define motivic cohomology.

QUESTION. Is it possible to construct a series of mixed motives in this sense whose points with values in K yield Bloch's higher Chow group complexes? For a *nonsingular* curve Y, the codimension-one Bloch complex yields the *cohomological* 1-motive of Y in the same sense as the Suslin motive yields the homological 1-motive.

Appendix: Brief review of spectra

The closest analogue of our algebraic situation is the category of simplicial sets rather than topological spaces. We give a brief resumé of the basic facts about spectra of simplicial pointed sets (see **[C]** for basic simplicial set information).

If $T^\bullet$ is a pointed simplicial set, we can define the *suspension* $\Sigma T^\bullet$ by $(\Sigma T)^n = T^n \wedge S^1$, where S^1 is the simplicial 1-sphere. The face maps and degeneracies are clear. A spectrum is a sequence $\{E_n\}_{n\geq 0}$ where each E_r is a pointed simplicial set, and we are given maps ϕ_n of pointed simplicial sets from ΣE_n to E_{n+1} for all n. We define the *loop space* $\Omega T^\bullet$ of a pointed simplicial set so that we have $\mathrm{Hom}(\Sigma X, Y) = (X, \Omega Y)$. Equivalently, in the definition of spectrum, we could have required maps ψ_n of pointed simplicial sets from E_n to ΩE_{n+1}. If the ϕ_n's are isomorphisms (up to homotopy), we say that $\{R_n\}$ is the *suspension spectrum* of E_0. If the ψ_n's are all isomorphisms, we say $\{E_n\}$ is an Ω-spectrum. If $A^\bullet$ is a simplicial abelian group, the *bar-construction* $BA^\bullet$ provides a delooping of $A^\bullet$; i.e., A is homotopic to $\Omega BA^\bullet$. For our purposes we define $(BA)_n = A_{n-1} \times A_{n-2} \times \cdots \times A_0$, and the face and degeneracy operators are given by the formulas $d_i(a_0, \ldots, a_n) = (d_{i-1}a_0, d_{i-2}a_1, \ldots, d_1 a_{i-1}, d_0 a_i \cdot a_{i+1}, a_{i+2}, \ldots, a_n)$ and $s_j(a_0, \ldots, a_n) = (s_{i-1}a_0, s_{i-2}a_1, \ldots, s_0 a_i, e_n, a_{i+1}, a_{i+2}, \ldots, a_n)$, where e_n is the identity element of A_n and $\bullet$ denotes the group multiplication. $BA^\bullet$ then becomes a simplicial abelian group.

So the sequence $\{B^n A^\bullet\}$ is an Ω-spectrum. (Note. This is *not* the same as the suspension spectrum of A.) If we only require that $A^\bullet$ be a simplicial abelian monoid rather than a group, then $\{B^n A^\bullet\}$ is still a spectrum, because there are still maps from $A^\bullet$ to $\Omega BA^\bullet$, but it is no longer an Ω-spectrum. In fact, by an unpublished result of Quillen, $\Omega BA^\bullet$ is homotopic to the *naive group completion* $(A^\bullet)^+$ of $A^\bullet$, where $(A^+)_n = A_n^+ =$ the set of ordered pairs (a, b), $a, b \in A_n$, modulo the equivalence relation $(a, b) \sim (c, d)$ iff $\exists e\colon a + d + e = b + c + e$.

REFERENCES

[A] J. F. Adams, *Stable Homotopy and Generalized Homotopy*, The University of Chicago Press, Chicago, 1974.

[Be] A. A. Beilinson, *Height pairing between algebraic cycles*, Current Trends in Arithmetical Algebraic Geometry, Contemp. Math **67** (1987), 1–24.

[BMS] A. Beilinson, R. MacPherson, and V. Schechtman, *Notes on motivic cohomology*, Duke Math. J. **54** (1987), 679–710.

[BK] S. Bloch and K. Kato, *L-Functions and Tamagawa Numbers of Motives*, The Grothendieck-Festschrift, vol. I, Birkhäuser, Boston, MA, 1990, pp. 333–401.

[Bl1] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[Bl2] ——, *Higher regulators, algebraic K-theory, and zeta-functions of elliptic curves*, Lecture notes, U. C. Irvine, 1977.

[C] E. B. Curtis, *Simplicial homotopy theory*, Adv. Math. **6** (1971), 107–209.

[D] P. Deligne, *Théorie de Hodge*. III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.

[Den] C. Deninger, *Duality in the étale cohomology of one-dimensional proper schemes and generalizations*, Math. Ann. **277** (1987), 529–541.

[DS] J. L. Dupont and C.-H. Sah, *Scissors congruences*. II, J. Pure Appl. Algebra **25** (1982), 159–195.

[Ge] W. Gerdes, *Grassmannian homology, algebraic K-theory, and higher Chow groups for an infinite field*, M.I.T. thesis, 1989.

[Go1] A. B. Goncharov, *Geometry of configurations, polylogarithms and motivic cohomology*, preprint of the Max-Planck-Institut für Mathematik, 1991, Adv. Math., submitted.

[Go2] ——, *Polylogarithms and motivic Galois groups*, these Proceedings, vol. 2, pp. 43–96.

[Le] M. Levine, *The derived motivic category*, preprint.

[L1] S. Lichtenbaum, *Values of zeta-functions at non-negative integers*, Number Theory, Noordwijkerhout 1983, Lecture Notes in Math., vol. 1068, Springer-Verlag, New York, 1984, pp. 127–138.

[L2] ——, *The construction of weight-two arithmetic cohomology*, Invent. Math. **88** (1987), 183–215.

[L3] ——, *New results on weight-two motivic cohomology*, The Grothendieck Festschrift, vol. III, Birkhäuser, Boston, MA, 1990, pp. 35–55.

[L4] ——, *Suslin complexes and Deligne* 1*-motives* (to appear).

[Ma] Ju. I. Manin, *Correspondences, motifs, and monoidal transformations*, Math. USSR-Sb. **6** (1968), 439–469; Russian, Mat. Sb. **77** (1968), 475–507.

[Mi] J. Milne, *Motivic cohomology and values of zeta-functions*, Compositio Math. **68** (1988), 59–102.

[N] E. Nart, *The Bloch complex in codimension one and arithmetic duality*, J. Number Theory **32** (1989), 321–331.

[So] C. Soulé, *Operations en K-théorie algébrique*, Canad. J. Math. **37** (1985), 488–550.

[Su1] A. A. Suslin, Lecture at K-theory conference, Luminy, 1988.

[Su2] ——(with Yu. P. Nesterenko), *Homology of the general linear group over a local ring, and Milnor K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 121–146; English transl., Math. USSR-Izv. **34** (1990), 121–145.

[T] B. Totaro, *K-theory and algebraic cycles, K*-Theory (to appear).

[V] V. Voevodsky, *Homology of schemes and covariant motives*, Harvard Ph.D. Thesis, 1992.

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On the Bijectivity of Some Cycle Maps

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Let X be a smooth projective variety over an algebraic number field k. It has been conjectured (or hoped) by Beilinson **[1]** and others that there would exist a (triangulated) category of *mixed motivic sheaves* on X with a constant object $\mathbb{Q}_X^M$ such that the cycle map

$$\text{(0.1)} \qquad \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}(\mathbb{Q}_X^M, \mathbb{Q}_X^M(p))$$

is an isomorphism, where $\mathrm{CH}^p(X)_{\mathbb{Q}}$ is the Chow group of codimension p algebraic cycles on X with rational coefficients and (p) is the Tate twist. (See **[18]** for the relation with the Hodge type conjecture.)

In **[15]**, we constructed a category such that (0.1) is an isomorphism for $p \le 2$ in the case $k = \mathbb{C}$. Modifying the argument and using **[16]** with results of **[3]**, we show:

(0.2) THEOREM. *If k is embeddable in $\mathbb{C}$, there exists a category such that* (0.1) *is surjective for $p \le 3$ and injective for $p \le 4$. Furthermore,* (0.1) *is surjective also for $p = 4$ if $k = \mathbb{C}$.*

See (3.3) and (3.5). The main idea is that we use sheaf-theoretic operations on X to control the *geometric level* (see (1.3)). The category is defined so that this kind of argument can apply as much as possible. More precisely, let $\{\mathscr{M}(X)\}$ be a theory of $\mathbb{Q}$-mixed sheaves satisfying the conditions (i) and (ii) in (1.2). The category in (0.2) is defined to be the derived category of a full subcategory of $\mathscr{M}(X)$ satisfying some good properties (1.5). The key point is the notion of geometric level which is used to control the (Yoneda) extension which appears in the target of (0.1). For example, an object of the full subcategory should have geometric level $\le n$ if its support has dimension n and codimension ≤ 2. The restriction on codimension, which implies the bound of codimension p in the theorem, comes from the condition that the

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category should be stable by some standard sheaf-theoretic operations (see (1.5)). For the proof of (0.2), we introduce also another full subcategory of $\mathscr{M}(X)$ for X equidimensional, to apply an inductive argument. These full subcategories are actually constructed using strict support decomposition and geometric level. See (2.1).

In §1, we study the properties of the full subcategories. These categories are constructed in §2, and Theorem (0.2) is proved in §3.

In this paper, varieties are defined over a field k and assumed separated.

1. Mixed sheaves

(1.1) Let k be a subfield of $\mathbb{C}$. Let $\{\mathscr{M}(X)\}$ be a theory of $\mathbb{Q}$-mixed sheaves on varieties X over k (see **[16]**). By definition, $\mathscr{M}(X)$ is an abelian category with a forgetful functor $\mathrm{For} : \mathscr{M}(X) \to \mathrm{Perv}(X(\mathbb{C}), \mathbb{Q})$ which is exact and faithful, and each object M of $\mathscr{M}(X)$ has a functorially defined *weight filtration* W such that $M \to \mathrm{Gr}_i^W M$ is an exact functor and $\mathrm{Gr}_i^W M$ are semisimple. Furthermore, there should exist the dual functor $\mathbb{D}$, the external product $\boxtimes$, the pull-back j^* by an open embedding j, and the (cohomological) direct image $H^j f_*$ by an affine morphism f between the categories $\mathscr{M}(X)$. These functors should satisfy some natural compatibilities and should be compatible with the corresponding functors of perverse sheaves by the functor For. If X is smooth, there should exist a constant object $\mathbb{Q}_X^M[\dim X]$ in $\mathscr{M}(X)$.

Then we can construct the standard functors $f_*, f_!, f^*, f^!, \psi_g, \varphi_{g,1}$, $\mathbb{D}, \boxtimes, \otimes, \mathscr{H}om$ between the bounded derived categories $D^b\mathscr{M}(X)$ so that they are compatible with the corresponding functors on the underlying $\mathbb{Q}$-complexes by the forgetful functor $\mathrm{For} : D^b\mathscr{M}(X) \to D_c^b(X(\mathbb{C}), \mathbb{Q})$. See [loc. cit.]. Furthermore, we have the constant object $\mathbb{Q}_X^M = (a_X)^*\mathbb{Q}_{\mathrm{Spec}\, k}^M$ in $D^b\mathscr{M}(X)$ for any X, where $a_X : X \to \mathrm{Spec}\, k$ is a natural morphism. See **[16,** (3.9)].

REMARK. The above functors satisfy some good properties. For example, we have an equivalence of categories

$$i_* : \mathscr{M}(X) \xrightarrow{\sim} \mathscr{M}_X(Y) \tag{1.1.1}$$

for a closed embedding $i : X \to Y$, where $\mathscr{M}_X(Y)$ is the full subcategory of $\mathscr{M}(Y)$ consisting of objects whose supports are contained in X. See **[16,** (1.4.6)]. We have also natural isomorphisms

$$\psi_g f_* = f_*\psi_{gf}, \qquad \varphi_{g,1} f_* = f_*\varphi_{gf,1} \tag{1.1.2}$$

for a proper morphism $f : X \to Y$ and a function g on Y. See **[16,** (5.10)]. If $M \in \mathscr{M}(X)$ is pure and f is proper, we have a (noncanonical) decomposition

$$f_*M \simeq \bigoplus_j (H^j f_* M)[-j] \quad \text{in } D^b\mathscr{M}(Y) \tag{1.1.3}$$

by **[14,** II, (4.5)], because f_*M is pure by **[16,** (6.7)].

(1.2) Besides the axiom of mixed sheaves, we shall assume also that $\{\mathscr{M}(X)\}$ satisfies the following conditions:

(i) The forgetful functor For : $\mathscr{M}(X) \to \mathrm{Perv}(X(\mathbb{C}), \mathbb{Q})$ factors naturally through the category of mixed Hodge Modules $\mathrm{MHM}(X_{\mathbb{C}})$.

(ii) $\mathrm{Hom}(\mathbb{Q}_X^M, \mathbb{Q}_X^M) = \mathbb{Q}$ for X smooth connected.

The condition (i) is trivially satisfied for the examples in **[16,** (1.8)], where $A = \mathbb{Q}$. As to condition (ii), we have to assume that k is algebraically closed (e.g., $k = \mathbb{C}$) in the case of example (i) in [loc. cit.]. Since condition (ii) is equivalent to $\mathrm{Hom}(\mathbb{Q}^M, H^0(X, \mathbb{Q}_X^M)) = \mathbb{Q}$ (i.e., the $\mathscr{M}$-Hodge conjecture for $p = 0$ in **[16,** (8.5)]) by the adjunction for $X \to \mathrm{Spec}\, k$, it is checked using the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ on the ℓ-adic part in the case of examples (iii)–(v) in [loc. cit.]. (In the case of example (ii), condition (ii) is reduced to the case $X = \mathrm{Spec}\, K$ for a finite extension K of k using the pull-back to a closed point of X, and we check it using the fact that $K \otimes_k \mathbb{C}$ is a direct sum of copies of $\mathbb{C}$ so that the projections to the direct factors induce the embeddings of K in $\mathbb{C}$.) For the examples in **[16,** (1.8)], the conditions of mixed sheaves are verified using **[3]** (and also **[17,** (1.7.2)] for the duality). Here we can also use its full subcategory $\mathscr{M}(X)^{\mathrm{go}}$ consisting of objects of geometric origin.

Remark. In example (v) of **[16,** (1.8)], the objects of $\mathscr{M}(X)$ for a smooth variety X consist of $((M, F), K_\sigma, K_l; W)$ where (M, F) is a filtered regular holonomic $\mathscr{D}$-module on X, K_σ is a $\mathbb{Q}$-perverse sheaf on $X_\sigma(\mathbb{C})$ with $X_\sigma := X \otimes_{k,\sigma} \mathbb{C}$ for each embedding $\sigma : k \to \mathbb{C}$, K_l is an étale $\mathbb{Q}_l$-perverse sheaf **[3]** on $\overline{X} := X \otimes_k \overline{k}$ with an action of $\mathrm{Gal}(\overline{k}/k)$ (compatible with the natural action on $\overline{X}$) for each prime number l, and W is a finite filtration on M, K_σ, K_l, such that they have comparison isomorphisms (compatible with W) $\mathrm{DR}(M \otimes_{k,\sigma} \mathbb{C}) = K_\sigma \otimes_{\mathbb{Q}} \mathbb{C}$ and $\varepsilon^* \overline{\sigma}^* K_l = K_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_l$ for $\overline{\sigma} : \overline{k} \to \mathbb{C}$ extending σ (compatible with the action of $\mathrm{Gal}(\overline{k}/k)$) as in **[6, 7, 9, 11]** (see **[3]** for the functor ε^*). We assume that $((M \otimes_{k,\sigma} \mathbb{C}, F), K_\sigma; W)$ is a mixed Hodge module on X_σ for any σ, and each graded piece $\mathrm{Gr}_k^W((M, F), K_\sigma, K_l)$ has a pairing that induces a polarization of the mixed Hodge module on X_σ to assure the semisimplicity of pure objects. (It might be possible to improve the construction of $\mathscr{M}(X)$ if one uses a theory of Ekedahl and Jannsen in **[10]**.)

(1.3) We say that $M \in \mathscr{M}(X)$ has *geometric level* $\leq n$ (see **[16,** (7.3)]) if each graded piece $\mathrm{Gr}_i^W M$ is isomorphic to a direct factor of $H^j \pi_* \mathbb{Q}_Y^M(r)$ for a projective morphism $\pi : Y \to X$ of a smooth variety Y of dimension $\leq n$ (where $i = j - 2r$). Using the hard and weak Lefschetz theorems, we may assume $j = \dim Y$ so that

$$n - w \in 2\mathbb{Z}, \tag{1.3.1}$$

if M is pure of weight w (i.e., $\mathrm{Gr}_i^W M = 0$ for $i \neq w$) and M has level n

(i.e., level $\leq n$ and not level $\leq n-1$). See Remark (ii) after [loc. cit.].

We will denote by $\mathscr{M}(X)_{\mathrm{gl}\leq n}$ the full subcategory consisting of objects of geometric level $\leq n$. This full subcategory is stable by the dual functor $\mathbb{D}$, because π_* commutes with $\mathbb{D}$ by duality. See [**16,** (2.4.2) and (4.12)].

We say that a pure object M has *strict support* Z for an irreducible closed variety Z, if $\operatorname{supp} M = Z$ and M has no nontrivial sub- or quotient objects with smaller supports. By the semisimplicity of pure objects, every pure object M has a (unique) decomposition by strict support :

$$M = \bigoplus_Z M_Z \tag{1.3.2}$$

where M_Z has strict support Z or $\varnothing$, and is called the direct factor of M with strict support Z. Note that the functor $M \to M_Z$ is exact.

Let X be an irreducible variety of dimension n and $j : U \to X$ denote the inclusion of a dense open subvariety U of X. Let M be a pure object with strict support X and $M' = j^*M$. Then we have a natural isomorphism

$$M = j_{!*}M' \ (:= \operatorname{Im}(H^0 j_! M' \to H^0 j_* M')), \tag{1.3.3}$$

because the right-hand side has no nontrivial sub- or quotient objects whose support is contained in $X \setminus U$. (Here $j_{!*}$ is called the intermediate direct image following [**3**].) In particular, we get

$$\operatorname{For}(M) = \mathrm{IC}_X L \ (:= j_{!*}(L[n])), \tag{1.3.4}$$

if U is smooth and $L := \operatorname{For}(M')[-n]$ is a local system. Here $\mathrm{IC}_X L$ is called the *intersection complex* with coefficients L. If $M' = \mathbb{Q}_U^M[n]$ (in particular, L is the constant sheaf $\mathbb{Q}_U$), then $j_{!*}(\mathbb{Q}_U^M[n])$ is denoted by $\mathrm{IC}_X\mathbb{Q}^M$ and is called the *intersection complex* in $\mathscr{M}(X)$. It is pure of weight n and has strict support X. We have the self-duality

$$\mathbb{D}(\mathrm{IC}_X\mathbb{Q}^M) = \mathrm{IC}_X\mathbb{Q}^M(n), \tag{1.3.5}$$

because the definition of the intermediate direct image in (1.3.3) is self-dual.

Remark. (i) If M is pure with strict support Z and $M|_U$ has geometric level $\leq n$ for a dense open subvariety U of Z, then M has geometric level $\leq n$. This follows from Nagata-Hironaka, which we apply to the morphism π in the definition of geometric level.

(ii) If M has geometric level $\leq n$ and is pure with strict support Z of dimension n, then M is isomorphic to a direct factor of $H^0\pi_*\mathbb{Q}_Y^M(r)$ for π as above, so that Y is generically finite over Z.

(1.4) For a variety X of dimension n, let X_j $(1 \leq j \leq r)$ be the irreducible components of X with dimension n, $X' = \bigcup_{1\leq j\leq r} X_j$, and $\mathrm{IC}_{X'}\mathbb{Q}^M = \bigoplus_{1\leq j\leq r}\mathrm{IC}_{X_j}\mathbb{Q}^M$. By [**16,** (3.9)], we have $\mathbb{Q}_X^M \in D^b\mathscr{M}(X)$ and its dual $\mathbb{D}_X^M = \mathbb{D}(\mathbb{Q}_X^M) \in D^b\mathscr{M}(X)$. By **[16**, (7.15)], we have

$$H^i\mathbb{Q}_X^M = 0 \quad \text{for} \quad i > n,$$
$$\mathrm{Gr}_i^W H^n\mathbb{Q}_X^M = 0 \quad \text{for} \quad i > n, \tag{1.4.1}$$
$$\mathrm{Gr}_n^W H^n\mathbb{Q}_X^M = \mathrm{IC}_{X'}\mathbb{Q}^M.$$

This implies natural morphisms

$$\mathbb{Q}_X^M[n] \to \mathrm{IC}_{X'}\mathbb{Q}^M \to \mathbb{D}_X^M(-n)[-n], \tag{1.4.2}$$

using (1.3.5) for the second morphism. If condition (ii) of (1.2) is satisfied, we get isomorphisms

$$\begin{aligned} \mathrm{Hom}(\mathbb{Q}_X^M[n], \mathbb{D}_X^M(-n)[-n]) &\xleftarrow{\sim} \mathrm{Hom}(\mathrm{IC}_{X'}\mathbb{Q}^M, \mathbb{D}_X^M(-n)[-n]) \\ &\xleftarrow{\sim} \mathrm{End}(\mathrm{IC}_{X'}\mathbb{Q}^M) = \bigoplus_{1\le j\le r} \mathrm{End}(\mathrm{IC}_{X_j}\mathbb{Q}^M) = \bigoplus^r \mathbb{Q}, \end{aligned} \tag{1.4.3}$$

because the first isomorphism follows from (1.4.1), the second from its dual, and the last from condition (ii) in (1.2) together with (1.3.3). Note that the morphisms of (1.4.2) are isomorphisms if X is smooth.

REMARK. The isomorphisms of (1.4.3) hold also for an abelian full subcategory of $\mathscr{M}(X)$ such that $\mathbb{Q}_X^M$, $\mathbb{D}_X^M$ are naturally defined in its derived category.

(1.5) With the notation of (1.3), we shall consider full subcategories $\mathscr{M}(V)'$ (resp. $\mathscr{M}(V)''$) of $\mathscr{M}(V)$ for equidimensional (resp. equidimensional and smooth) varieties V, which satisfy the following conditions.

(i) $\mathscr{M}(V)'$ (resp. $\mathscr{M}(V)''$) is stable by subquotients and extensions in $\mathscr{M}(V)$.

(ii) $\mathscr{M}(V)'$ (resp. $\mathscr{M}(V)''$) is stable by the Tate twist (r) for $r \in \mathbb{Z}$, the dual functor $\mathbb{D}$, the pull-back j^* for any open embedding j, and the direct images $j_!$, j_* for any affine open embedding j.

(iii) $\mathscr{M}(V)'$ is stable by the cohomological direct image $H^j\pi_*$ for any generically finite projective morphism $\pi : X' \to X$ (i.e., its restriction to a dense open subvariety of X' induces a finite morphism over the image).

(iv) $\mathscr{M}_X(V)'$ (resp. $\mathscr{M}_X(V)''$) $\subset \mathscr{M}(V)_{\mathrm{gl}\le\dim X}$ for a closed subvariety X of V with $\operatorname{codim} X \le 1$ (resp. 2), where $\mathscr{M}_X(V)'$ is the full subcategory of $\mathscr{M}(V)'$ consisting of objects whose supports are contained in X (and the same for $\mathscr{M}_X(V)''$).

(v) $\mathbb{Q}_V^M[\dim V] \in \mathscr{M}(V)'$ (resp. $\mathscr{M}(V)''$) if V is smooth.

(vi) $(\mathrm{Gr}_i^W M)_Z$ is isomorphic to a direct sum of $\mathbb{Q}_Z^M(r)[\dim Z]$ if $M \in \mathscr{M}(V)''$, $\dim Z = \dim V$, and $r := (\dim Z - i)/2 \in \mathbb{Z}$, where $(\mathrm{Gr}_i^W M)_Z$ is the component of $\mathrm{Gr}_i^W M$ with strict support Z (see (1.3.2)).

(vii) If X is a divisor on a smooth equidimensional variety V, then (1.1.1) induces an equivalence of categories $\mathscr{M}(X)' \xrightarrow{\sim} \mathscr{M}_X(V)''$.

(viii) There exist full subcategories $\mathrm{MHM}(X_{\mathbb{C}})'$ and $\mathrm{MHM}(X_{\mathbb{C}})''$ of $\mathrm{MHM}(X_{\mathbb{C}})$ satisfying the above conditions (i)–(vii) and the functor $\mathscr{M}(X) \to \mathrm{MHM}(X_{\mathbb{C}})$ in (i) of (1.2) induces $\mathscr{M}(X)' \to \mathrm{MHM}(X_{\mathbb{C}})'$ and $\mathscr{M}(X)'' \to \mathrm{MHM}(X_{\mathbb{C}})''$.

We will denote by $\mathscr{D}_X(V)$, $\mathscr{D}_X(V)'$, $\mathscr{D}_X(V)''$ the bounded derived categories of $\mathscr{M}_X(V)$, $\mathscr{M}_X(V)'$, $\mathscr{M}_X(V)''$, and similarly for $\mathscr{D}(V)$, $\mathscr{D}(V)'$, $\mathscr{D}(V)''$.

(1.6) Remarks. (i) For $M, N \in \mathscr{D}(V)'$, the natural isomorphism

$$\mathrm{Ext}^p_{\mathscr{D}(V)'}(M, N) \to \mathrm{Ext}^p_{\mathscr{D}(V)}(M, N) \tag{1.6.1}$$

is an isomorphism if $\max\{i : H^iM \neq 0\} - \min\{i : H^iN \neq 0\} + p \leq 1$. In fact, this can be reduced to the case $M, N \in \mathscr{M}(V)'$ and $p \leq 1$ (using the truncation σ) and then follows from condition (i) of (1.5). (Same for $\mathscr{D}(V)''$.)

(ii) Let $M \in \mathscr{M}(V)'$ (resp. $\mathscr{M}(V)''$). Then (1.3.1) and condition (iv) of (1.5) imply:

$(\mathrm{Gr}_i^W M)_Z = 0$ if $\operatorname{codim} Z \leq 1$ (resp. $\operatorname{codim} Z \leq 2$) and $\dim Z - i$ is odd.

(iii) Let X be a closed subvariety of V with codimension ≤ 1 (resp. 2). Let $M, N \in \mathscr{M}_X(V)'$ (resp. $\mathscr{M}_X(V)''$) such that M and N are pure of weight w and $w-k$ respectively for $k > 0$. Let $\xi \in \mathrm{Ext}^k(M, N)$, where Ext is defined in the derived category of $\mathscr{M}_X(V)'$ (resp. $\mathscr{M}_X(V)''$). Then there exists a closed subvariety Y of X such that $\dim Y < \dim X$, and the restriction of ξ to the complement of Y is zero. In fact, ξ is the composition of

$$\xi_i \in \mathrm{Ext}^1(M_{i-1}, M_i) \qquad (1 \leq i \leq k),$$

where $M_i \in \mathscr{M}_X(V)'$ (resp. $\mathscr{M}_X(V)''$) are pure of weight $w-i$, and $M_0 = M$, $M_k = N$. See [**14**, II, (4.5)]. Then the assertion follows from remark (ii), because $\operatorname{codim} X \leq 1$ (resp. 2).

(1.7) Proposition. *For any function g on V, $\mathscr{M}(V)'$ and $\mathscr{M}(V)''$ are stable by the unipotent monodromy part of the nearby and vanishing cycle functors $\psi_{g,1}$, $\varphi_{g,1}$.*

Proof. This follows from (i), (ii) of (1.5). In fact, let $S = \operatorname{Spec} k[t]$, $S^* = S \setminus \{0\}$, and $U = g^{-1}(S^*)$ with a natural inclusion $j : U \to V$. By definition (see [**16**, (5.2.8–5.2.9) and (5.4.1)]), we have isomorphisms

$$\begin{aligned} \psi_{g,1}M &= H^{-1}(C(j_!(j^*M \otimes g^*L_i) \to j_*(j^*M \otimes g^*L_i)), \\ \varphi_{g,1}M &= H^{-1}(C(C(j_!j^*M \to M) \to C(j_!(j^*M \otimes g^*L_i) \\ &\qquad \to j_*(j^*M \otimes g^*L_i)))), \end{aligned} \tag{1.7.1}$$

for i sufficiently large, where $L_i \in D^b\mathscr{M}(S^*)$ such that $\mathrm{For}(L_i)$ is an indecomposable local system of rank $i+1$ with unipotent monodromy. Furthermore, $L_i[1]$ belongs to $\mathscr{M}(S^*)$ and $\mathrm{Gr}_k^W(L_i[1]) = \mathbb{Q}_{S^*}^M(-r)[1]$ if $k = 2r+1$

for $r \in \mathbb{Z} \cap [0, i]$ and zero otherwise. Then $j^*M \otimes g^*L_i$ has a finite filtration whose graduation is $j^*M(-r)$, and the assertion follows from (i), (ii) of (1.5).

(1.8) PROPOSITION. *Let* $i : X \to Y$ *be a closed embedding of closed subvarieties of* V. *With the notation of* (1.5), *the natural functors*

$$(1.8.1) \qquad i_* : \mathscr{D}_X(V)' \to \mathscr{D}_Y(V)', \qquad i_* : \mathscr{D}_X(V)'' \to \mathscr{D}_Y(V)''$$

are fully faithful, and their essential images are the objects whose cohomological supports are contained in X.

PROOF. This follows from (i), (ii) of (1.5) and (1.7). In fact, it is enough to show that the functor $N \to \operatorname{Ext}^k(i_*M, i_*N)$ for $M, N \in \mathscr{M}_X(V)'$ is effaceable for $k > 0$. See [2]. (The argument is the same for $\mathscr{M}_X(V)''$.) Using an affine open covering of V with the stability by affine open direct images (see (ii) of (1.5)), the assertion is reduced to the case of an affine V. Factoring i, we may assume $X = Y \cap g^{-1}(0)$ for a function g on V. Then a quasi-inverse of (1.8.1) is given by $\varphi_{g,1}$. See [16, (5.6)] for the details. Note that $\mathscr{M}(V)'$ is stable by the functor ξ_g in [loc. cit.] using (i) of (1.5) and (1.7).

REMARK. The functor i_* will be sometimes omitted to simplify the notation, because it is induced by the natural inclusion $\mathscr{M}_X(V) \to \mathscr{M}_Y(V)$, and is fully faithful.

(1.9) PROPOSITION. *Let* X *be a closed subvariety of* V, *and* $U = V \setminus X$ *with natural inclusions* $i : X \to V$, $j : U \to V$. *Then the functors*

$$(1.9.1) \qquad j^* : \mathscr{D}(V)' \to \mathscr{D}(U)', \qquad i_* : \mathscr{D}_X(V)' \to \mathscr{D}(V)'$$

have left and right adjoint functors

$$(1.9.2) \qquad j_!, j_* : \mathscr{D}(U)' \to \mathscr{D}(V)', \qquad i^*, i^! : \mathscr{D}(V)' \to \mathscr{D}_X(V)',$$

with distinguished triangles of functors

$$(1.9.3) \qquad \to j_!j^* \to \mathrm{id} \to i_*i^* \to, \qquad \to i_*i^! \to \mathrm{id} \to j_*j^* \to .$$

Furthermore, these functors commute with the natural functor $\mathscr{D}(V)' \to \mathscr{D}(V)$. *We have the same for* $\mathscr{D}(V)''$.

PROOF. This follows from (i), (ii) of (1.5) and (1.8). In fact, the functors in (1.9.2) and the triangles in (1.9.3) are constructed using an affine open covering together with the stability by the affine open direct images (see (ii) of (1.5)) and (1.8). See also [13] and [16, (3.3)] for the details.

REMARK. By construction, we have

$$(1.9.4) \qquad \begin{aligned} &H^k j_*M = 0, \quad H^k i^!M = 0 \qquad \text{for } k < 0, \\ &H^k j_!M = 0, \quad H^k i^*M = 0 \qquad \text{for } k > 0, \end{aligned}$$

for $M \in \mathscr{M}(V), \mathscr{M}(V)'$, or $\mathscr{M}(V)''$. See also [3].

(1.10) **Proposition.** *For an equidimensional variety* V, $\mathbb{Q}_V^M$ *and* $\mathbb{D}_V^M$ *are naturally defined in* $\mathscr{D}(V)', \mathscr{D}(V)''$.

Proof. This follows from (i), (v) of (1.5) and (1.9). The assertion is clear for $\mathscr{D}(V)''$, because we assume always V smooth when we consider $\mathscr{M}(V)''$ or $\mathscr{D}(V)''$. The assertion for $\mathscr{D}(V)'$ is checked using the same argument as in **[16,** (7.13)] (and using (v) of (1.5)). In fact, we have a t-structure on $\mathscr{D}(V)', \mathscr{D}(V)''$ compatible with the classical (i.e., not perverse) t-structure on the underlying $\mathbb{Q}$-complexes by the forgetful functor, using (1.9) and **[3]**. See also **[16,** (7.12)].

(1.11) **Remarks.** (i) For a closed subvariety X of V and the natural inclusion $i : X \to V$, let

$$\mathbb{Q}_X^M = i^*\mathbb{Q}_V^M, \quad \mathbb{D}_X^M = i^!\mathbb{D}_V^M \quad \text{in } \mathscr{D}_X(V)' \text{ or } \mathscr{D}_X(V)''. \tag{1.11.1}$$

For a closed subvariety Y of X, we have the restriction and Gysin morphisms

$$\mathbb{Q}_X^M \to \mathbb{Q}_Y^M, \qquad \mathbb{D}_Y^M \to \mathbb{D}_X^M \tag{1.11.2}$$

induced by (1.9.3), because the functors i^*, $i^!$ are functorial for the composition of morphisms. They induce a morphism

$$\operatorname{Ext}^{-2d}_{\mathscr{D}(V)'}(\mathbb{Q}_Y^M, \mathbb{D}_Y^M(-d)) \to \operatorname{Ext}^{-2d}_{\mathscr{D}(V)'}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)) \tag{1.11.3}$$

by composition. Let $U = X \setminus Y$, and let $j : V \setminus Y \to V$ denote a natural inclusion. Then we have a distinguished triangle in $\mathscr{D}_X(V)'$ or $\mathscr{D}_X(V)''$:

$$\to \mathbb{D}_Y^M \to \mathbb{D}_X^M \to j_*\mathbb{D}_U^M \to, \tag{1.11.4}$$

by (1.9.3), and this induces a long exact sequence

$$\to E(Y, p, q) \to E(X, p, q) \to E(U, p, q) \to E(Y, p+1, q) \to, \tag{1.11.5}$$

using the adjunction for $Y \to V$ and j, where

$$E(X, p, q) = \operatorname{Ext}^p_{\mathscr{D}(V)'}(\mathbb{Q}_X^M, \mathbb{D}_X^M(q)) \quad \text{or} \quad \operatorname{Ext}^p_{\mathscr{D}(V)''}(\mathbb{Q}_X^M, \mathbb{D}_X^M(q)).$$

Here the morphism $E(Y, p, q) \to E(X, p, q)$ coincides with (1.11.3) if $(p, q) = (-2d, -d)$.

(ii) With the above notation, the isomorphisms of (1.4.3) hold also for $\mathrm{IC}_X\mathbb{Q}^M$, $\mathbb{Q}_X^M$, and $\mathbb{D}_X^M$ in $\mathscr{D}_X(V)'$ or $\mathscr{D}_X(V)''$.

(1.12) **Proposition.** *Let* $\pi : V' \to V$ *be a generically finite morphism of purely m-dimensional varieties (i.e., its restriction to a dense open subvariety of* V' *is quasi finite). Then we have the direct image functors*

$$\pi_!, \pi_* : \mathscr{D}(V')' \to \mathscr{D}(V)', \tag{1.12.1}$$

and the restriction and Gysin morphisms

$$\mathbb{Q}_V^M \to \pi_*\mathbb{Q}_{V'}^M, \qquad \pi_!\mathbb{D}_{V'}^M \to \mathbb{D}_V^M \tag{1.12.2}$$

are naturally defined in $\mathscr{D}(V)'$. *Furthermore, for a closed embedding* $i : X \to V$, *we have natural isomorphisms*

$$(1.12.3) \qquad \pi_! i'^* = i^* \pi_!, \qquad \pi_* i'^! = i^! \pi_*,$$

where $i' : X' := \pi^{-1}(X) \to V'$ *is a natural morphism, and* $\pi_!$, π_* *denote also the induced functors* $\mathscr{D}_{X'}(V')' \to \mathscr{D}_X(V)'$.

PROOF. This follows from (i), (iii) of (1.5) and (1.9), (1.10). For the direct image, we can apply the construction in **[2]** using (iii) of (1.5) and (1.9). Then (1.12.3) follows from the construction. See **[16,** (3.10)**]**. The remaining assertion (1.12.2) is clear if π is finite étale, and the general case is reduced to this using the t-structure as in (1.10). See the proof of **[16,** (7.13)**]** for the details.

(1.13) REMARK. With the above notation and assumption, assume further π proper. Then we have the restriction and Gysin morphisms

$$(1.13.1) \qquad \mathbb{Q}_X^M \to \pi_* \mathbb{Q}_{X'}^M, \quad \pi_* \mathbb{D}_{X'}^M \to \mathbb{D}_X^M \quad \text{in } \mathscr{D}_X(V)',$$

by (1.12.2), (1.12.3). They induce the direct image

$$(1.13.2) \qquad \pi_* : \mathrm{Ext}^{-2d}_{\mathscr{D}(V')'}(\mathbb{Q}_{X'}^M, \mathbb{D}_{X'}^M(-d)) \to \mathrm{Ext}^{-2d}_{\mathscr{D}(V)'}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d))$$

by composition. Let Y be a closed subvariety of X and $Y' = \pi^{-1}(Y)$. Then we have commutative diagrams

$$(1.13.3) \qquad \begin{array}{ccc} \pi_*\mathbb{Q}_{X'}^M & \longrightarrow & \pi_*\mathbb{Q}_{Y'}^M \\ \uparrow & & \uparrow \\ \mathbb{Q}_X^M & \longrightarrow & \mathbb{Q}_Y^M \end{array} \qquad \begin{array}{ccc} \pi_*\mathbb{D}_{Y'}^M & \longrightarrow & \pi_*\mathbb{D}_{X'}^M \\ \downarrow & & \downarrow \\ \mathbb{D}_Y^M & \longrightarrow & \mathbb{D}_X^M \end{array}$$

applying the functorial morphisms in (1.9.3) to (1.12.2). Note that, by the proofs of (1.9.3) and (1.12.3), the direct image of the restriction morphism $\mathbb{Q}_{X'}^M \to \mathbb{Q}_{Y'}^M$ in (1.11.2) by π_* coincides with the morphism $\pi_*\mathbb{Q}_{X'}^M \to (\tilde{i}_*\tilde{i}^*)\pi_*\mathbb{Q}_{X'}^M = \pi_*\mathbb{Q}_{Y'}^M$ obtained by (1.9.3) and (1.12.3), where $\tilde{i} : Y \to X$ denotes a natural inclusion, and $\tilde{i}_*$ before $\pi_*\mathbb{Q}_{Y'}^M$ is omitted (see Remark after (1.8)). We have the dual assertion for the Gysin morphism.

2. Construction

(2.1) DEFINITION. Let V be a variety of pure dimension m. With the notation of (1.3), we define the abelian full subcategories $\mathscr{M}(V)'$, $\mathscr{M}(V)''$ of $\mathscr{M}(V)_{\mathrm{gl}\le m}$ by the following conditions respectively :

(i) $(\mathrm{Gr}_i^W M)_Z$ has geometric level $\le m-1$ if $\operatorname{codim} Z \ge 1$.

(ii) $(\mathrm{Gr}_i^W M)_Z$ has geometric level $\le m-1$ (resp. $m-2$) if $\operatorname{codim} Z \ge 1$ (resp. 2) and is isomorphic to a direct sum of $\mathbb{Q}_Z^M(r)[\dim Z]$ if $\dim Z = \dim V$ and $r := (\dim Z - i)/2 \in \mathbb{Z}$.

Here we assume always V smooth, when we consider $\mathscr{M}(V)''$.

REMARK. We have to show that $\mathcal{M}(V)', \mathcal{M}(V)''$ satisfy the conditions of (1.5). Except for the stability by $j_!$, j_*, $H^k\pi_*$, this follows easily from the properties of geometric level and strict support decomposition. See (1.3). The remaining stability will be shown in this section.

(2.2) LEMMA. *Let V be a smooth variety of pure dimension m, and let X be a divisor with normal crossings. Let $U = V \setminus X$ with a natural inclusion $j: U \to V$. Then $\mathrm{Gr}_i^W(j_! \mathbb{Q}_U^M[m])$ is a direct sum of constant objects supported on the intersections of $m - i$ irreducible components of X. Furthermore, assume $X = g^{-1}(0)_{\mathrm{red}}$ for a function g on V. Then the primitive part of the graduation of the nearby (resp. vanishing) cycles $P\mathrm{Gr}_i^W(\psi_{g,1}\mathbb{Q}_V^M[m])$ (resp. $P\mathrm{Gr}_i^W(\varphi_{g,1}\mathbb{Q}_V^M[m])$) is a direct sum of constant objects supported on the intersections of $i - m + 2$ irreducible components of X for $i \geq m - 1$ (resp. m) and 0 otherwise. In particular, $\varphi_{g,1}\mathbb{Q}_V^M[m]$ has geometric level $\leq m - 2$.*

PROOF. This follows from **[16,** (7.6) and (7.7)]. For the assertion on $\varphi_{g,1}$, we use the isomorphism

$$P\mathrm{Gr}_i^W(\varphi_{g,1}\mathbb{Q}_V^M[m]) = \begin{cases} P\mathrm{Gr}_i^W(\psi_{g,1}\mathbb{Q}_V^M[m]) & \text{for } i \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

For the proof of this, we have the isomorphism

$$\text{(2.2.1)} \qquad \varphi_{g,1}\mathbb{Q}_V^M[m] = \mathrm{Coim}(N : \psi_{g,1}\mathbb{Q}_V^M[m] \to \psi_{g,1}\mathbb{Q}_V^M(-1)[m])$$

by the surjectivity of can in **[16,** (5.2.2)] and the injectivity of Var in **[16,** (5.8.1)].

(2.3) LEMMA. *Let $f: X \to Y$ be a proper surjective morphism of irreducible varieties such that X is smooth. Let $n = \dim X$ and $m = \dim Y$. Then $H^j(f_*\mathbb{Q}_X^M[n])_Z = 0$ for $|j| > n - m$, $Z = Y$ or $|j| > n - \dim Z - 2$, $Z \neq Y$. Furthermore, $H^j(f_*\mathbb{Q}_X^M[n])_Z$ has geometric level $\leq n - 2$ for $Z \neq Y$.*

PROOF. We proceed by induction on $\dim Y$. If $Z = Y$, the assertion is clear. Let g be a function locally defined on Y such that f is smooth on the complement of $g^{-1}(0)$. Then $Z \subset g^{-1}(0)$ if $H^j(f_*\mathbb{Q}_X^M[n])_Z \neq 0$ and $Z \neq Y$. We may assume $f^{-1}g^{-1}(0)$ is a divisor with normal crossings using the decomposition theorem (1.1.3) for the embedded resolution of $f^{-1}g^{-1}(0)$. Then $\varphi_{gf,1}\mathbb{Q}_X^M[n]$ has geometric level $\leq n - 2$ by (2.2), and the assertion follows from (1.1.2) using the inductive hypothesis, because $\varphi_{g,1}M = M$ if $\mathrm{supp}\, M \subset g^{-1}(0)$.

(2.4) PROPOSITION. *For a generically finite proper morphism of equidimensional varieties $f: X' \to X$, we have $H^j f_* M \in \mathcal{M}(X)'$ if $M \in \mathcal{M}(X')'$.*

PROOF. By the weight spectral sequence, we may assume M pure with strict support Z. Then M is isomorphic to a direct factor of $H^j\pi_*\mathbb{Q}_Y^M(r)$ as in (1.3), where $\dim Y$ satisfies some condition according to the codimension of Z. Then the assertion follows from (2.3), where we use the decomposition (1.1.3) for π to show that the Leray spectral sequence for the composition of the morphisms π and f degenerates at E_1.

REMARK. The proposition is not true for $\mathcal{M}(X)''$ even if f is birational. In fact, the image of a divisor on X' may have codimension ≥ 2.

(2.5) LEMMA. *Let V be a purely m-dimensional variety such that $\mathbb{Q}_{V(\mathbb{C})}[m]$ is a perverse sheaf on $V(\mathbb{C})$. Then $\mathbb{Q}_V^M[m] \in \mathcal{D}(V)$ belongs to $\mathcal{M}(V)$ and, furthermore, to $\mathcal{M}(V)'$.*

PROOF. The first assertion is clear, because $\mathrm{For}(\mathbb{Q}_V^M[m]) = \mathbb{Q}_{V(\mathbb{C})}[m]$. Let U be the smooth part of V, $X = V \setminus U$ with natural inclusions $j : U \to V$, $i : X \to V$. Then the natural morphism

(2.5.1) $$H^0 j_! \mathbb{Q}_U^M[m] \to \mathbb{Q}_V^M[m]$$

is surjective by the long exact sequence associated with the distinguished triangle $\to j_! j^* \to \mathrm{id} \to i_* i^* \to$. In fact, we have

(2.5.2) $$\mathrm{Hom}(\mathbb{Q}_V^M[m], i_* M) = \mathrm{Hom}(\mathbb{Q}_X^M[m], M) = 0$$

for any $M \in \mathcal{M}(X)$ by adjunction for i and (1.4.1) applied to X. So the assertion follows from (2.2) and (2.4) using a resolution of singularities.

REMARK. The hypothesis of the Lemma is satisfied if V is a divisor on a smooth variety of pure dimension $m+1$. See, for example, [**15**, (2.12)].

(2.6) PROPOSITION. *Let $j : U \to V$ be an affine open embedding of equidimensional varieties. Then the functors $j_!$, j_* induce*

(2.6.1) $$j_!, j_* : \mathcal{M}(U)' \to \mathcal{M}(V)', \qquad j_!, j_* : \mathcal{M}(U)'' \to \mathcal{M}(V)''.$$

PROOF. We show the assertion for $j_!$, because we have the duality $\mathbb{D} j_! = j_* \mathbb{D}$. Since $j_!$ is an exact functor, it is enough to show $j_! M \in \mathcal{M}(V)'$ or $\mathcal{M}(V)''$ if $M \in \mathcal{M}(U)'$ or $\mathcal{M}(U)''$ is pure with strict support Z. By definition of geometric level and (2.4), the assertion for $\mathcal{M}(X)'$ is reduced to the case V is smooth and purely m-dimensional, $X := V \setminus U$ is a divisor with normal crossings, and $M = \mathbb{Q}_U^M[m]$, using Nagata-Hironaka (which we apply to π in the definition of geometric level). Then the assertion follows from (2.2). The argument is the same for $\mathcal{M}(X)''$ if $\dim Z < m$ (using (2.3)). In the case $\dim Z = m$, we may assume $M = \mathbb{Q}_U^M[m]$ by definition (2.1). We have the short exact sequence

(2.6.2) $$0 \to \mathbb{Q}_X^M[m-1] \to j_! \mathbb{Q}_U^M[m] \to \mathbb{Q}_V^M[m] \to 0$$

because $X = V \setminus U$ is a divisor by hypothesis. So the assertion follows from (2.5) applied to X.

(2.7) REMARKS. (i) The stability by the nearby and vanishing cycle functors in (1.7) is not true if it is not restricted to the unipotent monodromy part. For example, consider $g = x^d + y^d + z^d$ on $X = \mathbb{A}^3$ for $d \geq 4$. Then $\psi_{g,\neq 1}\mathbb{Q}_X^M[3]$ has geometric level 2 and its support is the origin, where $\psi_{g,\neq 1}$ is the nonunipotent monodromy part of ψ_g.

(ii) The conditions of (2.1) may be replaced respectively by

(i)′ $(\mathrm{Gr}_i^W M)_Z$ is isomorphic to a direct sum of $\mathrm{IC}_Z\mathbb{Q}^M(r)$ if $\operatorname{codim} Z = 0$ and $r := (\dim Z - i)/2 \in \mathbb{Z}$, and has geometric level $\leq m-1$ otherwise.

(ii)′ $(\mathrm{Gr}_i^W M)_Z$ is isomorphic to a direct sum of $\mathrm{IC}_Z\mathbb{Q}^M(r)$ if $\operatorname{codim} Z \leq 1$ and $r := (\dim Z - i)/2 \in \mathbb{Z}$, and has geometric level $\leq m-2$ otherwise.

Then we get smaller categories. We can check that they satisfy the conditions in (1.5), where we assume that the restriction of the morphism π in (iii) to each irreducible component of X' induces a birational morphism onto the image.

3. Cycle maps

In this section, $\mathscr{M}(V)'$, $\mathscr{M}(V)''$, $\mathscr{D}(V)'$, $\mathscr{D}(V)''$ are as in (1.5).

(3.1) PROPOSITION. *For a variety X, we have a cycle map*

$$cl^M : \mathrm{CH}_d(X)_{\mathbb{Q}} \to \mathrm{Ext}^{-2d}_{\mathscr{D}(X)}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)). \tag{3.1.1}$$

Let V be a variety of pure dimension m, and X a closed subvariety. Then we have cycle maps

$$cl^M : \mathrm{CH}_d(X)_{\mathbb{Q}} \to \mathrm{Ext}^{-2d}_{\mathscr{D}(V)'}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)), \tag{3.1.2}$$

$$cl^M : \mathrm{CH}_d(X)_{\mathbb{Q}} \to \mathrm{Ext}^{-2d}_{\mathscr{D}(V)''}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)), \tag{3.1.3}$$

where we assume V smooth in (3.1.3). *Furthermore, their compositions with the natural morphisms* (*induced by* (1.8.1))

$$\mathrm{Ext}^{-2d}_{\mathscr{D}(V)'}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)) \to \mathrm{Ext}^{-2d}_{\mathscr{D}(X)}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)), \tag{3.1.4}$$

$$\mathrm{Ext}^{-2d}_{\mathscr{D}(V)''}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)) \to \mathrm{Ext}^{-2d}_{\mathscr{D}(X)}(\mathbb{Q}_X^M, \mathbb{D}_X^M(-d)) \tag{3.1.5}$$

coincide with (3.1.1).

PROOF. The first assertion (3.1.1) is due to **[16,** (8.2)]. The cycle maps (3.1.2), (3.1.3) are induced by (1.4.2) and (1.11.2). Its well-definedness is reduced to the divisor case by factoring the cycle map and replacing X with a closed subvariety. Then the restriction of the morphisms (3.1.4), (3.1.5) to the images of (3.1.2), (3.1.3) are injective for $d = n-1$ using the long exact sequence (1.11.5) (together with (1.4.3) and (i) in (1.6)), where U is a smooth open subvariety of X. See **[15,** (2.14)]. So the assertion follows from the well-definedness of (3.1.1).

REMARK. (i) If X is smooth (e.g., $X = V$), then (3.1.3) becomes

$$cl^M : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{\mathscr{D}(V)''}(\mathbb{Q}_X^M, \mathbb{Q}_X^M(p)), \tag{3.1.6}$$

where $p = \dim X - d$.

(ii) By (1.8), Ext in (3.1.2), (3.1.3) can be taken in $\mathscr{D}_X(V)'$, $\mathscr{D}_X(V)''$. The cycle maps (3.1.2), (3.1.3) are compatible with the inclusions of closed subvarieties X of V using the direct image (1.11.3).

(3.2) PROPOSITION. *Let $\pi : V \to V'$, X and X' be as in* (1.12), *and assume π proper. Then the push-down of cycles*

$$\pi_* : \mathrm{CH}_d(X')_{\mathbb{Q}} \to \mathrm{CH}_d(X)_{\mathbb{Q}}. \tag{3.2.1}$$

corresponds to the direct image (1.13.2) *by the cycle map* (3.1.2).

PROOF. This follows from the same argument as in [**14**, II, (2.4)], using the commutative diagrams (1.13.3) with the restriction and Gysin morphisms $\mathbb{Q}^M_{Y'} \to \mathbb{Q}^M_Z$, $\mathbb{D}^M_Z \to \mathbb{D}^M_{Y'}$, where Z is an irreducible closed subvariety of X' and $Y = \pi(Z)$.

(3.3) THEOREM. *Assume the condition* (ii) *in* (1.2). *Then the cycle maps* (3.1.2) *and* (3.1.3) *are surjective for $m - d \leq 2$ and 3 respectively, where $m = \dim V$. Furthermore, they are also surjective for $m - d = 3$ and 4 respectively if $k = \mathbb{C}$ and $\{\mathscr{M}(X)\} = \{\mathrm{MHM}(X)\}$.*

PROOF. We proceed by induction on $p := n - d$, where $n = \dim X$. We first show the assertion for (3.1.2). If $p = 0$, the assertion follows from (1.4.3) (and Remark (ii) of (1.11)).

If $p > 0$, we have $\operatorname{codim} X \leq 1$, because $\dim V - d \leq 2$. Let $\xi \in \mathrm{Ext}^{-2d}(\mathbb{Q}^M_X, \mathbb{D}^M_X(-d))$. For a smooth equidimensional open subvariety U of X, let $Y = X \setminus U$, and assume $\dim Y < n$. Since $\mathbb{D}^M_U = \mathbb{Q}^M_U(n)[2n]$ and $\mathbb{Q}^M_U[n]$ is pure of weight n, we can apply (iii) of (1.6) to $M = \mathbb{Q}^M_U[n]$, $N = \mathbb{D}^M_U(-d)[-n] = \mathbb{Q}^M_U(p)[n]$, and $k = 2p$ by (iv) of (1.5) and (1.8), because $\operatorname{codim} X \leq 1$. So the restriction of ξ to a smaller dense open subvariety, which is also denoted by U, is zero.

So ξ belongs to the image of (1.11.3) by the long exact sequence (1.11.5). Then the assertion follows from the inductive hypothesis applied to Y.

For the surjectivity of (3.1.3), we may assume $n = m$, because the case $n < m$ is reduced to the assertion for (3.1.2) by (vii) of (1.5) and (1.8). Then we repeat the above argument using (iv) of (1.5), and the assertion is reduced to the case $n < m$.

If $k = \mathbb{C}$ and $\{\mathscr{M}(X)\} = \{\mathrm{MHM}(X)\}$, the assertion is reduced to the case $d = m - 3$ and $n = m - 2$ for (3.1.2) by the same argument as above (where we use (vii) of (1.5) and (1.8) for (3.1.3)). By (1.4.3) and the long exact sequence (1.11.5), it is enough to show that for any $\xi \in \mathrm{Ext}^{2-2n}(\mathbb{Q}^M_X, \mathbb{D}^M_X(1-n))$, the restriction of ξ to a dense open subvariety is zero, where Ext is taken in the derived category of the abelian full subcategory of $\mathscr{M}(X)_{\mathrm{gl} \leq n+1}$ which is equivalent to $\mathscr{M}_X(V)'$. Here we may restrict X to an open subvariety and assume X smooth and equidimensional so that ξ belongs to $\mathrm{Ext}^2(\mathbb{Q}^M_X[n], \mathbb{Q}^M_X(1)[n])$. Then the assertion follows from the next proposition.

(3.4) Proposition. *Let X be a smooth equidimensional variety over $\mathbb{C}$ with* $\dim X = n$, *and* $\xi \in \mathrm{Ext}^2(\mathbb{Q}_X^M[n], \mathbb{Q}_X^M(1)[n])$, *where* Ext *is taken in the bounded derived category of a full subcategory of* $\mathrm{MHM}(X)_{\mathrm{gl}\le n+1}$ *which is stable by subquotients and extensions in* $\mathrm{MHM}(X)_{\mathrm{gl}\le n+1}$. *Then the restriction of ξ to a dense open subvariety of X is zero.*

Proof. By [**14**, II, (4.5)], ξ is the composition of

$$\xi_1 \in \mathrm{Ext}^1(\mathbb{Q}_X^M[n], M), \qquad \xi_2 \in \mathrm{Ext}^1(M, \mathbb{Q}_X^M(1)[n]),$$

such that M is pure of weight $n-1$ and has geometric level $\le n+1$. So M is a direct factor of $H^0 f_* \mathbb{Q}_Y^M(1)[n+1]$ for a smooth projective morphism $f: Y \to X$ of relative dimension 1 by restricting X if necessary. Then it is enough to show that ξ_1, ξ_2 are induced by divisors of $Y = Y \times_X X$ using the relative correspondence [**15**, (1.1)] (e.g., the mixed Hodge module associated with the extension class ξ_1 is a subquotient of $H^0 f_*(j_* \mathbb{Q}_{Y'}^M(1)[n+1])$ for an affine open embedding $j: Y' \to Y$). We may consider the extension classes ξ_1, ξ_2 in $\mathrm{MHM}(X)$, and the assertion follows from (3.5.1). See also the proof of [**15**, (3.9)].

Remark. By the same argument, the Proposition holds also for $\xi \in \mathrm{Ext}^2(\pi_* \mathbb{Q}_{X'}^M[n], \mathbb{Q}_X^M(1)[n])$, where $\pi : X' \to X$ is a finite étale morphism. Note that the proof of the Proposition uses the mixed Hodge theoretic description of the Picard variety (using the extension group).

(3.5) Theorem. *Assume the condition* (i) *in* (1.2). *If X is a purely n-dimensional quasi-projective variety and* $\mathbb{Q}_{X(\mathbb{C})}[n]$ *is a perverse sheaf, then the cycle maps* (3.1.2) *and* (3.1.3) *are injective for* $\dim V - d \le 3$ *and* 4 *respectively.*

Proof. Since the natural morphism $\mathrm{CH}_d(X)_{\mathbb{Q}} \to \mathrm{CH}_d(X_{\mathbb{C}})_{\mathbb{Q}}$ is injective (see also [**18**, (1.10)]), we may assume $k = \mathbb{C}$ and $\{\mathscr{M}(X)\} = \{\mathrm{MHM}(X)\}$ by the conditions (i) of (1.2) and (viii) of (1.5). Then, for any variety X, we have

(3.5.1) The cycle map (3.1.1) is bijective for $d = n-1$,

by [**15**, (3.8)], where $n = \dim X$.

We proceed by induction on $p := n - d$. We first show the assertion for (3.1.2). The assertion is clear for $p = 0$ by (1.4.3) (and Remark (ii) of (1.11)). If $p = 1$, it follows from (3.5.1), because the cycle map (3.1.1) is factorized by (3.1.2).

So it remains to show the case $p = 2$ or 3, because $\dim V - d \le 3$. Let ζ be a codimension p cycle on X with rational coefficients. By (1.10) and [**15**, (2.12), (2.13)], we have pure-dimensional closed subvarieties X_1, X_2 of X such that $X_1 \supset X_2 \supset |\zeta|$, $\dim X_i = n - i$, and $\mathbb{Q}_{X_i}^M[n-i] \in \mathscr{M}(V)'$ (i = 1, 2). Let

$$e_i \in \mathrm{Ext}^1_{\mathscr{D}(V)'}(\mathbb{Q}_{X_{i-1}}^M[n-i+1], \mathbb{Q}_{X_i}^M[n-i]) \qquad (i = 1, 2)$$

denote the natural extension class induced by the restriction morphism in (1.11.2) with $X_0 = X$. Then we have

$$\eta_i \in \mathrm{Ext}^{2p-i}_{\mathscr{D}(V)'}(\mathbb{Q}^M_{X_i}[n-i], \mathbb{D}^M_X(p-n)[-n])$$

such that $cl^M(\zeta) = \eta_1 \circ e_1 = \eta_2 \circ e_2 \circ e_1$ (see **[15,** (3.3)]).

Assume $cl^M(\zeta) = 0$ in $\mathscr{D}(V)'$. By the same argument as in the proof of **[15,** (3.4)], there exist a locally principal divisor Y on X containing X_2 and a pure object $M \in \mathscr{M}_X(V)'$ of weight n with

$$\widetilde{\gamma} \in \mathrm{Ext}^1_{\mathscr{D}(V)'}(\mathbb{Q}^M_Y[n-1], M(1)), \qquad \widetilde{\eta} \in \mathrm{Ext}^{2p-2}_{\mathscr{D}(V)'}(M(1), \mathbb{D}^M_X(p-n)[-n]),$$

such that $\widetilde{\eta} \circ \widetilde{\gamma} = \eta_2 \circ f_2$, $\widetilde{\gamma} \circ f_1 = 0$, and $M_Z = 0$ for $\dim Z \neq n$, where

$$f_1 \in \mathrm{Ext}^1_{\mathscr{D}(V)'}(\mathbb{Q}^M_X[n], \mathbb{Q}^M_Y[n-1]), \quad f_2 \in \mathrm{Ext}^1_{\mathscr{D}(V)'}(\mathbb{Q}^M_Y[n-1], \mathbb{Q}^M_{X_2}[n-2])$$

are induced by the restriction morphisms. See also Remark (i) of (3.6).

Since $\dim V - d \leq 3$ and $n - d = 2$ or 3, we have $\dim V - n \leq 1$, and M has geometric level $\leq n$ by (iv) of (1.5). So M is a direct factor of $H^0\pi_*(\mathbb{Q}^M_{X'}[n])$ for a generically finite projective morphism $\pi : X' \to X$. Since we have a (noncanonical) decomposition in $\mathscr{D}(V)'$:

$$\pi_*(\mathbb{Q}^M_{X'}[n]) \simeq \bigoplus_j H^j(\pi_*\mathbb{Q}^M_{X'}[n])[-j]$$

by **[14,** II, (4.5)] (see also (1.1.3)), we get morphisms

$$\text{(3.5.2)} \qquad \alpha : M \to \pi_*(\mathbb{Q}^M_{X'}[n]), \qquad \beta : \pi_*(\mathbb{Q}^M_{X'}[n]) \to M$$

such that $\beta \circ \alpha = \mathrm{id}$. Let $\gamma = \alpha \circ \widetilde{\gamma}$, $\eta = \widetilde{\eta} \circ \beta$ so that

$$\text{(3.5.3)} \qquad \eta \circ \gamma = \widetilde{\eta} \circ \widetilde{\gamma} = \eta_2 \circ f_2, \qquad \gamma \circ f_1 = 0.$$

If $\dim V = n$, we may assume $X = V$ replacing V (and using (1.8)). Let $Y' = \pi^{-1}(Y)$. Then, by (3.7), (3.8), we have

$$\text{(3.5.4)} \qquad \zeta' \in \mathrm{CH}^{p-1}(X')_{\mathbb{Q}}, \qquad \gamma' \in \mathrm{Ext}^1_{\mathscr{D}(X')'}(\mathbb{Q}^M_{Y'}[n-1], \mathbb{Q}^M_{X'}(1)[n])$$

such that $\widetilde{\eta}$ $(= \eta \circ \alpha)$ coincides with the image of ζ' by (3.7.1) and γ $(= \alpha \circ \widetilde{\gamma})$ with the image of γ' by (3.8.1).

If $\dim V \neq n$, we have $p = 2$, $n = \dim V - 1$ because $p = 2$ or 3, and $\dim V - d \leq 3$. Then we have (3.5.4) such that the above assertion holds in $\mathscr{D}(X')$, $\mathscr{D}(X)$ instead of $\mathscr{D}(X')'$, $\mathscr{D}(X)'$ (e.g., γ' is defined in $\mathscr{D}(X')$) by a similar argument. In this case we will use $\mathscr{D}(X')$, $\mathscr{D}(X)$ in place of $\mathscr{D}(X')'$, $\mathscr{D}(X)'$ in the later arguments, because we will treat only divisors, and we can apply (3.5.1).

By (3.5.4), the composition of the direct image of $cl^M(\zeta')\circ\gamma'$ with the restriction and Gysin morphisms $\mathbb{Q}_Y^M \to \pi_*\mathbb{Q}_{Y'}^M$, $\pi_*\mathbb{D}_{X'}^M \to \mathbb{D}_X^M$ coincides with $\eta\circ\alpha\circ\widetilde{\gamma}$ and then with $\eta_2\circ f_2$ by (3.5.3). Furthermore, Propositions (3.5), (3.6) in **[15]** hold also for $\mathscr{D}(X')'$ in the case $\dim V = n$. (See the Remark (ii) of (3.6).) So γ' is represented by a cycle $\sum_i r_i[Y'_i] \in \mathrm{CH}^0(Y')_{\mathbb{Q}}$ (see [loc. cit.]), and $cl^M(\zeta')\circ\gamma'$ coincides with the composition of $cl^M(\zeta'')$ with the Gysin morphism $\mathbb{D}_{Y'}^M \to \mathbb{D}_{X'}^M$, where Y'_i are the irreducible components of Y' and $\zeta'' = \sum_i r_i(\zeta'\cdot[Y'_i]) \in \mathrm{CH}^{p-1}(Y')_{\mathbb{Q}}$ (the intersection is defined using the pull-back of ζ' by $Y'_i \to X'$). So the composition of $\pi_* cl^M(\zeta'')$ with the Gysin morphism $\mathbb{D}_Y^M \to \mathbb{D}_X^M$ coincides with $\eta_2\circ f_2$ using the second commutative diagram of (1.13.3) (see (1.13.2) for π_*).

By definition of η_2 (see **[15**, (3.3.3)]), $\eta_2\circ f_2$ is the composition of the cycle class of ζ as a cycle on Y with the Gysin morphism $\mathbb{D}_Y^M \to \mathbb{D}_X^M$. So the cycle class of ζ as a cycle on Y coincides with $\pi_* cl^M(\zeta'')$ by the adjunction for $Y \to X$, and we have $\pi_* cl^M(\zeta'') = cl^M(\pi_*\zeta'')$ by (3.2). Since (3.1.2) is injective for Y and $p-1$ by the inductive hypothesis, ζ coincides with $\pi_*\zeta''$ in $\mathrm{CH}^{p-1}(Y)_{\mathbb{Q}}$ (where we use (3.5.1) in the case $\dim V \neq n$). So the assertion is reduced to

$$\zeta_1 := \sum_i r_i[Y'_i] = 0 \quad \text{in } \mathrm{CH}^1(X')_{\mathbb{Q}}, \tag{3.5.5}$$

because ζ'' in $\mathrm{CH}^p(X')_{\mathbb{Q}}$ coincides with $\zeta'\cdot\zeta_1$.

By definition (see **[15**, (3.5)]), $cl^M(\zeta_1)$ is the composition of the restriction morphism $\mathbb{Q}_{X'}^M \to \mathbb{Q}_{Y'}^M$ with γ' and corresponds to $\gamma\circ f_1$ by the adjunction isomorphism for π, using the first commutative diagram of (1.13.3). Here we consider the morphisms in $\mathscr{D}(X')$, $\mathscr{D}(X)$. Since $\gamma\circ f_1 = 0$ by (3.5.3), we have $cl^M(\zeta_1) = 0$, and the assertion follows from (3.5.1).

For the injectivity of (3.1.3), we may assume V connected and $X = V$, because the case $n < \dim V$ is reduced to the assertion for (3.1.2) by (vii) of (1.5) and (1.8). Then we repeat the above argument for (3.1.2) so that the assertion is reduced to the case $n < \dim V$. Here we may assume that X' in (3.5.2) is a disjoint union of X by (vi) of (1.5), and we do not need (3.7), (3.8) below (and (iii) of (1.5)) in this case.

(3.6) Remarks. (i) The variant of **[15**, (3.4)] used in the above proof (where $\pi_*\mathbb{Q}_{X'}^M[n]$ is replaced by M) holds for any full subcategory of $\mathrm{MHM}(X)_{\mathrm{gl}\le n}$ such that (i), (ii), and (v) of (1.5) hold. Note that M is denoted by A_1 in [loc. cit.], and (3.5.13) in [loc. cit.] should mean $(A_1)_Z = 0$ for $\dim Z \neq n$.

(ii) Propositions (3.5), (3.6) in **[15]** hold for any full subcategory of $\mathrm{MHM}(X)$ such that (i), (ii), and (v) of (1.5) hold.

(3.7) LEMMA. *Let $\pi : X' \to X$ and $\alpha : M \to \pi_*(\mathbb{Q}^M_{X'}[n])$ be as in the proof of (3.5), where we assume $X = V$. Then the composition*

$$(3.7.1)\quad \begin{aligned} \mathrm{CH}^q(X')_{\mathbb{Q}} &\to \mathrm{Ext}^{2q}_{\mathscr{D}(X')'}(\mathbb{Q}^M_{X'}[n], \mathbb{D}^M_{X'}(q-n)[-n]) \\ &\to \mathrm{Ext}^{2q}_{\mathscr{D}(X)'}(\pi_*\mathbb{Q}^M_{X'}[n], \mathbb{D}^M_X(q-n)[-n]) \\ &\to \mathrm{Ext}^{2q}_{\mathscr{D}(X)'}(M, \mathbb{D}^M_X(q-n)[-n]), \end{aligned}$$

is surjective for $q \le 2$, where the morphisms are induced by the cycle map (3.1.2), the composition with the Gysin morphism $\pi_*\mathbb{D}^M_{X'}[n] \to \mathbb{D}^M_X$, and α respectively.

PROOF. The assertion is clear for $q \le 0$ by the dual of (1.4.1), and we may assume $q = 1$ or 2.

Let $\xi \in \mathrm{Ext}^{2q}_{\mathscr{D}(X)'}(M, \mathbb{D}^M_X(q-n)[-n])$. We first show that there exists an open subvariety U of X such that $Z = X \setminus U$ has codimension $\ge q$, and the restriction of ξ to U is zero. Replacing X with an open subvariety whose complement has codimension ≥ 2, we may assume π finite so that $\pi_*(\mathbb{Q}^M_{X'}[n])$ belongs to $\mathscr{M}(X)'$ and is pure of weight n. Then, if we forget the condition on the codimension of Z, the assertion follows from (iii) of (1.6), and ξ belongs to the image of

$$(3.7.2)\quad \mathrm{Ext}^{2q}_{\mathscr{D}(X)'}(M, \mathbb{D}^M_Z(q-n)[-n]) \to \mathrm{Ext}^{2q}_{D(X)'}(M, \mathbb{D}^M_X(q-n)[-n])),$$

by the long exact sequence associated with the distinguished triangle (1.11.4) (applied to $Y = Z$). So we have to show that for any element of the source of (3.7.2), there exists a dense open subvariety of Z on which it vanishes. Let $Z' = \pi^{-1}(Z)$, and let $i : Z \to X$ denote a natural morphism. By restricting X to the complement of a closed subvariety of codimension ≥ 2, we may assume that Z is smooth and the restriction of π over Z is finite étale. Then the assertion follows from the Remark after (3.4) using the adjunction for i, because i_*i^*M is a direct factor of $\pi_*\mathbb{Q}^M_{Z'}[n]$ by (1.12.3).

So ξ belongs to the image of (3.7.2) with $\operatorname{codim} Z \ge q$. We have a natural morphism of

$$(3.7.3)\quad \begin{aligned} \mathrm{CH}_{n-q}(Z')_{\mathbb{Q}} &\to \mathrm{Ext}^{2q}_{\mathscr{D}(X')'}(\mathbb{Q}^M_{X'}[n], \mathbb{D}^M_{Z'}(q-n)[-n]) \\ &\to \mathrm{Ext}^{2q}_{\mathscr{D}(X)'}(\pi_*\mathbb{Q}^M_{X'}[n], \mathbb{D}^M_Z(q-n)[-n]) \\ &\to \mathrm{Ext}^{2q}_{\mathscr{D}(X)'}(M, \mathbb{D}^M_Z(q-n)[-n]), \end{aligned}$$

to (3.7.1) induced by the Gysin morphisms $\mathbb{D}^M_{Z'} \to \mathbb{D}^M_{X'}$, $\mathbb{D}^M_Z \to \mathbb{D}^M_X$, and it gives a commutative diagram, using the second commutative diagram of (1.13.3). So it is enough to show the surjectivity of (3.7.3).

By adjunction for i (and using (1.12.3)), we may replace $\mathbb{Q}^M_{X'}$ in (3.7.3) by $\mathbb{Q}^M_{Z'}$. Then we have the surjectivity if we take Ext in $\mathscr{D}(X')$ or $\mathscr{D}(X)$. In fact, the first morphism is surjective by (3.5.1) (because $n - q \ge \dim Z' - 1$), the second by the adjunction for π, and the last is clear. So it is enough to show that the natural morphism

(3.7.4) $\operatorname{Ext}^q_{\mathscr{D}(X)'}(M, \mathbb{D}^M_Z(q-n)[q-n]) \to \operatorname{Ext}^q_{\mathscr{D}(X)}(M, \mathbb{D}^M_Z(q-n)[q-n])$

is an isomorphism. Here we may replace M by i_*i^*M by adjunction. Since M has no nontrivial quotient whose support has dimension $< n$, we have $H^k i^*M = 0$ for $k \geq 0$ by (1.9.4). So the assertion follows from (1.6.1), because $H^j(\mathbb{D}^M_Z[q-n]) = 0$ for $j < 0$ by the dual of (1.4.1).

(3.8) LEMMA. *Let $\pi : X' \to X$ and $Y' \to Y$ be as in the proof of* (3.5), *where we assume $X = V$. Then the morphism*

(3.8.1)
$$\operatorname{Ext}^1_{\mathscr{D}(X')'}(\mathbb{Q}^M_{Y'}[n-1], \mathbb{Q}^M_{X'}(1)[n]) \to \operatorname{Ext}^1_{\mathscr{D}(X)'}(\mathbb{Q}^M_Y[n-1], \pi_*\mathbb{Q}^M_{X'}(1)[n]),$$

induced by the composition with the restriction morphism $\mathbb{Q}^M_Y \to \pi_*\mathbb{Q}^M_{Y'}$, is an isomorphism.

PROOF. If we take Ext in $\mathscr{D}(X')$ or $\mathscr{D}(X)$, the assertion follows from adjunction. So it is enough to show

$$\operatorname{Ext}^1_{\mathscr{D}(X')'}(\mathbb{Q}^M_{Y'}[n-1], \mathbb{Q}^M_{X'}(1)[n]) = \operatorname{Ext}^1_{\mathscr{D}(X')}(\mathbb{Q}^M_{Y'}[n-1], \mathbb{Q}^M_{X'}(1)[n]),$$
$$\operatorname{Ext}^1_{\mathscr{D}(X)'}(\mathbb{Q}^M_Y[n-1], \pi_*\mathbb{Q}^M_{X'}(1)[n]) = \operatorname{Ext}^1_{\mathscr{D}(X)}(\mathbb{Q}^M_Y[n-1], \pi_*\mathbb{Q}^M_{X'}(1)[n]).$$

Since X' is smooth and Y' is a divisor, the first assertion follows from (1.6.1) using (1.4.1). Similarly, we get the second using the adjunction for $i : Z \to X$ together with Lemma (2.3) which we apply to $\pi : X' \to X$. In fact, we have

$$H^j(i^*\mathbb{Q}^M_Y[n-1]) = 0 \quad \text{for } j > \dim Z - n + 1,$$
$$H^j(\pi_*\mathbb{Q}^M_{X'}[n])_Z = 0 \quad \text{for } j < \dim Z - n + 2,$$

if $Z \neq Y$ by (1.4.1) and (2.3).

(3.9) REMARK. In [**16,** (8.2)], we defined a cycle map

(3.9.1) $$cl^M : \mathrm{CH}_d(X)_{\mathbb{Q}} \to \operatorname{Ext}^{-2d}_{\mathscr{D}(X)^{\mathrm{go}}}(\mathbb{Q}^M_X, \mathbb{D}^M_X(-d)),$$

where $\mathscr{D}(X)^{\mathrm{go}}$ is the bounded derived category of the objects of geometric origin. For this cycle map, we can show the adjunction isomorphism

(3.9.2) $$\operatorname{Ext}^{-2d}(\mathbb{Q}^M_X, \mathbb{D}^M_X(-d)) \xrightarrow{\sim} \operatorname{Ext}^{-2d}(\mathbb{Q}^M_{\operatorname{Spec} k}, (a_X)_*\mathbb{D}^M_X(-d))$$

for $a_X : X \to \operatorname{Spec} k$. However, we cannot prove this for the cycle maps (3.1.2), (3.1.3). (The bijectivity of (3.9.2) would be related with conjectures in [**4, 8**] and also with [**12**].)

As to the relation of (3.9.1) with (3.1.2), (3.1.3), we have the natural morphisms (3.1.4), (3.1.5) where we assume $\mathscr{M}(X) = \mathscr{M}(X)^{\mathrm{go}}$ as in [**16**]. The surjectivity of (3.1.4), (3.1.5) is related with the $\mathscr{M}$-Hodge type conjecture. See [loc. cit.]. The injectivity is related with the surjectivity of the cycle map of Bloch's higher Chow group $\mathrm{CH}^p(Y, 1)_{\mathbb{Q}}$ (or the graduation of $K_1(Y)_{\mathbb{Q}}$) to $\operatorname{Ext}^{2p-1}(\mathbb{Q}^M_Y, \mathbb{Q}^M_Y(p))$ for smooth (locally closed) subvarieties Y of X. See [**15**].

REFERENCES

1. A. Beilinson, *Height pairing between algebraic cycles*, Lecture Notes in Math., vol. 1289, Springer-Verlag, New York and Berlin, 1987, pp. 1–26.
2. ——, *On the derived category of perverse sheaves*, Lecture Notes in Math., vol. 1289, Springer-Verlag, New York and Berlin, 1987, pp. 27–41.
3. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100**, Soc. Math. France, Paris, 1982.
4. S. Bloch, *Lectures on algebraic cycles*, Duke Univ. Math. Ser., vol. 4, Duke Univ., Durham, NC, 1980.
5. P. Deligne, *Théorie de Hodge* I, Actes Congrès Intern. Math. I (1970), 425–430; II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–58; III, ibid. **44** (1974), 5–77.
6. ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 313–346.
7. ——, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over **Q**, Springer-Verlag, New York, 1989, pp. 79–297.
8. ——, *A quoi servent les motifs?*, these Proceedings, vol. 1, pp. 143–161.
9. P. Deligne, J. Milne, A. Ogus, and K. Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math., vol. 900, Springer-Verlag, New York and Berlin, 1982.
10. T. Ekedahl, *On the adic formalism*, Grothendieck Festschrift II, Birkhäuser, Boston, MA, 1990, pp. 197–218.
11. U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, New York and Berlin, 1990.
12. J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.
13. M. Saito, *Mixed Hodge Modules*, Publ. Res. Inst. Math. Sci. **26** (1990), Kyoto Univ., pp. 221–333.
14. ——, *Hodge conjecture and mixed motives*. I, Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, pp. 283–303; II, Lecture Notes in Math., vol. 1479, Springer-Verlag, New York and Berlin, 1991, pp. 196–215.
15. ——, *On the injectivity of cycle maps*, Publ. Res. Inst. Math. Sci. **28** (1992), Kyoto Univ., pp. 99–127.
16. ——, *On the formalism of mixed sheaves*, preprint RIMS-784, 1991.
17. ——, *Extension of mixed Hodge Modules*, Compositio Math. **74** (1990), 209–234.
18. ——, *Some remarks on the Hodge type conjecture*, these Proceedings, vol. 1, pp. 85–100.

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Tannakian Categories

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Classical Pontryagin duality asserts that for any abelian locally compact topological group G, the character group $G^\frown$ of G is again locally compact and that the canonical map $\rho: G \to (G^\frown)^\frown$ is a continuous isomorphism, so that the group G is entirely determined by its character group. A similar assertion is true in the case of a commutative algebraic group (or commutative group scheme); the appropriate character group scheme in this case is the Cartier dual of G. Passing to noncommutative (topological) groups, it is clear from the outset that one cannot hope to recover a group G from its character group $G^\frown$, since the group G and its abelianization G^{ab} both have the same group of characters. The observation that a character is just a one-dimensional unitary representation of G suggests that we associate instead to a group G all of its unitary finite-dimensional representations. When G is compact, all of the finite-dimensional representations are unitary, and they define a discrete (i.e., algebraic) object, the tensor category $\mathrm{Rep}(G)$ of such representations. The classical theorem of Tannaka-Krein **[T]** explains how the group G may be recovered from the category $\mathrm{Rep}(G)$ as the group of tensor-preserving automorphisms of the forgetful functor from $\mathrm{Rep}(G)$ to the underlying category of vector spaces. The part of the theorem due to Krein **[Kr]** then characterizes those categories $\mathscr{C}$, from which it is possible to reconstruct a group G for which $\mathscr{C}$ is equivalent to the category $\mathrm{Rep}(G)$ (see for example **[Ch]**). It is the analogous theorem for algebraic groups (or group schemes) that will be discussed here.

Let us begin with the so-called neutral case. Here the group G is defined over a base field k, and the forgetful functor provides a so-called fibre functor (i.e., an exact, k-linear tensor functor) ω from the k-linear category $\mathscr{C} = \mathrm{Rep}(G)$ to the category of finite-dimensional k-vector spaces. The proof follows classical lines and may be found, in the algebraic context, in

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Saavedra's thesis **[Saa]**. We shall discuss it in some detail since the main features of the proof in the general case are already apparent in this simpler context. The full theorem is built up here from a much more primitive assertion for appropriate subcategories of $\mathscr{C}$, which follows from a purely categorical assertion known as the Barr-Beck theorem.

We now turn to the general, nonneutral case. In this instance, we are given a k-linear tensor category $\mathscr{C}$ for which there no longer exists a fibre functor from $\mathscr{C}$ into the category (k-vect), as in the neutral case, but merely into the category (K-vect) for some appropriate extension K of k. Such a category is said to be Tannakian. An interesting example of such a category is the following one. Let $\mathscr{C}$ be the appropriately defined $\mathbb{Q}$-tensor category $\mathscr{M}_p$ of motives over a finite field $\mathbb{F}_p$, for a full discussion of which we refer to **[Mi1]**. In this case, ℓ-adic cohomology for $\ell \neq p$ (resp. crystalline cohomology) provides a fibre functor $h_\ell \colon \mathscr{M}_p \to (\mathbb{Q}_\ell\text{-vect})$ (resp., $h_{\text{crys}} \colon \mathscr{M}_p \to (K(\mathbb{F}_p)\text{-vect})$, where $K(\mathbb{F}_p)$ stands for the field of fractions of the ring $W(\mathbb{F}_p)$ of Witt vectors of $\mathbb{F}_p$). However, there no longer exists a fibre functor $h \colon \mathscr{M}_p \to (\mathbb{Q}\text{-vect})$ from which h_ℓ and h_{crys} could be deduced by base change. It turns out that the obstruction to descending fibre functors such as these to ones with value in ($\mathbb{Q}$-vect) is cohomological in nature and lives in an appropriately defined nonabelian H^2 of the scheme $\operatorname{Spec}(\mathbb{Q})$. More precisely, one shows that the collection of fibre functors from the tensor category $\mathscr{C}$ to categories (K-vect) forms, for varying extensions K of the base scheme $\mathbb{Q}$, a gerbe $\mathscr{G}$ over $\operatorname{Spec}(\mathbb{Q})$ in the sense of Giraud **[Gi]**. By definition, $\mathscr{G}$ would have a global section (and so would define a so-called neutral cohomology class in H^2) if and only if there existed a ($\mathbb{Q}$-vect)-valued fibre functor on $\mathscr{C}$.

With this in mind, we review the theory of gerbes and its interpretation both in terms of bitorsors as in **[B1–2]** and in terms of transitive groupoids, as in **[Du1–2]** and **[De2]**. We do not, however, describe in full detail the surprisingly difficult proof by Deligne (following an inadequate attempt in **[Saa]**) that the collection of fibre functors on a Tannakian category $\mathscr{C}$ actually do form an affine gerbe. Once this fact has been established, the manner in which $\mathscr{C}$ may be reconstructed from its associated gerbe $\mathscr{G}$ is entirely analogous, as is explained in **[De2]**, to that by which we recovered $\mathscr{C}$ in the neutral case by classical Tannaka duality from the corresponding group G. Here, however, instead of focussing on the group G of automorphisms of a fixed fibre functor ω, one has to take into account the transitive groupoid of isomorphisms between the various possible fibre functors and, therefore, systematically replace groups by groupoids in the discussion.

In the last sections, we consider two alternate descriptions of gerbes. We begin by showing that the usual dictionary by which one describes sheaves over $\operatorname{Spec}(k)$ in the étale topology by continuous $\operatorname{Gal}(k^s/k)$-sets allows one to recover from the groupoid associated to a locally neutralized gerbe the corresponding Galois gerbe introduced by Langlands and Rapoport in **[L-R]**. We also give, in this situation, a much more explicit recipe than in **[L-R]** for recovering the gerbe from the corresponding Galois gerbe. We also explain, in the manner of **[B1]**, but in a more general context, how one can pass, by further local trivialization, from a groupoid (or bitorsor) description of a gerbe to an essentially equivalent definition given in terms of genuine nonabelian cocycles. When this process is applied to the Galois gerbe associated to a smooth affine gerbe $\mathscr{G}$ over $\mathrm{Spec}(k)$ in the étale topology, it yields nonabelian Galois cocycles along the lines of those defined by Springer at the very beginning of this subject **[Sp]**. We believe that both the definition of the nonabelian cocycles (at the level of generality occurring here) and the manner in which Galois gerbes may be compared to gerbes and to their cocycles are new (though in the latter case, the line of reasoning is not very far removed from that of Ulbrich in **[Ul2]**, which we feel is somewhat clarified by our discussion).

One of our purposes in writing this paper is to provide an introduction to this somewhat arcane subject and more particularly to Deligne's **[De2]**, to which the reader is referred for a complete proof of the Tannaka theorem (for another survey of this approach, with greater emphasis on groupoids, see **[Mi2,** Appendix A]). Some other interesting references for various aspects of the Tannakian theory are **[D-M]**, **[J-S2]**, and of course **[Saa]**. Our second aim is to present a short account of the theory of the nonabelian H^2, which stresses the description in terms of explicit cocycles, along the lines of **[B1–2]**. Nonabelian cohomology is a topic that has been studied quite independently by a number of authors over the years. We feel that it is now possible to give a unified view of this material, which emphasizes the compatibilities between various possible approaches and their relation to the theory of Tannakian categories. It must be said that it is A. Grothendieck who first saw the relation between the two topics under discussion here: it is he who realized that a Tannakian category could be described, in cohomological terms, by the gerbe of its fibre functors. A striking expression of this will be found in **[Gr]**.

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1. Neutral Tannaka duality

Let G be an affine group scheme defined over a ring k. A (finite-dimensional) representation of G defined over k is a group scheme homomorphism $\rho\colon G \to \mathrm{Aut}(V)$, where V is a projective k-module of finite rank (and so, when k is a field, a finite-dimensional k-vector space). As noted above, the representations of G form a category $\mathscr{C} = \mathrm{Rep}(G)$, a morphism $u\colon (\rho, V) \to (\rho', V')$ being an intertwining operator, i.e., a G-equivariant k-linear map $u\colon V \to V'$. The category $\mathscr{C}$ has a very rich structure, based on that of the underlying category of k-modules, which we will now review.

(1.1.1) *Tensor product.* $\mathscr{C}$ is endowed with a product

$$\mathscr{C} \times \mathscr{C} \to \mathscr{C},$$
$$(\rho_1, \rho_2) \mapsto \rho_1 \otimes \rho_2$$

which associates to every pair of representations $\rho_i: G \to \mathrm{Aut}(V_i)$ the representation

$$\rho_1 \otimes \rho_2: G \to \mathrm{Aut}(V_1 \otimes V_2)$$

defined by the action of G on V_1 and V_2. This tensor product is endowed with functorial coherent associativity and commutativity isomorphisms

(1.1.1.1) $$\rho_1 \otimes (\rho_2 \otimes \rho_3) \to (\rho_1 \otimes \rho_2) \otimes \rho_3,$$

(1.1.1.2) $$\rho_1 \otimes \rho_2 \to \rho_2 \otimes \rho_1.$$

For the associativity, this coherence means that the well-known pentagon **[Mac1]**, **[Mac2,** VII, §1], which can be constructed from any four representations ρ_i $(1 \le i \le 4)$ of G, is commutative. This property follows from the corresponding property for the tensor product in the category of k-modules. In practice, this means **[Mac1–2]** that we can neglect the parentheses in any expression involving iterated tensor products, and just as in ordinary group theory, consider expressions such as $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$. A further property of the commutativity condition (1.1.1.2) is its symmetry. This means that the composite map

(1.1.1.3) $$\rho_1 \otimes \rho_2 \to \rho_2 \otimes \rho_1 \to \rho_1 \otimes \rho_2$$

is the identity. The coherence of (1.1.1.2) may then simply be stated as the commutativity for the hexagonal diagram **[Mac1,** VII, §8] (actually a triangle since the associativity isomorphisms have been neglected).

(1.1.2) *Unity.* The trivial representation

$$I: G \to \mathrm{Aut}(1)$$

of G in the free one-dimensional k-module 1 is a unit in the category $\mathscr{C}$ for the tensor product. This means that there exist, for any object ρ of $\mathscr{C}$, functorial isomorphisms

(1.1.2.1) $$\rho \otimes I \to \rho \to I \otimes \rho$$

compatible with the symmetry isomorphisms (1.1.1.2).

A category with a coherently associative tensor product and a unit object I is said to be monoidal and is also often called a tensor category. In such a category, the unit isomorphisms (1.1.2.1) may be chosen to be coherent in an appropriate sense **[J-S1]**, **[Saa]**, **[D-M,** 1.3]. A tensor category $\mathscr{C}$ is said to be symmetric monoidal **[Mac2]** or ACU **[Saa]** (i.e., Associative, Commutative, Unitary), if it is endowed with a commutativity isomorphism (1.1.1.2) satisfying (1.1.1.3) and the hexagon condition. Following **[J-S1]**, a monoidal category $\mathscr{C}$ is called braided when it possesses a commutativity isomorphism (1.1.1.2) that no longer satisfies (1.1.1.3) but only the hexagon condition.[1]

A functor $F: \mathscr{C}_1 \to \mathscr{C}_2$ between monoidal (resp. symmetric monoidal) categories is a functor between the underlying categories together with natural isomorphisms

$$FA \otimes FB \to F(A \otimes B) \tag{1.1.2.2}$$

compatible in the sense of **[Saa]**, **[D-M,** 1.8] with the associativity, unity (resp. associativity, unity, and commutativity) constraints. Such a functor is also often called a tensor functor. A natural transformation $u: F \Rightarrow G$ between two such tensor functors is a natural transformation between the underlying ordinary functors which preserves the isomorphism (1.1.2.2) **[D-M,** 1.1.2].

(1.1.3) *Inverses.* We say that a symmetric monoidal category has inverses (or duals) if we are given in a functorial manner, for every object $\rho \in \mathscr{C}$, an inverse object $\rho^\vee$ of $\mathscr{C}$ and an evaluation map

$$\varepsilon: \rho \otimes \rho^\vee \to I. \tag{1.1.3.1}$$

The choice of such an inverse is automatically coherent. Indeed, there exists a canonical method **[J-S1,** Proposition 7], for associating to such maps (1.1.3.1) coevaluation maps

$$\delta: I \to \rho^\vee \otimes \rho \tag{1.1.3.2}$$

such that the composites

$$\begin{gathered}(\varepsilon \otimes \rho) \circ (\rho \otimes \delta): \rho \to \rho \otimes \rho^\vee \otimes \rho \to \rho, \\ (\rho^\vee \otimes \varepsilon) \circ (\delta \otimes \rho^\vee): \rho^\vee \to \rho^\vee \otimes \rho \otimes \rho^\vee \to \rho^\vee\end{gathered} \tag{1.1.3.3}$$

are the identity map. A symmetric monoidal category with pairs of maps ε and δ satisfying (1.1.3.3) is called a rigid $ACU\otimes$-category **[Saa]** (category theorists call this a symmetric compact closed monoidal category **[Ke-La]**). Inverses in such categories are coherent in the very strong sense that we then have a functor $X \mapsto X^\vee$ that is compatible with the symmetric monoidal structure (a compatibility condition can often still be satisfied without any symmetry condition on the tensor product **[F-Y2]**). Such a rigid $ACU\otimes$-category possesses internal Hom's, the object $\mathrm{Hom}(\rho_1, \rho_2)$ of $\mathscr{C}$ being defined by $\mathrm{Hom}(\rho_1, \rho_2) = \rho_1^\vee \otimes \rho_2$. Furthermore, to every endomorphism $f: \rho \to \rho$ in $\mathscr{C}$ there corresponds its trace $\mathrm{Tr}(f)$. This is the endomorphism of I given by

$$\mathrm{Tr}(f) = \varepsilon \circ (1 \otimes f) \circ \delta. \tag{1.1.3.4}$$

The trace can still be defined without any commutativity data on the tensor product, so long as the functor $X \mapsto X^\vee$ is compatible with the monoidal structure (see **[F-Y1,** 1.5]).

[1] There are actually, in this case, two separate hexagon conditions to be verified which coalesce whenever axiom (1.1.1.3) is satisfied.

For any group G, the category $\mathrm{Rep}(G)$ satisfies these conditions, where the inverse of a representation $\rho\colon G \to \mathrm{End}(V)$ is the contragredient representation $\rho^\vee\colon G \to \mathrm{End}(V^\vee)$, defined for any linear form $f \in V^\vee$ by

$$\rho^\vee(g)(f) = (f \circ \rho)(g^{-1}).$$

(1.1.4) *Tensoriality.* The following additional axioms, of a somewhat different nature, are also satisfied by the category $\mathscr{C} = \mathrm{Rep}(G)$:

(i) $\mathscr{C}$ is an abelian category;
(ii) we are given an isomorphism $k \approx \mathrm{End}(I)$.

In an abelian rigid $ACU\otimes$-category $\mathscr{C}$, condition (ii) determines, for every pair of objects X and Y, a k-module structure on $\mathrm{Hom}(X, Y)$ (with underlying abelian group structure given by (i)), for which the product on Hom-sets

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(X' \otimes Y') \to \mathrm{Hom}(X \otimes X', Y \otimes Y') \tag{1.1.4.1}$$

is k-bilinear. Let us say that a rigid $ACU\otimes$-category satisfying (i) and (ii) is tensorial, as opposed to tensor, over k (or is k-tensorial). Morphisms between such categories (resp. natural transformations between such morphisms) are those between the underlying symmetric monoidal categories. In such a category, the tensor product functor is exact in each variable (and this remains true without the assumption that the tensor product is symmetric or that $\mathrm{End}(I) \approx k$). Every object X of $\mathscr{C}$ has a dimension: this is the element $\dim(X)$ of k defined by the formula

$$\dim(X) = \mathrm{Tr}(1_X). \tag{1.1.4.2}$$

(1.1.5) *Fibre functors.* Let

$$\begin{aligned} \omega\colon \mathscr{C} &\to (k\text{-mod}), \\ (\rho, V) &\mapsto V \end{aligned} \tag{1.1.5.1}$$

be the forgetful functor taking its values in the category of finite rank projective k-modules. The functor ω obviously respects the tensor structure, i.e., is a functor between symmetric monoidal categories. It is also k-linear (on Hom-sets) and exact. An arbitrary k-linear exact tensor functor $\omega\colon \mathscr{C} \to (k\text{-mod})$ whose source is a k-tensorial category $\mathscr{C}$ is called a k-valued fibre functor.[2] For any k-algebra R, we denote by $\omega^{(R)}$ the tensor functor obtained from ω by composing it with the base change functor from k to R

$$\omega^{(R)}\colon \mathscr{C} \to (k\text{-mod}) \to (R\text{-mod}). \tag{1.1.5.2}$$

The category $\mathscr{C} \otimes_k R$ is defined in **[Mil,** 1.1.3], where it is shown that the functor $\omega^{(R)}$ factors uniquely through a fibre functor $\omega_R\colon \mathscr{C}\otimes R \to (R\text{-mod})$.

[2]The terminology presumably stems from the idea that such a functor ω defines some sort of "vector bundle" over the category $\mathscr{C}$ whose fibre at the object X is the k-module of $\omega(X)$.

It may therefore be viewed as the functor obtained from ω by the base change from k to R.

DEFINITION 1.2. A neutral Tannakian category over a ring k is a k-tensorial category $\mathscr{C}$ for which there exists a k-valued fibre functor ω. Such a pair $(\mathscr{C}, \omega)$ will be called a neutralized Tannakian category over k.

With such a pair $(\mathscr{C}, \omega)$ is associated the k-group scheme $G = \mathrm{Aut}(\omega)$ of natural transformations of the tensor functor ω into itself, whose R-valued points are defined, for any k-algebra R, by

$$G(R) = \mathrm{Aut}(\omega^{(R)}). \tag{1.2.1}$$

Starting from an arbitrary k-group scheme G and setting $\mathscr{C} = \mathrm{Rep}(G)$, with ω defined by the forgetful functor, there is an obvious map of R-valued functors

$$u\colon G \to \mathrm{Aut}(\omega) \tag{1.2.2}$$

which associates to a fixed element g of $G(R)$ and to varying objects (V, ρ) of $\mathrm{Rep}(G)$ the family of linear maps $\rho(g)\colon V \otimes R \to V \otimes R$. In the present algebraic context, Tannaka's theorem is the following statement:

PROPOSITION 1.3. *Let G be an affine group scheme defined over a field k. The induced morphism* (1.2.2) *is an isomorphism of functors.*

PROOF (for more details, see **[D-M Proposition 2.8, p. 129]**). For any object X of $\mathscr{C} = \mathrm{Rep}(G)$, let $\langle X\rangle_{\otimes}$ be the full subcategory of $\mathscr{C}$ whose objects are subquotients of direct sums of the representations $T^{m,n} = X^{\otimes m} \otimes (X^{\vee})^{\otimes n}$ generated by X. We denote by $\omega|\langle X\rangle_{\otimes}$ the restriction to this subcategory of the functor ω. Let G_X be the image of G in $\mathrm{GL}(X)$. We then have a natural inclusion

$$G_X \hookrightarrow \mathrm{Aut}(\omega|\langle X\rangle_{\otimes}) \hookrightarrow \mathrm{GL}(X),$$

and the group $G'_X = \mathrm{Aut}(\omega|\langle X\rangle_{\otimes})$ may be described as the subgroup of those linear transformations of X that leave invariant, for some pair of integers m, n, some tensor v occurring in some subquotient of $T^{m,n}$ that is fixed under the action by all elements of G_X. A theorem of Chevalley states that any subgroup G_X of $\mathrm{GL}(X)$ may be characterized as the stabilizer in $\mathrm{GL}(X)$ of a one-dimensional subspace D of some representation $V \in \mathscr{C}_X$, and we may then easily find a vector v' in the subquotient $D \otimes D^{\vee}$ of $V \otimes V^{\vee}$ such that G_X is the subgroup in $\mathrm{GL}(X)$ that leaves v' fixed. Since the conditions defining G'_X are more stringent than that by which we have just characterized its subgroup G_X, we see that the inclusion $u_X\colon G_X \hookrightarrow \mathrm{Aut}(\omega|\langle X\rangle_{\otimes})$ is an isomorphism. The isomorphism u (1.2.2) is then obtained by passing to the limit over larger and larger objects $X \in \mathscr{C}$.

The Tannaka theorem states that the construction $G \mapsto \mathrm{Rep}(G)$ has as left inverse the map that associates to the pair $(\mathrm{Rep}(G), \omega)$ the group $\mathrm{Aut}(\omega)$. The Krein-type theorem, which asserts that this is also a right-inverse, is the main statement of Tannakian duality. We begin by observing that, by the definition (1.2.1) of the group G associated to an arbitrary tensor functor $\omega: \mathscr{C} \to (k\text{-mod})$, the latter automatically factors as $\omega = \omega_0 \circ \lambda$ through the forgetful functor $\omega_0: \operatorname{Rep}(G) \to (k\text{-mod})$.

THEOREM 1.4. *Let* $(\mathscr{C}, \omega)$ *be a neutralized Tannakian category over a field* k, *and let* $G = \operatorname{Aut}(\omega)$ *be the associated group. The group* G *is an affine flat* k*-group scheme, and the functor* $\lambda: \mathscr{C} \to \operatorname{Rep}(G)$ *is an equivalence of (symmetric tensor) categories.*

EXAMPLES 1.4.1. (i) Let $\mathscr{C}$ be the category of $\mathbb{Z}$-graded k-vector spaces $(V^n)_{n\in\mathbb{Z}}$, with tensor product induced by the usual tensor product of vector spaces and with

$$\omega: (V^n) \mapsto \bigoplus V^n$$

as fibre functor. The Tannakian theorem identifies $\mathscr{C}$ with the category $\operatorname{Rep}(G_m)$ of representations of the multiplicative group $\mathbb{G}_m$ in $(k\text{-vect})$. The action of $\mathbb{G}_m$ on the total space V of the representation defines a grading on V for which $\mathbb{G}_m$ acts on the factor V^n through the character $\lambda \mapsto \lambda^n$.

(ii) Let $\mathscr{C}$ be the category $\operatorname{Hod}_{\mathbb{R}}$ of real Hodge structures. An object in $\mathscr{C}$ is a real vector space V together with an isomorphism

$$V \otimes \mathbb{C} \simeq \bigoplus V^{p,q},$$

where $V^{p,q}$ and $V^{q,p}$ are conjugate complex subspaces of $V \otimes \mathbb{C}$. A fibre functor $\omega: \mathscr{C} \to (\mathbb{R}\text{-Vect})$ is defined by $\omega(V, (V^{p,q})) = V$. The Tannaka theorem identifies $\operatorname{Hod}_{\mathbb{R}}$ with the category of representations of the real algebraic group $S = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ deduced by Weil restriction from $\mathbb{C}$ to $\mathbb{R}$ from the multiplicative group over $\mathbb{C}$. An element $\lambda \in S(\mathbb{R}) = \mathbb{C}^*$ acts on the factor $V^{p,q}$ of $V \otimes \mathbb{C}$ by multiplication by $\lambda^{-p}\bar{\lambda}^{-q}$.

(iii) Let $\mathscr{C}$ be the category of Artin motives. This is the subcategory of the category of the $\mathbb{Q}$-Tannakian category M_k of motives over a field k of characteristic zero whose objects are defined by zero-dimensional varieties. To such a variety X corresponds the sheaf on the small étale site $\operatorname{Spec}(k)_{\text{ét}}$ which it represents, and this is described, as we shall see again in 4.1, by the finite set $X(\bar{k})$ of its points with values in the algebraic closure $\bar{k}$ of k, viewed as a discrete set with continuous (left) action of the Galois group Γ of k, endowed with the Krull topology. If we associate to X the finite-dimensional $\mathbb{Q}$-vector space $\mathbb{Q}^{X(\bar{k})}$, with the induced action of Γ, we obtain an equivalence between the category $\mathscr{C}$ and the category of $\operatorname{Rep}(\Gamma)$ of finite-dimensional representations of Γ (see **[D-M,** p. 211] for additional discussion of this example).

1.4.2. The following sketch of a proof of Theorem 1.4 follows, in the neutral case, the proof given in **[De2]** (see also **[D-M,** Theorem 2.1.1]). We begin, as in **[Saa]**, by considering a much weaker statement. Dropping the tensor structure on $\mathscr{C}$, we suppose here that $\mathscr{C}$ is merely an abelian k-linear category in which every object is of finite length and every k-vector space

$\mathrm{Hom}(X, Y)$ is finite dimensional. Since these properties are satisfied in the category (k-mod), the existence of a fibre functor ω implies that they also are whenever $\mathscr{C}$ is neutral Tannakian. A categorical lemma then asserts that, for every object X of $\mathscr{C}$, the full subcategory $\langle X \rangle$ of $\mathscr{C}$ whose objects are subquotients of the X^n has a projective generator P_X, and therefore the functor $\mathrm{Hom}(P_X, -)$ is an equivalence between $\langle X \rangle$ and the category $\mathscr{A}_X$ of finite-type right modules over the finite-type k-algebra $A_X = \mathrm{End}(P_X)$. Under this equivalence, the object P_X is transformed into A_X itself (viewed as a right A_X-module via right multiplication and denoted henceforth in this context by A_d) while the functor $\omega|\langle X \rangle : \langle X \rangle \to (k\text{-mod})$ becomes an exact faithful functor

$$S: \mathscr{A}_X \to (k\text{-mod}) \tag{1.4.2.1}$$

between two categories of modules. We shall say that such a functor $S = \omega|\langle X \rangle$ is a weak fibre functor on $\langle X \rangle$ (it is not, however, a fibre functor in the traditional sense since the subcategory $\langle X \rangle$ of $\mathscr{C}$ does not have a tensor structure). A Morita-type statement asserts that any such functor between two categories of modules is equivalent to the functor of tensorization by the (A_X, k)-bimodule $M = S(A_d) = \omega(P_X)$

$$\begin{aligned} S: \mathscr{A}_X &\to (k\text{-mod}), \\ E &\mapsto E \otimes M. \end{aligned} \tag{1.4.2.2}$$

Note that the (faithfully flat) left A_X-module structure of M is particularly apparent in the second of the two descriptions of M which have just been given, since the ring $A_X = \mathrm{End}(P_X)$ acts in an obvious manner on $M = \omega(P_X)$ through the functor ω. Returning to the first description of M, we may now identify A_X with the ring of endomorphisms of the object A_d of $\mathscr{A}_X$, so that A_X acts on M on the left through the functor S.

Viewing the dual module $M^\vee = \mathrm{Hom}_k(M, k)$ as a right A_X-module, for the action induced by that of A_X on M, we may also introduce the functor

$$\begin{aligned} U: (k\text{-mod}) &\to \mathscr{A}_X, \\ V &\mapsto V \otimes M^\vee, \end{aligned}$$

which is a left adjoint of S. Since S is an exact faithful functor between abelian categories, we may call on the Barr-Beck theorem (for a dual version of this theorem, see [**Mac2**, VI, §7]). This gives us, for any such functor $S: \mathscr{A}_1 \to \mathscr{A}_2$ with left adjoint U, a very pleasant description of objects of the category $\mathscr{A}_1$ in terms of those of $\mathscr{A}_2$, together with some additional structure. To be somewhat more specific, we note that one can build out of the pair of natural transformations

$$\varepsilon: SU \to I, \qquad \eta: I \to US \tag{1.4.2.3}$$

defined by the adjointness of S and U, a co-monad [**Mac2**]. Setting

$$B_X = SU(k) = M^\vee \otimes_A M, \tag{1.4.2.4}$$

this yields a diagram of k-modules

$$B_X \to B_X \otimes B_X \rightrightarrows B_X^{\otimes 3}.$$

This, together with the augmentation on B_X given by (1.4.2.3), defines a k-coalgebra structure on the k-module B_X. To any object $E \in \mathscr{A}_X$ we then associate the k-module $F = SE$, together with an appropriate diagram

$$F \to F \otimes B_X \rightrightarrows F \otimes B_X \otimes B_X$$

which defines a B_X-comodule structure on F. The Barr-Beck theorem asserts that we have obtained in this manner an equivalence $S \colon E \mapsto F$ between the category of finite type A_X-modules and that of finite type B_X-comodules.

1.4.3. This Barr-Beck description of the category $\langle X \rangle$ is a very primitive version of the Tannaka theorem. It may be viewed as asserting that a category such as $\langle X \rangle$, which is generated by a single object X and possesses a weak fibre functor $\omega (= \omega|_X) \colon \langle X \rangle \to (k\text{-mod})$, may be reconstructed as the category $(B_X\text{-comod})$ of finite-type comodules under the k-coalgebra B_X. This interpretation can even be carried further: for every k-module R, ω induces by base change as in (1.1.5.2) a functor

$$\omega^{(R)} \colon E \mapsto E \otimes M \otimes R,$$

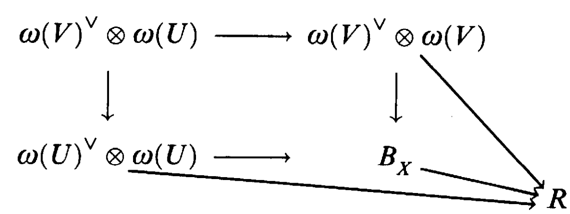
and any natural transformation from ω to $\omega^{(R)}$ is given by an element of $\mathrm{Hom}_A(M, M \otimes R) = \mathrm{Hom}_k(M^\vee \otimes_A M, R)$. The k-module B_X can therefore be interpreted as representing the endomorphism of the weak fibre functor ω:

$$\mathrm{End}(\omega) \colon R \mapsto \mathrm{Hom}(\omega, \omega^{(R)}).$$

Note, however, that we are representing this endomorphism functor on the category $(k\text{-mod})$ rather than on the category of commutative k-algebras on which we traditionally represent functors in algebraic geometry.

Another way of viewing B_X is to observe that an element of $\mathrm{End}(\omega)(R)$ consists in the collection, for varying objects U in $\langle X \rangle$, of the corresponding sets $\mathrm{Hom}(\omega(U), \omega(U) \otimes R) = \mathrm{Hom}(\omega(U)^\vee \otimes \omega(U), R)$, with the obvious compatibility for every morphism $\varphi \colon U \to V$ in $\langle X \rangle$ so that we may view the representing object B_X of $\mathrm{End}(\omega)$ as the co-end of ω, that is, as the universal object in k-mod for the problem

(1.4.3.1)



1.4.4. From this lowest level Tannaka theorem we can progressively build more elaborate ones. First of all, we may pass from $\langle X \rangle$ to the category

$\operatorname{Ind}\langle X\rangle$ of ind-objects of $\langle X\rangle$ and so obtain an equivalence between this category and that of all B_X-comodules. We can then view our original category $\mathscr{C}$ as the inductive limit (over a filtering set I of objects X of T) of the categories $\langle X\rangle$. Introducing the large k-coalgebra $B = \lim_{X\in I} B_X$, it then follows that $\mathscr{C}$ is equivalent to the category of B-comodules.

In order to get the full Tannakian statement, we must now consider the additional given structure on $\mathscr{C}$. Since $\mathscr{C}$ is monoidal symmetric and $\omega\colon \mathscr{C} \to (k\text{-mod})$ is a genuine (i.e., tensor product preserving) fibre functor, we have, for any pair of objects X and Y of $\mathscr{C}$, an induced pairing of categories

$$\langle X\rangle \times \langle Y\rangle \to \langle X \otimes Y\rangle,$$

and the universal property (1.4.3.1) ensures that this induces a ring homomorphism $B_X \otimes B_Y \to B_{X\otimes Y}$. By passing to the limit, this gives a pairing

$$B \otimes B \to B. \tag{1.4.4.1}$$

The various properties of the tensor product on $\mathscr{C}$ are then reflected in the corresponding properties of the k-module B: the associativity (resp., commutativity) axiom in $\mathscr{C}$ implies that the pairing (1.4.4.1) is associative (resp., commutative), while the existence of a unit in $\mathscr{C}$ determines a unit section of B. Finally, the inverse law (1.1.3) in $\mathscr{C}$ corresponds to an antipodal map $i\colon B \to B$. The k-coalgebra B is therefore a commutative (but not necessarily cocommutative) Hopf algebra representing the group functor $G = \operatorname{End}(\omega)$ on the category of commutative k-algebras. Since a B-comodule structure $V \to V \otimes B$ on a k-module V defines a representation $\rho\colon G \to \operatorname{Aut}(V)$ on V of the group G represented by B, this finishes the proof that the category $\mathscr{C}$ is equivalent to the category $\operatorname{Rep}(G)$.

Remarks 1.5. (i) Observe that the proof just outlined still made sense under weaker hypotheses on the category $\mathscr{C}$, at the expense of weakening the corresponding properties of the ring B. For example, if the tensor product on $\mathscr{C}$ does not satisfy any commutativity conditions at all, then B is a noncommutative, noncocommutative Hopf algebra, and it was pointed out by Pareigis **[P]**, before the advent of quantum groups, that this is an interesting way of constructing nonobvious Hopf algebras. In the quantum groups terminology, the previous proof then asserts that the monoidal category $\mathscr{C}$ is equivalent to the category $\operatorname{Rep}(G)$ of representations of the quantum group G defined by B **[Ul1, Y1]**. In this direction, one should also refer to **[Ka-Lu]** where the category of representations of quantum deformations of the Lie group associated to a finite-dimensional simple Lie algebra $\mathbf{g}$ is interpreted as the category of representations of $\mathbf{g}$ itself, but endowed with a nontrivial tensor structure derived, following **[Dr2]**, from the Knizhnik-Zalomodchikov equations. Finally, the intermediate case of a category $\mathscr{C}$ that is braided rather than monoidal, or monoidal symmetric, is also worthy of interest: categories arising from topology, such as the category $\mathscr{C}$ of (framed) tangles, are of this type **[F-Y1, Ca]**.

(ii) Another way of weakening the Tannaka duality is to consider on $\mathscr{C}$ functors $\omega: \mathscr{C} \to k$-mod that no longer respect the associativity (resp., the commutativity) conditions. The pairing (1.4.4.1) on B is then no longer strictly commutative (resp. associative) but only up to some conjugation. B is then said to be a quasi-triangular-Hopf (resp. quasi-Hopf) algebra. The dual notion for coalgebras was originally introduced by Drinfeld in **[Dr1]**. In the braided case, the link invariants of Turaev-Reshetikin provide an interesting example of this sort of functor on the category of framed tangles, with values in $(\mathbb{C}\text{-vect})$. We refer to **[Ca, Ly, Y2, Maj]** for further discussion along these lines.

(iii) Various properties of the category $\mathscr{C}$ are reflected in corresponding properties of the group G. For example, over a field of characteristic zero, the group G is pro-reductive (i.e., a projective limit of reductive groups) if and only if the corresponding category $\mathscr{C}$ is semisimple. Similarly, a generalization of Example 1.4.1(i) says that the category $\mathscr{C}$ is graded if and only if we are given a central homomorphism of groups $\mathbb{G}_m \to G$. We refer to **[Saa]** (particularly to II.4.3) for additional examples in this direction.

2. Nonabelian cohomology

2.1. Just as an H^1 classifies principal bundles, the objects that an H^2 classifies are known as gerbes, and they were studied in a very general setting in **[Gi]**. Let us consider a category $\mathscr{C}$ fibered by a functor $p: \mathscr{C} \to \mathscr{S}$ over a category $\mathscr{S}$ (we have in mind the case in which the base category $\mathscr{S}$ is the category of schemes over some base scheme Y, or even, for $Y = \operatorname{Spec}(R)$ affine, the category of affine schemes over Y). We then have, for every object $S \in \mathscr{S}$, a subcategory $\mathscr{C}_S$ of $\mathscr{C}$ whose objects (resp. arrows) project to S (resp. to id_S). For each morphism $f: S' \to S$ in $\mathscr{S}$, an inverse image functor

$$f^*: \mathscr{C}_S \to \mathscr{C}_{S'} \tag{2.1.1}$$

may be defined, and for each pair of composable arrows f, g a coherent natural transformation

$$\varphi_{f,g}: (fg)^* \simeq g^* \circ f^* \tag{2.1.2}$$

may be defined. The coherence means that we can in practice neglect the transformations (2.1.2) and pretend that it is an equality, as we shall always do henceforth. We shall assume that we have products in $\mathscr{C}$ and that (2.1.1) respects them. $\mathscr{C}$ is said to be fibered in groupoids over $\mathscr{S}$ whenever the fibre categories $\mathscr{C}_S$ are groupoids, in other words, categories in which every arrow is an isomorphism.

Supposing now that $\mathscr{S}$ is a site, and so defines a Grothendieck topology on its final object e, we say that the fibered categories $\mathscr{C}_S$ form a stack (in groupoids) whenever objects and morphisms in $\mathscr{C}$ glue properly. For an object x_S in $\mathscr{C}_S$ and some covering map $p: S \to T$ of an object T in $\mathscr{S}$, this means that descent data for x_S with respect to p is always effective, so that such an object x_S, with given descent data, always descends to an object x of $\mathscr{C}_T$. On morphisms, the glueing condition then merely asserts that for any pair of objects x and y of $\mathscr{C}_T$, the presheaf of sets $\mathrm{Hom}(x, y)$ is in fact a sheaf on T.

A fibred category $\mathscr{C}$ over the site $\mathscr{S}$ for which the presheaf of objects $S \mapsto \mathrm{ob}(\mathscr{C}_S)$ and the presheaves of morphisms $\mathrm{Hom}(x, y)$ satisfy the sheaf condition is called a prestack above $\mathscr{S}$ (in other words, when the maps (2.1.2) are neglected, such a prestack is nothing else than a sheaf of categories $S \mapsto \mathscr{C}_S$). This is a weaker condition on $\mathscr{C}$ than being a stack over $\mathscr{S}$, but if we apply a categorical sheafification process **[Gi, II, 2.1.3]**, analogous to the usual one which transforms a presheaf of sets into the associated sheaf, one can force the effectivity of the descent to be satisfied and so construct from the prestack $\mathscr{C}$ an associated stack $\mathscr{C}^{\mathrm{a}}$, together with a morphism of prestacks

$$\mathscr{C} \to \mathscr{C}^{\mathrm{a}},$$

which is universal with respect to maps from $\mathscr{C}$ to stacks over $\mathscr{S}$.

DEFINITION 2.2. A stack in groupoids $\mathscr{G}$ over a site $\mathscr{S}$ is a gerbe whenever the following two conditions are satisfied:

(i) $\mathscr{G}$ is *locally nonempty*: there exists a covering map $S \to e$ in $\mathscr{S}$ such that $\mathrm{ob}(\mathscr{G}_S) \neq \varnothing$.

(ii) $\mathscr{G}$ is *locally connected*: for every pair of objects $x, y \in \mathscr{G}_T$, there exists a covering map $p: S \to T$ such that $\mathrm{Hom}(p^*x, p^*y)$ is nonempty.

The gerbes in the topos T of sheaves on a site $\mathscr{S}$ form a 2-category, for which 1-morphisms $\rho: \mathscr{G} \to \mathscr{H}$ between two gerbes are the so-called cartesian functors. These may just be viewed as functors $\rho: \mathscr{G} \to \mathscr{H}$ that preserve the inverse image functor (2.1.1), up to an isomorphism compatible with the transformations (2.1.2). Similarly, a 2-arrow $\eta: \rho \to \sigma$ between two such 1-morphisms is a cartesian natural transformation, i.e., one that is compatible with the maps (2.1.1).

The simplest example of a gerbe over $\mathscr{S}$ is the gerbe $\mathrm{Tors}(G)$ associated to a group G in the topos T of sheaves on $\mathscr{S}$, whose fibre over $S \in \mathscr{S}$ is the category $\mathrm{Tors}(S, G)$ of right G-torsors (i.e, of right G-principal bundles) on S. Indeed, principal bundles certainly satisfy effective descent, so form a stack, and any one of them is locally isomorphic to the trivial one whence the local connectivity. Finally, $\mathrm{Tors}(G)$ is even globally nonempty: the category $\mathscr{C}_e$ of G-torsors on e is nonempty, since it contains the trivial G-torsor. A gerbe is called neutral whenever the fibre category $\mathscr{G}_e$ over the final object e of $\mathscr{S}$ is nonempty. The choice of an object x in $\mathscr{G}_e$ identifies a neutral gerbe $\mathscr{G}$ with the gerbes of G-torsors, where $G = \mathrm{Aut}(x)$ by the equivalence

$$\begin{aligned} \mathscr{G}_S &\to \mathrm{Tors}(S, G), \\ y &\mapsto \mathrm{Isom}(p^*x, y), \end{aligned} \tag{2.2.1}$$

where $p: S \to e$ is the canonical projection.

2.3. The nonabelian H^2 of the final object e of $\mathscr{S}$ is by definition the set of equivalence classes of gerbes over $\mathscr{S}$. It is a set $H^2(e)$ with a distinguished subset $H^2(e)'$ consisting of the classes of the neutral gerbes. It is not immediately apparent from this definition that such a set has the usual features of a cohomology set, such as possessing a (nonabelian) group of coefficients or having a description in terms of cocycles. Indeed, there does not turn out to be a (globally defined) group of coefficients G in the topos T of sheaves on $\mathscr{S}$ for which we could set $H^2(e) = H^2(e, G)$. However, for a given gerbe $\mathscr{G}$, the choice of a local object x in $\mathscr{G}_S$ (for some covering map $S \to e$) determines, in the same manner as above, an S-group $G = \mathrm{Aut}(x)$ which we may think of as the (locally defined) coefficient group[3] of $\mathscr{G}$. The gerbe $\mathscr{G}$ is said to be affine (resp. smooth) when the corresponding group G is affine (resp. smooth) over S. The choice of a local object x also determines, as in (2.2.1), an equivalence

$$\Phi\colon \mathscr{G}|_S \to \mathrm{Tors}(G) \tag{2.3.1}$$

between the restriction $\mathscr{G}|_S$ to S of the stack $\mathscr{G}$ and the neutral gerbe defined by G. This equivalence is the key to a cocycle description of the gerbe $\mathscr{G}$. Indeed, $\mathscr{G}$ may now be constructed from the local data $\mathrm{Tors}(G)$ determined by the local group G, once we give ourselves appropriate transition data. This consists in the equivalence of stacks

$$\varphi = p_1^*\Phi \circ (p_2^*\Phi)^{-1}\colon\ \mathrm{Tors}(p_2^*G) \to \mathrm{Tors}(p_1^*G) \tag{2.3.2}$$

over $S \times S$, which restricts to the identity on the diagonal. Since Φ is an equivalence rather than an isomorphism of fibred categories, the functor φ does not satisfy the usual transition condition between the pull-backs to the triple product S^3. Instead, we are merely given a natural transformation ψ between morphisms of stacks on S^3

$$\psi\colon p_{12}^*\varphi \circ p_{23}^*\varphi \Rightarrow p_{13}^*\varphi \tag{2.3.3}$$

which in turn satisfies an obvious compatibility condition [**B1**, (6.2.7)–(6.2.8)] when pulled back in the four possible manners to S^4. It is this arrow ψ that must be viewed as the nonabelian 2-cochain associated to $\mathscr{G}$, and the compatibility condition just mentioned says that it is a 2-cocycle. However, it will be apparent that, in this nonabelian situation, the cocycle ψ does not make much sense on its own, and it is preferable to think of the full cocycle as consisting in the triplet (G, φ, ψ) with ψ satisfying the compatibility condition.

2.4. Here is an alternate description of the data (G, φ, ψ). Let us begin by recalling that, for any pair of groups G and H of a topos T, a (G, H)-bitorsor (or (G, H)-biprincipal bundle) T consists of an object E, endowed

[3] We differ here from [**Gi**] by not taking as coefficients for the cohomology groups the lien associated to the locally given group G (see 5.6 for a further discussion of this point).

with a left action of G and a right action of H on E that commute with each other and such that the action of G (resp. of H) makes E into a left G-torsor (resp. a right H-torsor). The corresponding right action of G (resp. left action of H) on E defines an associated (H, G)-torsor which is called the opposite bitorsor of E and is denoted E^0. When $G = H$, we say that E is a G-bitorsor. The opposite G-bitorsor is now called the inverse G-bitorsor of E. Given an equivalence

$$\varphi\colon \operatorname{Tors}(G) \to \operatorname{Tors}(H) \tag{2.4.1}$$

between categories of torsors, we may associate to it the right H-torsor $F = \varphi(G_d)$, the image by φ of the trivial right G-torsor G_d. The isomorphism $G \to \operatorname{Aut}(G_d)$ yields a left action of G on F, which in turn defines a (G, H)-bitorsor structure on F, to which corresponds an opposite (H, G)-bitorsor $E = F^0$. Conversely, if we start from an arbitrary bitorsor E, the contracted product over G with E defines a function (2.4.1):

$$P \mapsto P \overset{G}{\wedge} E^0 \tag{2.4.2}$$

since the right-hand terms have a right H-action induced by the given H-action on E^0. These two constructions are quasi-inverse of each other, so that we have the Morita-type statement that any equivalence (2.4.1) is equivalent to one defined as in (2.4.2), and in fact the category

$$\operatorname{Eq}(\operatorname{Tors}(G), \operatorname{Tors}(H))$$

of such equivalences is equivalent to the category $\operatorname{Bitors}(H, G)$ of (H, G)-bitorsors [**Gi**, IV 5.2.5].

This gives us an alternate way of displaying the nonabelian cocycle (2.3.3) associated to the gerbe $\mathcal{G}$ with local trivializing object x. The equivalence (2.3.2) may now be described by the (p_1^*G, p_2^*G)-bitorsor

$$E = \operatorname{Isom}(p_2^*x, p_1^*x) \tag{2.4.3}$$

above $S \times S$, whose set of sections above $(f, g)\colon T \to S \times S$ is the set $\operatorname{Isom}(g^*x, f^*x)$ of arrows between the corresponding objects in the fibre groupoid $\mathcal{G}_T$ of $\mathcal{G}$ over T. The left and right actions of the groups $p_i^*G = \operatorname{Aut}(p_i^*x)$ on E are given by composition of the corresponding arrows, and the pull-back $G = \operatorname{Aut}(x)$ of E over the diagonal Δ_S of $S \times S$ has a canonical section (corresponding to the identity map 1_x in $\operatorname{Aut}(x)$). The natural transformation ψ now reads as the isomorphism of (p_1^*G, p_3^*G)-bitorsors over S^3

$$\psi\colon p_{12}^*E \wedge p_{23}^*E \to p_{13}^*E \tag{2.4.4}$$

defined by composing arrows $f \in \operatorname{Isom}(p_2^*x, p_1^*x)$, $g \in \operatorname{Isom}(p_3^*x, p_2^*x)$. Finally, the compatibility condition for ψ translates into the pleasant statement that the two possible induced arrows

$$p_{12}^*E \wedge p_{23}^*E \wedge p_{34}^*E \to p_{14}^*E \tag{2.4.5}$$

on S^4 coincide. The pair (E, ψ) through which we now have described the gerbe $\mathcal{G}$ endowed with its locally defined neutralizing object x was introduced in **[B1]** in a slightly more restrictive setting and also in **[U12–4]**, where it is called the cocycle bitorsor associated to $(\mathcal{G}, x)$. We prefer to call it the *bitorsor cocycle*, in order to emphasize that it is just a (Čech) 1-cocycle with values in the stack of bitorsors of T. Note that since such an E is in particular a p_1^*G-torsor, E is by descent theory an affine scheme over $S \times S$ whenever G is affine over S, and it is also smooth whenever G is smooth over S. The converse is also true, since $G = \Delta_S^* E$.

2.5. The information that the bitorsor cocycle encapsulates has in fact been given in a somewhat redundant manner. We have already observed that the S-group $G = \mathrm{Aut}(x)$ may be retrieved as the pull-back $\Delta^* E$ of $E = \mathrm{Isom}(p_2^* x, p_1^* x)$ under the diagonal map $\Delta \colon S \to S \times S$ and its group structure is that obtained by pulling back the composition pairing (2.4.4) along the iterated diagonal $\Delta^{(2)} \colon S \to S^3$. The left and right actions of the groups $p_i^* G$ on E can be retrieved by pulling back the arrow (2.4.3) by the appropriate diagonal maps $S^2 \to S^3$. The upshot of this discussion is the statement that the minimal information required in order to recover the entire bitorsor cocycle structure is an epimorphism

(2.5.1) $$p \colon E \to S \times S$$

of T, together with a canonical section

(2.5.2) $$\delta \colon S \to E$$

of E over the diagonal, an S^3-map ψ (2.4.4) satisfying the compatibility condition (2.4.5), and an inverse map

(2.5.3) $$E \to E$$

which lives above the permutation of factors map $\sigma \colon S \times S \to S \times S$. These maps are required to satisfy appropriate axioms.

A convenient way of remembering the axioms in question on E is to introduce the source map $s \colon E \to S$ (resp. the target map $t \colon E \to S$) defined by composition of the epimorphism p with the projection of S^2 onto its second (resp. first factor). E (resp. S) may then be viewed as the object of arrows (resp. object of objects) of a groupoid object

(2.5.4) $$\Gamma \colon (E \rightrightarrows S)$$

in T, in which composition of composable arrows $e_1, e_2 \in E$ is given by the map ψ (2.4.3), the canonical section (2.5.2) associates to an object s of S the corresponding identity arrow 1_s in E, and (2.5.3) reverses the arrows in E. We refer to **[I, VI, 2.6.1]**, **[De2, 1.6]** for a diagrammatic description of the groupoid axioms in terms of E and S. The groupoid Γ is said to be transitive, since p is an epimorphism. A transitive groupoid Γ for which the structure map $S \to e$ is an epimorphism (as is the case here) is what Duskin calls a bouquet **[Du1–2]**.

2.6. Let us now consider a morphism

$$\rho: \mathcal{G} \to \mathcal{H} \tag{2.6.1}$$

between two gerbes which are respectively locally neutralized by objects x and y defined over a common cover $S \to e$ of e. We denote by G (resp. H) the S-group of automorphisms of x (resp. y). The morphism ρ may be described in terms of the bitorsors cocycles E and F associated as in 2.4 to the pair of objects x and y. A full description requires the introduction of the (H, G) bitorsor

$$X = \mathrm{Isom}_{\mathcal{H}}(\rho(x), y) \tag{2.6.2}$$

over S and is somewhat involved. It is more expedient to make the additional choice of a splitting χ of X over some extension S' of S, in other words, of an arrow

$$\chi: \rho(x) \to y \tag{2.6.3}$$

defined over S'. This induces a homomorphism

$$\tilde{\rho}_\chi: G_{S'} \to H_{S'} \tag{2.6.4}$$

between the pull-backs $G_{S'}$ and $H_{S'}$ of G and H to S' which sends an element u of $G = \mathrm{Aut}(x)$ to $\chi \circ \rho(u) \circ \chi^{-1} \in H = \mathrm{Aut}(y)$. The pull-backs E' (resp. F') over $S' \times S'$ of the bitorsor cocycles E (resp. F) are related by the $(p_1^* \rho_\chi, p_2^* \rho_\chi)$-equivariant bitorsor map

$$\rho_\chi: E' \to F' \tag{2.6.5}$$

which sends a section $u: p_2^* x \to \eta_1^* x$ of E' to the corresponding section $\rho_\chi(u) = p_1^*(\chi) \circ \rho(u) \circ p_2^*(\chi)^{-1}$:

$$p_2^* y \to p_2^*(\rho(x)) \to p_1^*(\rho(x)) \to p_1^* y \tag{2.6.6}$$

of F'. The restriction of (2.6.5) to the diagonal of $S' \times S'$ is the map (2.6.4). It is readily seen that the map (2.6.5) is compatible with the cocycle mappings ψ (2.4.4) associated to E' and F' and that it gives a full description of the morphism (2.6.1).

REMARK 2.7. Suppose that the gerbes $\mathcal{G}$ and $\mathcal{H}$ are affine. We say that a morphism (2.6.1) between such gerbes respects the affine structure if the induced map of sheaves (2.6.4) is a morphism of schemes. This need not be the case if the site $\mathcal{S}$ is small, for example, in the case of the small étale site which will be discussed in detail in §4. In such instances, the 2-category of affine gerbes (in which 1-morphisms respect the affine structure) is not a full 2-subcategory of the 2-category of gerbes.

2.8. Let us now suppose that we are given a second morphism $\sigma: (\mathcal{G}, x) \to (\mathcal{H}, y)$ and that we have chosen an arrow $\chi': \sigma(x) \to y$ defined over the same base S' as χ (2.6.3). The pair (σ, χ') defines a morphism (2.6.5) of bitorsors

(2.8.1) $$\sigma_{\chi'}: E' \to F'.$$

A 2-morphism $\eta: \rho \to \sigma$ defines an arrow $\eta(x): \rho(x) \to \sigma(x)$ in $\mathscr{H}$ which may be described by the element h of the group $H_{S'}$ defined by

(2.8.2) $$h = \chi' \circ \eta(x) \circ \chi^{-1}.$$

It then follows from the previous definitions that the morphism $\sigma_{\chi'}$ is obtained from ρ_χ (2.6.5) by conjugating ρ_χ by h. More precisely, one has, for every section u of E',

(2.8.3) $$\sigma_{\chi'}(u) = (p_1^* h)\rho_\chi(u)(p_2^* h)^{-1}.$$

The formulas (2.6.5)–(2.6.6) (resp. (2.8.3)) give us a complete description of the 1-morphisms (resp. 2-morphisms) in the 2-category of gerbes of the topos T. It should, however, be emphasized that this description depends on the choice of a pair of arrows χ and χ' (2.6.3), and it would now be in order to examine the manner in which the elements ρ_χ and h are modified when we make alternate choices for the maps χ and χ'. We return to this question, in a slightly less general context, in 5.5.

2.9. We have so far been concerned with extracting from a locally neutralized gerbe $(\mathscr{G}, x)$ the bitorsor cocycle $E \to S \times S$ which describes it. Conversely, we must examine the manner in which a gerbe $\mathscr{G}$ may be reconstructed from the induced bitorsor cocycle triplet (G, E, ψ). One way of doing this is simply to reverse the previous discussion and to apply the Morita theory of 2.4 in order to glue local categories of torsors into a gerbe. A more direct approach is possible which requires the introduction of the notion of a torsor under a groupoid $\Gamma: (E \rightrightarrows S)$ in a topos T. In the set case, this is due to Haefliger, who made use of this concept in his classification of foliations [H]. For Γ a groupoid in a topos T (which we may suppose defined by a site $\mathscr{S}$), let us consider objects X of T together with a projection map

(2.9.1) $$q: X \to S.$$

A right action of Γ on (X, q) is a map

(2.9.2) $$\mu: X \underset{S,t}{\times} E \to X, \quad (x, \gamma) \mapsto x\gamma$$

in T which associates to each element x of X and to every arrow γ in Γ whose target is $q(x)$ an element $x\gamma$ of X such that $q(x\gamma) = s(\gamma)$. In other words, we require that the diagram

$$\begin{array}{ccc} X \underset{S,t}{\times} E & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle q} \\ E & \xrightarrow{s} & S \end{array}$$

commutes. For a pair of composable arrows γ_2, γ_1 and any object x such that $t(\gamma_1) = q(x)$, we ask that

$$x(\gamma_1\gamma_2) = (x\gamma_1)\gamma_2$$

and that $x\gamma = x$ whenever $\gamma = 1_{p(x)}$ is an identity arrow. When this action of Γ on the object X is principal, we say that the projection $p\colon X \to e$ defines a Γ-pseudotorsor.[4] When p is an epimorphism, (X, q) is called a Γ-torsor over e. Such Γ-torsors form a category $\mathrm{Tors}(e, \Gamma)$, morphisms being Γ-equivariant S-maps $(X, q) \to (X', q')$. When the groupoid Γ has a single object e and thus is entirely described by the group G of its arrows, the concept of right Γ-torsor boils down to that of right G-torsor.

REMARK 2.10. When T is the topos associated to one of the usual categories of schemes over k, we may mimic the definition just given of the action of a groupoid $\Gamma \to S \times S$ on a set X above S, in order to define a representation of $\Gamma \to S \times S$ on a (quasi-coherent) S-module V. By this we mean an $(S \times S)$-morphism $\rho\colon \Gamma \to \mathrm{End}(p_1^* V, p_2^* V)$ compatible with the composition laws (i.e., a functor between the groupoids $\Gamma \to S^2$ and $\mathrm{End}(p_1^* V, p_2^* V) \to S^2$ which is the identity on objects). In other words, for each point g of Γ we now have a morphism $\rho(g)\colon V_{s(g)} \to V_{t(g)}$ satisfying the obvious compatibility for a pair of composable arrows in Γ and such that $\rho(e) = \mathrm{id}_{V_e}$ for every object e of the groupoid.

2.11. For any map $\lambda\colon U \to e$ of the site $\mathcal{S}$, we may base-change Γ to the groupoid $\Gamma_U\colon (\lambda^* E \rightrightarrows \lambda^* S)$ in the topos $T|_U$. Denoting by $\mathrm{Tors}(U, \Gamma)$ the category of Γ_U-torsors in $T|_U$ we observe that the pullback $\lambda^* X$ of a Γ-torsor X is an object of this category, so that we have in fact constructed a fibred category $\mathrm{Tors}(\Gamma)$ over $\mathcal{S}$, whose fibre at $U \in \mathcal{S}$ is the category $\mathrm{Tors}(U, \Gamma)$. The usual descent properties ensure that Γ-torsors glue properly, so that $\mathrm{Tors}(\Gamma)$, just as $\mathrm{Tors}(G)$, is in fact a stack.

Another feature that carries over from the group to the groupoid case is the notion of a trivial Γ-torsor, with the key difference that trivial Γ-torsors are no longer all isomorphic. Let σ be an object of Γ. The groupoid Γ acts on the right by composition of arrow on the set $X = \mathrm{Ar}^{\sigma}(\Gamma)$ of arrows of Γ whose target is σ, and this action makes X into a Γ-torsor called the trivial right Γ-torsor associated to σ. It follows that the stack $\mathrm{Tors}(\Gamma)$ is locally nonempty, since its fibre over S contains the trivial torsor $\mathrm{Ar}^{\Delta_S}(\Gamma_S)$ associated to the diagonal section of $S \times S$.

Now let X be a Γ-torsor. The choice of a section x of X determines a trivialization of X, defined by the isomorphism of Γ-torsors

$$\varphi_x\colon \mathrm{Ar}^{p(x)}(\Gamma) \to X, \quad \gamma \mapsto x\gamma. \tag{2.11.1}$$

[4] It is important to view X as an object over e, despite the fact that it is endowed with a map to S.

*Since such a section x does not in general exist over e, but it does after base-*change since $p\colon X \to e$ is an epimorphism, we do have local trivialisations (2.11.1). It is, furthermore, easily verified that any morphism $\varphi\colon \mathrm{Ar}^{\sigma}(\Gamma) \to \mathrm{Ar}^{\tau}(\Gamma)$ is given by

$$\varphi(\gamma) = \gamma'\gamma \tag{2.11.2}$$

for some arrow γ' in Γ whose source is σ and whose target is τ. In particular, the stack $\mathrm{Tors}(\Gamma)$ is locally connected whenever Γ is transitive.

A convenient way of summarizing the previous discussion is the following: let us identify the groupoid Γ of T with the prestack that it represents, in other words the groupoid of T whose sheaf of objects (resp. of morphisms) is the sheaf of sections of S (resp. of E). We then have a functor between fibred categories

$$a\colon \Gamma \to \mathrm{Tors}(\Gamma),$$
$$\sigma \mapsto \mathrm{Ar}^{\sigma}(\Gamma),$$

and the previous discussion may now be restated as follows.

PROPOSITION 2.12. *For any groupoid Γ of T, $(\mathrm{Tors}(\Gamma), a)$ is the stack associated to the prestack represented by Γ. It is a gerbe whenever Γ is transitive.*

When Γ is the groupoid $(G \to e)$ defined by a group G, we recover the familiar statement that $\mathrm{Tors}(G)$ is the stack associated to the groupoid defined by G **[De1]**.

Returning now to the construction given in Proposition 2.12, we now show that in fact it allows one to construct arbitrary gerbes.

PROPOSITION 2.13 **[De2, Du1–2]**. *The construction $\Gamma \mapsto \mathrm{Tors}(\Gamma)$ is quasi-inverse to that which associates to a locally neutralized gerbe $\mathcal{G}$ its bitorsor cocycle (viewed as a transitive groupoid $\Gamma\colon (E \rightrightarrows S)$).*

PROOF. We have already observed that we have a local object $x = \mathrm{Ar}^{\Delta_S}(\Gamma)$ in the fibre category $\mathrm{Tors}(\Gamma)_S$, and, as we know, the sections of

$$\mathrm{Isom}(p_2^*x, p_1^*x) = \mathrm{Isom}(\mathrm{Ar}^{p_2}(\Gamma), \mathrm{Ar}^{p_1}(\Gamma))$$

are uniquely determined by arrows in Γ. The groupoid Γ is therefore the bitorsor cocycle for the gerbe $\mathrm{Tors}(\Gamma)$ with local neutralizing object x. Conversely, given a gerbe $\mathcal{G}$ and a local neutralizing object x over S, to which are associated a groupoid $\Gamma\colon (E \rightrightarrows S)$, we have a fully faithful morphism of prestacks

$$\lambda\colon \Gamma \to \mathcal{G}$$

defined on sections $s\colon U \to S$ of $\mathrm{ob}(\Gamma)$ (resp. on U-sections $t\colon s_2 \to s_1$ of $\mathrm{Isom}(p_2^*x, p_1^*x) = \mathrm{Ar}(\Gamma)$) by

$$\lambda(s) = s^*x, \qquad \lambda(t) = (s_1, s_2)^*(t).$$

By Proposition (2.12), this factors through a fully faithful morphism of stacks

$$\lambda_a\colon \operatorname{Tors}(\Gamma) \to \mathcal{G}.$$

This map is essentially surjective, since every object y of $\mathcal{G}$ is locally isomorphic to some pull-back of the given object x, i.e., to an object in the image of λ.

3. General Tannakian duality

3.1. We can now discuss Tannakian duality in the nonneutral case. Let $\mathcal{C}$ be a tensorial category over a field k. $\mathcal{C}$ is said to be Tannakian if it possesses an (exact k-linear) fibre functor $\omega\colon \mathcal{C} \to (S\text{-mod})$ taking its values in the category $(S\text{-mod})$ of quasi-coherent sheaves on some nonempty k-scheme S. The categories $\operatorname{FIB}(\mathcal{C})_S$ of S-valued fibre functors on $\mathcal{C}$ may be assembled, for varying S, into a category $\operatorname{FIB}(\mathcal{C})$ fibred over $\operatorname{Spec}(k)$. Deligne, correcting an incomplete proof of **[Saa]**, proves the following

THEOREM 3.2 (**[De2]**). *Let $\mathcal{C}$ be a Tannakian category over* $\operatorname{Spec}(k)$. *Then* $\operatorname{FIB}(\mathcal{C})$ *is an affine gerbe over* $\operatorname{Spec}(k)$ *in the fpqc topology.*

PROOF. $\operatorname{FIB}(\mathcal{C})$ certainly satisfies the requisite properties for a stack, and since $\mathcal{C}$ is Tannakian, $\operatorname{FIB}(\mathcal{C})_S$ is by definition nonempty for some covering S. The nonformal part of the proof consists in showing that for any two fibre functors $\omega_1, \omega_2\colon \mathcal{C} \to S\text{-mod}$ are locally isomorphic in the flat topology. For this, one must verify the assertion that the corresponding groupoid $\Gamma\colon \operatorname{Isom}(p_2^*\omega, p_1^*\omega) \to S \times S$ is (affine) faithfully flat and therefore transitive in the fpqc topology. It will then follow that there exists a section of this map over a *finite* flat extension T of S, so that any two fibre functors derived by pull-back from ω are in fact isomorphic after a finite flat extension. The proof relies on the statement, which is surprisingly difficult to prove and for which we refer to **[De2]**, that the tensor product (in an appropriate sense) of two Tannakian categories is itself tensorial. This allows one to apply, in a universal situation, the elementary fact that, for any fiber functor ω on a tensorial category $\mathcal{C}$ and any $X \neq 0$ in $\mathcal{C}$, $\omega(X)$ is a locally free S-module, since $\omega(X^\vee)$ inverts it.

In the opposite direction, one can associate to any fpqc gerbe $\mathcal{G}$ over $\operatorname{Spec}(k)$ the Tannakian category $\operatorname{Rep}(\mathcal{G})$ of its representations. When $\mathcal{G}$ is the neutral gerbe $\operatorname{Tors}(G)$, its category of representations is just the Tannakian category $\operatorname{Rep}(G)$. When $\mathcal{G}$ is viewed as the associated stack of a transitive groupoid $\Gamma\colon (E \to S^2)$ elements of $\operatorname{Rep}(\mathcal{G})$ are simply representations of the groupoid Γ on quasi-coherent sheaves V on S. The main statement of Tannakian duality is the following generalization of Theorem 1.4.

THEOREM 3.3. *Let $\mathcal{C}$ be a Tannakian category over a field k, and let $\mathcal{G}$ be the associated gerbe of fibre functors. The natural functor $\mathcal{C} \to \operatorname{Rep}(\mathcal{G})$ is an equivalence of Tannakian categories.*

PROOF. The proof is essentially the one given above in the neutral case, whose notation we use. Instead of dealing with the k-coalgebra $B = \lim B_X$ whose coalgebra structure reflected the tensor structure of $\mathscr{C}$, and with the category of B-comodules given by the Barr-Beck theorem, we are now in a more complicated situation in which $G = \mathrm{Spec}(B)$, with its group structure defined by the coalgebra structure of B, must be replaced by the affine groupoid (2.5.1)

$$\Gamma\colon E = \mathrm{Spec}(L) \rightrightarrows S = \mathrm{Spec}(R).$$

To be somewhat more specific, the induced map $E \to S \times S$ defines an $(R \otimes R)$-module structure on the commutative ring L, and we view this as defining an (R, R)-bimodule structure on the underlying k-module of L. The composition law in the groupoid Γ defines, on the ring level, a partial comultiplication

$$c\colon L \to L \otimes_{R\otimes R} L$$

while the identity section $S \to E$ gives us an augmentation $e\colon L \to R$. Starting with an R-valued (rather than k-valued) fibre functor $\omega|_{\langle X\rangle}\colon \langle X\rangle \to (R\text{-mod})$, we may view it, as in the neutral case, as a functor $S\colon (A\text{-mod}) \to (R\text{-mod})$. This corresponds, by Morita, to tensorization by an (A, R)-bimodule M, and the R-bimodule $B_X = M^\vee \otimes M$ may then be identified with the ring of sections L of $E = \mathrm{Isom}(p_2^*\omega, p_1^*\omega)$. We now have a partial comultiplication and an augmentation on L which are induced by the composition law and the unit section of the groupoid Γ and which define a so-called coalgebroid structure on L. When the Barr-Beck theorem is applied in this context, it yields an equivalence between the category $\langle X\rangle$ and that of comodules, in an appropriate sense, under the coalgebroid L. The proof then continues exactly as in the neutral case, by passing to the limit over X, and yields, as expected, an identification of the category $\mathscr{C}$ with that of representations of the groupoid Γ.

EXAMPLE 3.4. Let us make explicit, when $\mathscr{C}$ is a k-tensorial category of motives, the various objects which have been introduced so far. Viewing in the neutral case, as we did in the introduction, a fibre functor $x\colon \mathscr{C} \to (k\text{-vect})$ as defining a cohomological functor on $\mathscr{C}$, the k-group $G = \mathrm{Aut}(x)$ yielded by the Tannaka theory is nothing else than the group of (multiplicative) cohomological transformations of the cohomology theory x into itself. This is sometimes called in this context the motivic Galois group associated to the theory (see for example **[D-M**, p. 213]). It had already been observed a long time ago in **[A-H]** that the $\mathbb{F}_p$-group associated in this manner to the ordinary mod p cohomology functor on the category of topological spaces is nothing else than the automorphism group of the additive group $\mathbb{G}_{a,\mathbb{F}_p}$.

Similarly, in the nonneutral case, we have attached to any given fibre functor $y\colon \mathscr{C} \to (K\text{-vect})$, viewed as a K-valued cohomology theory on $\mathscr{C}$, the K-group of multiplicative self-transformations $\mathrm{Aut}(y)$ of the theory y, and seen that this is the group of locally defined coefficients of the corresponding gerbe $\mathcal{G}$. It still deserves to be called a motivic Galois group, though it no longer really has a preferred representative. The bitorsor $E = \mathrm{Isom}(p_2^* x, p_1^* x)$ (2.4.3) of natural transformations between the pair of cohomology theories $p_2^* x$ and $p_1^* x$ might then be called, by extrapolation from the situation considered in [**De5,** Theorem 2], the (bi)torsor of periods between these two theories. An explicit description of such a torsor (in an admittedly neutral situation) will be found in [**W**]. Note that, while the local motivic Galois groups $\mathrm{Aut}(p_i^* G)$ associated to the two theories $p_2^* x$ and $p_1^* x$ are in general not equal, they are locally isomorphic and in fact inner forms of each other; indeed, as will be seen in (5.2.3), a local section u of the torsor of periods conjugates these two groups into one another.

REMARK 3.5. (i) In fact, slightly more has been proven than is asserted in Theorem 3.3: since all the constructions are functorial, what has actually been defined is a pair of 2-functors FIB and REP between the 2-category of k-Tannakian categories and that of affine gerbes over $\mathrm{Spec}(k)$, which are quasi-inverse (see [**Saa,** Theorem 2.3.2], or, for a more down to earth formulation, [**Mi2,** Theorem A.20]).

(ii) It may be considered a small blemish on the given definition of a Tannakian category $\mathcal{C}$ that it is not internal to $\mathcal{C}$ (since it involves an auxiliary fibre functor). The following alternate description of a Tannakian category over a field k of characteristic 0 is given in [**D2,** §7] (see also [**Do-R 1, 2**]). A k-tensorial category $\mathcal{C}$ is Tannakian if and only if every object X has a dimension (1.1.4.2) which is a nonnegative integer n. It is also equivalent to require that, for each object X, a sufficiently high exterior power $\bigwedge^n X$ of X be trivial. The exterior power is obtained in the usual manner from the corresponding tensor power $\bigotimes^n X$ by antisymmetrization.

4. Gerbes in the étale topology

4.1. Let us consider Tannakian categories $\mathcal{C}$ over a field k, and for which there exists a K-valued fibre functor x, with values in some finite separable field extension K of k. In this case, the corresponding gerbe $\mathcal{G} = \mathrm{FIB}(\mathcal{C})$ of fibre functors on $\mathcal{C}$ is a gerbe over $\mathrm{Spec}(k)$ in the étale topology. We set $\Gamma_k = \mathrm{Gal}(k^s/k)$, where k^s is a separable closure of k. Γ_k is a topological group, for the Krull topology defined by taking as a fundamental system of neighborhoods of 1 the normal subgroups $\Gamma_K = \mathrm{Gal}(k^s/K)$ of Γ_k associated to the various finite normal separable extensions K of k. We denote as usual by $\Gamma_{K/k}$ the finite Galois group $\mathrm{Gal}(K/k) = \Gamma_k/\Gamma_K$. Setting $S = \mathrm{Spec}(K)$ (resp. $S_{k^s} = \mathrm{Spec}(k^s)$), we assume that the gerbe $\mathcal{G}$ over $\mathrm{Spec}(k)_{\text{ét}}$ is neutralized over S by an object x in $\mathcal{G}_S$ and also, for simplicity, that S is a Galois cover of $\mathrm{Spec}(k)$, so that that K is as above a Galois extension of k defined by the normal subgroup Γ_K of Γ_k. Finally, we suppose that the S-group $G = \mathrm{Aut}(x)$ associated as in 2.3 to x is affine smooth, i.e., is a K-algebraic group, so that $\mathcal{G}$ is an affine smooth gerbe over k.

We may now translate into the setting of Γ_k-sets the description of the bitorsor cocycle E associated to such a pair $(\mathcal{G}, x)$. Let us begin by considering the map (2.5.1), viewed as an S-map via the projection onto the first factor of $S \times S$. The right action of the group $\Gamma_{K/k}$ on S induced by the left action of $\Gamma_{K/k}$ on K yields an isomorphism

$$\begin{aligned} S \times \Gamma_{K/k} &\to S \times S, \\ (x, \gamma) &\mapsto (x, x\gamma). \end{aligned} \tag{4.1.1}$$

Let

$$\pi \colon E' \to S \times \Gamma_{K/k} \tag{4.1.2}$$

be the pull-back of (2.5.1) by (4.1.1). We know that an object F of the topos T of sheaves on the small étale site $\mathrm{Spec}(k)_{\text{ét}}$ may be described by the set $F(k^s)$ of its sections over $\mathrm{Spec}(k^s)$, endowed with the discrete topology and viewed as a continuous (left) Γ_k-set for the natural action of Γ_k on $F(k^s)$ (see, for example, **[De4,** II, Proposition 4.4]). In particular, the S-map π is described in such Galois-theoretic terms by the surjective Γ_K-equivariant map between the corresponding sets of k^s-valued points

$$\pi(k^s) \colon E'(k^s) \to \Gamma_{k/K}, \tag{4.1.3}$$

with trivial action of Γ_K on the target. Rather than working with this map, we find it is somewhat more convenient to describe the $(S \times \Gamma_{K/k})$-object E' in terms of its pull-back E'' to $S \times \Gamma_k$, together with the descent data from $S \times \Gamma_k$ to $S \times \Gamma_{K/k}$. The Γ_K-set $\mathcal{E} = E''(k^s)$ associated to E'' is the pull-back of (4.1.3) by the projection of Γ_k onto $\Gamma_{K/k}$, and the induced surjective Γ_K-map

$$\varphi \colon \mathcal{E} \to \Gamma_k \tag{4.1.4}$$

now describes the projection of E' on $S \times \Gamma_k$. The groupoid composition law (2.4.4) translates, at the level of E'', to a family of morphisms

$$E''_\sigma \times \sigma^* E''_\tau \to E''_{\sigma\tau} \tag{4.1.5}$$

for each pair of elements $\sigma, \tau \in \Gamma_k$, where E''_σ denotes the fibre of E'' over $S \times \sigma$. The Γ_K-set associated to E''_σ is just the fibre $\mathcal{E}_\sigma$ of φ at σ, while to $\sigma^* E''_\tau$ corresponds the set $\mathcal{E}_\tau$ with Γ_K-action twisted by σ (as in (5.7.1)). With these identifications, the groupoid composition law (4.1.5) now reads as a map

$$\mathcal{E}_\sigma \times \mathcal{E}_\tau \to \mathcal{E}_{\sigma\tau}$$

and so defines a multiplication

$$\mathcal{E} \times \mathcal{E} \to \mathcal{E} \tag{4.1.6}$$

on $\mathcal{E}$. To the groupoid inverse law (2.5.3) corresponds in a similar manner an inverse law on $\mathcal{E}$, while to the canonical section δ (2.5.2) of E is associated a Γ_K-invariant element e of $\mathcal{E}$. We obtain in this manner a group structure on $\mathscr{E}$, with e as unit element, and for which the projection (4.1.4) is a homomorphism. Furthermore, the descent data controlling the descent from the $(S \times \Gamma_k)$-scheme E'' to the $(S \times \Gamma_{K/k})$-scheme E' may be viewed as being given, for each $\sigma \in \Gamma_k$, by a family of maps

$$\psi_{\sigma,\gamma}: E''_\sigma \to ((1 \times \gamma)^* E'')_\sigma = E''_{\sigma\gamma}$$

for varying $\gamma \in \Gamma_K$, satisfying the usual transitivity conditions for pairs γ, $\gamma' \in \Gamma_K$, and which induce corresponding maps

$$\psi_{\sigma,\gamma}: \mathscr{E}_\sigma \to \mathscr{E}_{\sigma\gamma} \tag{4.1.7}$$

on k^s-valued points. For each element $\gamma \in \Gamma_K$, let us define $j(\gamma) \in \mathscr{E}_\gamma$ by the formula

$$j(\gamma) = \psi_{1,\gamma}(e). \tag{4.1.8}$$

The compatibility of the groupoid composition map (4.1.5) with the descent datum $\psi_{\sigma,\gamma}$ ensures that $\psi_{\sigma,\gamma}$ coincides with the restriction to $\mathscr{E}_\sigma$ of right multiplication by $j(\gamma)$ in the group $\mathscr{E}$. The transitivity condition for the descent data now ensures that the splitting

$$j: \Gamma_K \to \mathscr{E} \tag{4.1.9}$$

of the projection φ (4.1.4) of $\mathscr{E}$ on Γ_k defined by (4.1.8) is a homomorphism.

We may now describe the precise relationship between the action of Γ_k on the set $\mathscr{E}$ and the multiplication (4.1.6) in $\mathscr{E}$. In order to do so, it is convenient to think of the bitorsor cocycle E, as we may always do in view of the discussion in (2.3)–(2.4), as being associated to a gerbe $\mathscr{G}$ over $\mathrm{Spec}(k)_{\text{ét}}$, locally neutralized by an object x in $\mathscr{G}_S$. It then follows from (2.4.3) that an element of $\mathscr{E}_\sigma$ is given by a section of the sheaf $\mathrm{Isom}(\sigma^* x_{k^s}, x_{k^s})$, where x_{k^s} is the pull-back of x to S_{k^s}. The neutral element e of $\mathscr{E}$ corresponds to the identity section of $\mathrm{Aut}(x_{k^s})$ and the multiplication induced in $\mathscr{E}$ by (4.1.5) now translates, in view of the discussion following (4.1.5), into the twisted composition map

$$\begin{aligned} \mathrm{Isom}(\sigma^* x_{k^s}, x_{k^s}) \times \mathrm{Isom}(\tau^* x_{k^s}, x_{k^s}) &\to \mathrm{Isom}((\sigma\tau)^* x_{k^s}, x_{k^s}) \\ (u, v) &\mapsto u \circ \sigma^*(v), \end{aligned} \tag{4.1.10}$$

as in **[L-R**, p. 153] while $j(\gamma) \in \mathrm{Isom}(\gamma^* x_{k^s}, x_{k^s})$ becomes the descent data describing the object x in terms of its pull-back x_{k^s}. The Galois action on $\mathscr{E}$ of an element $\gamma \in \Gamma_K$ now describes the descent from S_{k^s} to S of a section u of the sheaf $\mathrm{Isom}(\sigma^* x_{k^s}, x_{k^s})$. It therefore sends u to the unique section u' of $\mathrm{Isom}(\sigma^* x_{k^s}, x_{k^s})$ such that the following diagram commutes:

$$\begin{array}{ccc} \gamma^*\sigma^* x_{k^s} & \xrightarrow{\gamma^* u} & \gamma^* x_{k^s} \\ \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} \\ \sigma^* x_{k^s} & \xrightarrow[u']{} & x_{k^s} \end{array}$$

where γ' is the descent data describing the pull-back $\tilde{\sigma}^* x$ of the object $x \in \mathcal{G}_S$ by the automorphism of S described by the class $\tilde{\sigma}$ of $\sigma \bmod \Gamma_K$. It readily follows from the Galois-theoretic description of the category $\tilde{\sigma}^*(\mathcal{G}_S)$ that $\gamma' = \sigma^*(j(\gamma^\sigma))$, where

$$j(\gamma^\sigma)\colon (\sigma^{-1}\gamma\sigma)^* x_{k^s} \to x_{k^s}$$

is the descent data on x_{k^s} associated to the conjugate $\gamma^\sigma = \sigma^{-1}\gamma\sigma$ of γ. The upshot of this discussion is, in view of the description (4.1.10) of the multiplication in $\mathcal{E}$, that the action of Γ_K on the fibre $\mathcal{E}_\sigma$ of $\mathcal{E}$ is given in terms of the multiplication (4.1.6) by the twisted conjugacy rule

$$\text{(4.1.12)} \qquad \begin{aligned} &\Gamma_K \times \mathcal{E}_\sigma \to \mathcal{E}_\sigma \\ &(\gamma, u) \mapsto j(\gamma) u (j(\gamma^\sigma))^{-1}. \end{aligned}$$

In particular, this action reduces on the fibre $\mathcal{E}_1 = \mathrm{Aut}(x_{k^s}) = G(k^s)$ to ordinary conjugation by $j(\gamma)$, so the latter defines the descent data for G from k^s to K, in other words, the K-structure on the algebraic group $G = \mathrm{Aut}(x)$.

Definition 4.2 (**[L-R]**). Let K be a finite Galois extension of a field k. Consider a short exact sequence of groups

$$\text{(4.2.1)} \qquad 1 \to \mathcal{A} \to \mathcal{E} \xrightarrow{\varphi} \Gamma_k \to 1$$

split by a homomorphism $j\colon \Gamma_K \to \mathcal{E}$ over the open normal subgroup Γ_K of Γ_k, and let G be the étale sheaf on $S = \mathrm{Spec}(K)$ defined by the Γ_K-group structure on $\mathcal{A}$ given by conjugation through j. The partially split exact sequence $(\mathcal{E}, \varphi, j)$ will be called a K/k-Galois gerbe (or simply a Galois gerbe) when G is represented by an affine smooth group scheme on S.

Remark 4.3. It follows from the previous discussion that a necessary and sufficient condition for the partially split exact sequence $(\mathcal{E}, \varphi, j)$ to define a K/k-gerbe is that $\mathcal{A} = G(k^s)$, for some algebraic group G defined over k^s, and that the action on $\mathcal{A}$ through j of each element $\sigma \in \Gamma_K$, which may then be viewed as a descent isomorphism of sheaves $\sigma^* G \to G$ in $\mathrm{Spec}(k^s)_{\text{ét}}$, extends to a descent isomorphism between the corresponding k^s-algebraic groups. It can be verified that such an isomorphism $\sigma^* G \to G$ of k^s-algebraic groups is what Langlands-Rapoport call a σ-linear automorphism of the group $\mathcal{A}$. We may now consider, for varying finite Galois extensions K of the field k, the inductive system of Galois K/k-gerbes obtained by restricting the splitting j of the fixed exact sequence $(\mathcal{E}, \varphi)$ to the corresponding open subgroups Γ_K of Γ_k. It follows that this inductive system yields, in the limit, a Galois gerbe in the sense of **[L-R]**.

4.4. In fact, the entire bitorsor cocycle structure of E can be recovered from the associated Galois gerbe. Indeed, starting from a Galois gerbe $(\mathcal{E}, \varphi, j)$, we may define a Γ_K-group structure on $\mathcal{E}$ by (4.1.1.2) and so recover a sheaf E'' together with a map to $S \times \Gamma_k$ determined by φ. Descent from E'' to an object E' above $S \times \Gamma_{K/k}$ is merely, at the Γ_K-set level, passing from $\mathscr{E}$ to its quotient $\mathscr{E}/\Gamma_K$ by the right action of Γ_K on $\mathscr{E}$ through j. In particular, the induced Γ_K-group structure on $\mathscr{E}/\Gamma_K$ is just given by left translation through j. The multiplication $m: \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ on $\mathscr{E}$ descends to a map of left Γ_K-sets

$$\mathscr{E} \wedge^{\Gamma_K} (\mathscr{E}/\Gamma_K) \to \mathscr{E}/\Gamma_K \tag{4.4.1}$$

which is compatible with the multiplication in $\Gamma_{K/k}$ and which therefore defines a family of maps

$$E'_\sigma \times \sigma^* E'_\tau \to E'_{\sigma\tau} \tag{4.4.2}$$

for each $\sigma, \tau \in \Gamma_{K/k}$, descended from (4.1.5). The pull-back of E' by the inverse of (4.1.1) then yields an étale sheaf E on $S \times S$, on which (4.4.2) determines a groupoid structure, for which the class $\operatorname{mod} \Gamma_K$ of the identity element of $\mathscr{E}$ defines the canonical section of E over the diagonal. The discussion of 2.5 then shows that E is endowed with a left (resp. right) principal action of $p_1^* G$ (resp. $p_2^* G$). Since G is by hypothesis an affine smooth S-group scheme, it follows that E is a smooth affine scheme over $S \times S$, so the structure map p (2.5.1) is an epimorphism. In other words:

PROPOSITION 4.5. *Let K be a finite Galois extension of a field k. The construction given above provides a bijection between the set of smooth affine gerbes over* $\operatorname{Spec}(k)_{\text{ét}}$ *neutralized over* $\operatorname{Spec}(K)$ *and the set of K/k-Galois gerbes.*

4.6. We now examine in what manner morphisms between smooth affine gerbes over $\operatorname{Spec}(k)_{\text{ét}}$ which respect the affine structure (resp. natural transformations between such morphisms) can be viewed in the Galois gerbe context. We have just seen that to a pair of smooth affine gerbes $\mathscr{G}$ and $\mathscr{H}$ over $\operatorname{Spec}(k)$, neutralized in the étale topology over a common finite Galois extension K of k by objects x and y, is associated a pair of extensions

$$\begin{gathered} 1 \to \mathscr{A} \to \mathscr{E} \xrightarrow{\varphi} \Gamma_k \to 1, \\ 1 \to \mathscr{B} \to \mathscr{F} \xrightarrow{\psi} \Gamma_k \to 1 \end{gathered} \tag{4.6.1}$$

and a pair of partial splittings $j: \Gamma_K \to \mathscr{E}$, $j': \Gamma_K \to \mathscr{F}$ of φ and ψ, determining a pair of K/k-Galois gerbes, with local K-groups A and B such that $A(k^s) = \mathscr{A}$ and $B(k^s) = \mathscr{B}$. To a morphism of gerbes $\rho: \mathscr{G} \to \mathscr{H}$ is associated, once we choose an arrow χ (2.6.3) which we take here to be defined over some finite Galois extension K' of K, a homomorphism of group schemes $\tilde{\rho}_\chi$ (2.6.4) defined over $S' = \operatorname{Spec}(K')$. The corresponding map between the associated étale sheaves on S' is described by the $\Gamma_{K'}$-equivariant homomorphism

$$\tilde{\rho}_\chi : \mathscr{A} \to \mathscr{B} \tag{4.6.2}$$

induced by (2.6.4) on k^s-valued points. Similarly, the bitorsor map ρ_χ (2.6.5) is described by a $\Gamma_{K'}$ -equivariant homomorphism

$$\rho_\chi : \mathscr{E} \to \mathscr{F} \tag{4.6.3}$$

which restricts to $\tilde{\rho}_\chi$ and which induces the identity on the quotient group Γ_k. Note that the map ρ_χ is not in general compatible with the splittings j and j' of φ and ψ but merely with the restrictions of j and j' to the open subgroup $\Gamma_{K'}$ of Γ_K, since the map (2.6.5) exists only above $S' \times S'$. The following is a finite level variant of the definition in **[L-R]** (where, however, the requirement that ρ induces the identity on Γ_K has inadvertently been omitted).

DEFINITION 4.7. A homomorphism ρ (4.6.3) inducing the identity on Γ_k, and which is compatible with the restrictions of the sections j and j' to the subgroup $\Gamma_{K'}$, is called a K'-morphism between the corresponding K/k-Galois gerbes (4.6.1) when the restriction (4.6.2) of ρ to the kernels is induced by homomorphism of S'-groups schemes.

We have seen that a 2-arrow $\eta: \rho \to \sigma$ between a pair of such morphisms of gerbes $\rho, \sigma: \mathscr{G} \to \mathscr{H}$ is described by an element h (2.8.2). This may now be viewed as an element h of $\mathscr{B}$ that is invariant under the action of $\Gamma_{K'}$. Formula (2.8.3) now tells us that the homomorphism ρ_χ (4.6.3) differs from the corresponding homomorphism $\sigma_{\chi'}$ associated to $\sigma: \mathscr{G} \to \mathscr{H}$ and to the chosen arrow $\chi': \sigma(x) \to y$ by inner conjugation by h:

$$\sigma_{\chi'} = i_h \circ \rho_\chi . \tag{4.7.1}$$

The following proposition summarizes the previous discussion.

PROPOSITION 4.8. *With the previous notation, let $\mathscr{G}$ and $\mathscr{H}$ be a pair of affine smooth gerbes over* $\mathrm{Spec}(k)_{\text{ét}}$, *and let x (resp. y) be a neutralizing object of $\mathscr{G}$ (resp. $\mathscr{H}$) over a finite Galois extension K of k, to which corresponds the K/k Galois gerbes $(\mathscr{E}, \varphi, j)$ (resp. $\mathscr{F}, \psi, j'$)* (4.6.1). *Let $\chi: z \to y_{K'}$ be an arrow in $\mathscr{H}_{S'}$, whose target is the inverse image of y in $\mathscr{H}_{S'}$, for some finite Galois extension K' of K. There exists a bijection between the set of morphisms of affine smooth gerbes $\rho: \mathscr{G} \to \mathscr{H}$ (respecting the affine structure) for which $\rho(x_{K'}) = z$ and the set of K'-morphisms ρ_χ between the corresponding Galois gerbes. A natural transformation between two such morphisms ρ and σ is described in terms of the corresponding arrows χ and χ' of $\mathscr{H}_{S'}$ by a $\Gamma_{K'}$-invariant element h in $\mathscr{B}$ such that the equation* (4.7.1) *holds.*

4.9. The dictionary that translates between smooth affine gerbes over $\mathrm{Spec}(k)_{\text{ét}}$ and Galois gerbes will be complete once we observe that Proposition 2.13 yields a direct method for reconstructing the gerbe $\mathscr{G}$ over $\mathrm{Spec}(k)_{\text{ét}}$ out of the associated Galois gerbe. The construction is analogous to that which is given in **[U12]** (for a much less direct construction of the gerbe associated to a Galois gerbe, requiring Tannakian duality, see **[L-R,** p. 153**]**).

Let $(\mathscr{G}, x)$ be such a gerbe, neutralized by an object x over a k-scheme $S = \operatorname{Spec}(K)$ and to which corresponds a groupoid $\Gamma\colon (E \rightrightarrows S)$. We have seen in 2.9 that if T is a k-scheme, described by the associated set $\mathscr{T} = T(k^s)$ with the discrete topology, and on which Γ_K acts continuously, then the fibre $\mathscr{G}_T$ of $\mathscr{G}$ consists of the category $\operatorname{Tors}(T, \Gamma_T)$ of right torsors over T under the groupoid $\Gamma_T\colon (E_T \rightrightarrows S_T)$ which is the pull-back of Γ by T. To an object X in this category is associated the discrete left Γ_K -set $\mathscr{X} = X(k^s)$, together with the surjective map $q''\colon \mathscr{X} \to \mathscr{T}$ corresponding to the epimorphism of schemes $q'\colon X \to T$. The right action of the groupoid Γ_T on X is now described by a right Γ_K -equivariant action

$$\mathscr{X} \times \mathscr{E} \to \mathscr{X} \tag{4.9.1}$$

of the group $\mathscr{E}$ on $\mathscr{X}$, such that $xe = x$ for all $x \in \mathscr{X}$, and for which the diagram

$$\begin{array}{ccc} \mathscr{X} \times \mathscr{E} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathscr{T} \times \Gamma_k & \longrightarrow & \mathscr{T} \end{array} \tag{4.9.2}$$

commutes, the lower horizontal map being given by the right action of Γ_k on $\mathscr{T}$. The object X is a Γ_T-torsor if and only if the action (4.9.1) of $\mathscr{E}$ on $\mathscr{X}$ is principal, in other words, if, given a pair of elements x_1 and x_2 of $\mathscr{X}$ such that $q''(x_2) = q''(x_1)\gamma$ for some γ in Γ_k, there exists a unique ε in $\mathscr{E}$ such that $x_2 = x_1\varepsilon$ and $\varphi(\varepsilon) = \gamma$. Furthermore, if we associate to another Γ_T-torsor Y the corresponding set $\mathscr{Y} = Y(k_s)$, a morphism $v\colon X \to Y$ in $\operatorname{Tors}(T, \Gamma_T)$ is described by an $\mathscr{E}$-equivariant $\mathscr{T}$-map $v = u(k_s)\colon \mathscr{X} \to \mathscr{Y}$. This completes our Galois-theoretic description of the gerbe $\operatorname{Tors}(\Gamma)$.

5. Explicit cocycles

5.1. A more explicit description of a gerbe $\mathscr{G}$ may be obtained if we do not merely choose, as we did until now, a local trivialization x of $\mathscr{G}$ over S, but also a section u of the induced bitorsor cocycle E. Once this second choice has been made, we shall see that we obtain a nonabelian 2-cocycle with values in the S-group $G = \operatorname{Aut}(x)$, which gives, up to equivalence, a very concrete and complete description of the gerbe $\mathscr{G}$. The price to pay, however, for such explicitness is the complication of the formulas obtained. It will also be necessary to investigate the dependence of the cocycles in question on the pair of choices (x, u). Naturally, a section u of E over $S \times S$ does not in general exist, but we may always find one by passing to a refinement $T \to S \times S$. This is consistent with the fact that we cannot, in general, expect to represent a class in H^2 by a Čech cocycle associated to some covering $S \to e$ of e but rather by some hypercovering. Specifically, the construction that follows will yield a class in the $H^2(e)$ defined as the cohomology of the hypercover of e constructed as the 1-coskeleton of the truncated simplicial object $(T \rightrightarrows S)$ (we refer for an explicit description of this simplicial object, whose ith stage we denote henceforth by $(T/S)_i$, to [**B1**, 6.3]). When $T = S \times S$, so that E has a global section over $S \times S$, this coskeleton just reduces to the nerve of the covering $S \to e$, and we do obtain, in this case, ordinary Čech cocycles for the covering in question. We shall continue, however, to use the notation p_{ij}, etc., in the general situation for the various face maps between the terms of the hypercover, despite the fact that they no longer are, as in the Čech case, partial projections from products of copies of S onto the corresponding (i, j)th factors.

Just as we have been so far considering gerbes $\mathcal{G}$ over e which were locally neutralized, for some covering map $S \to e$, by the choice of an object x in the fiber $\mathcal{G}_S$, we now have

DEFINITION 5.2. Let $S \to e$ and $T \to S \times S$ be a pair of covering maps in the site $\mathcal{S}$. A gerbe $\mathcal{G}$ in the topos of sheaves on $\mathcal{S}$ is decomposed over (T/S) by the choice of an object x in $\mathcal{G}_S$ and of a section u above T of the associated bitorsor cocycle $E = \mathrm{Isom}(p_2^*x, p_1^*x)$.

Such a section u of E may be viewed as an arrow

$$u: p_2^*x \to p_1^*x \tag{5.2.1}$$

in $\mathcal{G}$. It trivializes E as a left p_1^*G-torsor on $S \times S$ (resp. on the covering T of $S \times S$). The entire (p_1^*G, p_2^*G)-bitorsor structure on E is then described[5] by the section $\lambda = \lambda_u$

$$\lambda = \lambda_u: p_2^*G \to p_1^*G \tag{5.2.2}$$

of $\mathrm{Isom}(p_2^*G, p_1^*G)$ such that $\lambda(g)u = ug$ for all sections g of p_2^*G, and λ is a isomorphism of affine $(S \times S)$-group schemes whenever the gerbe $\mathcal{G}$ (and hence the $(S \times S)$-scheme E) is affine. Viewed within $\mathcal{G}$, λ is the familiar map

$$\begin{aligned} \lambda: \mathrm{Aut}(p_2^*x) &\to \mathrm{Aut}(p_1^*x), \\ g &\mapsto ugu^{-1} \end{aligned} \tag{5.2.3}$$

which conjugates a loop g based at p_2^*x by a path u into a loop based at p_1^*x. The isomorphism ψ (2.4.4) may now be described by the section $\gamma = \gamma_u$ of the pull-back p_1^*G of G over $S \times S \times S$ (resp. over the corresponding stage $(T/S)_2$ of the hypercovering defined by T) for which the equality

$$\psi(p_{12}^*(u), p_{23}^*(u)) = \gamma p_{13}^*(u) \tag{5.2.4}$$

holds in p_{13}^*E. When ψ is viewed as composing arrows in $\mathcal{G}$ pulled back from the arrow (5.2.1), γ is characterized as the unique section over $S \times S \times S$

[5]In [**B1**], the emphasis was placed on the opposite map $\lambda^{-1}: p_1^*G \to p_2^*G$.

(resp. over $(T/S)_2$) of $p_1^*G = \mathrm{Aut}(p_1^*x)$ such that the diagram

(5.2.5)
$$\begin{array}{ccccc} p_3^*x & \xrightarrow{p_{23}^*u} & p_2^*x & \xrightarrow{p_{12}^*u} & p_1^*x \\ & {\scriptstyle p_{13}^*u}\searrow & & \nearrow{\scriptstyle \gamma} & \\ & & p_1^*x & & \end{array}$$

commutes. Setting $\lambda_{ij} = p_{ij}^*\lambda$, it follows that λ and γ are related by the equation

(5.2.6)
$$\lambda_{12} \circ \lambda_{23} = i_\gamma \circ \lambda_{13}$$

in $\mathrm{Isom}(p_3^*G, p_1^*G)$, where

$$\begin{aligned} i_\gamma : p_1^*G &\to p_1^*G, \\ g &\mapsto \gamma g \gamma^{-1} \end{aligned}$$

is the inner automorphism of p_1^*G defined by γ. Comparing the various pull-backs $\gamma_{ijk} = p_{ijk}^*\gamma$ of γ to S^4 (resp. to the corresponding stage $(T/S)_3$ of the hypercovering), it then follows from a diagram built up from four copies of (5.2.5) that the γ_{ijk} are related by the following relation between sections of the pull-back p_1^*G of G to S^4:

(5.2.7)
$$\lambda(\gamma_{234})\gamma_{124} = \gamma_{123}\gamma_{134}.$$

Here $\lambda(\gamma_{234})$ really means $p_{12}^*(\lambda)(\gamma_{123})$: it is the image in p_1^*G of the section γ_{234} of p_2^*G by the pull-back by $p_{12}: S^4 \to S^2$ (resp. $p_{12}: (T/S)_3 \to T$) of the map (5.2.2).

The cocycle γ satisfying (5.2.7) is a sort of 2-cocycle twisted by λ with values in G. It is more appropriate to say, along the lines of the discussion in §2.3, that to a triple $(\mathscr{G}, x, u)$ which decomposes $\mathscr{G}$ over (T/S) corresponds a nonabelian cocycle (G, λ, γ), with λ a section (5.2.2) of $\mathrm{Isom}(p_2^*G, p_1^*G)$ over $(T/S)_2$ and γ a section (5.2.4) over $(T/S)_3$ of the S-group $G = \mathrm{Aut}(x)$, satisfying the cocycle conditions (5.2.6)–(5.2.7). Conversely, the decomposed gerbe $\mathscr{G}$ may be reconstructed, up to equivalence, from the cocycle triplet (G, λ, γ) in the following manner. The section λ determines, as in **[B1,** §4], a split (p_1^*G, p_2^*G) bitorsor E over T. Let us suppose for simplicity that $T = S \times S$ (in the general case, the technique of **[B1,** 6.5] for descending a bitorsor from a hypercover to Čech cover must be used). It is easy to verify (see **[B1,** Theorem 4.5] in the particular case of G-bitorsors) that, since the section γ of G satisfies (5.2.6), it determines an isomorphism (2.4.4), and this determines, in view of (5.2.7), a bitorsor cocycle structure on E. We can then recover the gerbe $\mathscr{G}$ from E in the manner indicated in §2.9, either by using E to glue trivial gerbes as in (2.3.2) or more directly by applying the construction (2.11) to the groupoid that E defines.

Another section v of the bitorsor E over $S \times S$ (resp. T) differs from the section u (5.2.1) by a section δ of p_1^*G over $S \times S$ (resp. T):

(5.2.8) $$v = \delta u.$$

The arrow $\lambda' = \lambda_v$ (5.2.3) associated to v is then related to the corresponding arrow $\lambda = \lambda_u$ by the identity

(5.2.9) $$\lambda' = i_\delta \lambda$$

in $\mathrm{Isom}(p_2^*G, p_1^*G)$, and the section $\gamma' = \gamma_v$ of p_1^*G over $S \times S \times S$ (resp. over $T/S)_2$) associated as in (5.2.4)–(5.2.5) to v then differs from the corresponding section $\gamma = \gamma_u$ by the identity

(5.2.10) $$\gamma' \delta_{13} = \delta_{12} \lambda_{12}(\delta_{23})\gamma.$$

The triple (G, λ', γ') is cohomologous to (G, λ, γ) whenever there exists a section δ of p_1^*G over $S \times S$ (resp. over T) for which the identities (5.2.9) and (5.2.10) are satisfied. In a much more restrictive context, cocycles such as (5.2.7) (resp. coboundaries such as (5.2.10)) are to be found in the work of Dedecker (see **[Ded2]** for a survey of some of his results).

5.3. In fact, the relations (5.2.9) and (5.2.10) are not the most general cohomology relations that should be allowed between cocycle triples: in order to obtain the most general coboundary conditions, one should not only allow the section u (5.2.1) of E to vary, but also the neutralizing object x of $\mathcal{G}$. In this case, the first term G of the cohomology triple (G, λ, γ) also varies, and the cohomology relations (5.2.9)–(5.2.10) must be modified to take this extra element of complication into account. A closely related phenomenon occurs in **[B1,** Definition 7.1], where it is mistakenly postulated that a morphism between gerbes should respect the local trivializations. We shall now carry through the required discussion, noting that it is merely a version of §2.6 in which the map (2.6.1) is the identity map and trivializations of the bitorsors that occur have been chosen.

We suppose we are given a gerbe $\mathcal{G}$ and a pair of decompositions (x, u) and (x', v) of $\mathcal{G}$ over (T/S). We have seen that to each of these decompositions is associated a cocycle triple (G, λ, g) and (G', λ', γ') defined by (5.2.2) and (5.2.5) and which each satisfy the cocycle conditions (5.2.6)–(5.2.7). After a possible base change from S to S_1 (and correspondingly the extension of T to T_1) we may choose an arrow

(5.3.1) $$\chi : x \to x'$$

defined over S_1. The map χ induces an isomorphism

(5.3.2) $$\mu : G \to G', \quad \alpha \mapsto \chi \circ \alpha \circ (\chi)^{-1}$$

defined over S_1. To u is associated the arrow

(5.3.3) $$\tilde{u} : p_2^* x' \to p_1^* x'$$

over T_1 defined by $\tilde{u} = p_1^*\chi \circ u \circ (p_2^*\chi)^{-1}$. This differs from the section v by a T_1-section δ of $\mathrm{Aut}(p_1^*x') = p_1^*G'$, so that δ is defined by the identity

(5.3.4) $$v = \delta\tilde{u}$$

which replaces (5.2.8). It follows from the definition (5.2.3) that the section $\tilde{\lambda}$ of $\mathrm{Isom}(p_2^*G', p_1^*G')$ associated to $\tilde{u}$ is obtained from the section λ defined by u by "conjugation by μ":

$$\tilde{\lambda} = (p_1^*\mu) \circ \lambda \circ (p_2^*\mu)^{-1}.$$

The identity (5.3.4) thus induces the relation

(5.3.5) $$\lambda' \circ (p_2^*\mu) = i_\delta \circ (p_1^*\mu) \circ \lambda$$

in $\mathrm{Isom}(p_2^*G, p_1^*G')$ between the cocycle elements λ and λ', which generalizes (5.2.9). Similarly, comparing the diagrams (5.2.5) associated to the pairs (x, u) and (x', u'), one verifies that the corresponding sections γ and γ' of p_1^*G and of p_1^*G' are related by the identity

(5.3.6) $$\gamma'\delta_{13} = \delta_{12}\tilde{\lambda}_{12}(\delta_{23})\mu(\gamma)$$

between sections of p_1^*G' over $(T_1/S_1)_2$, which is the appropriate generalization of (5.2.10). In this formula, the element $\tilde{\lambda} \in \mathrm{Isom}(p_2^*G', p_1^*G')$ is obtained from $\tilde{u}$ (5.3.3) as the corresponding element $\lambda \in \mathrm{Isom}(p_2^*G, p_1^*G)$ was obtained from u. It follows from (5.3.2) that $\tilde{\lambda}$ is related to λ by "conjugation by μ", in other words, that $\tilde{\lambda}$ is defined by

(5.3.7) $$\tilde{\lambda} = (p_1^*\mu) \circ \lambda \circ (p_2^*\mu)^{-1}.$$

We may now summarize the previous discussion as follows.

PROPOSITION 5.4. *Let $\mathcal{G}$ be a gerbe, decomposed by the pairs (x, u), (x', v), to which correspond the nonabelian 2-cocycle triples (G, λ, γ) and (G', λ', γ'). The choice of a local isomorphism χ (5.3.1) determines a local isomorphism $\mu: G \to G'$ and a section δ of p_1^*G' such that the identities (5.3.5)–(5.3.7) are satisfied. We then say that the pair (μ, δ) is a coboundary relation between the cocycle triples (G, λ, γ) and (G', λ', γ'). Conversely, a 2-cocycle triple (G, λ, γ) determines a gerbe $\mathcal{G}$ with a canonical decomposition, and two cohomologous triples determine equivalent gerbes.*

5.5. To be complete, one must then examine, in the spirit of the description given in §2.8 of a 2-morphism, in what manner the coboundary pair (μ, δ) is affected when the arrow χ is replaced by another arrow $\chi_1: x \to x'$ (which we may suppose is also defined over S_1). In that case, χ_1 is related to χ by the identity

(5.5.1) $$\chi_1 = \theta \circ \chi$$

for some S_1-section θ of G'. It follows that the automorphism μ_1 of G' defined by χ_1 differs from μ (5.3.2) by inner conjugation

(5.5.2) $$\mu_1 = i_\theta \circ \mu.$$

It can then be checked from the formula (5.3.4) defining δ and the corresponding formula for the element δ_1 associated in a similar manner to χ_1, and δ and δ_1 are related by the formula

$$\delta_1(p_1^*\theta) = \delta({}^{\mu}\lambda(p_2^*\theta)) \tag{5.5.3}$$

where ${}^{\mu}\lambda$ is the element of $\mathrm{Isom}(p_2^*G', p_1^*G')$ determined by $\lambda \in \mathrm{Isom}(p_2^*G, p_1^*G)$ and μ (5.3.2); in other words,

$${}^{\mu}\lambda = p_1^*\mu \circ \lambda \circ (p_2^*\mu)^{-1}. \tag{5.5.4}$$

REMARK 5.6. (i) Similar nonabelian cocycles were attached to gerbes in **[B1]** in the following more restrictive situation: given a locally trivialized gerbe $(\mathcal{G}, x)$, one makes the additional hypothesis that the locally defined group $G_1 = \mathrm{Aut}(x)$ associated to the object x descends to a group G defined over the final object e of T. One then says that $\mathcal{G}$ is a G-gerbe. A pleasant feature, in this case, is that such a G-gerbe may be interpreted (see **[B1,** §6]) as a torsor (in an appropriate sense) under the gr-stack BITORS(G) of G-bitorsors. It is shown in **[B1]** that the set of equivalence classes of G-gerbes is classified by the set $H^1(e, G \to \mathrm{Aut}(G))$ of (hyper-)cohomology classes with values in the crossed module $G \to \mathrm{Aut}(G)$ defined by the inner conjugation homomorphism of the group G. Returning to the explicit cocycles introduced above, we see that such a 1-cocycle consists again of the pair (λ, γ) given above with γ a section of G over S^3 and with λ simply a section of $\mathrm{Aut}(G)$ over $S \times S$. The cocycle and coboundary conditions (5.2.6)–(5.2.7) and (5.2.9)–(5.2.10) remain unchanged, and the terms i_γ (resp. i_δ) of formulas (5.2.6), (5.2.9) are now the images by the inner conjugation map of the sections γ (resp. δ) of G. It is possible to give a geometric description of an arbitrary gerbe, analogous to that just mentioned of a G-gerbe as a torsor under a gr-stack, but this requires the introduction, in the spirit of (2.9) of the rather cumbersome notion of a torsor under the groupoid-stack BITORS of all bitorsors, and will not be pursued here.

(ii) If we are willing to neglect some information, we may, following **[Gi]**, introduce for any pair of S-groups G and H, the sheaf $\mathrm{Isex}(G, H)$ of outer isomorphisms of G with H, by taking the quotient of the sheaf $\mathrm{Isom}(G, H)$ by H, for the left action of the group H on $\mathrm{Isom}(G, H)$, defined by

$$\begin{aligned} H \times \mathrm{Isom}(G, H) &\to \mathrm{Isom}(G, H), \\ (h, u) &\mapsto i_h \circ u. \end{aligned} \tag{5.6.1}$$

If we view the map λ (5.2.2) as defining a section $\underline{\lambda}$ of $\mathrm{Isex}(p_2^*G, p_1^*G)$, it follows from (5.2.6) that $\underline{\lambda}$ satisfies the ordinary 1-cocycle condition

$$\underline{\lambda}_{12} \circ \underline{\lambda}_{23} = \underline{\lambda}_{13}. \tag{5.6.2}$$

The section $\underline{\lambda}$ thus defines descent data from S to e for the local group G, but this merely descends G to an e-object in the category of liens (also known as bands) **[Gi,** IV, §1], not in the category of groups. The map

$$(G, \lambda, \gamma) \mapsto (G, \underline{\lambda}) \tag{5.6.3}$$

thus describes at the cocycle level the 2-functor which associates to a gerbe its lien. In the simpler case of a G-gerbe associated to a globally defined group G, this reduces to the right-hand arrow of **[B1,** 5.2.3] (except for a slightly different choice of conventions).

It should be noted that Giraud defines his cohomology groups $H^2(e, L)$ with values in a lien L as consisting in equivalence classes of pairs $(\mathcal{G}, \varphi)$, where $\mathcal{G}$ is a gerbe defined over e and φ: lien$(\mathcal{G}) \to L$ an isomorphism in the category of liens. From our point of view, it would be preferable, if we wanted to introduce liens as coefficients, to consider the set $H^2(L)$ of equivalence classes of gerbes $\mathcal{G}$ over e for which there exists an isomorphism such as φ, but without specifying this isomorphism. Indeed, this new set $H^2(L)$ is the fibre above L of the map induced on cohomology classes by the cocycle map (5.6.3), so it is in particular a subset of the set of cohomology classes described by Proposition 5.4. It is the quotient of Giraud's $H^2(e, L)$, by the natural action of the group $\mathrm{Aut}(L)$ of automorphisms of L in the category of liens. Giraud's set $H^2(e, L)$ is closer to an abelian H^2 than ours, since he shows **[Gi,** IV, Theorem 3.3.3] that it is either the empty set, or a principal homogeneous set under the abelian cohomology group $H^2(e, ZL)$ (the abelian group ZL being the center of the lien L). Such a distinction between two possible sorts of nonabelian H^2 sets had already been noted in his context by Dedecker, who referred to them, respectively, as thick and meager cohomology **[Ded1]**, **[Do,** §4, 5]. At the level of Galois gerbes, the distinction between the two notions is the following. Two Galois gerbes (4.4.1) with $\ker(\varphi) = \ker(\varphi') = \mathcal{A}$ only define the same class in $H^2(e, L)$, when the two extensions of Γ_k by $\mathcal{A}$ defined by φ and φ' are equivalent. On the other hand, the discussion in (4.6) shows that any two Galois gerbes related by an isomorphism ρ_χ (4.6.3) define the same cohomology class in $H^2(L)$, even when the automorphism $\tilde{\rho}_\chi$ (4.6.2) of $\mathcal{A}$ induced by ρ_χ is not the identity.

5.7. Returning to the particular case of smooth affine gerbes over $\mathrm{Spec}(k)$ in the étale topology examined in §4, and with the same notation, we may now translate the foregoing discussion into Galois cohomological terms. Let $\mathcal{G}$ be a smooth affine gerbe with trivialization x over $S = \mathrm{Spec}(K)$, the corresponding bitorsor E certainly splits over $k^s \otimes K$, and, a fortiori, over $k^s \otimes k^s$. We are therefore in the pleasant situation in which hypercoverings are not necessary, so long as we take into account questions of continuity. Specifically, the canonical left action

$$\mathrm{Aut}(S_{k^s}) \times S_{k^s} \to S_{k^s}$$

on $S_{k^s} = \mathrm{Spec}(k^s)$ of the group of automorphisms of S_{k^s} induces a right action on S_{k^s} of the opposite group $\Gamma_k = \mathrm{Gal}(k^s/k)$. This yields the standard identification

$$S_{k^s} \times (\Gamma_k)^r \to (S_{k^s})^{r+1}$$
$$(x, \gamma_1, \gamma_2, \dots, \gamma_r) \to (x, x\gamma_1, x\gamma_1\gamma_2, \dots, x\gamma_1\gamma_2\cdots\gamma_r)$$

by which one passes from Čech to Galois cohomology. To (5.2.2) corresponds the continuous Galois cochain $\sigma \mapsto \alpha_\sigma$, taking its value α_σ in $\mathrm{Isom}(\sigma^*G, G)$. Let σ be an element of Γ_k, and let s be the automorphism of S corresponding to the class of $\sigma \bmod \Gamma_K$. The pull-back s^*G of the S-group G may be described by the group $G(k^s)$ of its k^s-rational points, but on which the Galois group Γ_K acts through the conjugation by σ:

$$\Gamma_K \times G(k^s) \to G(k^s), \quad (\rho, g) \mapsto ({}^{\sigma^{-1}\rho\sigma}g). \tag{5.7.1}$$

It follows that the continuous cochain

$$\Gamma_k \to \mathrm{Aut}(G(k^s)), \quad \sigma \mapsto a_\sigma \tag{5.7.2}$$

has its values in the Γ_K-equivariant automorphisms of $G(k^s)$, where the source is endowed with the twisted Γ_K-module structure (5.7.1). This means that the automorphism a_σ satisfies the condition

$$a_\sigma({}^\rho g) = {}^{(\sigma\rho\sigma^{-1})}\, a_\sigma(g) \tag{5.7.3}$$

for all $\rho \in \Gamma_K$. In fact, since $\mathcal{G}$ is affine, a_σ extends to an isomorphism $\sigma^*G \to G$ of algebraic groups over k^s; so the discussion in Remark 4.3 shows that the cochain a_σ is in fact a σ-linear automorphism of the group $G(k^s)$ (such an algebraicity condition on the cocycle a_σ is already to be found in Springer's original definition on the nonabelian H^2 in Galois cohomology; see [**Sp**, 2.4–2.5], [**Bo**]). Similarly, the element γ (5.2.5) now becomes a continuous 2-cochain

$$\Gamma_k \times \Gamma_k \to G(k^s), \quad \sigma, \tau \mapsto h_{\sigma,\tau}, \tag{5.7.4}$$

and the cocycle conditions (5.2.6) and (5.2.7) become

$$a_\sigma a_\tau = i(h_{\sigma,\tau}) a_{\sigma\tau}, \tag{5.7.5}$$

$$a_\sigma(h_{\tau,\chi}) h_{\sigma,\tau\chi} = h_{\sigma,\tau} h_{\sigma\tau,\chi}. \tag{5.7.6}$$

It now follows from 5.6(ii) that the lien of $\mathcal{G}$ is described by the 1-cocycle

$$\sigma \mapsto \tilde{a}_\sigma \tag{5.7.7}$$

induced by a_σ with values in the outer automorphisms of $G(k^s)$. Translating in a similar manner the coboundary relations (5.2.9)–(5.2.10), we see that the cocycle pairs $(a_s, h_{\sigma,\tau})$ and $(a'_s, h'_{\sigma,\tau})$ are cohomologous when there exists a continuous $G(k^s)$-valued 1-cochain

$$\Gamma_K \to G(k^s), \quad \sigma \mapsto \delta_\sigma \tag{5.7.8}$$

such that the relations

$$a'_\sigma = i_{\delta_\sigma} a_\sigma, \tag{5.7.9}$$

$$h'_{\sigma,\tau}\delta_{\sigma\tau} = \delta_\sigma a_\sigma(\delta_\tau) h_{\sigma,\tau} \tag{5.7.10}$$

are satisfied. If we wish to consider the cohomology classes of Proposition 5.4, we must introduce new coboundary relations between the cocycle pairs $(a_s, h_{\sigma,\tau})$ and $(a'_s, h'_{\sigma,\tau})$ satisfying conditions (5.7.5)–(5.7.6). These will now be said to be cohomologous whenever there is a pair (v, δ_σ), with δ_σ a 1-cochain (5.7.8) and v a Γ_K-equivariant automorphism of $G(k^s)$ such that the conditions

$$va'_\sigma = i_{\delta_\sigma} a_\sigma v, \tag{5.7.9$'$}$$

$$v(h'_{\sigma,\tau})\delta_{\sigma\tau} = \delta_\sigma a_\sigma(\delta_\tau) h_{\sigma,\tau} \tag{5.7.10$'$}$$

(which reduce to (5.7.9)–(5.7.10) for $v = 1$) are satisfied.

We leave to the reader the exercise of translating in a similar manner the "coboundary between coboundaries" conditions (5.5.2)–(5.5.4).

5.8. We chose here to derive these Galois cocycle and coboundary formulas by specializing the general theory to $\mathrm{Spec}(k)_{\text{ét}}$. It should, however, be pointed out that they could have been obtained somewhat more directly by starting out from the Galois gerbe (4.1.4) associated to the given locally neutralized gerbe $(\mathcal{G}, x)$: reverting to the notation of §4, let us observe that a section u (5.2.1) of E over some Galois extension $S' \times S'$ of $S \times S$ translates to a continuous section $t: \Gamma_k \to \mathcal{E}$ of the associated Galois gerbe, which is left $\Gamma_{K'}$-equivariant (for the action of the open subgroup $\Gamma_{K'}$ of Γ_K on $\mathcal{E}$ via (4.1.12)), and therefore, satisfies

$$t(\sigma) = j(\gamma)t(\sigma)j(\gamma^\sigma)^{-1} \tag{5.8.1}$$

for all $\sigma \in \Gamma_K$, $\gamma \in \Gamma_{K'}$. Since u determines, in the notation of 4.1.2, a section of E', t descends to a section of $\mathcal{E}/\Gamma_K$ over $\Gamma_{K/k}$, so the condition

$$t(\sigma\gamma) = t(\sigma)j(\gamma) \tag{5.8.2}$$

is also satisfied. In view of (5.8.2), (5.8.1) simplifies to condition

$$t(\gamma\sigma) = j(\gamma)\tau(\sigma). \tag{5.8.1$'$}$$

We can assume that the chosen section u of E lifts the diagonal map of S' to the pull-back to S' of the unit section (2.5.2). In this case it follows from (5.8.1)$'$ that the restrictions to $\Gamma_{K'}$ of t and of j coincide. The cocycles (5.7.2)–(5.7.4) now correspond to the classical Schreier cocycles for the splitting t of the Galois gerbe extension

$$1 \to G(k^s) \to \mathscr{E} \to \Gamma_k \to 1,$$

as defined for example in **[B1,** 8.10.3–8.10.4], together with a trivialisation of their restriction to $\Gamma_{K'}$. In **[Ul2,** Proposition 7.2], a $G(k^s)$-valued 1-cocycle is described, whose vanishing controls the obstruction to extending the partial splitting $j\colon \Gamma_K \to \mathscr{E}$ to a full section t of φ over Γ_k. Its existence is not surprising, since the bitorsor E does not always have a global section over $S \times S$. More relevant, from our point of view, is the fact, mentioned above, that the restriction of the homomorphism j to some open subgroup $\Gamma_{K'}$ of Γ_K can always be extended to a section of $\mathscr{E}$ over the full Galois group Γ_k of k.

REMARK 5.9. In the situation discussed in Example 3.4, in which $\mathscr{G}$ is the gerbe of fibre functors of some Tannakian category of motives $\mathscr{C}$ over k, Proposition 5.4 explains how the gerbe $\mathscr{G}$ (and hence, by the Tannaka duality Theorem 3.3, the category $\mathscr{C}$ itself) may be recovered from purely abstract data, consisting of an algebraic K-group G (which may, as we have seen, be thought of as the motivic Galois group of self-transformations of a K-valued cohomology theory x on $\mathscr{C}$), together with families of abstractly given "period" morphisms a_σ (5.7.2) and sections $h_{\sigma,\tau}$ (5.7.4) of G over k^s, satisfying conditions (5.7.5)–(5.7.6).

BIBLIOGRAPHY

[A-H] M. Atiyah and F. Hirzebruch, *Kohomologie-Operationen und charakteristische Klassen*, Math. Z. **77** (1961), 149–187.

[Bo] M. Borovoĭ, *Abelianization of the second nonabelian Galois cohomology*, preprint.

[B1] L. Breen, *Bitorseurs et cohomologie nonabélienne*, in the Grothendieck Festschrift I, Progr. Math. **86**, Birkhäuser, Boston, 1990, pp. 401–476.

[B2] ——, *Théorie de Schreier supérieure*, Ann. Sci. École Norm. Sup. (4) **25** (1992), 465–514.

[Ca] P. Cartier, *Développments récents sur les groupes de tresses. Applications à la topologie et à l'algèbre*, Sém. Bourbaki **716** (1989).

[Ch] C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton, NJ, 1946.

[Ded1] P. Dedecker, *Les foncteurs* Ext_Π, H^2_Π *et* H^2_Π *nonabéliens*, C. R. Acad. Sci. Paris **258** (1964), 4891–4894.

[Ded2] ——, *Three dimensional non-abelian cohomology for groups*, Category Theory, Homology Theory and Their Applications. II, Lecture Notes in Math., vol. 92, Springer-Verlag, Berlin and New York, 1969, pp. 32–64.

[De1] P. Deligne, *La formule de dualité globale*, Exposé XVIII of SGA4, Lecture Notes in Math. **305**, Springer-Verlag, Berlin and New York, 1973.

[De2] ——, *Catégories Tannakiennes*, in the Grothendieck Festschrift II, Progr. Math. **87**, Birkhäuser, Boston, 1990, pp. 111–195.

[De3] ——, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$, Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989.

[De4] ——, *Cohomologie étale*: *Les points de départ*, Cohomologie Etale (SGA 4 1/2), Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin and New York, 1977.

[De5] P. Deligne, *Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes*, notes by J.-L. Brylinski, Mémoires de la S.M.F. (2nd series) **2** (1980), 23–33.

[D-M] P. Deligne and J. S. Milne, *Tannakian categories*, Hodge Cycles, Motives and Shimura Varieties, Lectures Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982.

[Do] J.-C. Douai, 2-*Cohomologies Galoisienne des groups semi-simples*, Doctoral thesis, Université de Lille I, Lille, France, 1976.

[Do-R1] S. Doplicher and J. Roberts, *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. of Math. (2) **130** (1989), 75–119.

[Do-R2] ——, *A new duality theorem for compact groups*, Invent. Math. **98** (1989), 175–218.

[Dr1] V. G. Drinfel'd, *Quasi-Hopf algebras*, Algebra i Analiz **1** (1989); English transl., Leningrad Math. J. **1** (1990), 1419–1457.

[Dr2] ——, *On quasi-triangular quasi-Hopf algebras closely related to* $\mathrm{Gal}(\overline{Q}/Q)$, Algebra i Analiz **2** (1990), 149–181; English transl., Leningrad. Math. J. **2** (1991), 829–860.

[Du1] J. Duskin, *An outline of non-abelian cohomology in a topos*:(I) *The theory of bouquets and gerbes*, Cahiers Topologie Géom. Différentielle **23** (1983), 165–191.

[Du2] ——, *Non-abelian cohomology in a topos*, preprint.

[F-Y1] P. Freyd and D. Yetter, *Braided compact closed categories with applications to low-dimensional topology*, Adv. Math. **77** (1989), 156–182.

[F-Y2] ——, *Coherence theorems via knot theory*, J. Pure Appl. Algebra **78** (1992), 49–76.

[Gi] J. Giraud, *Cohomologie non abélienne*, Grundlehren Math. Wiss., vol. 179, Springer-Verlag, New York, 1971.

[Gr] A. Grothendieck, extract from pp. 206–207 of *Récoltes et Semailles*, Montpellier (1985), quoted on p. 345 of J.-P. Serre, *Motifs*, in Journées Arithmétiques de Luminy 17–21 Juillet 1989, Astérisque **198–200** (1991), 333–349.

[H] A. Haefliger, *Structures feuilletées et cohomologie à valeurs dans un faisceau de groupoïdes*, Comment. Math. Helv. **32** (1958), 248–329.

[I] L. Illusie, *Complexe cotangent et déformations*. II, Lecture Notes in Math., vol. 283, Springer-Verlag, Berlin and New York, 1972.

[J-S1] A. Joyal and R. Street, *Braided monoidal categories*, Macquarie Reports 86008 (1986).

[J-S2] ——, *An introduction to Tannaka duality and quantum groups*, Category Theory (Proceedings Como 1990) (A. Carboni, M. C. Pedicchio, and G. Rosolini, eds.), Lecture Notes in Math., vol. 1488, 1991, pp. 413–492.

[Ka-Lu] D. Kazhdan and G. Lusztig, *Affine Lie algebras and quantum groups*, preprint.

[Ke-La] G. M. Kelley and M. L. Laplaza, *Coherence for compact closed categories*, J. Pure Appl. Algebra **19** (1980), 193–213.

[Kr] M. Krein, *A principle of duality for a bicompact group and square block algebra*, Dokl. Akad. Nauk. SSSR **69** (1949), 725–728.

[L-R] R. P. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378** (1987), 113–220.

[Ly] V. V. Lyubashenko, *Tensor categories and RCFT*:I *Hopf algebras in rigid categories*, II *Modular transformations*, preprints ITP-90-30E, ITP-90-59E, Kiev 1990.

[Mac1] S. Mac Lane, *Natural associativity and commutativity*, Rice University Studies, vol. 49, 1963, pp. 28–46; reprinted in the Selected Papers (I. Kaplansky, ed.), Springer-Verlag, New York, 1979.

[Mac2] ——, *Categories for the working mathematician*, Graduate Texts in Math., vol. 5, Springer-Verlag, New York, 1972.

[Maj] S. Majid, *Tannaka-Krein theorems for quasi-Hopf algebras and other results*, Contemp. Math. (to appear).

[Mi1] J. S. Milne, *Motives over finite fields*, these Proceedings, vol. 1, pp. 401–459.

[Mi2] ——, *The points on a Shimura variety modulo a prime of good reduction*, The Zeta Functions of Picard Modular Surfaces, Les Publications du Centre Rech. Math., Montréal, 1992, pp. 153–255.

[P] B. Pareigis, *A non-commutative, non-cocommutative Hopf algebra in nature*, J. Algebra **70** (1981), 356–374.

[Saa] N. Saavedra Rivano, *Catégories tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin and New York, 1972.

[Sp] T. A. Springer, *Nonabelian H^2 in Galois Cohomology*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., vol. IX, Amer. Math. Soc., Providence, RI, 1966, pp. 164–182.

[T] T. Tannaka, *Ueber den Dualitätssatz der nichtkommutativen topologischen Gruppen*, Tôhoku Math. J. **45** (1939), 1–12.

[Ul1] K.-H. Ulbrich, *On the Hopf algebras and rigid monoidal categories*, Israel J. Math. **72** (1990), 252–256.

[Ul2] ——, *On nonabelian* $\mathbf{H}^2$ *for profinite groups*, Canad. J. Math. **43** (1991), 213–224.

[Ul3] ——, *On the correspondence between gerbes and bouquets*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 1–5.

[Ul4] ——, *On cocycle bitorsors and gerbes over a Grothendieck topos*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 49–55.

[W] J.-P. Wintenberger, *Torseurs pour les motifs et pour les repésentations p-adiques potentiellement de type CM*, Math. Ann. **288** (1990), 1–8.

[Y1] D. Yetter, *Quantum groups and representations of monoidal categories*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 261–290.

[Y2] ——, *Coalgebras, coends and reconstruction*, preprint.

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Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques

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> "...Dix choses soupçonnées seulement, dont aucune (la conjecture de Hodge disons) n'entraîne conviction, mais qui mutuellement s'éclairent et se complètent et semblent concourir à une même harmonie encore mystérieuse, acquièrent dans cette harmonie force de vision. Alors même que toutes les dix finiraient par se révéler fausses, le travail qui a abouti à cette vision provisoire n'a pas été fait en vain ... " (A. Grothendieck, *Récoltes et Semailles*, cité dans **[35]**).

Le présent exposé rassemble une série de questions et de conjectures portant sur les *groupes de Galois motiviques*, le corps de base étant de caractéristique zéro.

Je me suis limité à deux thèmes:

1. Structure des groupes de Galois motiviques (composante neutre, partie abélienne, partie semi-simple);
2. Propriétés des représentations ℓ-adiques ccorrespondantes lorsque le corps de base est de type fini sur $\mathbf{Q}$ (image du groupe de Galois, éléments de Frobenius).

Parmi les questions importantes laissées de côté figurent:

- les relations avec la "philosophie de Langlands", cf. **[3, 7, 21]**;
- l'aspect "torsion galoisienne", cf. **[10, 25, 41, 42]**;
- l'aspect "cristallin", cf. **[13]**;
- l'aspect "périodes et transcendance", cf. **[7]**.

Le point de vue adopté est celui de Grothendieck: décrire le "paradis motivique", avec les énoncés conjecturaux les plus optimistes possibles. Pour alléger, ces énoncés sont rédigés sous forme affirmative; seul, un point d'interrogation au début de la phrase indique qu'il s'agit d'une assertion non démontrée. Ainsi "10.3? L'indice de $G_{\ell,E}$ dans $G_L(\mathbf{Z}_\ell)$ reste borné quand

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ℓ varie" est une conjecture, non un théorème. J'espère que cette convention n'induira pas le lecteur en erreur. D'ailleurs la plupart des énoncés en question font partie du folklore. Ainsi, l'essentiel des n^{os} 1, 2, ..., 7, 9, 12 était connu de Grothendieck dès 1964–1965, et j'étais familier avec le contenu du n° 11 en 1977, et du n° 13 en 1966. Ne sont récents, à ma connaissance, que 7.5, une partie du n° 8, le n°10, et la fin du n° 12.

1. Structure des groupes de Galois motiviques

1. Notations. On note k un corps de caractéristique 0, que l'on suppose plongeable dans $\mathbf{C}$; on choisit un tel plongement $\sigma: k \to \mathbf{C}$.

On suppose vraies les *conjectures standard*, ainsi que la *conjecture de Hodge*, cf. [**14, 17, 18**]. On note M_k, ou simplement M, la catégorie des *motifs* sur k, définie au moyen de l'équivalence numérique des cycles algébriques (ou de l'équivalence homologique—c'est la même chose d'après l'une des conjectures standard). Précisons qu'il s'agit de motifs qui sont sommes directes de motifs *purs*; nous ne nous occupons pas de motifs *mixtes*. La catégorie M est semi-simple, cf. [**16**].

Si E est un motif sur k, on note $M(E)$ la plus petite sous-catégorie tannakienne de M contenant E. Si E et E' sont des motifs, on dit que E' est *dominé* par E, et l'on écrit $E' \prec E$, si E' appartient à $M(E)$. On a

$$M = \varinjlim_{E} M(E),$$

la limite inductive étant prise suivant l'ensemble préordonné filtrant des motifs.

2. Les groupes G_M et $G_{M(E)}$. Soit $\mathrm{Vect}_{\mathbf{Q}}$ la catégorie des $\mathbf{Q}$-espaces vectoriels de dimension finie. Le choix du plongement $\sigma : k \to \mathbf{C}$ permet de définir sur M un *foncteur fibre* à valeurs dans $\mathrm{Vect}_{\mathbf{Q}}$, à savoir la réalisation de Betti $H_\sigma : M \to \mathrm{Vect}_{\mathbf{Q}}$. Ce foncteur est fidèle. Son schéma d'automorphismes $\mathrm{Aut}^{\otimes}(H_\sigma)$ est le *groupe de Galois motivique de* k, au sens de Grothendieck (cf. Deligne–Milne [**10**] et Saavedra [**25**]). Nous le noterons $G_{M,k}$ ou simplement G_M. C'est un $\mathbf{Q}$-groupe proalgébrique linéaire. La catégorie $\mathrm{Rep}_{\mathbf{Q}} G_M$ des $\mathbf{Q}$-représentations linéaires de G_M est équivalente à la catégorie M (comme catégorie tannakienne).

REMARQUE. Le groupe G_M dépend du choix du plongement σ (il est d'ailleurs noté $G(\sigma)$ dans [**10**, p. 213]). Toutefois, il n'en dépend qu'à torsion intérieure près (et même par torsion par un cocycle à valeurs dans sa composante neutre G_M^0, cf. n° 6); la plupart des propriétés que nous en donnerons sont invariantes par une telle torsion.

Du fait que M est semi-simple, toutes les représentations linéaires de G_M sont semi-simples. D'où:

2.1? *Le groupe* G_M *est proréductif* (i.e., limite projective de $\mathbf{Q}$-groupes linéaires réductifs).

De façon plus précise, on a

$$G_M = \lim\operatorname*{proj}_{E} G_{M(E)}, \tag{2.2}$$

où $G_{M(E)} = G_{M(E),k}$ désigne le groupe motivique attaché à la catégorie tannakienne $M(E)$, et la limite projective est prise sur les k-motifs ordonnés par la relation de domination. Les morphismes de transition

$$G_{M(E')} \to G_{M(E)} \quad (\text{pour } E' \succ E)$$

sont surjectifs. Les $G_{M(E)}$ sont des **Q**-groupes linéaires réductifs, non nécessairement connexes (à moins que k ne soit algébriquement clos, cf. n° 6).

3. Caractérisations de $G_{M(E)}$. Soit E un motif. Notons $\mathbf{GL}_E$ le groupe linéaire du **Q**-espace vectoriel $H_\sigma(E)$. Le groupe $G_{M(E)}$ se plonge de façon naturelle dans $\mathbf{GL}_E$. Le sous-groupe de $\mathbf{GL}_E$ ainsi obtenu peut être caractérisé de plusieurs manières:

(a) *Tenseurs invariants.* Convenons de noter **1** le motif trivial de rang 1 (cohomologie de dimension 0 de la variété $\operatorname{Spec}(k)$). Si r et s sont des entiers ≥ 0, notons $\mathbf{T}^{r,s}(E)$ le produit tensoriel de r copies de E et de s copies du dual E^* de E. Un élément de $\mathbf{T}^{r,s}(H_\sigma(E))$ est dit *invariant* (ou *motivé*) s'il provient d'un morphisme de motifs $\mathbf{1} \to \mathbf{T}^{r,s}(E)$; il revient au même de dire qu'il est invariant par l'action de $G_{M(E)}$. Inversement:

3.1? *Le groupe $G_{M(E)}$ est le sous-groupe de $\mathbf{GL}_E$ formé des éléments qui fixent tous les éléments invariants de tous les $\mathbf{T}^{r,s}(H_\sigma(E))$.*

Cela provient du fait qu'un groupe réductif est caractérisé par ses invariants tensoriels, cf. e.g. [**8**, p. 40, prop. 3.1].

(b) *Représentations ℓ-adiques.* On suppose que k est de type fini sur **Q**. Il résulte alors de conjectures de Grothendieck et de Tate que l'on a:

3.2? *Soit ℓ un nombre premier. Le groupe $G_{M(E)/\mathbf{Q}_\ell}$ est l'adhérence pour la topologie de Zariski de l'image de la représentation ℓ-adique associée à E.*

Pour un énoncé plus précis, voir §2, n° 9.

(c) *Tores de Hodge et groupes de Mumford-Tate.* La bigraduation de $H_\sigma(E) \otimes \mathbf{C}$ donnée par la théorie de Hodge peut s'interpréter (grâce au dictionnaire: **Z**-graduations $\Leftrightarrow$ actions de $\mathbf{G}_m$) comme un homomorphisme

$$h_E : \mathbf{G}_m \times \mathbf{G}_m \to \mathbf{GL}_{E/\mathbf{C}} \quad (\text{défini sur } \mathbf{C}). \tag{3.3}$$

L'image de h_E est contenue dans la composante neutre $G^0_{M(E)}$ du groupe $G_{M(E)}$, et l'on a, d'après Mumford-Tate (cf. [**23**]):

3.4? *Le groupe* $G^0_{M(E)}$ *est le plus petit* **Q**-*sous-groupe algébrique de* $\mathbf{GL}_E$ *qui, après extension des scalaires à* **C**, *contient le tore* $\mathrm{Im}(h_E)$.

On obtient ainsi une caractérisation, sinon de $G_{M(E)}$, du moins de sa composante neutre, le *groupe de Mumford-Tate.*

4. Exemples.

4.1. Le groupe $G_{M(E)}$ associé au *motif* **1** est $\{1\}$.

4.2. Soit T le *motif de Tate*, défini par exemple comme l'homologie de dimension 2 de la droite projective $\mathbf{P}_1$. On a

$$G_{M(T)} = \mathbf{G}_m .$$

4.3. Soit E une *courbe elliptique sans multiplications complexes*, vue comme motif de poids -1 (i.e. identifiée à son homologie de dimension 1). On a:

$$G_{M(E)} = \mathbf{GL}_E \simeq \mathbf{GL}_2 .$$

4.4. Soit E une *courbe elliptique à multiplications complexes.* Soit K le corps quadratique imaginaire correspondant. Supposons que les multiplications complexes de E soient définies sur k (resp. ne le soient pas). On a:

$$G_{M(E)} = T_K \quad (\text{resp. } G_{M(E)} = N_K),$$

où $T_K = R_{K/\mathbf{Q}}\mathbf{G}_m$ est le tore de dimension 2 défini par K, et N_K est le normalisateur de T_K dans $\mathbf{GL}_2$; on a $(N_K : T_K) = 2$.

4.5. Soit E une *variété abélienne* de dimension $n \geq 1$ dont l'anneau des $\overline{k}$-endomorphismes est réduit à $\mathbf{Z}$. Si n est impair (ou si $n = 2$ ou 6), on peut montrer que $G_{M(E)}$ est le groupe $\mathbf{GSp}_{2n}$ des *similitudes symplectiques* relativement à une forme bilinéaire alternée de rang $2n$. (Par contre, lorsque $n = 4$, $G_{M(E)}$ peut être strictement contenu dans $\mathbf{GSp}_{2n}$, cf. Mumford **[24]**.)

4.6. Soit E la somme directe du *motif de Ramanujan* et du motif de Tate. Le groupe $G_{M(E)}$ est le sous-groupe de $\mathbf{GL}_2 \times \mathbf{G}_m$ formé des couples (u, x) tels que $\det(u) = x^{11}$.

5. Les homomorphismes $\mathbf{G}_m \xrightarrow{\mathbf{w}} G_M \xrightarrow{\mathbf{t}} \mathbf{G}_m$. L'homomorphisme $\mathbf{t}$ est l'homomorphisme canonique $G_M \to G_{M(T)} = \mathbf{G}_m$, où T est le motif de Tate, cf. n° 4.2.

L'homomorphisme $\mathbf{w} : \mathbf{G}_m \to G_M$ est celui qui est associé à la graduation par le *poids*. Après extension des scalaires à **C**, il est égal au composé

$$\mathbf{G}_m \xrightarrow{\delta} \mathbf{G}_m \times \mathbf{G}_m \xrightarrow{h_E} G_{M/\mathbf{C}},$$

où δ est l'homomorphisme diagonal, et h_E est l'homomorphisme de Hodge, cf. 3.3.

L'image de $\mathbf{w}$ est contenue dans le centre de G_M. On a la formule

$$\mathbf{t} \circ \mathbf{w} = -2 \quad \text{dans } \operatorname{Hom}(\mathbf{G}_m, \mathbf{G}_m) = \mathbf{Z}, \tag{5.1}$$

autrement dit $\mathbf{t}(\mathbf{w}(x)) = x^{-2}$ pour tout point x de $\mathbf{G}_m$. Cela traduit le fait que le motif de Tate est de poids -2.

Si E est un motif, on notera encore $\mathbf{w}$ l'homomorphisme de $\mathbf{G}_m$ dans $G_{M(E)}$ obtenu en composant $\mathbf{w} : \mathbf{G}_m \to G_M$ et $G_M \to G_{M(E)}$. De même, si $E \succ T$, on notera $\mathbf{t}$ l'homomorphisme de $G_{M(E)}$ dans $\mathbf{G}_m$ par lequel se factorise $\mathbf{t}$: $G_M \to \mathbf{G}_m$.

Exemple. Dans la situation du n° 4.6, on a $\mathbf{w}(x) = (x^{-11}, x^{-2})$ et $\mathbf{t}(u, x) = x$.

Remarque. On pourrait éviter le signe "moins" de 5.1 en remplaçant $\mathbf{w}$ par son opposé, c'est-à-dire en donnant la préséance à *l'homologie* et non à la cohomologie. La convention suivie ici est celle de Saavedra ([**25**, chap. VI, n°4.2]) et de Deligne-Milne [**10**].

6. Composante neutre de G_M et changement de base. Soit $\overline{k}$ une clôture algébrique de k, et soit $\overline{\sigma} : \overline{k} \to \mathbf{C}$ un plongement de $\overline{k}$ dans $\mathbf{C}$ prolongeant σ. Soit $\Gamma_k = \operatorname{Gal}(\overline{k}/k)$ le groupe de Galois de $\overline{k}$ sur k. Le groupe Γ_k est un groupe profini. On peut l'identifier à un $\mathbf{Q}$-*groupe proalgébrique de dimension* 0 qui est "constant", i.e., dont tous les points sont rationnels sur $\mathbf{Q}$. Avec cette convention, on a une suite exacte:

$$1 \to G_M^0 \to G_M \to \Gamma_k \to 1, \tag{6.1}$$

où G_M^0 désigne la composante neutre de G_M. Cela provient de ce que Γ_k est le groupe motivique attaché à la catégorie tannakienne des *motifs d'Artin* sur k, cf. [**10**, p. 211 et p. 214].

Soit k' une extension de k munie d'un plongement de $\overline{k}'$ dans $\mathbf{C}$ prolongeant celui de $\overline{k}$. L'extension des scalaires de k à k' définit un foncteur $M_k \to M_{k'}$, d'où un homomorphisme des groupes motiviques correspondants: $G_{M,k'} \to G_{M,k}$. On a un diagramme commutatif:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G^0_{M,k'} & \longrightarrow & G_{M,k'} & \longrightarrow & \Gamma_{k'} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G^0_{M,k} & \longrightarrow & G_{M,k} & \longrightarrow & \Gamma_k & \longrightarrow & 1 \end{array}$$

où la flèche $\Gamma_{k'} \to \Gamma_k$ est la flèche naturelle.

6.2? *L'homomorphisme* $G^0_{M,k'} \to G^0_{M,k}$ *est surjectif.*

Cela provient de ce que, si E est un motif sur k, l'homomorphisme canonique $G^0_{M(E),k'} \to G^0_{M(E),k}$ est un isomorphisme (cf. 3.4).

De plus:

6.3? *Lorsque* k' *est algébrique sur* k, *l'homomorphisme* $G^0_{M,k'} \to G^0_{M,k}$ *est un isomorphisme* (autrement dit, $G_{M,k'}$ s'identifie à l'image réciproque de $\Gamma_{k'}$ dans $G_{M,k}$).

En particulier, on a $G^0_{M,k} = G_{M,\bar{k}}$.

Cela résulte de ce que tout k'-motif est dominé par un motif provenant par extension des scalaires d'un k-motif (utiliser une restriction des scalaires à la Weil).

REMARQUE. Lorsque k' n'est pas algébrique sur k, on peut montrer (contrairement à ce qui est affirmé dans [**10**, p. 214, prop. 6.22 (b)]) que $G^0_{M,k'} \to G^0_{M,k}$ *n'est pas un isomorphisme.* De façon plus précise, il existe des homomorphismes de $G^0_{M,k'}$ dans $\mathbf{GL}_2$ qui ne se factorisent pas par $G^0_{M,k}$, par exemple ceux associés aux courbes elliptiques sur k' dont l'invariant modulaire est transcendant sur k.

Toutefois, il est possible de démontrer le résultat suivant (sous réserve des conjectures admises ci-dessus):

6.4? *Soit* E *un motif sur* k. *Il existe un corps de nombres* k_1 *et un motif* E_1 *sur* k_1 *tels que les* **Q**-*groupes* $G_{M(E),k}$ *et* $G_{M(E_1),k_1}$ *soient isomorphes.*

Voici le principe de la démonstration. On peut supposer k de type fini sur $\mathbf{Q}$. Soit S un ensemble fini non vide de nombres premiers. En utilisant une variante du théorème d'irréductibilité de Hilbert (cf. [**34**, p. 149]), on montre qu'il est possible de spécialiser (E, k) en (E_1, k_1) de telle sorte que k_1 soit fini sur $\mathbf{Q}$, et que les images des représentations ℓ-adiques de E sur k et de E_1 sur k_1 soient *les mêmes* pour tout $\ell \in S$. On a alors $G_{M(E),k} = G_{M(E_1),k_1}$ d'après 3.2.

7. La partie torique de G^0_M. Le groupe G^0_M est proréductif connexe. Il est donc isogène au produit d'un groupe de type multiplicatif connexe (protore) par un groupe prosemi-simple. De façon plus précise, posons

$$C = \text{composante neutre du centre de } G^0_M;$$
$$D = \text{groupe dérivé de } G^0_M.$$

On a

$$G^0_M = C \cdot D, \qquad (7.1?)$$

et C (resp. D) est un protore (resp. un groupe prosemi-simple).

Soit $S = (G^0_M)^{\mathrm{ab}}$ le plus grand quotient commutatif de G^0_M. On a

$$S = G^0_M/D = C/(C \cap D). \qquad (7.2?)$$

La projection $C \to S$ est une isogénie.

Structure du groupe S. Le protore S est limite projective des tores notés T_{m} dans [27] (voir aussi [21, 26]). Son groupe des caractères

$$X = X(S) = \mathrm{Hom}_{\overline{\mathbf{Q}}}(S, \mathbf{G}_m)$$

est un $\Gamma_{\mathbf{Q}}$-module que l'on peut décrire explicitement (cf. [22, n°1.4]):

Notons c la conjugaison complexe, vue comme élément de $\Gamma_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Soit I le plus petit sous-groupe distingué fermé de $\Gamma_{\mathbf{Q}}$ contenant tous les $ucu^{-1}c$, avec $u \in \Gamma_{\mathbf{Q}}$. Le sous-corps de $\overline{\mathbf{Q}}$ fixé par I est le corps $\mathbf{Q}^{\mathrm{cm}}$, réunion de tous les sous-corps de $\overline{\mathbf{Q}}$ de type CM. Par construction, c est un élément central d'ordre 2 de $\Gamma_{\mathbf{Q}}/I = \mathrm{Gal}(\mathbf{Q}^{\mathrm{cm}}/\mathbf{Q})$.

7.3? *Le groupe X des caractères du protore S est le groupe des fonctions localement constantes $f: \Gamma_{\mathbf{Q}} \to \mathbf{Z}$ satisfaisant aux deux conditions suivantes*:

7.3.1. *f est constante* $\mathrm{mod}\, I$ (i.e., f se factorise par $\Gamma_{\mathbf{Q}}/I$) ; cela équivaut à dire que $f(sct) = f(cst)$ quels que soient s, $t \in \Gamma_{\mathbf{Q}}$;

7.3.2. *il existe $n(f) \in \mathbf{Z}$ tel que $f(s) + f(sc) = n(f)$ pour tout $s \in \Gamma_{\mathbf{Q}}$.* (Noter que $f(sc) = f(cs)$ d'après 7.3.1.)

La structure de $\Gamma_{\mathbf{Q}}$-module de X est donnée par la formule

$$(tf)(s) = f(st) \quad \text{si } s, t \in \Gamma_{\mathbf{Q}}. \tag{7.3.3}$$

Dans cette description de X, l'homomorphisme $\mathbf{w} : \mathbf{G}_m \to G_M^0 \to S$ a pour dual l'homomorphisme $X \to \mathbf{Z}$ donné par $f \mapsto n(f)$. Quant à $\mathbf{t}: S \to \mathbf{G}_m$, il correspond à $f = -1$.

Remarque. Le groupe G_M/D relatif à $k = \mathbf{Q}$ est le groupe appelé par Langlands *groupe de Taniyama*, cf. [9, 21]. La catégorie tannakienne correspondante est celle des $\mathbf{Q}$-motifs qui sont *potentiellement de type* CM. On a une suite exacte

$$1 \to S \to G_M/D \to \Gamma_{\mathbf{Q}} \to 1. \tag{7.3.4}$$

L'action de $\Gamma_{\mathbf{Q}}$ sur S déduite de cette suite exacte correspond à l'action de $\Gamma_{\mathbf{Q}}$ (à droite) sur $X = X(S)$ par $f(t)(s) = f(ts)$ si $f \in X$ et s, $t \in \Gamma_{\mathbf{Q}}$. (Pour une construction directe de G_M/D à partir de 7.3.4, voir [22].)

Structure du groupe C. Soit $X(C)$ le groupe des caractères de C. Du fait que C est isogène à S, on a $X(C) \subset \mathbf{Q} \otimes X(S)$. D'autre part, on sait que l'homomorphisme $\mathbf{w} : \mathbf{G}_m \to S$ se relève en $\mathbf{w} : \mathbf{G}_m \to C$. Ces deux renseignements montrent que $X(C)$ est contenu dans le groupe $\widetilde{X}$ des fonctions localement constantes

$$f: \Gamma_{\mathbf{Q}} \to \mathbf{Q},$$

satisfaisant aux conditions 7.3.1 et 7.3.2, autrement dit:

7.4.1 *f est constante* $\mathrm{mod}\, I$;

7.4.2 *il existe $n(f) \in \mathbf{Z}$ tel que $f(s) + f(sc) = n(f)$ pour tout $s \in \Gamma_{\mathbf{Q}}$.*

(Noter que $n(f)$ doit appartenir à $\mathbf{Z}$, et pas seulement à $\mathbf{Q}$; cela traduit l'existence de $\mathbf{w}: \mathbf{G}_m \to C$.)

7.5? *On a* $X(C) = \widetilde{X}$.

(Autrement dit, C est le plus grand revêtement connexe de S dans lequel $\mathbf{w}$ se relève.)

Modulo les conjectures déjà admises, on peut démontrer 7.5 en utilisant les motifs associés aux variétés abéliennes construites par Shimura dans **[38]** et **[39]**. [Il s'agit de variétés abéliennes "de type IV", dont l'anneau des endomorphismes est un ordre d'un corps K de type CM, l'action de cet ordre sur l'espace tangent étant décrite par une famille d'entiers (r_ν, s_ν) soumise à des conditions que l'on trouvera dans Shimura, *loc. cit.* (De telles variétés existent sur $\overline{\mathbf{Q}}$: cela se déduit de **[39]** en appliquant 6.4.) Les motifs ainsi obtenus fournissent des éléments de $X(C)$ que l'on peut expliciter; en variant K et les (r_ν, s_ν) on constate que l'on obtient suffisamment d'éléments pour engendrer $\widetilde{X} \bmod X$. D'où 7.5.]

EXEMPLE. Voici, d'après Anderson **[1]**, comment on peut construire des éléments de $\widetilde{X}$ n'appartenant pas à X. Soit m un entier > 1 et soit

$$\chi_m : \Gamma_{\mathbf{Q}} \to (\mathbf{Z}/m\mathbf{Z})^*$$

le m-ème caractère cyclotomique. Si $a \in \mathbf{Z}/m\mathbf{Z}$, notons $\langle a \rangle$ l'unique représentant de a dans $\mathbf{Z}$ qui est compris entre 1 et m. Définissons

$$f_m : \Gamma_{\mathbf{Q}} \to \mathbf{Q}$$

par

$$f_m(s) = \frac{1}{m}\langle \chi_m(s) \rangle .$$

On a

$$f_m(s) + f_m(sc) = \frac{1}{m}(\langle \chi_m(s) \rangle + \langle -\chi_m(s) \rangle) = 1 \quad \text{si } s \in \Gamma_{\mathbf{Q}},$$

ce qui montre que f_m appartient à $\widetilde{X}$. Comme $f_m(1) = 1/m$, f_m n'appartient pas à X.

Structure du groupe $C \cap D$. Le groupe $C \cap D$ est un $\mathbf{Q}$-groupe proalgébrique commutatif de dimension 0. Son dual est $\widetilde{X}/X$. Vu 7.5, cela donne

7.6? *Le dual* $X(C \cap D)$ *de* $C \cap D$ *est isomorphe au* $\Gamma_{\mathbf{Q}}$*-module formé des fonctions localement constantes* $f : \Gamma_{\mathbf{Q}}/I \to \mathbf{Q}/\mathbf{Z}$ *telles que* $f(s) + f(sc) = 0$ *pour tout* $s \in \Gamma_{\mathbf{Q}}/I$.

Noter que les groupes S, C, et $C \cap D$ ne dépendent pas de k: les motifs de type CM sont rigides. Il n'en est pas de même du groupe prosemi-simple D dont nous allons maintenant nous occuper: ce groupe dépend effectivement de k.

8. La partie semi-simple de G_M^0. La structure du groupe dérivé D de G_M^0 est mal connue, même conjecturalement. Par exemple:

8.1. *Est-il vrai que D est simplement connexe* (i.e. produit direct de groupes semi-simples simplement connexes)?

La question suivante est liée à 8.1 (et même *équivalente* à 8.1, comme me l'a montré Deligne):

Soit $\widetilde{G} \to G$ une isogénie de $\mathbf{Q}$-groupes réductifs, et soit $f : G_M^0 \to G$ un homomorphisme surjectif. On désire relever f en $\tilde{f} : G_M^0 \to \widetilde{G}$. Une condition nécessaire est que l'homomorphisme de Hodge

$$\mathbf{G}_m \times \mathbf{G}_m \to (G_M^0)_{/\mathbf{C}} \to G_{/\mathbf{C}} \qquad \text{(cf. n° 3.3)}$$

soit relevable à $\widetilde{G}_{/\mathbf{C}}$.

8.2. *Cette condition est-elle suffisante?*

A la place des isogénies on peut considérer des extensions centrales quelconques. Ainsi, une réponse positive à 8.2 entraînerait:

8.3? *Tout homomorphisme surjectif* $G_M^0 \to \mathbf{PGL}_2$ *se relève en* $G_M^0 \to \mathbf{GL}_2$.

Autrement dit, tout motif à groupe $\mathbf{PGL}_2$ proviendrait, après extension finie du corps de base, d'un motif à groupe $\mathbf{GL}_2$.

Autre question:

8.4. *Quels sont les groupes semi-simples* (ou, plus généralement, réductifs connexes) *qui sont du type* $G_{M(E)}$ (sur un corps k convenable, donc aussi sur $\overline{\mathbf{Q}}$, d'après 6.4)?

Noter que l'existence de *polarisations* sur la catégorie des motifs impose certaines restrictions aux groupes semi-simples G de type $G_{M(E)}$:

8.5? L'action de $\Gamma_{\mathbf{Q}}$ sur le graphe de Dynkin de G se factorise par une action de $\Gamma_{\mathbf{Q}}/I$, où I est le sous-groupe de $\Gamma_{\mathbf{Q}}$ défini au n° 7. De plus, la conjugaison complexe c agit sur le graphe en question par *l'involution d'opposition* (celle notée $-w_0$ dans **[31]**, fin du n° 3.1).

8.6? Il existe $\gamma \in G(\mathbf{R})$ ayant les deux propriétés suivantes:

(i) $\gamma^2 = 1$;

(ii) le centralisateur de γ dans $G(\mathbf{R})$ est un sous-groupe compact maximal.

(Avec les notations du n° 3.3, on peut prendre $\gamma = h_E(i, -i)$; c'est un *élément de Hodge* au sens de **[10,** §§4, 6] et **[25,** V.3.3.1])

La propriété 8.6 entraîne par exemple que $\mathbf{SL}_2$ *n'est pas de la forme* $G_{M(E)}$. En effet, un élément γ de $\mathbf{SL}_2(\mathbf{R})$ satisfaisant à (i) est égal à ± 1, et son centralisateur est $\mathbf{SL}_2(\mathbf{R})$, qui n'est pas compact.

Voici un cas particulier de 8.4 où l'on s'attend à une réponse positive:

8.7. (cf. Langlands **[21,** p. 216]) *Soit* H *une structure de Hodge polarisable et soit* G *son groupe de Mumford-Tate. Supposons que* $\mathrm{Lie}(G)$ *soit de type* $\{(1,-1),(0,0),(-1,1)\}$. *Existe-t-il un motif sur* $\mathbf{C}$ *dont* H *soit la réalisation de Hodge*?

(Si oui, on conclurait par 6.4 à l'existence d'un $\overline{\mathbf{Q}}$-motif E tel que $G_{M(E)} \simeq G$.)

La question suivante paraît plus hasardeuse:

8.8. *Existe-t-il un motif* E *tel que* $G_{M(E)}$ *soit un groupe simple de type exceptionnel* G_2 *(ou* E_8*)*?

2. Représentations ℓ-adiques

A partir de maintenant, on suppose que le corps de base k est *une extension de type fini de* $\mathbf{Q}$.

9. Images des représentations ℓ-adiques (ℓ **fixé).** Soit E un motif sur k. Si ℓ est un nombre premier, la cohomologie ℓ-adique de E (sur $\overline{k}$) sera notée $V_\ell(E)$; on peut l'identifier à $\mathbf{Q}_\ell \otimes H_\sigma(E)$. L'action de Γ_k sur $V_\ell(E)$ définit un homomorphisme $\Gamma_k \to \mathbf{GL}_E(\mathbf{Q}_\ell)$ dont l'image est contenue dans le groupe des $\mathbf{Q}_\ell$-points du groupe $G_{M(E)}$. On obtient ainsi un homomorphisme

$$\rho_{\ell,E}\colon \Gamma_k \to G_{M(E)}(\mathbf{Q}_\ell),$$

qui est appelé la *représentation ℓ-adique associée à* E. Cette représentation est continue. Son image $\mathrm{Im}(\rho_{\ell,E})$ est un sous-groupe compact du groupe de Lie ℓ-adique $G_{M(E)}(\mathbf{Q}_\ell)$. D'après des conjectures de Grothendieck et de Mumford-Tate, on a:

9.1? *Le groupe* $\mathrm{Im}(\rho_{\ell,E})$ *est ouvert dans* $G_{M(E)}(\mathbf{Q}_\ell)$. (Il revient au même de dire que son algèbre de Lie est égale à celle de $G_{M(E)}$ sur $\mathbf{Q}_\ell$.)

De plus:

9.2? *Le groupe* $\mathrm{Im}(\rho_{\ell,E})$ *rencontre chacune des composantes connexes de* $G_{M(E)}$.

Lorsqu'on passe à la limite sur E, on obtient un homomorphisme

$$\rho_\ell : \Gamma_k \to G_M(\mathbf{Q}_\ell),$$

et le composé

$$\Gamma_k \to G_M(\mathbf{Q}_\ell) \to \Gamma_k \qquad \text{(cf. 6.1)}$$

est l'identité, ce qui précise 9.2.

Les propriétés 9.1 et 9.2 entraînent:

9.3? *Le* Γ_k*-module* $V_\ell(E)$ *est semi-simple.*

9.4? *Le groupe* $\mathrm{Im}(\rho_{\ell,E})$ *est dense dans* $G_{M(E)/\mathbf{Q}_\ell}$ *pour la topologie de Zariski* (cf. 3.2).

Inversement, si l'on savait démontrer 9.3 et 9.4, il ne serait pas difficile d'en déduire 9.1 (et bien sûr aussi 9.2).

REMARQUE. On peut vérifier 9.1, ..., 9.4 dans chacun des exemples du n° 4 (pour le cas de 4.5, voir [2] et [33]).

10. Images des représentations ℓ-adiques (ℓ variable). On désire préciser la façon dont varie $\mathrm{Im}(\rho_{\ell,E})$ lorsque ℓ parcourt l'ensemble P des nombres premiers. Cela peut se faire (conjecturalement) de plusieurs points de vue:

Indépendance des représentations ℓ-adiques. Posons

$$G_{\ell,E} = \mathrm{Im}(\rho_{\ell,E}) \subset G_{M(E)}(\mathbf{Q}_\ell).$$

La famille des représentations

$$\rho_{\ell,E} : \Gamma_k \to G_{\ell,E} \quad (\ell \in P)$$

définit un homomorphisme

$$\rho_E : \Gamma_k \to \prod_{\ell \in P} G_{\ell,E}.$$

Nous dirons que les $\rho_{\ell,E}$ sont *indépendantes sur* k si cet homomorphisme est *surjectif*. Cela revient à dire que les extensions galoisiennes de k correspondant aux groupes $G_{\ell,E}$ sont linéairement disjointes. (Exemple: $k = \mathbf{Q}$, $E =$ motif de Tate, ou motif de Ramanujan.)

10.1? *Il existe une extension finie* k' *de* k (dépendant du motif E considéré) *telle que les* $\rho_{\ell,E}$ *soient indépendantes sur* k'.

EXEMPLE. On peut démontrer cette conjecture lorsque E est une variété abélienne et k un corps de nombres, cf. [33].

Les Z-formes du motif E. Soit L un *réseau* du $\mathbf{Q}$-espace vectoriel $H_\sigma(E)$. Si $\ell \in P$, $L_\ell = \mathbf{Z}_\ell \otimes L$ est un $\mathbf{Z}_\ell$-réseau de $V_\ell(E)$; pour ℓ assez grand, L_ℓ est stable par l'action de Γ_k via $\rho_{\ell,E}$. Nous dirons que L est une *Z-forme de* E si cela se produit *pour tout* $\ell \in P$.

EXEMPLE. Si E est le motif attaché à une variété abélienne A, les Z-formes de E correspondent aux variétés abéliennes qui sont k*-isogènes* à A.

Le groupe $\mathrm{Aut}(E)$ des automorphismes de E opère sur l'ensemble des Z-formes de E.

10.2? *Les* Z*-formes de* E *sont en nombre fini, modulo l'action de* $\mathrm{Aut}(E)$.

Dans le cas des variétés abéliennes, cette conjecture dit que, si A est une telle variété, il n'y a (à k-isomorphisme près) qu'un *nombre fini de variétés abéliennes sur* k *qui sont* k*-isogènes à* A (ce qui a été démontré par Faltings, cf. [11, 12]).

Quelques propriétés des groupes $G_{\ell,E} = \mathrm{Im}(\rho_{\ell,E})$.
Choisissons une **Z**-forme L de E au sens ci-dessus (il en existe). Notons $G_L(\mathbf{Z}_\ell)$ le sous-groupe de $G_{M(E)}(\mathbf{Q}_\ell)$ formé des éléments qui laissent stable le $\mathbf{Z}_\ell$-réseau $\mathbf{Z}_\ell \otimes L$; c'est le groupe des $\mathbf{Z}_\ell$-points du **Z**-schéma en groupes G_L défini par le réseau L et le **Q**-groupe $G_{M(E)}$. Par construction, on a $G_{\ell,E} \subset G_L(\mathbf{Z}_\ell)$; d'après 9.1, $G_{\ell,E}$ est ouvert dans $G_L(\mathbf{Z}_\ell)$, donc d'indice fini dans ce groupe.

10.3? *L'indice de* $G_{\ell,E}$ *dans* $G_L(\mathbf{Z}_\ell)$ *reste borné quand* ℓ *varie.*

Cette conjecture entraîne:

10.4? *Pour* ℓ *assez grand* (dépendant de k et de L), $G_{\ell,E}$ *contient tous les éléments de* $G_l(\mathbf{Z}_\ell)$ *qui sont congrus à* 1 $(\mathrm{mod}\,\ell)$, *i.e. qui agissent trivialement sur* $L/\ell L$.

De plus:

10.5? *Pour* ℓ *assez grand,* $G_{\ell,E}$ *contient* $\mathbf{w}(\mathbf{Z}_\ell^*)$, *où* $\mathbf{w}$ *est l'homomorphisme de* $\mathbf{G}_m$ *dans* $G_{M(E)}$ *défini au n°*5.

Lorsque E est une variété abélienne, 10.5 (combiné à 10.1) équivaut à une conjecture de Lang (cf. [**19**]) affirmant que $\mathrm{Im}(\rho_E)$ *contient un sous-groupe ouvert du groupe* $\widehat{\mathbf{Z}}^* = \prod \mathbf{Z}_\ell^*$ *des homothéties.* Il est possible de démontrer un résultat un peu plus faible (cf. [**33**]): il existe un entier $c > 0$ tel que $\mathrm{Im}(\rho_E)$ contienne les puissances c-ièmes des homothéties.

10.6? *Supposons* $G_{M(E)}$ *connexe. Alors*:

(a) *Pour* ℓ *assez grand,* $G_{\ell,E}$ *contient le groupe dérivé de* $G_L(\mathbf{Z}_\ell)$.
(b) *Il existe un entier* $n > 0$, *ne dépendant que de* E, *tel que, pour* ℓ *assez grand,* $G_{\ell,E}$ *contienne toutes les puissances* n*-ièmes des éléments de* $G_L(\mathbf{Z}_\ell)$.

(Avec les notations de 11.6, on devrait pouvoir prendre pour n tout entier > 0 tel que Y_H contienne nY.)

Images des représentations $(\mathrm{mod}\,\ell)$. En réduisant $(\mathrm{mod}\,\ell)$ la représentation $\rho_{\ell,E}$ on obtient une action de Γ_k sur $L/\ell L$, d'où un homomorphisme

$$\tilde{\rho}_{\ell,E} : \Gamma_k \to G_L(\mathbf{Z}/\ell\mathbf{Z}) \subset \mathrm{Aut}(L/\ell L).$$

Notons $\widetilde{G}_{\ell,E}$ l'image de cet homomorphisme.
Les conjectures ci-dessus entraînent:

10.7? *L'indice de* $\widetilde{G}_{\ell,E}$ *dans* $G_L(\mathbf{Z}/\ell\mathbf{Z})$ *reste borné quand* ℓ *varie.*

10.8? *Pour* ℓ *assez grand, l'action de* $\widetilde{G}_{\ell,E}$ *sur* $L/\ell L$ *est semi-simple.*

10.9? *Pour* ℓ *assez grand, tout élément de* $L/\ell L$ *invariant par* $\widetilde{G}_{\ell,E}$ *est réduction* $(\mathrm{mod}\,\ell)$ *d'un élément motivé de* L (i.e., d'un élément invariant par $G_{M(E)}$, cf. n°3).

En appliquant 10.9 au motif $E \otimes E^*$, on obtient

10.10? *Pour* ℓ *assez grand, le commutant de* $\widetilde{G}_{\ell,E}$ *dans* $\mathrm{End}(L/\ell L)$ *est la réduction* $(\mathrm{mod}\,\ell)$ *de* $\mathrm{End}(E) \cap \mathrm{End}(L)$; *c'est une algèbre semi-simple sur* $\mathbf{Z}/\ell\mathbf{Z}$.

EXEMPLE. Lorsque E est une variété abélienne, 10.7 n'est pas connue, 10.9 est immédiate, et 10.8 et 10.10 ont été démontrées par Faltings, cf. **[11, 12]**.

Ultraproduits. L'une des façons d'exploiter les représentations $(\mathrm{mod}\,\ell)$, pour ℓ variable, est d'utiliser les ultraproduits des corps $\mathbf{Z}/\ell\mathbf{Z}$, qui ont l'avantage d'être des corps de caractéristique 0. Rappelons comment on procède (cf. e.g. Bourbaki A I.156, exerc.17 d):

On choisit un ultrafiltre non trivial $\mathfrak{U}$ sur l'ensemble P des nombres premiers; on associe à $\mathfrak{U}$ l'idéal $\mathfrak{m}$ de l'anneau produit $R = \prod_{\ell \in P} \mathbf{Z}/\ell\mathbf{Z}$ formé des (x_ℓ) tels que l'ensemble des ℓ tels que $x_\ell = 0$ appartienne à $\mathfrak{U}$. L'idéal $\mathfrak{m}$ est un idéal maximal de R, et le corps quotient $F = R/\mathfrak{m}$ est un corps de caractéristique 0, appelé *l'ultraproduit des* $\mathbf{Z}/\ell\mathbf{Z}$ relativement à $\mathfrak{U}$. Si l'on pose

$$V_F(E) = F \otimes H_\sigma(E),$$

on a $V_F(E) = (\prod_{\ell \in P} L/\ell L)/\mathfrak{m}(\prod_{\ell \in P} L/\ell L)$. On en déduit une action (non continue en général) de Γ_k sur $V_F(E)$, d'où une représentation

$$\rho_{F,E} : \Gamma_k \to \mathbf{GL}(V_F(E)),$$

dont l'image est contenue dans $G_{M(E)}(F)$. Il paraît naturel de conjecturer:

10.11? *Le groupe* $\mathrm{Im}(\rho_{F,E})$ *est dense dans* $G_{M(E)/F}$ *pour la topologie de Zariski.*

En particulier:

10.12? *Le* Γ_k*-module* $V_F(E)$ *est semi-simple et son commutant est* $F \otimes \mathrm{End}(E)$.

11. Images des représentations ℓ-adiques (ℓ variable): suite. Les notations sont celles du n° 10: E est un motif et L est une $\mathbf{Z}$-forme de E.

Le point de vue adélique. On a vu au n° 10 que la famille des $\rho_{\ell,E}$ définit un homomorphisme continu

$$\rho_E = (\rho_{\ell,E}) : \Gamma_k \to \prod_{\ell \in P} G_{\ell,E} \subset \prod_{\ell \in P} G_L(\mathbf{Z}_\ell).$$

Or le produit $\prod_{\ell \in P} G_L(\mathbf{Z}_\ell)$ est un *sous-groupe ouvert compact* du groupe adélique $G_{M(E)}(\mathbf{A}^f)$, où $\mathbf{A}^f = \mathbf{Q} \otimes \widehat{\mathbf{Z}}$ est l'anneau des adèles finis de $\mathbf{Q}$. On peut ainsi voir ρ_E comme une représentation adélique

$$\rho_E : \Gamma_k \to G_{M(E)}(\mathbf{A}^f),$$

et l'on peut se demander si l'image de cette représentation est *ouverte*. Vu la définition de la topologie adélique, cela signifie que $\mathrm{Im}(\rho_E)$ *contient un produit* $\prod_{\ell\in P} U_\ell$, *où* U_ℓ *est un sous-groupe ouvert de* $G_L(\mathbf{Z}_\ell)$, *égal à* $G_L(\mathbf{Z}_\ell)$ *pour tout* ℓ *assez grand*. Ceci ne peut se produire que *si* $G_{M(E)}$ *est connexe*, comme on le voit aisément. Nous ferons désormais cette hypothèse. La propriété pour $\mathrm{Im}(\rho_E)$ d'être ouverte est alors invariante par extension de type fini de k : elle ne dépend que du motif E. Nous allons voir (conjecturalement) dans quel cas elle a lieu.

Motifs maximaux. Puisque $G_{M(E)}$ est supposé connexe, l'homomorphisme $G_M^0 \to G_{M(E)}$ est surjectif. Nous dirons que E est *maximal* si

11.1. *Le noyau de l'homomorphisme* $G_M^0 \to G_{M(E)}$ *est connexe.*

Cela équivaut à la propriété suivante:

11.2. *Si* $G' \to G_{M(E)}$ *est un revêtement connexe non trivial de* $G_{M(E)}$, *l'homomorphisme* $G_M^0 \to G_{M(E)}$ *ne se relève pas en* $G_M^0 \to G'$.

Une autre façon de formuler 11.1 et 11.2 est:

11.3. *Si* k' *est une extension finie de* k, *et si* E' *est un* k'*-motif dominant* E *tel que* $G_{M(E')} \to G_{M(E)}$ *soit un revêtement connexe, alors* $G_{M(E')} \to G_{M(E)}$ *est un isomorphisme* (i.e. on a à la fois $E' \succ E$ et $E \succ E'$).

La formulation 11.3 justifie le terme de "maximal".

Enoncé de la conjecture. C'est le suivant:

11.4? *Supposons* $G_{M(E)}$ *connexe. Les deux propriétés suivantes sont équivalentes*:

(i) E *est maximal*;

(ii) $\mathrm{Im}(\rho_E)$ *est ouvert dans le groupe adélique* $G_{M(E)}(\mathbf{A}^f)$.

L'implication (ii) $\Rightarrow$ (i) ne présente pas de difficultés. En effet, si E n'est pas maximal, on peut supposer (après extension finie de k) qu'il existe E' dominant E tel que $G_{M(E')} \to G_{M(E)}$ soit un revêtement connexe non trivial, cf. 11.3. L'homomorphisme ρ_E se factorise alors en

$$\Gamma_k \to G_{M(E')}(\mathbf{A}^f) \to G_{M(E)}(\mathbf{A}^f).$$

Mais on vérifie facilement que, si $G' \to G$ est une isogénie de degré > 1 de $\mathbf{Q}$-groupes réductifs connexes, *l'image de* $G'(\mathbf{A}^f)$ *dans* $G(\mathbf{A}^f)$ *a un intérieur vide*. En appliquant ce résultat à $G = G_{M(E)}$ et $G' = G_{M(E')}$, on en déduit bien que $\mathrm{Im}(\rho_E)$ n'est pas ouvert.

La véritable *conjecture* est donc l'implication (i) $\Rightarrow$ (ii). C'est une conjecture particulièrement optimiste: on peut montrer qu'elle entraîne toutes celles du n° 10.

Critère de maximalité. Disons que E est *H-maximal* si l'homomorphisme de Hodge

$$h_E : \mathbf{G}_m \times \mathbf{G}_m \to G_{M(E)/\mathbf{C}} \qquad \text{(cf. n° 3)}$$

ne peut se relever à aucun revêtement connexe non trivial $G' \to G_{M(E)}$. Il est clair que "H-maximal" entraîne "maximal" et la réciproque serait vraie si la réponse à la question 8.2 était "oui". La conjecture 11.4 entraîne donc (cf. [**30,** C.3.8]):

11.5? *Si E est H-maximal,* $\mathrm{Im}(\rho_E)$ *est ouvert dans le groupe adélique* $G_{M(E)}(\mathbf{A}^f)$.

Or la H-maximalité est facile à tester lorsque l'on connaît $G_{M(E)}$ et h_E. On peut par exemple utiliser le groupe Y des cocaractères du "tore canonique" de $G_{M(E)}$ sur $\mathbf{Q}$; cf. [**31,** n°31]. Ce groupe est un $\mathbf{Z}$-module libre de type fini; il contient l'ensemble $R^\vee$ des coracines; il est muni d'une action du groupe de Weyl W et d'une action de $\Gamma_\mathbf{Q}$ (*loc. cit.*) L'homomorphisme de Hodge donne un couple d'éléments (y_p, y_q) de Y, défini à W-conjugaison près. La somme de y_p et y_q est invariante par W et par $\Gamma_\mathbf{Q}$; elle correspond à $\mathbf{w}: \mathbf{G}_m \to G_{M(E)}$. Si $c \in \Gamma_\mathbf{Q}$ est la conjugaison complexe, on a $cy_p \in Wy_q$. Soit Y_H le sous-groupe de Y engendré par $R^\vee$ et par $\Gamma_\mathbf{Q} W y_p$, sous-groupe qui est stable par l'action de W et de $\Gamma_\mathbf{Q}$.

11.6? *L'indice de Y_H dans Y est fini.*

Sinon, il existerait un $\mathbf{Q}$-sous-groupe normal propre de $G_{M(E)}$ contenant le tore de Hodge $\mathrm{Im}(h_E)$, ce qui est impossible d'après 3.4 (pour un argument analogue, cf. [**31,** n°3.2, Lemme 3b]).

Les revêtements connexes $G' \to G_{M(E)}$ auxquels h_E se relève correspondent aux sous-$\Gamma_\mathbf{Q}$-modules Y' de Y contenant Y_H (noter que l'action de W sur Y/Y_H est triviale, cf. Bourbaki LIE VI, §1, prop. 27). En particulier:

11.7? *Le motif E est H-maximal si et seulement si Y_H est égal à Y.*

On voit également que, parmi tous les revêtements connexes $G' \to G_{M(E)}$ auxquels h_E se relève, il y en a un qui est plus grand que tous les autres, à savoir celui correspondant à $Y' = Y_H$. On en déduit:

11.8? *Après extension finie de k, il existe un motif E' dominant E, qui est maximal et tel que $G_{M(E')} \to G_{M(E)}$ soit un revêtement connexe.*

Il revient au même de dire que *le noyau de $G_M^0 \to G_{M(E)}$ n'a qu'un nombre finie de composantes connexes*.

Exemples.

11.9. Prenons pour E la puissance tensorielle n-ième $T(n)$ du motif de Tate, où n est un entier $\neq 0$. On a $G_{M(E)} = \mathbf{G}_m$, $Y = \mathbf{Z}$, $R^\vee = \varnothing$,

$y_p = y_q = -n$, et $Y_H = nY$. On en conclut que

$$T(n) \text{ est } H\text{-maximal} \Leftrightarrow n = \pm 1 \Leftrightarrow T(n) \text{ est maximal},$$

auquel cas 11.4 se vérifie immédiatement (irréductibilité des polynômes cyclotomiques).

11.10. Soit E le motif de Ramanujan. On a $G_{M(E)} = \mathbf{GL}_2$, $Y = \mathbf{Z} \times \mathbf{Z}$, $y_p = (-11, 0)$, $y_q = (0, -11)$; les deux coracines sont $(-1, 1)$ et $(1, -1)$; le groupe $\Gamma_{\mathbf{Q}}$ agit trivialement sur Y, et le groupe W agit en permutant les deux facteurs. Le groupe Y_H est formé des couples (y, y') tels que $y + y' \equiv 0 \pmod{11}$; il est d'indice 11 dans Y. Il en résulte que E *n'est pas H-maximal*. Il n'est d'ailleurs pas maximal non plus. En effet, si l'on pose $E' = E \oplus T$, où T est le motif de Tate (cf. 4.6), on constate que E' est H-maximal et que $G_{M(E')} \to G_{M(E)}$ est une isogénie de degré 11. Ici encore, il est possible de démontrer 11.4, cf. [**29**, n°3.1].

11.11. Prenons pour E une *variété abélienne* de dimension $n \geq 1$, avec n impair (ou $n = 2$, ou $n = 6$), et supposons que l'anneau des $\overline{k}$-endomorphismes de E soit réduit à $\mathbf{Z}$. On a alors $G_{M(E)} = \mathbf{GSp}_{2n}$, cf. n° 4.5, et l'on vérifie facilement que E est H-maximal, donc maximal. La conjecture 11.4 est vraie pour un tel motif: le groupe $\mathrm{Im}(\rho_E)$ est *ouvert* dans le groupe adélique $\mathbf{GSp}_{2n}(\mathbf{A}^f)$. Ce résultat est énoncé (sans démonstration) dans [**33**] en supposant que k est un corps de nombres; le cas général s'en déduit par un argument de spécialisation.

12. Eléments de Frobenius. Dans ce n°, ainsi que dans le suivant, on suppose que k est un *corps de nombres algébriques*, autrement dit une extension finie de $\mathbf{Q}$.

On note E un k-motif.

Places non ramifiées. Soit v une place non archimédienne de k. Notons $k(v)$ son corps résiduel, p_v sa caractéristique résiduelle, et Nv le nombre d'éléments de $k(v)$. Soit w un prolongement de v à $\overline{k}$.

Soit ℓ un nombre premier, et soit $\rho_{\ell,E} : \Gamma_K \to G_{M(E)}(\mathbf{Q}_\ell)$ la représentation ℓ-adique correspondante. Soit $D_{\ell,E,w}$ (resp. $I_{\ell,E,w}$) l'image par $\rho_{\ell,E}$ du *groupe de décomposition* (resp. *d'inertie*) de w.

On a

$$I_{\ell,E,w} \subset D_{\ell,E,w} \subset G_{\ell,E} = \mathrm{Im}(\rho_{\ell,E}). \tag{12.1}$$

A conjugaison près, ces groupes sont indépendants du choix de la place w prolongeant v. On dit que $\rho_{\ell,E}$ est *non ramifiée en* v si $I_{\ell,E,w} = 1$. On conjecture:

12.2? *S'il existe un nombre premier $\ell \neq p_v$ tel que $\rho_{\ell,E}$ soit non ramifiée en v, ceci est vrai pour tout $\ell \neq p_v$* (et, pour $\ell = p_v$, $\rho_{\ell,E}$ est *admissible* en w, au sens de Fontaine [**13**]).

Lorsque c'est le cas, on dit que E a *bonne réduction* en v ; cela se produit pour toutes les places v sauf un nombre fini.

EXEMPLES. Le motif de Tate et le motif de Ramanujan ont bonne réduction partout. Le motif défini par une variété abélienne A a bonne réduction en v si et seulement si A a bonne réduction en v (critère de Néron-Ogg-Shafarevich, cf. **[36]** et **[15,** exposé IX, §5]).

Eléments de Frobenius. Supposons que E ait bonne réduction en v. Soit ℓ un nombre premier $\neq p_v$. Le groupe $D_{\ell,E,w}$ est topologiquement engendré par *l'élément de Frobenius* $F_{\ell,E,w}$; la classe de conjugaison de $F_{\ell,E,w}$ dans $G_{\ell,E}$ ne dépend que de v.

REMARQUE. Précisons que $F_{\ell,E,w}$ est l'élément de Frobenius "arithmétique", inverse de celui dit "géométrique". Ainsi, si E est le motif de Tate, $F_{\ell,E,w}$ est égal à Nv, vu comme élément de $\mathbf{Q}_\ell^* = G_{M(E)}(\mathbf{Q}_\ell)$.

D'après le théorème de densité de Chebotarev, les $F_{\ell,E,w}$ sont *équirépartis* (et a fortiori *denses*) dans l'espace des classes de conjugaison de $G_{\ell,E}$. Autrement dit:

12.3? *Soit μ la mesure de Haar de $G_{\ell,E}$, normalisée de telle sorte que sa masse totale soit égale à* 1. *Si U est une partie ouverte et fermée de $G_{\ell,E}$, stable par conjugaison, l'ensemble des v tels que $F_{\ell,E,w}$ appartienne à U a une densité égale à $\mu(U)$.*

(Le même énoncé vaut, plus généralement, pour toute partie U de $G_{\ell,E}$, stable par conjugaison, dont la frontière est de mesure nulle pour μ.)

On conjecture:

12.4? *L'élément de Frobenius $F_{\ell,E,w}$ est semi-simple* (comme automorphisme de $V_\ell(E)$).

12.5? *Le polynôme caractéristique de $F_{\ell,E,w}$ est à coefficients dans* $\mathbf{Q}$; *il ne dépend pas de* ℓ (pourvu que $\ell \neq p_v$); *si E est pur de poids i, ses racines dans $\overline{\mathbf{Q}}$ sont de valeur absolue $Nv^{-i/2}$* (ce sont des "Nv-nombres de Weil" de poids $-i$).

EXEMPLE. D'après Weil, 12.4 et 12.5 sont vrais pour les motifs des variétés abéliennes, i.e., pour l'homologie (ou la cohomologie) de dimension 1. En dimension supérieure, on sait peu de chose sur 12.4; la situation est meilleure pour 12.5: grâce à Deligne (cf. **[4, 5]**), on sait que 12.5 est vraie pour la cohomologie de dimension i quelconque d'une variété propre et lisse ayant bonne réduction en v.

La variété des classes de conjugaison de $G_{M(E)}$. Si G est un **Q**-groupe algébrique réductif, d'algèbre affine A, on peut faire agir G sur A par automorphismes intérieurs. Notons A^G la sous-algèbre de A formée des éléments fixés par G; c'est l'algèbre des *fonctions centrales* sur G. La **Q**-

variété $\operatorname{Spec} A^G$ sera notée $\mathrm{Cl}\, G$. On l'appelle la *variété des classes de conjugaison* de G; si Ω est une extension algébriquement close de $\mathbf{Q}$, les points de $\mathrm{Cl}\, G$ à valeurs dans Ω correspondent bijectivement aux *classes de conjugaison semi-simples* de $G(\Omega)$. La dimension de $\mathrm{Cl}\, G$ est égale au *rang* de G, c'est-à-dire à la dimension d'un tore maximal de G.

Appliquons ceci à $G = G_{M(E)}$. Si E a bonne réduction en v, et si $\ell \neq p_v$, l'élément de Frobenius $F_{\ell,E,w}$ définit un élément $F_{\ell,E,v}$ de $\mathrm{Cl}\, G_{M(E)}(\mathbf{Q}_\ell)$ ne dépendant que de v. On déduit de 12.5 (appliqué aux divers motifs de $M(E)$):

12.6? *L'élément* $F_{\ell,E,v}$ *de* $\mathrm{Cl}\, G_{M(E)}(\mathbf{Q}_\ell)$ *est rationnel sur* $\mathbf{Q}$, *et est indépendant de* ℓ.

Cet élément sera noté $F_{E,v}$; il appartient à $\mathrm{Cl}\, G_{M(E)}(\mathbf{Q})$.

REMARQUE. La variété $\mathrm{Cl}\, G_{M(E)}$ ne change pas par torsion intérieure de $G_{M(E)}$; à isomorphisme canonique près, elle ne dépend donc pas du plongement σ de k dans $\mathbf{C}$ choisi au début; il en est de même de $F_{E,v}$.

EXEMPLES.

(i) Si E est le *motif de Tate*, on a $\mathrm{Cl}\, G_{M(E)} = G_{M(E)} = \mathbf{G}_m$, et $F_{E,v}$ est égal à Nv, vu comme élément de $\mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^*$.

(ii) Si $k = \mathbf{Q}$, et si E est le *motif de Ramanujan*, on a $G_{M(E)} = \mathbf{GL}_2$. L'algèbre A^G correspondante est $\mathbf{Q}[t, d, d^{-1}]$, où t est la trace et d le déterminant; cela permet d'identifier $\mathrm{Cl}\, \mathbf{GL}_2$ à la variété des polynômes quadratiques $X^2 - tX + d$, où d est inversible. Avec cette identification, la classe de Frobenius $F_{E,p}$ associée à un nombre premier p est égale au couple

$$(t, d) = (\tau(p), p^{11}),$$

où τ est la fonction de Ramanujan, cf. [**29**].

Tores de Frobenius. Avec les notations ci-dessus, soit $H_{\ell,E,w}$ le plus petit sous-groupe algébrique de $G_{M(E)/\mathbf{Q}_\ell}$ qui contienne $F_{\ell,E,w}$. Il résulte de 12.4 que $H_{\ell,E,w}$ est un groupe de type multiplicatif; son groupe des caractères est le groupe $X_{E,v}$ engendré par les valeurs propres de $F_{\ell,E,w}$. D'après 12.5, ce groupe a une **Q**-*structure* naturelle, qui correspond à l'action de $\Gamma_{\mathbf{Q}}$ sur $X_{E,v}$ (c'est l'unique **Q**-structure pour laquelle l'élément de Frobenius est rationnel). A isomorphisme près, ce groupe ne dépend, ni du choix de ℓ, ni du choix de w; nous le noterons $H_{E,v}$. Sa composante neutre $T_{E,v} = (H_{E,v})^0$ est un tore, le *tore de Frobenius* de v; on a $H_{E,v} = T_{E,v}$ si et seulement si $X_{E,v}$ ne contient aucune racine de l'unité $\neq 1$. On peut considérer $H_{E,v}$ et $T_{E,v}$ comme des sous-groupes de $G_{M(E)}$, définis sur $\overline{\mathbf{Q}}$, à conjugaison près par $G_{M(E)}(\overline{\mathbf{Q}})$. Les conjectures faites plus haut entraînent:

12.7? *On a* $\mathbf{w}(\mathbf{G}_m) \subset T_{E,v}$.

12.8? *Lorsque* v *varie, les groupes* $H_{E,v}$ *et* $T_{E,v}$ *sont en nombre fini, modulo conjugaison par* $G_{M(E)}(\overline{\mathbf{Q}})$.

On peut se demander dans quel cas $H_{E,v}$ (ou $T_{E,v}$) est un *tore maximal* de $G_{M(E)}$. Cela se produit souvent. Plus précisément:

12.9? *L'ensemble des* v *tels que* $H_{E,v}$ *soit un tore maximal de* $G_{M(E)}$ (auquel cas $T_{E,v} = H_{E,v}$) *est un ensemble infini de densité* $1/c_E$, *où* c_E *est l'ordre du groupe* $G_{M(E)}/G^0_{M(E)}$.

En particulier, cet ensemble est de densité 1 si et seulement si $G_{M(E)}$ est connexe.

Exemple. Si E est une courbe elliptique, ses tores de Frobenius sont de deux types:

(i) $T_{E,v}$ est un tore de dimension 2 (tore maximal) si la réduction de E en v est ordinaire;

(ii) $T_{E,v} = \mathbf{G}_m$ (tore des homothéties, cf. 12.7) si la réduction de E en v est supersingulière.

On a $H_{E,v} = T_{E,v}$ dans le cas (i), et $(H_{E,v} : T_{E,v}) = 1, 2, 3, 4$ ou 6 dans le cas (ii).

Remarque. Supposons satisfaite la propriété suivante:

12.10. Il existe une $\mathbf{Z}$-forme L de E (cf. n° 10) et un nombre premier ℓ tels que l'action de Γ_k sur $L/\ell L$ (resp. sur $L/4L$ si $\ell = 2$) soit triviale.

On peut alors montrer que l'on a $H_{E,v} = T_{E,v}$ pour tout v, et que le groupe $G_{M(E)}$ est connexe (la démonstration est la même que celle de [**2**, prop. 3.6]).

Places ramifiées. Une bonne partie de ce qui précède peut s'étendre aux places v qui sont ramifiées:

Soit v une telle place, et soit $\ell \neq p_v$. Notons $D^{\mathrm{alg}}_{\ell,E,w}$ et $I^{\mathrm{alg}}_{\ell,E,w}$ les adhérences pour la topologie de Zariski des groupes de décomposition et d'inertie $D_{\ell,E,w}$ et $I_{\ell,E,w}$ introduits au début de ce n°. Ce sont des sous-groupes algébriques de $G_{M(E)/\mathbf{Q}_\ell}$. D'après un argument de Grothendieck (unipotence de la monodromie, cf. [**36**, Appendice]), on a:

12.11. *La composante neutre du groupe d'inertie* $I^{\mathrm{alg}}_{\ell,E,w}$ *est un groupe unipotent de dimension* 0 *ou* 1.

Il paraît raisonnable de conjecturer:

12.12? *Le quotient* $D^{\mathrm{alg}}_{\ell,E,w}/I^{\mathrm{alg}}_{\ell,E,w}$ *est un groupe de type multiplicatif; il possède une* $\mathbf{Q}$*-structure et une seule pour laquelle l'image de l'élément de Frobenius de* $D_{\ell,E,w}/I_{\ell,E,w}$ *est rationnelle sur* $\mathbf{Q}$; *le* $\mathbf{Q}$*-groupe algébrique ainsi défini ne dépend pas de* ℓ (pourvu que $\ell \neq p_v$).

Pour énoncer une généralisation de 12.5, il est commode d'introduire la terminologie suivante: soit G un $\mathbf{Q}$-groupe algébrique, soient K_1 et K_2

deux extensions de $\mathbf{Q}$, et soient H_1 et H_2 des sous-groupes algébriques de $G_{/K_1}$ et de $G_{/K_2}$ respectivement. On dira que H_1 est *géométriquement conjugué* à H_2 (relativement à $\mathbf{Q}$) si, pour tout corps algébriquement clos Ω, et tout couple de plongements $K_1 \to \Omega$, $K_2 \to \Omega$, les groupes $H_{1/\Omega}$ et $H_{2/\Omega}$ sont conjugués dans $G_{/\Omega}$.

On peut alors conjecturer:

12.13? *Si ℓ_1 et ℓ_2 sont deux nombres premiers distincts de p_v, les groupes $D^{\mathrm{alg}}_{\ell_1,E,w}$ et $D^{\mathrm{alg}}_{\ell_2,E,w}$ sont géométriquement conjugués; il en est de même pour les groupes d'inertie $I^{\mathrm{alg}}_{\ell_1,E,w}$ et $I^{\mathrm{alg}}_{\ell_2,E,w}$.*

REMARQUES.

1) Le groupe $I^{\mathrm{alg}}_{\ell,E,w}$ est *fini* si et seulement si E a *potentiellement bonne réduction en* v, i.e. acquiert bonne réduction après extension finie de k. Le cas où $I^{\mathrm{alg}}_{\ell,E,w}$ est *connexe* mérite d'être appelé *semi-stable.*
2) Si la condition 12.10 est satisfaite, les groupes $D^{\mathrm{alg}}_{\ell,E,w}$ et $I^{\mathrm{alg}}_{\ell,E,w}$ sont connexes; en particulier, E est semi-stable au sens ci-dessus.

13. Equirépartition des éléments de Frobenius. Les notations sont les mêmes qu'au n° 12: k est un corps de nombres, et E un k-motif.

Le groupe $G^1_{M(E)}$. Soient G^1_M le noyau de l'homomorphisme $\mathbf{t}: G_M \to \mathbf{G}_m$ du n° 5, et $G^1_{M(E)}$ l'image de G^1_M dans $G_{M(E)}$ par la projection $G_M \to G_{M(E)}$. Le groupe $G^1_{M(E)}$ est un $\mathbf{Q}$-sous-groupe normal de $G_{M(E)}$. On a

$$G_{M(E)} = \mathbf{w}(\mathbf{G}_m) \cdot G^1_{M(E)}.$$

Le quotient $G_{M(E)}/G^1_{M(E)}$ est, soit trivial, soit isomorphe à $\mathbf{G}_m$.

Lorsque E domine le motif de Tate, $G^1_{M(E)}$ est égal au noyau de l'homomorphisme

$$\mathbf{t}: G_{M(E)} \to \mathbf{G}_m, \qquad \text{cf. n° 5}.$$

EXEMPLES.

(i) Si E est un motif d'Artin, ou un motif tel que $G_{M(E)}$ soit semi-simple, on a $G^1_{M(E)} = G_{M(E)}$.
(ii) Si E est une courbe elliptique sans multiplications complexes (cf. 4.3), on a $G_{M(E)} \cong \mathbf{GL}_2$ et $G^1_{M(E)} \cong \mathbf{SL}_2$. Même chose pour le motif de Ramanujan.

Equirépartition. Soit Σ l'ensemble des places de k où E n'a pas bonne réduction. C'est un ensemble fini (cf. n° 12). Si $v \notin \Sigma$ (et si l'on accepte les conjecture déjà faites), on peut parler de la *classe de Frobenius* de v, qui est un élément $F_{E,v}$ de $G_{M(E)}(\mathbf{C})$, défini à conjugaison près, et dont la classe de conjugaison est rationnelle sur $\mathbf{Q}$, cf. 12.5. On peut multiplier cet élément par $\mathbf{w}(Nv^{1/2})$, qui appartient au centre de $G_{M(E)}(\mathbf{R})$. On obtient ainsi

$$\varphi_{E,v} = \mathbf{w}(Nv^{1/2}) \cdot F_{E,v}. \tag{13.1}$$

Cette normalisation "analytique" de la classe de Frobenius a l'avantage que l'on a

$$\varphi_{E,v} \in G^1_{M(E)}(\mathbf{C}), \tag{13.2}$$

du fait que $\mathbf{tw}(Nv^{1/2}) = Nv^{-1}$.

Comme l'application $\mathrm{Cl}\, G^1_{M(E)} \to \mathrm{Cl}\, G_{M(E)}$ est injective, la classe de $\varphi_{E,v}$ dans $G^1_{M(E)}(\mathbf{C})$ est bien définie. (Noter que cette classe n'est pas rationnelle sur $\mathbf{Q}$ en général; elle est seulement rationnelle sur $\mathbf{Q}(p_v^{1/2})$.)

Soit maintenant K un *sous-groupe compact maximal* de $G^1_{M(E)}(\mathbf{C})$, ou, ce qui revient au même, une *forme réelle compacte de* $G^1_{M(E)}$ au sens de Deligne [**6**, p. 255]. Soit $\mathrm{Cl}\, K$ l'espace des classes de conjugaison de K; c'est un sous-espace de $\mathrm{Cl}\, G^1_{M(E)}(\mathbf{C})$. On vérifie facilement:

13.3. *Soit $x \in G^1_{M(E)}(\mathbf{C})$. Supposons x semi-simple. Pour que la classe de conjugaison de x appartienne à $\mathrm{Cl}\, K$, il faut et il suffit que, pour toute représentation linéaire r de $G^1_{M(E)}$, les valeurs propres de $r(x)$ soient de module* 1.

(On peut prendre r définie sur $\mathbf{C}$, ou sur $\mathbf{Q}$: cela revient au même.)

En appliquant ceci à $\varphi_{E,v}$, et en utilisant 12.5, on en déduit:

13.4? *La classe de conjugaison de $\varphi_{E,v}$ appartient à $\mathrm{Cl}\, K$.*

Munissons $\mathrm{Cl}\, K$ de la mesure μ image par la projection $K \to \mathrm{Cl}\, K$ de la *mesure de Haar normalisée* du groupe compact K. La *conjecture d'équirépartition* (à la Sato-Tate) des classes de Frobenius s'énonce de la façon suivante:

13.5? *Les classes des $\varphi_{E,v}$ sont équiréparties dans $\mathrm{Cl}\, K$ pour la mesure μ.*

(Précisons que l'on ordonne les places de K non dans Σ de telle sorte que $v \mapsto Nv$ soit une fonction croissante.)

De même que dans le cas ℓ-adique (cf. 12.3), cet énoncé équivaut à:

13.6? *Soit U une partie de K, stable par conjugaison, et dont la frontière est de mesure nulle pour μ. Alors l'ensemble des v tels que $\varphi_{E,v}$ appartienne à U a une densité égale à $\mu(U)$.*

Des arguments standard (cf. e.g. [**27**, chap. I, App.]) permettent de transformer 13.4 et 13.5 en les deux formes équivalentes suivantes:

13.7? *Pour tout fonction continue centrale* f *sur* K, *on a*

$$\sum_{Nv \le X} f(\varphi_{E,v}) = \mu(f)X/\log X + o(X/\log X) \quad \textit{pour } X \to \infty,$$

où $\mu(f)$ *est l'intégrale de* f *pour* μ.

13.8? *Pour toute représentation linéaire complexe irréductible* r *de* $G^1_{M(E)}$, *distincte de la représentation unité, on a*

$$\sum_{Nv \le X} \operatorname{Tr} r(\varphi_{E,v}) = o(X/\log X) \quad \textit{pour } X \to \infty.$$

(Précisons que, dans cette formule comme dans celle de 13.7, la sommation porte sur les places v de k de norme $\le X$ *n'appartenant pas à l'ensemble fini* Σ.)

Relations avec les fonctions L. La conjecture 13.8 devrait résulter d'un énoncé plus précis, relatif à la fonction L_r définie par

$$L_r(s) = \prod_{v \notin \Sigma} 1/\det(1 - r(\varphi_{E,v})Nv^{-s}). \tag{13.9}$$

D'après 13.4, ce produit converge absolument pour $\mathrm{Re}(s) > 1$. La conjecture suivante entraîne 13.8 (cf. **[40]** et **[27]**, *loc. cit.*):

13.10? *La fonction* $L_r(s)$ *se prolonge en une fonction méromorphe dans tout le plan complexe, d'ordre* ≤ 1; *elle est holomorphe et* $\neq 0$ *sur la droite* $\mathrm{Re}(s) = 1$.

(Rappelons que la représentation irréductible r est supposée $\neq 1$.)

Pour plus de détails sur les fonctions L_r (facteurs locaux aux mauvaises places, termes exponentiels, facteurs gamma, équation fonctionnelle, relations avec la théorie des représentations, holomorphie, hypothèse de Riemann...), voir par exemple **[3, 7, 20, 21, 28, 37, 40]**.

Exemples.

13.11. Les conjectures 13.5 à 13.10 ci-dessus sont démontrées lorsque E est un *motif d'Artin* (grâce à Chebotarev et Artin), ou un *motif de type* CM (grâce à Hecke).

13.12. Soit $k = \mathbf{Q}$. Choisissons pour E le *motif de Ramanujan.* On a $\Sigma = \varnothing$, $G^1_{M(E)} \cong \mathbf{SL}_2$ et $K \cong \mathbf{SU}_2(\mathbf{C})$. Si p est un nombre premier, et si l'on écrit $\tau(p)$ sous la forme

$$\tau(p) = 2p^{11/2}\cos(\alpha_p), \qquad \text{avec } 0 \le \alpha_p \le \pi,$$

on peut prendre pour représentant de $\varphi_{E,p}$ la matrice unitaire

$$\begin{pmatrix} e^{i\alpha_p} & 0 \\ 0 & e^{-i\alpha_p} \end{pmatrix},$$

qui est de déterminant 1, et dont la trace est $a_p = \tau(p)/p^{11/2} = 2\cos(\alpha_p)$. L'espace $\mathrm{Cl}\,K$ s'identifie à l'intervalle $[0, \pi]$, la mesure μ étant la *mesure de Sato-Tate* $\frac{2}{\pi}\sin^2\alpha\, d\alpha$. La conjecture 13.5 dit que *les* α_p *sont équirépartis sur* $[0, \pi]$ *pour cette mesure.* En utilisant 13.8, on peut reformuler ceci en termes des "moments" de la suite des a_p :

13.13? *Pour tout entier* $n \geq 1$, *on a*

$$\sum_{p\leq X}(a_p)^n = c_n X/\log X + o(X/\log X) \quad \textit{pour } X \to \infty,$$

où c_n *est donné par*

$$c_n = \begin{cases} 0 & \textit{si } n \textit{ est impair} \\ (m+2)(m+3)\cdots(m+m)/m! & \textit{si } n = 2m. \end{cases}$$

(Les premières valeurs des c_n sont: 0,1,0,2,0,5,0,14,0,42,... .)

On sait démontrer 13.13 pour $n = 1, 2, 3, 4$ et il y a un résultat partiel pour $n = 5$, cf. Shahidi [**37**].

La situation est la même pour les courbes elliptiques sur $\mathbf{Q}$ (ce qui est le cas envisagé initialement par Sato et Tate, cf. [**40**]), pourvu que l'on sache que ces courbes sont "modulaires".

Bibliographie

1. G. Anderson, *Cyclotomy and an extension of the Taniyama group*, Comp. Math. **57** (1986), 153–217.
2. W. Chi, *ℓ-adic and λ-adic representations associated to abelian varieties defined over number fields*, Amer. J. Math. **114** (1992), 315–354.
3. L. Clozel, *Motifs et formes automorphes: Applications du principe de fonctorialité*, Automorphic Forms, Shimura Varieties and L-functions, vol. I, 77–159, Academic Press, 1990.
4. P. Deligne, *La conjecture de Weil* I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.
5. ——, *La conjecture de Weil* II, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252.
6. ——, *Variétés de Shimura: Interprétation modulaire, et techniques de construction de modèles canoniques*, Proc. Sympos. Pure Math. **33** (1979), vol. 2, 247–289.
7. ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math. **33** (1979), vol. 2, 313–346.
8. ——, *Hodge cycles on abelian varieties* (notes by J. S. Milne), Lect. Notes in Math. **900** (1982), 9–100.
9. ——, *Motifs et groupe de Taniyama*, Lect. Notes in Math. **900** (1982), 261–279.
10. P. Deligne et J. S. Milne, *Tannakian categories*, Lect. Notes in Math. **900** (1982), 101–228.
11. G. Faltings, *Endlichkeitssätze für abelschen Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366; *Erratum*, ibid. **75** (1984), 381.
12. G. Faltings, G. Wüstholz et al, *Rational Points*, Vieweg, 1984.
13. J.-M. Fontaine et W. Messing, *p-adic periods and p-adic étale cohomology*, Current Trends in Arithmetical Algebraic Geometry, Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 179–207.
14. A. Grothendieck, *Standard conjectures on algebraic cycles*, Bombay Colloquium on Algebraic Geometry, Oxford, 1969, pp. 193–199.

15. ——, *Groupes de Monodromie en Géométrie Algébrique* (SGA 7 I), Lect. Notes in Math. **288** (1972).
16. U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), 447–452.
17. S. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix exposés sur la cohomologie des schémas, North-Holland, 1968, pp. 359–386.
18. ——, *Motives*, Algebraic Geometry, Wolters-Noordhoff, Groningen, 1972, pp. 53–82.
19. S. Lang, *Division points on curves*, Ann. Mat. Pura Appl. **70** (1965), 229–234.
20. R. P. Langlands, *Euler Products*, notes polycopiées, Yale Univ., 1967.
21. ——, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, Proc. Sympos. Pure Math. **33** (1979), vol. 2, 205–246.
22. J. S. Milne et K-y. Shih, *Langlands's construction of the Taniyama group*, Lect. Notes in Math. **900** (1982), 229–260.
23. D. Mumford, *Families of abelian varieties*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. **9** (1966), 347–351.
24. ——, *A note of* (sic) *Shimura's paper "Discontinuous groups and abelian varieties"*, Math. Ann. **181** (1969), 345–351.
25. N. Saavedra Rivano, *Catégories Tannakiennes*, Lect. Notes in Math. **265** (1972).
26. N. Schappacher, *Periods of Hecke characters*, Lect. Notes in Math. **1301** (1988).
27. J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, Benjamin, New York, 1968; seconde édition, Addison-Wesley, Redwood City, 1989.
28. ——, *Facteurs locaux des fonctions zêta des variétés algébriques* (*définitions et conjectures*), Sém. DPP 1969–1970, exposé **19** (= Oe.87).
29. ——, *Congruences et formes modulaires* (*d'après H. P. F. Swinnerton-Dyer*), Sém. Bourbaki 1971/72, n° 416, Lect. Notes in Math. **317** (1973) (= Oe.95).
30. ——, *Représentations ℓ-adiques*, Kyoto Symposium on Number Theory, 1977, pp. 177–193 (= Oe.112),
31. ——, *Groupes algébriques associés aux modules de Hodge-Tate*, Astérisque **65** (1979), 155–188 (= Oe.119).
32. ——, *Résumés des cours au Collège de France*, Annuaire du Collège de France (1984–1985), 85–90.
33. ——, *Résumés des cours au Collège de France*, Annuaire du Collège de France (1985–1986), 95–99.
34. ——, *Lectures on the Mordell-Weil Theorem*, Vieweg, 1989.
35. ——, *Motifs*, Astérisque **198-199-200** (1992), 333–349.
36. J.-P. Serre et J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517 (= J.-P. Serre, Oe.79).
37. F. Shahidi, *Automorphic L-functions, A Survey*, Automorphic Forms, Shimura Varieties and L-Functions, vol. I, Acad. Press, 1990, pp. 415–437.
38. G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Ann. of Math. **78** (1963), 149–192.
39. ——, *Moduli and fibre systems of abelian varieties*, Ann. of Math. **83** (1966), 294–338.
40. J. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry, Harper and Row, New York, 1965, pp. 93–110.
41. J.-P. Wintenberger, *Torseur entre cohomologie étale p-adique et cohomologie cristalline: le cas abélien*, Duke Math. J. **62** (1991), 511–526.
42. ——, *Torseurs pour les motifs et pour les représentations p-adiques potentiellement de type CM*, Math. Ann. **288** (1990), 1–8.

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Motives over Finite Fields

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ABSTRACT. The category of motives over the algebraic closure of a finite field is known to be a semisimple $\mathbb{Q}$-linear Tannakian category, but unless one assumes the Tate conjecture there is little further one can say about it. However, once this conjecture is assumed, it is possible to give an almost entirely satisfactory description of the category together with its standard fibre functors. In particular, it is possible to list properties of the category that characterize it up to equivalence and to prove (without assuming any conjectures) that there does exist a category with these properties. The Hodge conjecture implies that there is a functor from the category of CM-motives over $\mathbb{Q}^{\text{al}}$ to the category of motives over $\mathbb{F}$. We construct such a functor.

Contents

Introduction

After sketching the construction of the category of motives over a finite field or its algebraic closure in §1, we develop the basic properties of the categories in §2 (under the assumption of the Tate conjecture). In particular we classify the simple objects up to isomorphism and compute their

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endomorphism algebras. We show that the category of motives over $\mathbb{F}$ has exactly two polarizations.

In §3, we list properties of the category of motives over $\mathbb{F}$ together with the structure provided by the Frobenius automorphisms sufficient to characterize it uniquely up to equivalence, and we show (without any assumptions) that there does exist a category with these properties. We also prove a similar result for the category together with its standard fibre functors.

There is one other category of motives for which there is a similarly explicit description, namely, the category of CM-motives over $\mathbb{Q}^{\mathrm{al}}$. Conjecturally, reduction modulo p defines a tensor functor from this Tannakian category to that of motives over $\mathbb{F}$. We construct such a reduction functor (assuming the Tate conjecture).

Beyond its intrinsic interest, the study of motives over finite fields gives a beautiful illustration of the power of the Tannakian formalism in a nonelementary (i.e., nonneutral) case. Also the theory of motives over $\mathbb{F}$ provides the philosophical underpinning for the conjecture of Langlands and Rapoport describing the points on the reduction of a Shimura variety to characteristic p, which is the starting point of Langlands's program to realize the zeta functions of such varieties as automorphic L-series.

Some philosophy. Since we shall be describing a category with varying degrees of definiteness, we discuss what this means.

Consider first an object X of a category. When we say that X (possibly plus additional data) is determined by a property P we may mean one of several things:

(a) The object X (plus data) is uniquely determined by P, i.e., X is the only object (plus data) satisfying P.

(b) The object X (plus data) is uniquely determined by P up to a unique isomorphism, i.e., if Y (plus data) is a second object satisfying P, then there is a unique isomorphism between X and Y (respecting the data) and any morphism from one to the other (respecting the data) is an isomorphism.

(c) The object X (plus data) is uniquely determined by P up to isomorphism, i.e., if Y (plus data) also satisfies P, then there exists an isomorphism between X and Y respecting the data, and any morphism from one to the other (respecting the data) is an isomorphism.

For example, the algebraic closure of a field is determined in the sense (c), whereas an object plus the data of a morphism is determined by a universal property in the sense (b). For all intents and purposes, (b) is as good as (a)—for example, it allows us to speak of a specific element of X—but (c) is much weaker.

Similarly, when we say that a category $\mathbf{C}$ (plus data) is determined by a property P we may mean one of several things:

(a) The category **C** (plus data) is uniquely determined by P.
(b) The category **C** (plus data) is uniquely determined by P up to a unique equivalence (respecting the data).
(c) The category **C** (plus data) is uniquely determined by P up to an equivalence (respecting the data) which itself is uniquely determined up to a unique isomorphism (respecting the data).
(d) The category **C** (plus data) is uniquely determined by P up to an equivalence (respecting the data) which is uniquely determined up to isomorphism (respecting the data).
(e) The category **C** (plus data) is uniquely determined by P up to an equivalence (respecting the data).

For example, a Tannakian category is determined by its gerb of fibre functors in the sense (b). For all intents and purposes, (c) is as good as (b) and (a)—for example, it allows us to speak of a specific object of **C**—but (d) is a little weaker than (c), and (e) is much weaker than (d).

Acknowledgments. The notes at the end of each section discuss sources. In addition, it should be mentioned that much of the content of this article was probably known to Grothendieck in the sixties. It is a pleasure to thank Deligne for his help with the article.

Notations

Throughout, $\mathbb{F}$ is an algebraic closure of the field $\mathbb{F}_p$, and $\mathbb{F}_q$ is the subfield of $\mathbb{F}$ with q elements. The letter ℓ denotes a prime of $\mathbb{Q}$, possibly p or ∞. The symbol k^{al} denotes an algebraic closure of a field k. For $\mathbb{Q}$, we take $\mathbb{Q}^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Complex conjugation on $\mathbb{C}$ or any subfield is denoted by ι or by $z \mapsto \bar{z}$. We often use $[*]$ to denote an equivalence class containing $*$.

The ring of adèles over $\mathbb{Q}$ is denoted by $\mathbb{A}$; a subscript f on $\mathbb{A}$ indicates that the infinite component has been omitted, and a superscript p indicates that the component at p has been omitted.

For a prime w of a number field K, $\|\cdot\|_w$ denotes the normalized valuation at w.

An algebraic variety over a field k is a geometrically reduced scheme of finite type (not necessarily connected) over k. When V is an algebraic variety over $\mathbb{F}_q$, π_V denotes the Frobenius automorphism of V relative to $\mathbb{F}_q$: it acts as the identity map on the underlying set of V, and it acts as $f \mapsto f^q$ on $\mathcal{O}_V$.

By a k-linear tensor category we mean a k-linear category **T** together with a k-bilinear functor $\otimes: \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ and sufficient constraints so that the tensor product of any (unordered) set of objects of **T** is well defined up to a canonical isomorphism. This means that there is an identity object, an associativity constraint, and a commutativity constraint satisfying certain axioms.

For an abelian category $\mathbf{T}$, $\Sigma(\mathbf{T})$ denotes the set of isomorphism classes of simple objects in $\mathbf{T}$, and $K(\mathbf{T})$ denotes the Grothendieck group of $\mathbf{T}$.

For a category $\mathbf{T}$, $\mathrm{Ind}(\mathbf{T})$ denotes the category of direct systems of objects (X_α) in $\mathbf{T}$ indexed by small directed sets with Hom defined by

$$\mathrm{Hom}((X_\alpha), (Y_\beta)) = \varprojlim_\alpha \varinjlim_\beta \mathrm{Hom}(X_\alpha, Y_\beta).$$

For a perfect field k of characteristic $p \neq 0$, $W(k)$ is the ring of Witt vectors with coefficients in k, and $K(k)$ is the field of fractions of $W(k)$. The Frobenius automorphism $x \mapsto x^p$ of k and its liftings to $W(k)$ and $K(k)$ are denoted by σ.

When K is a finite field extension of k, $(\mathbb{G}_m)_{K/k}$ is the torus over k obtained from $\mathbb{G}_m$ over K by restriction of scalars. We write $\mathbb{S}$ for $(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$. For any affine group scheme G over a field k, $X^*(G)$ denotes the group of characters of G defined over some algebraic closure of k.

When we say that a statement $P(N)$ holds for all $N >> 1$, we mean that it holds for all sufficiently divisible positive integers N, i.e., that there exists an N_0 such that

$$N > 0, \quad N \in \mathbb{N}, \; N_0 | N \implies P(N) \text{ is true.}$$

We use the following notations (see §1 for detailed definitions):

$\mathbf{CV}^0(k)$: category of correspondences of degree 0;

$\mathbf{Hdg}_\mathbb{Q}$: category of polarizable rational Hodge structures;

$\mathbf{Mot}(k)$: category of motives over k;

$\mathbf{Rep}_k(G)$: category of representations of G on finite-dimensional vector spaces over k;

$\mathbf{V}_\infty$: category of graded complex vector spaces with a semilinear endomorphism F such that $F^2 = (-1)^m$ on an object of weight m;

$\mathbf{V}_\ell(\mathbb{F}_q)$: category of semisimple continuous representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on finite-dimensional vector spaces over $\mathbb{Q}_\ell$;

$\mathbf{V}_\ell(\mathbb{F})$: category of germs of semisimple continuous representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on finite-dimensional vector spaces over $\mathbb{Q}_\ell$;

$\mathbf{V}_p(k)$: category of F-isocrystals over k.

1. Construction of the category of motives over a finite field

Algebraic correspondences. Fix a field k. For a smooth projective variety V over k, we define $Z^r(V)$ (*space of algebraic cycles of codimension* r *on* V) to be the $\mathbb{Q}$-vector space with basis the closed irreducible subvarieties of V of codimension r, and we define $A^r(V)$ to be the quotient of $Z^r(V)$ by the subspace of cycles numerically equivalent to zero. When all the irreducible components of V have dimension d and W is a second smooth projective variety over k, the elements of $A^d(V \times W)$ are called *algebraic correspondences from* V *to* W *of degree* 0. For example, the graph of a morphism from W to V defines an algebraic correspondence from V to W of degree zero.

The category $\mathbf{CV}^0(k)$ is constructed as follows: it has one object $h(V)$ for each smooth projective variety V over k, and a morphism from $h(V)$ to $h(W)$ in $\mathbf{CV}^0(k)$ is an algebraic correspondence of degree 0 from V to W. Composition of morphisms is defined by

$$A^{\dim U}(U \times V) \times A^{\dim V}(V \times W) \longrightarrow A^{\dim U}(U \times W),$$

$$(a, b) \mapsto b \circ a \underset{\mathrm{df}}{=} (p_{U \times W})_*(p^*_{U \times V}(a) \cdot p^*_{V \times W}(b)).$$

See Saavedra [**26**, p. 385]. It is an additive $\mathbb{Q}$-linear category, and $V \mapsto h(V)$ is a contravariant functor from the category of smooth projective varieties over k to $\mathbf{CV}^0(k)$. There is a tensor structure on $\mathbf{CV}^0(k)$ for which

$$h(V) \otimes h(W) = h(V \times W)$$

and for which the commutativity and associativity constraints are defined by the obvious isomorphisms

$$V \times W \approx W \times V, \qquad U \times (V \times W) \approx (U \times V) \times W.$$

On adding the images of projectors and inverting the Lefschetz motive, one obtains the false category of motives $\mathbf{M}(k)$ over k (ibid. VI.4). This is a $\mathbb{Q}$-linear tensor category with duals, but it can not be Tannakian: in any tensor category with duals there is a notion of the rank[1] (or dimension) of an object, which is intrinsic, and is therefore preserved by any tensor functor; hence, when a fibre functor exists, the dimension of an object is a positive integer; but the dimension of $h(V)$ in $\mathbf{M}(k)$ is the Euler-Poincaré characteristic $(\Delta \cdot \Delta)$ of V, which is often negative.

The category of motives over a finite field. In order to obtain a Tannakian category, we must define a gradation on $\mathbf{M}(k)$ and use it to modify the commutativity constraint. For a general field it has not been proved that this is possible, but for a finite field we can proceed as follows. Let V be a smooth projective variety of dimension d over $\mathbb{F}_q$, and let π_V be the Frobenius morphism of V over $\mathbb{F}_q$. It follows from the results of Deligne on the Weil conjectures [**4**] that for $i = 0, 1, \ldots, 2d$ there is a well-defined polynomial $P_i(T) \in \mathbb{Q}[T]$ which is the characteristic polynomial of π_V acting on the étale cohomology group $H^i(V \otimes \mathbb{F}, \mathbb{Q}_\ell)$ for any $\ell \neq p, \infty$ (or on the corresponding crystalline cohomology group (Katz and Messing [**17**])). These polynomials are relatively prime because their roots have different absolute values, and the graph of the map $\prod_{i=0}^{2d} P_i(\pi_V)$ is numerically equivalent to zero because it is homologically equivalent to zero for any $\ell \neq p, \infty$. The Chinese remainder

[1] In the notation of the proof of (1.1), the rank of X is $\mathrm{ev}_X \circ \delta$ regarded as an element of k; equivalently, in the notation introduced below, it is the trace of id_X.

theorem shows that there are polynomials $P^i(T) \in \mathbb{Q}[T]$ such that

$$P^i(T) \equiv \begin{cases} 1 & \bmod P_i(T), \\ 0 & \bmod P_j(T) \text{ for } j \neq i. \end{cases}$$

The graph of $p^i \underset{\text{df}}{=} P^i(\pi_V)$ is a well-defined projector in $C^d(V \times V)$, and

$$1 = p^0 + p^1 + \cdots + p^{2d}.$$

There is a unique gradation on $\mathbf{M}(k)$ for which

$$h(V) = \bigoplus h^i(V), \quad h^i(V) = \operatorname{Im}(p^i), \quad \text{all } V.$$

We can now modify the commutativity constraint in $\mathbf{M}(\mathbb{F}_q)$ as follows: write the given commutativity constraint

$$\dot{\psi}_{X,Y}\colon X \otimes Y \to Y \otimes X,$$

as a direct sum,

$$\dot{\psi}_{X,Y} = \bigoplus \dot{\psi}^{r,s}, \qquad \dot{\psi}^{r,s}\colon X^r \otimes Y^s \xrightarrow{\approx} Y^s \otimes X^r,$$

and define

$$\psi_{X,Y} = \bigoplus (-1)^{rs} \dot{\psi}^{r,s}.$$

Now

$$\operatorname{rank} h(V) = \sum h^i(V) \quad (\text{rather than } \sum (-1)^i h^i(V)).$$

Write $\mathbf{Mot}(\mathbb{F}_q)$ for $\mathbf{M}(\mathbb{F}_q)$ with this new commutativity constraint. Its objects are the *motives over* $\mathbb{F}_q$.

PROPOSITION 1.1. *The tensor category* $\mathbf{Mot}(\mathbb{F}_q)$ *is a semisimple Tannakian category over* $\mathbb{Q}$.

PROOF. By construction, it is a pseudo-abelian tensor category, and $\operatorname{End}(1) = \mathbb{Q}$. As is explained in Saavedra [**26**, VI.4.1.3.5], duals exist, i.e., for every object X of $\mathbf{Mot}(\mathbb{F}_q)$, there is an object $X^\vee$ and morphisms $\operatorname{ev}_X\colon X \otimes X^\vee \to 1$ and $\delta\colon 1 \to X^\vee \otimes X$ such that

$$(\operatorname{ev}_X \otimes \operatorname{id}_X) \circ (\operatorname{id}_X \otimes \delta) = \operatorname{id}_X, \qquad (\operatorname{id}_{X^\vee} \otimes \operatorname{ev}_X) \circ (\delta \otimes \operatorname{id}_{X^\vee}) = \operatorname{id}_{X^\vee}.$$

In fact, for an irreducible smooth projective variety V of dimension d, $h(V)^\vee = h(V)(d)$ and $\operatorname{ev}_{h(V)}$ is deduced from

$$h(V) \otimes h(V) = h(V \times V) \xrightarrow{h(\Delta)} h(V) \xrightarrow{p^{2d}} h^{2d}(V) = \mathbb{Q}(-d)$$

by tensoring with $\mathbb{Q}(d)$. Because we have worked with numerical equivalence, Jannsen [**16**, Theorem 1] shows that $\mathbf{Mot}(\mathbb{F}_q)$ is a semisimple abelian category. Finally, because of our modification of the commutativity constraint, the rank of every object of $\mathbf{Mot}(\mathbb{F}_q)$ is a positive integer, and so Deligne [7, Theorem 7.1] shows that $\mathbf{Mot}(\mathbb{F}_q)$ is Tannakian. □

Let $\mathbf{T}$ be a Tannakian category over a field k. A *fibre functor* on $\mathbf{T}$ over a k-algebra R is an exact k-linear tensor functor from $\mathbf{T}$ to the category of R-modules. It automatically takes values in the category of projective R-modules of finite rank and is faithful (unless $R = 0$), and for any $X, Y \in \mathrm{ob}(\mathbf{T})$ the map

$$\mathrm{Hom}(X, Y) \otimes R \longrightarrow \mathrm{Hom}_R(\omega(X), \omega(Y))$$

is injective Deligne [**8**, 2.10, 2.13].

Tate triples. Recall [**9**, 5.1], that to give a $\mathbb{Z}$-gradation on a Tannakian category $\mathbf{T}$ is the same as to give a homomorphism $w\colon \mathbb{G}_m \to \mathit{Aut}^{\otimes}(\mathrm{id}_{\mathbf{T}})$. A *Tate triple* over a field F is a system $(\mathbf{T}, w, T)$ consisting of a Tannakian category $\mathbf{T}$ over F, a $\mathbb{Z}$-gradation on $\mathbf{T}$ (called the *weight gradation*), and an invertible object T (called the *Tate object*) of weight -2. For an object X of $\mathbf{T}$ and an integer n, we set $X(n) = X \otimes T^{\otimes n}$. A *morphism of Tate triples*

$$(\mathbf{T}_1, w_1, T_1) \longrightarrow (\mathbf{T}_2, w_2, T_2)$$

is a morphism of tensor categories $\mathbf{T}_1 \to \mathbf{T}_2$ preserving the gradations together with an isomorphism $\eta(T_1) \to T_2$.

EXAMPLE 1.2. (a) The system $(\mathbf{Mot}(\mathbb{F}_q), w, T)$ with w the gradation defined above and T the dual of the Lefschetz motive, $T = h^2(\mathbb{P}^1)^\vee$, is a Tate triple over $\mathbb{Q}$.

(b) By a *rational Hodge structure* we mean a finite-dimensional vector space V over $\mathbb{Q}$ together with a homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V \otimes \mathbb{R})$ such that the corresponding weight map $w_h \underset{\mathrm{df}}{=} h^{-1}|\mathbb{G}_m$ is defined over $\mathbb{Q}$. The category of rational Hodge structures together with its natural weight gradation and Tate object $\mathbb{Q}(1) \underset{\mathrm{df}}{=} (2\pi i\mathbb{Q}, z \mapsto z\bar{z})$ is a Tate triple over $\mathbb{Q}$.

Extension of coefficients. Let $(\mathbf{T}, \otimes)$ be a tensor category over a field k, and let L be a field containing k. An *L-module* in $\mathbf{T}$ is an object X of $\mathbf{T}$ together with a k-linear homomorphism $L \to \mathrm{End}(X)$. A subobject of X is said to *generate* (X, i) if it is not contained in any proper L-submodule of X.

Now assume $\mathbf{T}$ to be Tannakian, and consider the category $\mathrm{Ind}(\mathbf{T})$ of small filtered direct systems of objects in $\mathbf{T}$. Identify $\mathbf{T}$ with a full subcategory of $\mathrm{Ind}(\mathbf{T})$, and define $\mathbf{T} \otimes L$ to be the category whose objects are the L-modules in $\mathrm{Ind}(\mathbf{T})$ generated by objects in $\mathbf{T}$.

PROPERTIES.

(1.3.1) The category $\mathbf{T} \otimes L$ has a natural tensor structure for which it is a Tannakian category over L.

(1.3.2) There is a canonical tensor functor

$$X \mapsto X \otimes L\colon \mathbf{T} \longrightarrow \mathbf{T} \otimes L$$

having the property that

$$\mathrm{Hom}(X, Y) \otimes L = \mathrm{Hom}(X \otimes L, Y \otimes L).$$

This functor is faithful, and when k has characteristic zero and $\mathbf{T}$ is semisimple, $\mathbf{T} \otimes L$ is the pseudo-abelian envelope of its image.

(1.3.3) A fibre functor ω of $\mathbf{T}$ over R extends uniquely to a fibre functor $\omega \otimes L$ of $\mathbf{T} \otimes L$ over $R \otimes_k L$ such that $(\omega \otimes L)(X) = \omega(X) \otimes_k L$ for X in $\mathbf{T}$. Moreover, the groupoid attached to $(\mathbf{T} \otimes L, \omega \otimes L)$ is obtained from that attached to $(\mathbf{T}, \omega)$ by base change. (For the notion of the groupoid attached to a Tannakian category, see [**3**], [**7**, 1.12], or Theorem 3.24 below.)

(1.3.4) Suppose that L is a finite extension of k. An L-module (X, i) of $\mathbf{T}$ is generated as an L-module by X itself, and so can be regarded as an object of $\mathbf{T} \otimes L$. In this way, $\mathbf{T} \otimes L$ becomes identified with the category of L-modules in $\mathbf{T}$ (cf. [**5**, p. 321]).

(1.3.5) The extension of scalars of a Tate triple is a Tate triple.

There is no good reference for these statements, but some can be obtained by realizing $\mathbf{T}$ as the category of representations of a groupoid and applying [**6**, 4.6iii]. See also [**26**, p. 201].

EXAMPLE 1.4. Let L be a field of characteristic zero, and replace $Z^r(V)$ in the construction of the category of motives over $\mathbb{F}_q$ with $Z^r(V) \otimes L$. We then obtain a semisimple Tannakian category $\mathbf{Mot}(\mathbb{F}_q)_L$ over L, called the *category of motives over* $\mathbb{F}_q$ *with coefficients in* L. The obvious tensor functor $\mathbf{Mot}(\mathbb{F}_q) \to \mathbf{Mot}(\mathbb{F}_q)_L$ extends canonically to an equivalence of tensor categories $\mathbf{Mot}(\mathbb{F}_q) \otimes L \to \mathbf{Mot}(\mathbb{F}_q)_L$.

EXAMPLE 1.5. Let $\mathbf{T}$ be a Tannakian category over $\mathbb{R}$. From $\mathbf{T}$ we obtain a Tannakian category $\mathbf{T} \otimes \mathbb{C}$ over $\mathbb{C}$, together with a semilinear tensor functor

$$X \mapsto \bar{X} : \mathbf{T} \otimes \mathbb{C} \to \mathbf{T} \otimes \mathbb{C},$$

and a functorial isomorphism of tensor functors $\mu_X : X \to \bar{\bar{X}}$ such that $\mu_{\bar{X}} = \bar{\mu}_X$. Conversely, every such triple $(\mathbf{T}', X \mapsto \bar{X}, \mu)$ arises from a Tannakian category $\mathbf{T}$ over $\mathbb{R}$ (the category $\mathbf{T}$ can be recovered from the triple as the category whose objects are the pairs $(X, a : X \to \bar{X})$ such that $\bar{a} \circ a = \mu_X$).

From the point of view (1.3.4), we can also regard $\mathbf{T} \otimes \mathbb{C}$ as the category of $\mathbb{C}$-modules (X, i) in $\mathbf{T}$. Then $\overline{(X, i)} = (X, i \circ \iota)$ and μ_X is the identity map. The functor $X \mapsto X \otimes \mathbb{C}$ sends X to $X \oplus X$ with $a + bi \in \mathbb{C}$ acting as $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Polarizations. Let $(\mathbf{T}, w, T)$ be a Tate triple over a subfield k of $\mathbb{R}$. A *bilinear form* on an object X of weight n of $\mathbf{T}$ is a morphism

$$\varphi : X \otimes X \longrightarrow T^{\otimes(-n)}.$$

It is said to be *nondegenerate* if the map $X \to X^\vee(-n)$ it defines is an isomorphism. The *parity* of a nondegenerate φ is the unique morphism $\varepsilon\colon X \to X$ such that[2]

$$\varphi(x, x') = \varphi(x', \varepsilon x).$$

Let $u \in \mathrm{End}(X)$; the *transpose* u^{t} of u with respect to a nondegenerate φ is defined by

$$\varphi(ux, x') = \varphi(x, u^{\mathrm{t}}x').$$

Then $(uv)^{\mathrm{t}} = v^{\mathrm{t}}u^{\mathrm{t}}$, $a^{\mathrm{t}} = a$ for $a \in k$, $\varepsilon^{\mathrm{t}} = \varepsilon^{-1}$, and if ε is in the centre of $\mathrm{End}(X)$, then $u^{\mathrm{tt}} = u$.

The evaluation map (see the proof of 1.1) allows us to define a trace map

$$\mathrm{Tr}\colon \mathrm{End}(X) = \mathrm{Hom}(1, X \otimes X^\vee) \xrightarrow{\mathrm{Hom}(1,\mathrm{ev})} \mathrm{Hom}(1, 1) = k.$$

A nondegenerate bilinear form φ is said to be a *Weil form* if its parity ε is central and if for all nonzero $u \in \mathrm{End}(X)$, $\mathrm{Tr}(u \cdot u^{\mathrm{t}}) > 0$. Two Weil forms φ and ψ are said to be *compatible* if $\varphi \oplus \psi$ is also a Weil form.

Suppose there is given for each homogeneous X in $\mathbf{T}$ an equivalence class (for the relation of compatibility) $\Pi(X)$ of Weil forms of parity $w_X(-1) = (-1)^{\mathrm{wt}(X)}$ on X; we say that Π is a *(graded) polarization* on $(\mathbf{T}, w, T)$ if

(1.6.1) for all homogeneous X and Y of the same weight,

$$\varphi \in \Pi(X),\ \psi \in \Pi(Y) \implies \varphi \oplus \psi \in \Pi(X \oplus Y);$$

(1.6.2) for all homogeneous X and Y,

$$\varphi \in \Pi(X),\ \psi \in \Pi(Y) \implies \varphi \otimes \psi \in \Pi(X \otimes Y);$$

(1.6.3) the identity map $T \otimes T \to T^{\otimes 2}$ lies in $\Pi(T)$.

The axioms have the consequence that

$$\varphi \in \Pi(X),\ X' \subset X \implies \varphi|X' \in \Pi(X');$$

in particular, $\varphi|X'$ is nondegenerate. A polarizable Tannakian category is semisimple. (See [**26**, V.2.4.1.1].)

Let Π_0 be a polarization on $(\mathbf{T}, w, T)$, and let z be an element of $\mathrm{Aut}^{\otimes}(\mathrm{id}_{\mathbf{T}})$ of order 2 that acts as the identity on T. If $\varphi \in \Pi_0(X)$, then $z\varphi \underset{\mathrm{df}}{=} ((x, y) \mapsto \varphi(x, zy))$ is also a Weil form, and $z \cdot \Pi_0 = \{z\varphi \mid \varphi \in \Pi_0\}$ is a polarization on $(\mathbf{T}, w, T)$. Every polarization on $(\mathbf{T}, w, T)$ is of the form $z \cdot \Pi_0$ for a unique z [**8**, 5.15].

EXAMPLE 1.7. Let $\mathbf{V}_\infty$ be the category of pairs (V, F) with V a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ and F a semilinear automorphism of V such that F^2 acts as $(-1)^m$ on the m^{th} graded piece of V. Then $\mathbf{V}_\infty$ has a

[2] Here, and elsewhere, we identify an object X with its functor of "points" $Z \mapsto \mathrm{Hom}(Z, X)$. The parity can also be described as the automorphism of X that measures the difference between the two isomorphisms $X \to X^\vee(-n)$, $x \mapsto \varphi(x \otimes \cdot)$, $x \mapsto \varphi(\cdot \otimes x)$.

natural tensor structure relative to which it is a nonneutral Tannakian category over $\mathbb{R}$. The pair $T = (\mathbb{C}, z \mapsto \bar{z})$, with $\mathbb{C}$ regarded as a homogeneous vector space of weight -2, is a Tate object for $\mathbf{V}_\infty$. For (V, F) homogeneous of degree m, define a $(-1)^m$*-symmetric form* on V to be a nondegenerate bilinear form $\varphi\colon V \otimes V \to T^{\otimes -m}$ with parity $(-1)^m$, i.e., such that $\varphi(x, y) = (-1)^m \varphi(y, x)$, and call such a form *positive-definite* if $\varphi(x, Fx) > 0$, all $x \neq 0$. For any (V, F) homogeneous of weight m, let $\Pi_{\text{can}}(V, F)$ be the set of all $(-1)^m$-symmetric positive-definite forms on V. Then Π_{can} is a polarization on $\mathbf{V}_\infty$. There is exactly one other polarization, namely, $w(-1) \cdot \Pi_{\text{can}}$.

EXAMPLE 1.8. A *polarization* of a rational Hodge structure (V, h) of weight m is a morphism $\varphi\colon V \otimes V \to \mathbb{Q}(-m)$ of rational Hodge structures such that $(x, y) \mapsto (2\pi i)^m \varphi(x, h(i)y)$ is a symmetric positive-definite form on $V \otimes \mathbb{R}$. The category $\mathbf{Hdg}_{\mathbb{Q}}$ of polarizable rational Hodge structures together with the weight gradation and the Tate object $\mathbb{Q}(1)$ is a Tate triple over $\mathbb{Q}$, and there is a polarization on $\mathbf{Hdg}_{\mathbb{Q}}$ such that $\Pi(V, h)$ comprises the polarizations of (V, h) in the sense just defined.

CONJECTURE 1.9. *The Tate triple* $(\mathbf{Mot}(\mathbb{F}_q), w, T)$ *has a polarization.*

In fact, Grothendieck's standard conjectures imply that $\mathbf{Mot}(\mathbb{F}_q)$ has a canonical polarization—see [**26**, VI.4.4]. Later (2.44) we shall see that the Tate conjecture implies that $\mathbf{Mot}(\mathbb{F})$ has a polarization which is unique up to multiplication by $w(-1)$.

PROPOSITION 1.10. *Let* Π *be a polarization on* $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$. *There exists an exact faithful tensor functor* $\omega_\infty\colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R} \to \mathbf{V}_\infty$ *of Tate triples carrying* Π *into* Π_{can}; *moreover,* ω_∞ *is unique up to multiplication by* $w(-1)$.

PROOF. Apply [**10**, 5.20]. □

The ℓ-adic fibre functors. Let V be a smooth projective variety over a field k, and let ℓ be a prime number not equal to the characteristic of k. For every r, there is a cycle map

$$cl^r\colon Z^r(V) \longrightarrow H^{2r}(V \otimes k^{\text{al}}, \mathbb{Q}_\ell(r)) \qquad \text{(étale cohomology)}.$$

Unfortunately, we do not know that this map factors through $A^r(V)$, i.e., that if an algebraic cycle is numerically equivalent to zero then its cohomology class is zero. This is equivalent to the following existence statement for algebraic cycles: if there exists a cohomology class c such that $cl(Z) \cdot c \neq 0$, then there exists an algebraic cycle Z' such that $Z \cdot Z' \neq 0$.

PROPOSITION 1.11. *Assume that for any smooth projective variety* V *over* $\mathbb{F}_q$ *the cycle maps* $Z^r(V) \to H^{2r}(V \otimes \mathbb{F}, \mathbb{Q}_\ell(r))$ *factor through* $A^r(V)$. *Then the functor*

$$V \mapsto H_\ell(V) \underset{\text{df}}{=} \bigoplus_r H^r(V \otimes \mathbb{F}, \mathbb{Q}_\ell)$$

extends uniquely to a fibre functor ω_ℓ *on* $\mathbf{Mot}(\mathbb{F}_q)$ *over* $\mathbb{Q}_\ell$.

Proof. Standard properties of étale cohomology (see for example Milne [21, VI.11.6] show that H_ℓ is a functor on $\mathbf{CV}^0(\mathbb{F}_q)$, and it is then obvious that it extends to $\mathbf{Mot}(\mathbb{F}_q)$. The Künneth formula implies that it is a tensor functor on $\mathbf{Mot}(\mathbb{F}_q)$. It is exact because it is additive. (For more details, see Demazure [11, §8].) □

Remark 1.12. If the hypothesis of (1.11) holds for all $\ell \neq p$, then there is a fibre functor ω^p over $\mathbb{A}_f^p$ such that $\omega^p \otimes_{\mathbb{A}_f^p} \mathbb{Q}_\ell = \omega_\ell$ for all ℓ.

The p-adic fibre functor. Let k be a perfect field of characteristic $p \neq 0$. For any smooth projective variety V over k, we set

$$H^r_{\mathrm{crys}}(V) = H^r(V/W(k)) \otimes_{W(k)} K(k),$$

where $H^r(V/W(k))$ is the r^{th} crystalline cohomology group of V with respect to $W(k)$ [1]. Then $H^r_{\mathrm{crys}}(V)$ is a finite-dimensional vector space over $K(k)$.

Proposition 1.13. *Assume that for any smooth projective variety V over $\mathbb{F}_q$ the cycle map $Z^r(V) \to H^{2r}_{\mathrm{crys}}(V)$ factors through $A^r(V)$. Then the functor*

$$V \mapsto H_{\mathrm{crys}}(V) \underset{\mathrm{df}}{=} \bigoplus H^r_{\mathrm{crys}}(V)$$

extends uniquely to a fibre functor ω_p on $\mathbf{Mot}(\mathbb{F}_q)$ over $K(k)$.

Proof. Standard properties of crystalline cohomology [1]; [22, 2.11]; [13] show that H_{crys} is a functor on $\mathbf{CV}^0(\mathbb{F}_q)$, and the same argument as in the proof of (1.11) shows that this functor then extends to a fibre functor on $\mathbf{Mot}(\mathbb{F}_q)$. □

The Tate conjecture and consequences. We write $\zeta(V, s)$ for the zeta function of a variety V over $\mathbb{F}_q$.

Conjecture 1.14 (Tate conjecture). *For all smooth projective varieties V over $\mathbb{F}_q$ and $r \geq 0$, the dimension of $A^r(V)$ is equal to the order of the pole of $\zeta(V, s)$ at $s = r$.*

Proposition 1.15. *Assume the Tate conjecture* (1.14). *For any smooth projective variety V over $\mathbb{F}_q$ and any $\ell \neq p, \infty$, the cycle map $Z^r(V) \to H^{2r}(V \otimes \mathbb{F}, \mathbb{Q}_\ell(r))$ defines an isomorphism*

$$A^r(V) \otimes \mathbb{Q}_\ell \longrightarrow H^{2r}(V \otimes \mathbb{F}, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)}, \quad \text{all } r.$$

Moreover, π_V acts semisimply on $H_\ell(V)$.

Proof. See [30, 2.9]; [22, 8.6]. □

In particular, the Tate conjecture implies that an algebraic cycle on V is numerically equivalent to zero if and only if its class in $H_\ell(V)$ is zero, and so we can apply (1.11). Let $\mathbf{V}_\ell(\mathbb{F}_q)$ be the category of semisimple continuous representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on finite-dimensional $\mathbb{Q}_\ell$-vector spaces. It is a Tannakian category over $\mathbb{Q}_\ell$.

COROLLARY 1.16. *Assume* (1.14). *For any* $\ell \neq p, \infty$, *the functor* ω_ℓ *defines a fully faithful tensor functor*

$$\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell \longrightarrow \mathbf{V}_\ell(\mathbb{F}_q).$$

PROOF. Proposition 1.15 says that H_ℓ is a fully faithful functor $\mathbf{CV}^0(\mathbb{F}_q)\otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}_q)$, and it follows that its extension to $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$ is also fully faithful. □

Let k be a perfect field of characteristic $p \neq 0$. An *F-isocrystal* over k is a finite-dimensional vector space M over $K(k)$ together with a σ-linear isomorphism $F: M \to M$. We shall drop the "F" and simply call them isocrystals over k. The isocrystals over k form a Tannakian category over $\mathbb{Q}_p$, which we denote by $\mathbf{V}_p(k)$.

PROPOSITION 1.17. *Assume* (1.14). *The functor* ω_p *defines a fully faithful tensor functor*

$$\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_p \longrightarrow \mathbf{V}_p(\mathbb{F}_q).$$

PROOF. There is an analogous statement to (1.15) for the crystalline cohomology, which can be applied as in the proof of (1.16) to obtain the proposition. □

The category of motives over $\mathbb{F}$. Everything in this section holds *mutatis mutandis* with $\mathbb{F}_q$ replaced by $\mathbb{F}$.

Let ρ_1 and ρ_2 be continuous semisimple representations of open subgroups U_1 and U_2 of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ on the same finite-dimensional $\mathbb{Q}_\ell$-vector space V. We say that ρ_1 and ρ_2 are *related* if they agree on an open subgroup of $U_1 \cap U_2$. This is an equivalence relation, and we call an equivalence class of representations a *germ of an ℓ-adic representation* of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$. With the obvious structure, the germs of ℓ-adic representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ form a Tannakian category $\mathbf{V}_\ell(\mathbb{F})$ over $\mathbb{Q}_\ell$.

THEOREM 1.18. *The category* $\mathbf{Mot}(\mathbb{F})$ *of motives over* $\mathbb{F}$ *is a semisimple Tannakian category over* $\mathbb{Q}$. *Assume the Tate conjecture* (1.14).

(a) *The functor* $V \mapsto \bigoplus_r H^r(V, \mathbb{Q}_\ell)$ (*étale cohomology*) *extends to a fully faithful tensor functor*

$$\omega_\ell\colon \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell \longrightarrow \mathbf{V}_\ell(\mathbb{F}).$$

(b) *The functor* $V \mapsto \bigoplus_r H^r(V/W(\mathbb{F})) \otimes K(\mathbb{F})$ (*crystalline cohomology*) *extends to a fully faithful tensor functor*

$$\omega_p\colon \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_p \longrightarrow \mathbf{V}_p(\mathbb{F}).$$

PROOF. Straightforward extension of previous results. □

Notes. This section reviews standard material, most of which can be found already in [**26**].

2. Basic properties of the category of motives over a finite field

Throughout this section, we assume the Tate conjecture (1.14). Then $\mathbf{Mot}(\mathbb{F}_q)$ and $\mathbf{Mot}(\mathbb{F})$ are semisimple Tannakian categories over $\mathbb{Q}$ with the fibre functors ω_ℓ, $\ell = 2, 3, 5, \ldots, \infty$, described in §1.

Characteristic polynomials. For a motive X and an integer r, consider the alternating map

$$a = \sum \operatorname{sgn}(\sigma) \cdot \sigma \colon X^{\otimes r} \to X^{\otimes r}$$

(sum over the elements of the symmetric group on r letters). Then $a/r!$ is a projector in $\operatorname{End}(X^{\otimes r})$, and we define $\Lambda^r X$ to be its image. For any fibre functor ω, $\omega(\Lambda^r X) = \Lambda^r \omega(X)$, and so

$$\operatorname{rank}(\Lambda^r X) = \binom{\operatorname{rank} X}{r}.$$

In particular, $\operatorname{rank}(\Lambda^r X) = 1$ if $r = \operatorname{rank}(X)$. For an endomorphism α of X, we define $\det(\alpha)$ to be $\Lambda^{\operatorname{rank} X}\alpha$ (regarded as an element of $\mathbb{Q}$).

PROPOSITION 2.1. *For any endomorphism α of a motive X, there is a unique polynomial $P_\alpha(t) \in \mathbb{Q}[t]$ such that*

$$P_\alpha(n) = \det(n - \alpha) \quad \text{all } n \in \mathbb{Q}.$$

Moreover, $P_\alpha(t)$ is monic of degree equal to the rank of X, and it is equal to the characteristic polynomial of α acting on $\omega(X)$ for any fibre functor ω.

PROOF. If $P(t)$ and $Q(t)$ both have the property, then their difference has infinitely many roots, and hence is zero. Thus there is at most one such polynomial $P_\alpha(t)$.

Let ω be a fibre functor over a field K. The characteristic polynomial $P(t)$ of $\omega(\alpha)$ acting on $\omega(X)$ is a monic polynomial of degree $r = \operatorname{rank} X$ with coefficients in K such that $P(n) = \det(n - \alpha)$ for all $n \in$ K. Write $P(t) = \sum c_i t^i$, $c_i \in K$. Choose r distinct elements n_j of $\mathbb{Q}$, and note that $(c_i)_{1 \le i \le r}$ is the unique solution of the system of linear equations

$$c_0 + c_1 n_j + c_2 n_j^2 + \cdots + c_{r-1} n_j^{r-1} + n_j^r = \det(n_j - \alpha), \qquad j = 1, 2, \ldots, r,$$

with coefficients in $\mathbb{Q}$. Therefore, each $c_i \in \mathbb{Q}$.

Alternatively, and more directly, we can simply set

$$c_{r-i} = (-1)^i \operatorname{Tr}(\alpha | \Lambda^i X) = (-1)^i \operatorname{Tr}\left(\frac{a}{i!} \circ \overset{i}{\otimes} \alpha\right). \quad \square$$

We call $P_\alpha(t)$ the *characteristic polynomial* of α and sometimes write it $P_\alpha(X, t)$.

The Frobenius endomorphism. Recall that for any variety V over $\mathbb{F}_q$, π_V denotes the Frobenius endomorphism of V relative to $\mathbb{F}_q$. These morphisms commute with all morphisms of varieties over $\mathbb{F}_q$ and, more generally, with algebraic correspondences of degree zero (see Kleiman [**18**, p. 80]). It follows that, for each motive X, there is a $\pi_X \in \operatorname{End}(X)$ such that

(a) if $X = h(V)$, then $\pi_X = h(\pi_V)$;
(b) $\pi_{X\otimes Y} = \pi_X \otimes \pi_Y$; $\pi_1 = \mathrm{id}_1$; $\pi_Y \circ \alpha = \alpha \circ \pi_X$ for all morphisms $\alpha\colon X \to Y$.

Condition (b) says that the π_X's form an endomorphism of the identity functor of $\mathbf{Mot}(\mathbb{F}_q)$ regarded as a tensor functor, i.e., $(\pi_X) \in \operatorname{End}^{\otimes}(\mathrm{id})$, which implies that each π_X is an automorphism [**8**, 1.13]. Note that π acts on $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$ as multiplication by q, and therefore it acts on the Tate motive as multiplication by q^{-1}.

PROPOSITION 2.2. *For a motive X over $\mathbb{F}_q$, $\mathbb{Q}[\pi_X] \subset \operatorname{End}(X)$ is a product of fields, and if X is homogeneous of weight m, then for every homomorphism $\rho\colon \mathbb{Q}[\pi_X] \to \mathbb{C}$, $|\rho\pi_X| = q^{m/2}$.*

PROOF. Because π_X acts semisimply on $\omega_\ell(X)$ (see 1.15, 1.16), $\mathbb{Q}[\pi_X]\otimes \mathbb{Q}_\ell$ is a product of fields, and this implies that the same is true of $\mathbb{Q}[\pi_X]$. If $X = h^m(V)$ for V a smooth projective variety over $\mathbb{F}_q$, the second assertion is part of the Weil conjectures [**4**], and the general case follows easily from this special case. □

REMARK 2.3. If X is effective, then (by definition)

$$X \oplus Y = h(V)$$

for some motive Y and smooth projective variety V. The eigenvalues of π_V are algebraic integers, and therefore, the same is true of π_X. If X is an arbitrary motive over $\mathbb{F}_q$, then $X(n)$ is effective for some n, and so $q^n\pi_X$ is an algebraic integer for some n.

Classification of the isomorphism classes of simple motives. By a *central division* (respectively, *simple) algebra* over a field K, we mean a division (respectively, simple) algebra having centre K and of finite dimension over K.

PROPOSITION 2.4. *Let X be a simple motive over $\mathbb{F}_q$. Then $\mathbb{Q}[\pi_X]$ is a field, and* $\operatorname{End}(X)$ *is a central division algebra over $\mathbb{Q}[\pi_X]$.*

PROOF. Because X is simple, any nonzero endomorphism α of X is an isomorphism, which shows that $\operatorname{End}(X)$ is a division algebra and that $\mathbb{Q}[\pi_X]$ is a subfield. The Tate conjecture (1.14) implies that $\operatorname{End}(X)\otimes\mathbb{Q}_\ell$ is the centralizing ring of $\mathbb{Q}[\pi_X]\otimes\mathbb{Q}_\ell$ in $\operatorname{End}(\omega_\ell(X))$, and because $\mathbb{Q}[\pi_X]\otimes\mathbb{Q}_\ell$ is semisimple the double centralizer theorem (Bourbaki [**2**, 5.4, Corollary 2, p. 50]) then implies that $\mathbb{Q}[\pi_X]\otimes\mathbb{Q}_\ell$ is the centre of $\operatorname{End}(X)\otimes\mathbb{Q}_\ell$. It follows that $\mathbb{Q}[\pi_X]$ is the centre of $\operatorname{End}(X)$. □

DEFINITION 2.5. An algebraic number π is said to be a *Weil q-number of weight m* if

(a) for every embedding $\rho: \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $|\rho(\pi)| = q^{m/2}$;
(b) for some n, $q^n\pi$ is an algebraic integer.

The set of Weil q-numbers in $\mathbb{Q}^{\mathrm{al}}$ is denoted by $W(q)$. It is a subgroup of $\mathbb{Q}^{\mathrm{al}\times}$ stable under the action of $\Gamma \underset{\mathrm{df}}{=} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. We can associate with an arbitrary Weil q-number π the orbit $[\pi] \in \Gamma\backslash W(q)$ consisting of the set of conjugates of π in $\mathbb{Q}^{\mathrm{al}}$, i.e., of the set of images of π under the embeddings $\mathbb{Q}[\pi] \hookrightarrow \mathbb{Q}^{\mathrm{al}}$.

Condition (2.5a) implies that $\pi \mapsto \pi' = q^m/\pi$ defines an involution $\alpha \mapsto \alpha'$ of $\mathbb{Q}[\pi]$ such that $\rho(\alpha') = \iota\rho(\alpha)$ for all embeddings $\rho: \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$. Hence, if π is a Weil q-number, then $\mathbb{Q}[\pi]$ is either a CM-field or a totally real field according to whether $\pi \neq \pi'$ or $\pi = \pi'$.

From (2.2, 2.3, 2.4) we know that, for a simple motive X of weight m over $\mathbb{F}_q$, π_X is a Weil q-number of weight m. Recall that $\Sigma(\mathbf{Mot}(\mathbb{F}_q))$ is the set of isomorphism classes of simple objects in $\mathbf{Mot}(\mathbb{F}_q)$.

PROPOSITION 2.6. *The map $X \mapsto [\pi_X]$ defines a bijection*

$$\Sigma(\mathbf{Mot}(\mathbb{F}_q)) \longrightarrow \Gamma\backslash W(q).$$

PROOF. Let X and X' be simple motives over $\mathbb{F}_q$ whose Weil numbers π and π' are conjugate. Then $\mathrm{Hom}(\omega_\ell(X), \omega_\ell(X'))^\Gamma \neq 0$, and so the Tate conjecture implies $\mathrm{Hom}(X, X') \neq 0$. Hence, X and X' are isomorphic.

Let π be a Weil q-number in $\mathbb{Q}^{\mathrm{al}}$; we have to prove that $[\pi]$ arises from a motive. For some $n \geq 0$, $q^n\pi$ will be an algebraic integer. If X is a simple motive with $[\pi_X] = [q^n\pi]$, then $X(n)$ will be a simple motive with $[\pi_{X(n)}] = [\pi]$. Therefore, we can assume that π is an algebraic integer. Let m be its weight. If $m = 0$, then π is a root of unity and it arises from an Artin motive. Otherwise, Honda's theorem [**29**, Theorem 1] shows that there is a simple abelian variety A over $\mathbb{F}_{q^m}$ such that $[\pi_A] = [\pi]$. Consider the abelian variety A_* over $\mathbb{F}_q$ obtained from A by restriction of scalars. Then $P(h^1(A_*), t) = P(h^1(A), t^m)$, and so π occurs as a root of $P(h^1(A_*)^{\otimes m}, t)$. For some simple factor X of $h^1(A_*)^{\otimes m}$, π will be conjugate to π_X. □

REMARK 2.7. The proof shows that, under the assumption of the Tate conjecture, the Tannakian category $\mathbf{Mot}(\mathbb{F}_q)$ is generated (as a Tannakian category) by the motives of abelian varieties and Artin motives.

Isotypic motives. An object in an abelian category is *isotypic* if it is isomorphic to a direct sum of copies of a single simple object. Proposition 2.4 shows that the endomorphism ring of an isotypic motive X over $\mathbb{F}_q$ is a matrix algebra over a central division algebra over the field $\mathbb{Q}[\pi_X]$, i.e., it is a central simple algebra over $\mathbb{Q}[\pi_X]$.

Let E be a central simple algebra of degree e^2 over a field F of finite degree f over $\mathbb{Q}$, and let K be an extension of $\mathbb{Q}$ that splits E, i.e., such that $E \otimes_{\mathbb{Q}} K$ is a product of matrix algebras over K. Write $\mathrm{Hom}(F, K) = \{\sigma_1, \dots, \sigma_f\}$. Then $E \otimes_{\mathbb{Q}} K = E_1 \times \cdots \times E_f$ where $E_i \underset{\mathrm{df}}{=} E \otimes_{E, \sigma_i} K$ is a matrix algebra of degree e^2 over K. Up to isomorphism there are exactly f nonisomorphic simple representations $V_1, \dots, V_f$ of E over K, each of dimension e over K, and their sum $V = \bigoplus V_i$ is called the *reduced representation* of E.

PROPOSITION 2.8. *Let X be an isotypic motive over $\mathbb{F}_q$, and let $E = \mathrm{End}(X)$.*

(a) *The rank of X is $[E : \mathbb{Q}[\pi_X]]^{1/2} \cdot [\mathbb{Q}[\pi_X] : \mathbb{Q}]$.*
(b) *For any fibre functor ω over a field K that splits E, the representation of E on $\omega(X)$ is isomorphic to the reduced representation.*
(c) *For $\alpha \in E$,*

$$P_\alpha(X, t) = Nm_{\mathbb{Q}[\pi_X]/\mathbb{Q}}(c_\alpha(t)),$$

where $c_\alpha(t)$ is the reduced characteristic polynomial of α in $E/\mathbb{Q}[\pi_X]$. In particular, $P_\pi(X, t) = m_\pi(t)^e$ where $m_\pi(t)$ is the minimum polynomial of π in the extension $\mathbb{Q}[\pi]/\mathbb{Q}$ and $e = [E : \mathbb{Q}[\pi]]^{1/2}$.

PROOF. (a) The number $[E : \mathbb{Q}[\pi]]^{1/2} \cdot [\mathbb{Q}[\pi] : \mathbb{Q}]$ is the degree over $\mathbb{Q}$ of a maximal commutative étale subalgebra of E. It is therefore also the degree over $\mathbb{Q}_\ell$ of a maximal commutative étale subalgebra of $E \otimes \mathbb{Q}_\ell$, $\ell \neq p, \infty$. But $E \otimes \mathbb{Q}_\ell$ is the centralizer in $\mathrm{End}(\omega_\ell(X))$ of the semisimple endomorphism $\omega_\ell(\pi)$, and so this degree is the dimension of $\omega_\ell(X)$ as a $\mathbb{Q}_\ell$-vector space, which equals the rank of X.

(b) Suppose the representation of E on $\omega(\mathrm{X})$ is isomorphic to $\bigoplus m_i V_i$, $m_i \geq 0$. For any $\alpha \in \mathbb{Q}[\pi]$, the characteristic polynomial of α on V_i is $(t - \sigma_i \alpha)^e$, and so $P_\alpha(t) = \prod_{1 \leq i \leq f} (t - \sigma_i \alpha)^{e m_i}$, where $f = [\mathbb{Q}[\pi] : \mathbb{Q}]$. Because $P_\alpha(t)$ has coefficients in $\mathbb{Q}$, the m_i's must be equal, and because $P_\alpha(t)$ has degree ef, each $m_i = 1$. (Alternatively, let L be a maximal commutative étale subalgebra of E. For any fibre functor ω over a field K, $L \otimes_{\mathbb{Q}} K$ acts faithfully on $\omega(X)$, and $[L \otimes_{\mathbb{Q}} K : K] = \dim_K \omega(X)$, and so $\omega(X)$ is a free $L \otimes_{\mathbb{Q}} K$-module of rank 1. When K splits E, this implies that $\omega(X)$ is isomorphic to the reduced representation.)

(c) Choose a fibre functor as in (b) and note that the two polynomials become equal in $K[t]$. On taking $\alpha = \pi_X$, we find that

$$P_{\pi_X}(X, t) = \mathrm{Nm}_{\mathbb{Q}[\pi_X]/\mathbb{Q}}(t - \pi)^e = (m_{\pi_X}(t))^e. \quad \square$$

The isocrystal of a motive. We first recall the Dieudonné-Manin classification of isocrystals (i.e., F-isocrystals) over an algebraically closed field k. For each pair of relatively prime integers (r, s) with $r \geq 1$,

$$N_{r,s} = \mathbb{Q}_p[T]/(T^r - p^s), \qquad F_N = \text{multiplication by } T,$$

is an isocrystal over $\mathbb{F}_p$, and we define

$$M_{r,s} = K(k) \otimes_{\mathbb{Q}_p} N_{r,s}, \qquad F_M = \sigma \otimes F_N.$$

It is an isocrystal over k of rank r. (In general, the rank of an isocrystal (M, F) as an element of the Tannakian category $\mathbf{V}_p(k)$ is the dimension of M as a vector space over $K(k)$.)

THEOREM 2.9. *Let k be an algebraically closed field of characteristic $p \neq 0$. The category $\mathbf{V}_p(k)$ is semisimple. For each pair of relatively prime integers (r, s) with $r \geq 1$, the isocrystal $M_{r,s}$ is simple, and every simple isocrystal over k is isomorphic to $M_{r,s}$ for exactly one pair (r, s).*

PROOF. See [**10**, IV]. □

Write $M_{s/r}$ for $M_{r,s}$. Every isocrystal M over k can be written uniquely as a direct sum

$$M = (M_{\lambda_1})^{r_1} \oplus \cdots \oplus (M_{\lambda_n})^{r_n}, \qquad \lambda_1 < \lambda_2 < \cdots < \lambda_n, \ r_i \geq 1.$$

The numbers λ_i are called the *slopes* of M, and r_i is the *multiplicity* of λ_i.

For an isocrystal M over $\mathbb{F}_{p^n}$, we let $\pi_M = F^n$. It is a $K(\mathbb{F}_{p^n})$-linear endomorphism of M. When k is not algebraically closed, the category $\mathbf{V}_p(k)$ need not be semisimple.

PROPOSITION 2.10. *The following conditions on an isocrystal (M, F) over $\mathbb{F}_q$ are equivalent:*

(a) *(M, F) is semisimple, i.e., it is a direct sum of simple isocrystals over $\mathbb{F}_q$;*
(b) *$\mathrm{End}(M, F)$ is semisimple;*
(c) *π_M is a semisimple endomorphism of M (regarded as a vector space over $K(\mathbb{F}_q)$).*

When these conditions hold, the centre of $\mathrm{End}(M, F)$ is $\mathbb{Q}_p[\pi_M]$.

PROOF. (a) $\Longrightarrow$ (b): If M is simple, then $\mathrm{End}(M, F)$ is a division algebra; if M is isotypic, then $\mathrm{End}(M, F)$ is a matrix algebra over a division algebra; if M is semisimple, then $\mathrm{End}(M, F)$ is a product of matrix algebras over division algebras.

(b) $\Longrightarrow$ (c): Because $\mathbb{Q}_p[\pi_M]$ is contained in the centre of $\mathrm{End}(M, F)$, it is a product of fields.

(c) $\Longrightarrow$ (b,a): Condition (c) implies that the centralizing ring C of $K(\mathbb{F}_q)[\pi_M]$ in the ring of endomorphisms of M (regarded as a $K(\mathbb{F}_q)$-vector space) is a semisimple $K(\mathbb{F}_q)$-algebra. The map

$$\mathrm{End}(M, F) \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q) \hookrightarrow C$$

is injective and, on counting dimensions, we see that it is an isomorphism. Therefore, $\mathrm{End}(M, F)$ must also be semisimple.

The category of all isocrystals over $\mathbb{F}_q$ satisfying (c) is therefore a $\mathbb{Q}_p$-linear abelian category such that the endomorphism ring of every object is a semisimple ring of finite dimension over $\mathbb{Q}_p$. It is well known that this implies that all the objects of the category are semisimple (see [**16**, Lemma 2].

Finally, because π_M is a semisimple endomorphism of M, the centre of the ring C defined above is $K(\mathbb{F}_q)[\pi_M]$. But $C = \mathrm{End}(M, F) \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q)$, and it follows that the centre of $\mathrm{End}(M, F)$ is $\mathbb{Q}_p[\pi_M]$. □

REMARK 2.11. The map $M \mapsto [\pi_M]$ defines a bijection from the set of isomorphism classes of simple isocrystals over $\mathbb{F}_q$ to the set of orbits of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p)$ acting on $\mathbb{Q}_p^{\mathrm{al}\times}$ (Kottwitz [**19**, 11.2, 11.4].

Let (M, F) be an isocrystal over a perfect field k. For any perfect field $k' \supset k$, $(M_{k'}, F_{k'}) \underset{\mathrm{df}}{=} (K(k')\otimes M, \sigma\otimes F)$ is an isocrystal over k'. The *slopes* (and *multiplicities*) of M are defined to be the slopes (and multiplicities) of $M_{k^{\mathrm{al}}}$.

Let ord_p denote the p-adic valuation $\mathbb{Q}_p^\times \twoheadrightarrow \mathbb{Z}$ on $\mathbb{Q}_p$ or its extension to any field algebraic over $\mathbb{Q}_p$.

PROPOSITION 2.12. *Let M be an isocrystal over $\mathbb{F}_q$ of rank d, and let $\{a_1, \dots, a_d\}$ be the family of eigenvalues of π_M. Then the family of slopes of M is $\{\mathrm{ord}_p(a_1)/\mathrm{ord}_p(q), \dots, \mathrm{ord}_p(a_d)/\mathrm{ord}_p(q)\}$.*

PROOF. See [**10**, p. 90]. □

THEOREM 2.13. *Let X be a motive over $\mathbb{F}_q$. Then $\omega_p(X)$ is a semisimple isocrystal over $\mathbb{F}_q$ of rank equal to* rank X. *The characteristic polynomial of π_X on X is equal to the characteristic polynomial of $\pi_{\omega_p(X)}$ on $\omega_p(X)$. If $\{a_1, \dots, a_d\}$ is the family of roots of $P_{\pi_X}(X, t)$, then the family of slopes of $\omega(X)$ is $\{\mathrm{ord}_p(a_1)/\mathrm{ord}_p(q), \dots, \mathrm{ord}_p(a_d)/\mathrm{ord}_p(q)\}$.*

PROOF. The Tate conjecture implies that $\mathrm{End}(\omega_p(X), F) = \mathrm{End}(X) \otimes \mathbb{Q}_p$ (see §1.17), and so it, and $\omega_p(X)$, are semisimple. It is clear from the definition of the action of F on the crystalline cohomology of a variety [**1**] that the Frobenius endomorphism π_X of a motive X induces the Frobenius endomorphism $\pi_{\omega_p(X)}$ of $\omega_p(X)$, i.e., that

$$\pi_{\omega_p(X)} = \omega_p(\pi_X),$$

and so they have the same characteristic polynomial. The final statement follows from (2.12). □

The endomorphism algebra of a simple motive. Let K be a non-Archimedean local field, and consider a central division algebra D over K. Choose a maximal subfield L of D that is unramified over K. The Skolem-Noether theorem [**2**, §10] shows that every automorphism of L is induced by an inner automorphism of D. In particular, there is a $\gamma \in D$ such that $\gamma x \gamma^{-1} = \mathrm{Frob}(x)$ for all $x \in L$, where Frob is the geometric Frobenius element in $\mathrm{Gal}(L/K)$ (it acts as $x \mapsto x^{q^{-1}}$ on the residue field). The valuation $\mathrm{ord}\colon L^{\times} \twoheadrightarrow \mathbb{Z}$ extends uniquely to a valuation $\mathrm{ord}\colon D^{\times} \to \mathbb{Q}$, and the invariant of D is defined by the rule

$$\mathrm{inv}_K(D) = \mathrm{ord}(\gamma) \in \mathbb{Q}/\mathbb{Z}.$$

The Wedderburn theorems imply that a central simple algebra E over K is isomorphic to a matrix algebra over a division algebra D over K, uniquely determined up to isomorphism, and the invariant of E is defined to be that of D.

In the proof of the next proposition, we shall need to use the following fact. Let K' be a field,

$$K \subset K' \subset D,$$

and let D' be the centralizing ring of K' in D. The double centralizer theorem shows that D' is a central division algebra over K'. When K' is unramified over K, then we can choose the field L in the definition of $\mathrm{inv}_K(D)$ to contain it, and then it is clear that

$$\mathrm{inv}_{K'} D' = [K' \colon K] \cdot \mathrm{inv}_K D.$$

This formula holds even when K' is ramified over K.

PROPOSITION 2.14. *Let (M, F) be a simple isocrystal over $\mathbb{F}_q$. Then $E \underset{\mathrm{df}}{=} \mathrm{End}(M, F)$ is a central division algebra over $\mathbb{Q}_p[\pi_M]$ with invariant*

$$-\frac{\mathrm{ord}_p(\pi_M)}{\mathrm{ord}_p(q)} \cdot [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p];$$

moreover,

$$\mathrm{rank}\, M = [E \colon \mathbb{Q}_p[\pi_M]]^{1/2} \cdot [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p].$$

PROOF. Because M is simple, $\mathbb{Q}_p[\pi_M]$ is a field, and so the term $\mathrm{ord}_p(\pi_M)$ is well defined, and is equal to $\mathrm{ord}_p(\pi)$ for any conjugate π of π_M.

Let $\lambda = \mathrm{ord}_p(\pi_M)/\mathrm{ord}_p(q)$. Then $M_{\mathbb{F}}$ is isomorphic to a direct sum of copies of M_λ, and so $\mathrm{End}(M_{\mathbb{F}}, F)$ is a matrix algebra over $\mathrm{End}(M_\lambda, F)$. But (see [**10**, p. 80]), $\mathrm{End}(M_\lambda, F)$ is a central division algebra over $\mathbb{Q}_p$ with invariant[3] $-\lambda$.

When we extend the action of π_M on M to $M_{\mathbb{F}} = K(\mathbb{F}) \otimes M$ by linearity so that $F_{\mathbb{F}}^n = \pi_M \circ \sigma^n$, where $n = \mathrm{ord}_p q$, then $\mathrm{End}(M, F)$ becomes the centralizing ring of $\mathbb{Q}_p[\pi_M]$ in $\mathrm{End}(M_{\mathbb{F}}, F_{\mathbb{F}})$. Hence,

$$\begin{aligned}\mathrm{inv}_{\mathbb{Q}_p[\pi_M]} \mathrm{End}(M, F) &= [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p] \cdot \mathrm{inv}_{\mathbb{Q}_p} \mathrm{End}(M_{\mathbb{F}}, F_{\mathbb{F}}) \\ &= [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p] \cdot (-\lambda),\end{aligned}$$

which proves the first statement.

[3] We are using a different sign convention for the invariant from Demazure.

Recall from the proof of (2.10) that

$$E \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q) \approx C$$

where C is the centralizing ring of $K(\mathbb{F}_q)[\pi_M]$ in $\mathrm{End}(M)$. The second statement in the proposition can be proved by noting that the right-hand side is equal to the degree over $\mathbb{Q}_p$ of a maximal commutative étale subalgebra of E, and that this and the left-hand side are both equal to the degree over $K(\mathbb{F}_q)$ of a maximal commutative étale subalgebra of C. □

For a central division algebra D over an Archimedean local field K, $\mathrm{inv}_K(D)$ is defined to be 0 or $\frac{1}{2}$ (mod 1) according to whether D is split or nonsplit. For a central division algebra over a number field K and a prime v of K, we set

$$\mathrm{inv}_v(D) = \mathrm{inv}_{K_v}(D \otimes_K K_v).$$

THEOREM 2.15. *Let K be an algebraic number field.*

(a) *Two central division algebras D and D' over K are isomorphic if and only if $\mathrm{inv}_v(D) = \mathrm{inv}_v(D')$ for all primes v of K.*

(b) *An element $(i_v) \in \bigoplus_v \mathbb{Q}/\mathbb{Z}$ (sum over all primes of K) is the family of invariants of a central division algebra over K if and only if $\sum_v i_v = 0$, $2i_v = 0$ if v is real, and $i_v = 0$ if v is complex.*

(c) *For a central division algebra over a number field K, $[D\colon K]^{1/2}$ is the least common denominator of the numbers $\mathrm{inv}_v(D)$.*

PROOF. This is a restatement of fundamental results in class field theory. For a discussion of the results, with references, see [**25**, Chapter 8] or [**24**, Chapter 18]. □

Since $\mathrm{End}(X)$ is a central division algebra over the field $\mathbb{Q}[\pi_X]$ when X is simple, to describe its isomorphism class we only have to give its invariants at the primes of $\mathbb{Q}[\pi_X]$.

THEOREM 2.16. *Let X be a simple motive over $\mathbb{F}_q$, and let $E = \mathrm{End}(X)$. For any prime v of $\mathbb{Q}[\pi_X]$, $\|\pi_X\|_v = q^{\mathrm{inv}_v(E)}$. Explicitly, this says that*

$$\mathrm{inv}_v(E) = \begin{cases} 1/2 & \textit{if } v \textit{ is real and } X \textit{ has odd weight}; \\ -\dfrac{\mathrm{ord}_v(\pi_X)}{\mathrm{ord}_v(q)} \cdot [\mathbb{Q}[\pi_X]_v : \mathbb{Q}_p] & \textit{if } v|p; \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. If $v|\ell$ with $\ell \neq p, \infty$, then $\omega_\ell(X)$ is a free module over $\mathbb{Q}_\ell \otimes \mathbb{Q}[\pi_X]$ of rank $e = [E\colon \mathbb{Q}[\pi_X]]^{1/2}$ (see the proof of Proposition 2.8), and so $E \otimes \mathbb{Q}_\ell$ is the ring of $e \times e$ matrices over $\mathbb{Q}[\pi_X] \otimes \mathbb{Q}_\ell$. Hence, in this case, $\mathrm{inv}_v(E) = 0$.

If $v|p$, then the statement follows from (2.13) and (2.14).

If v is a real, then it corresponds to an embedding $\mathbb{Q}[\pi_X] \hookrightarrow \mathbb{R}$, and we can regard π_X as a real number such that $\pi_X^2 = q^m$. If m is even, then

$X = \mathbb{Q}(\frac{-m}{2})$ or becomes isomorphic to it over $\mathbb{F}_{q^2}$ (depending on whether $\pi_X = q^{\frac{m}{2}}$ or $-q^{\frac{m}{2}}$). In either case, X has rank 1, and so $\mathrm{inv}_v(E) = 0$. Hence, we can assume that m is odd. If q is a square in $\mathbb{Q}$, then $\mathbb{Q}[\pi_X] = \mathbb{Q}$, and $\mathrm{inv}_v(E) = 1/2$ because $\mathrm{inv}_p(E) = 1/2$ and the sum of the invariants is 0 (mod 1). Suppose q is not a square in $\mathbb{Q}$, and let X' be the base change of X to $\mathbb{F}_{q^2}$. Then $\pi_{X'} = \pi_X^2 = q^m$, and so according to the case just considered, $\mathrm{End}(X')$ is a central simple algebra over $\mathbb{Q}$ with invariant 1/2 at ∞. Because $\mathrm{End}(X)$ is the centralizer in $\mathrm{End}(X')$ of $\mathbb{Q}[\pi_X]$, we see that it has invariant 1/2 at each of the two infinite primes of $\mathbb{Q}[\sqrt{q}]$. □

The tensor structure on Mot($\mathbb{F}_q$). Because $\mathbf{Mot}(\mathbb{F}_q)$ is semisimple, the Grothendieck group $K(\mathbf{Mot}(\mathbb{F}_q))$ of $\mathbf{Mot}(\mathbb{F}_q)$ is the free abelian group on the set of isomorphism classes of simple objects in $\mathbf{Mot}(\mathbb{F}_q)$. The tensor structure on $\mathbf{Mot}(\mathbb{F}_q)$ defines a multiplication on $K(\mathbf{Mot}(\mathbb{F}_q))$, which we now determine.

Let W be a set with an action of a group Γ, and let $\mathbb{Z}[\Gamma\backslash W]$ be the free abelian group generated by $\Gamma\backslash W$. Assume that every orbit is finite, and that W has a group structure compatible with the action of Γ, i.e., such that

$$g(ww') = (gw)(gw'), \qquad g \in \Gamma, \ w, w' \in W.$$

Then we can define a multiplication on $\mathbb{Z}[\Gamma\backslash W]$ as follows: for orbits $o = \{w_1, \dots, w_m\}$ and $o' = \{w'_1, \dots, w'_n\}$, write $\{w_i w'_j \mid 1 \le i \le m, \quad 1 \le j \le n\}$ as a disjoint union of orbits with multiplicities, $\coprod r_i o_i$, and define

$$o \cdot o' = \sum r_i o_i.$$

With this structure $\mathbb{Z}[\Gamma\backslash W]$ becomes a commutative ring (with 1 if the identity element of W is fixed by Γ). For example, to see that the associative law holds, note that if $o = \{w_1, \dots\}$, $o' = \{w'_1, \dots\}$, and $o'' = \{w''_1, \dots\}$, then both $o(o'o'')$ and $(oo')o''$ are obtained by decomposing the family $\{w_i w'_j w''_k\}$ into a disjoint union of orbits with multiplicities.

For $\pi \in W(q)$, let $d(\pi)$ be the least common denominator of the numbers $i_v(\pi)$ where $\|\pi\|_v = q^{i_v(\pi)}$, v a prime of $\mathbb{Q}[\pi]$. Note that $d(\pi') = d(\pi)$ if π' is conjugate to π.

Define

$$\gamma \colon K(\mathbf{Mot}(\mathbb{F}_q)) \to \mathbb{Z}[\Gamma\backslash W(q)]$$

to be the $\mathbb{Z}$-linear map that sends the isomorphism class of a simple object X to $d(\pi_X) \cdot [\pi_X]$.

PROPOSITION 2.17. *The map γ is an injective homomorphism of rings with image the set of elements $\sum n_{[\pi]} \cdot [\pi]$ such that $d(\pi) | n_{[\pi]}$ for all $[\pi]$.*

PROOF. For any object X of a semisimple Tannakian category over a field k, $\mathrm{End}(X)$ is a finitely generated semisimple k-algebra, and

$$X \text{ is isotypic} \iff \mathrm{End}(X) \text{ is simple} \iff \text{the centre of } \mathrm{End}\, X \text{ is a field.}$$

Let C be the centre of $\mathrm{End}(X)$. Then C is a product of fields, and X decomposes into a product of isotypic components according as C decomposes into a product of fields: if

$$C = C_1 \times \cdots \times C_r, \qquad 1 = (e_1, \ldots, e_r),$$

then

$$X = X_1 \oplus \cdots \oplus X_r, \qquad X_i = \mathrm{Im}(e_i)$$

with the X_i the isotypic components of X.

Choose a fibre functor ω for $\mathbf{Mot}(\mathbb{F}_q)$ over some large field K containing $\mathbb{Q}^{\mathrm{al}}$. For a motive X over $\mathbb{F}_q$, the centre of $\mathrm{End}(X)$ is $\mathbb{Q}[\pi_X]$, and the factors of $\mathbb{Q}[\pi_X]$ can be identified with the orbits of Γ acting on $\mathrm{Hom}(\mathbb{Q}[\pi_X], \mathbb{Q}^{\mathrm{al}})$. But this last set can be identified with the set of eigenvalues of π_X acting on $\omega(X)$, and so the isotypic components of X are in natural one-to-one correspondence with the orbits of Γ acting on this set of eigenvalues. Moreover, (2.8b) shows that if $m_{[\pi]}$ is the multiplicity with which an orbit $[\pi]$ occurs in the family of eigenvalues, then $\gamma(X) = \sum m_{[\pi]} \cdot [\pi]$. With this description of γ, it is clear that γ takes products to products, because the family of eigenvalues of $\pi_{X \otimes X'}$ acting on $\omega(X \otimes X') = \omega(X) \otimes \omega(X')$ is the family of products $\pi\pi'$ with π an eigenvalue of π_X and π' an eigenvalue of $\pi_{X'}$.

The remaining statements are obvious. □

Motives over $\mathbb{F}$. Let R be a ring and consider the set of pairs (a, n) where $a \in R$ and $n \geq 1$. We say that two pairs (a, n) and (a', n') are equivalent if $a^{n'N} = a'^{nN}$ for some $N \geq 1$. An equivalence class of such pairs will be called a *germ of an element* of R.

Suppose R is a $\mathbb{Q}$-algebra of finite dimension, and let α be a germ of an element of R represented by (a, n). For $N >> 1$, the algebra $\mathbb{Q}[a^N]$ is independent of the choice of (a, n) and N. We denote it by $\mathbb{Q}[\alpha]$.

Let X be a motive over $\mathbb{F}$. For any model X_n of X over a field $\mathbb{F}_{p^n}$ we obtain a Frobenius element $\pi_{X_n} \in \mathrm{End}(X_n) \subset \mathrm{End}(X)$. The germ of an element of $\mathrm{End}(X)$ represented by (π_{X_n}, n) is independent of the choice of X_n and will be called the *Frobenius endomorphism* π_X of X.

When $n | n'$, there is a homomorphism

$$\pi \mapsto \pi^{n'/n} : W(p^n) \to W(p^{n'}),$$

and we define $W(p^\infty) = \varinjlim W(p^n)$. Thus, an element of $W(p^\infty)$ is represented by a pair (π, n) with $\pi \in W(p^n)$, and (π, n) and (π', n') represent the same element of $W(p^\infty)$ if and only if $\pi^{n'N} = \pi'^{nN}$ for some $N \geq 1$. The Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts on $W(p^\infty)$, and we write $[\pi]$ for the orbit of an element π.

To a simple motive X over $\mathbb{F}$, we can attach an orbit $[\pi_X] \in \Gamma\backslash W(p^\infty)$ as follows: for any representative (π, n) of π_X, $[\pi_X]$ is the image of $[\pi] \in \Gamma\backslash W(p^n)$ in $\Gamma\backslash W(p^\infty)$.

THEOREM 2.18. *The map* $X \mapsto [\pi_X]$ *defines a bijection*

$$\Sigma(\mathbf{Mot}(\mathbb{F})) \to \Gamma\backslash W(p^\infty).$$

PROOF. This follows easily from (2.6). □

THEOREM 2.19. *Let* X *be a simple motive over* $\mathbb{F}$.

(a) *The endomorphism ring* $\mathrm{End}(X)$ *of* X *is a central division algebra over* $\mathbb{Q}[\pi_X]$.

(b) *If* π_X *is represented by* (π, n), *then the invariant of* $\mathrm{End}(X)$ *at a prime* v *of* $\mathbb{Q}[\pi_X]$ *is determined by the rule*:

$$\|\pi\|_v = (p^n)^{\mathrm{inv}_v(\mathrm{End}(X))}.$$

(c) *The rank of* X *is* $[\mathrm{End}(X)\colon \mathbb{Q}[\pi_X]]^{1/2} \cdot [\mathbb{Q}[\pi_X]\colon \mathbb{Q}]$.

PROOF. The motive X, together with all its endomorphisms, will be defined over some field $\mathbb{F}_q$, and so this theorem follows from (2.4), (2.8), and (2.16). □

Suppose π_X is represented by (π, n). Define $d(\pi)$ to be the least common denominator of the numbers $i_v(\pi)$, where $\|\pi_n\|_v = (p^n)^{i_v(\pi)}$.

COROLLARY 2.20. *The map*

$$[X] \mapsto d(\pi_X) \cdot [\pi_X]\colon \Sigma(\mathbf{Mot}(\mathbb{F})) \longrightarrow \mathbb{Z}[\Gamma\backslash W(p^\infty)]$$

extends by linearity to a homomorphism of rings

$$K(\mathbf{Mot}(\mathbb{F})) \longrightarrow \mathbb{Z}[\Gamma\backslash W(p^\infty)].$$

PROOF. The proof is the same as that of (2.17). □

The category $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$. Let L be a subfield of $\mathbb{Q}^{\mathrm{al}}$. As noted in (1.4), we can obtain $\mathbf{Mot}(\mathbb{F}_q) \otimes L$ by replacing $Z^r(V)$ with $Z^r(V) \otimes L$ in the definition of $\mathbf{Mot}(\mathbb{F}_q)$. Just as before, there is a bijection

$$\Sigma(\mathbf{Mot}(\mathbb{F}_q) \otimes L) \longrightarrow \Gamma_L\backslash W(q), \qquad \Gamma_L = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/L).$$

Moreover, if X is a simple object of $\mathbf{Mot}(\mathbb{F}_q)\otimes L$, then $E = \mathrm{End}(X)$ is a central division algebra over $L[\pi_X]$ with rank $[E\colon L[\pi_X]]^{1/2} \cdot [L[\pi_X]\colon L]$ whose invariant at a prime v of $L[\pi_X]$ is determined by the formula $\|\pi_X\|_v = q^{\mathrm{inv}_v(E)}$. There is a canonical homomorphism of rings

$$K(\mathbf{Mot}(\mathbb{F}_q) \otimes L) \longrightarrow \mathbb{Z}[\Gamma_L\backslash W(q)].$$

On applying these remarks in the case $L = \mathbb{Q}^{\mathrm{al}}$, we obtain the following result.

PROPOSITION 2.21. *The simple objects of* $\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}^{\mathrm{al}}$ *are all of rank* 1, *and the map* $X\mapsto\pi_X$ *is a bijection*

$$\Sigma(\mathbf{Mot}(\mathbb{F}_q))\longrightarrow W(q)$$

with the property that $\pi_{X\otimes X'}=\pi_X\cdot\pi_{X'}$.

Recall (Gabriel and Demazure [**12**, p. 472]) that with any abelian group Σ, there is associated an affine group scheme $D(\Sigma)$ over k such that for any k-algebra R,

$$D(\Sigma)(R)=\mathrm{Hom}(\Sigma, R^{\times}).$$

In fact $D(\Sigma)=\mathrm{Spec}\,A$ with $A=k[\Sigma]$, and the group structure on $D(\Sigma)$ is defined by the following co-algebra structure on A:

$$\Delta(\sigma)=\sigma\otimes\sigma,\quad \epsilon\sigma=1,\quad \mathrm{inv}(\sigma)=\sigma^{-1},\quad \sigma\in\Sigma.$$

Note that Σ can be recovered from $D(\Sigma)$ because $\Sigma=X^*(D)$. The group schemes of the form $D(\Sigma)$ are said to be *diagonalizable.*

PROPOSITION 2.22. *The Tannakian category* $\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}^{\mathrm{al}}$ *is neutral, and the group associated with any fibre functor over* $\mathbb{Q}^{\mathrm{al}}$ *is the diagonalizable group scheme* $P(q)$ *with* $X^*(P(q))=W(q)$.

PROOF. We first recall a general result on Tannakian categories.

LEMMA 2.23. *Let* $\mathbf{T}$ *be a semisimple Tannakian category over a field* k *of characteristic zero. If every simple object of* $\mathbf{T}$ *has rank* 1, *then for any fibre functor* ω *of* $\mathbf{T}$ *over* k, $Aut^{\otimes}(\omega)=D(\Sigma)$, *where* Σ *is the set of isomorphism classes of simple objects in* $\mathbf{T}$ *with the group structure given by tensor product.*

PROOF. Let $G=Aut^{\otimes}(\omega)$. Then G is a pro-reductive affine group scheme over k whose simple representations are all of dimension 1. This implies that G is diagonalizable and that the simple representations correspond to the characters of G. Therefore, $X^*(G)=\Sigma(\mathbf{T})$ and $G=D(\Sigma(\mathbf{T}))$. □

Because $\mathbf{Mot}(\mathbb{F}_q)$ is Tannakian, it has fibre functor over some field Ω, which we may assume to be algebraically closed and to contain $\mathbb{Q}^{\mathrm{al}}$. Then $\mathbf{Mot}(\mathbb{F}_q)\otimes\Omega$ is neutral, and (2.23) and the analogue of (2.21) for Ω show that the affine group scheme associated with any fibre functor is $P(q)$. This implies that the band of $\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}^{\mathrm{al}}$ is represented by the affine group scheme $P(q)$ over $\mathbb{Q}^{\mathrm{al}}$. The obstruction to the existence of a fibre functor over $\mathbb{Q}^{\mathrm{al}}$ is a class in $H^2(\mathbb{Q}^{\mathrm{al}}, P(q))$ (cohomology with respect to the fpqc topology) (see [**26**, III.3.2]). In contrast to the more common cohomology groups, those with respect to the fpqc topology commute with projective limits, and so $H^2(\mathbb{Q}^{\mathrm{al}}, P)=\varprojlim H^2(\mathbb{Q}^{\mathrm{al}}, P')$ where the limit is over the algebraic quotients of P. But for an algebraic group, the cohomology groups with respect to the fpqc and fppf topologies agree (ibid. III.3.1), and so $H^2(\mathbb{Q}^{\mathrm{al}}, P)=0$. □

REMARK 2.24. For each element $\pi \in W(q)$, choose a simple motive $X(\pi)$ over $\mathbb{F}_q$ with Weil number π. Let ω be a fibre functor, and choose a nonzero element $e_\pi \in \omega(X(\pi))$ for each π. Then

$$(f_\pi) \mapsto \sum f_\pi(e_\pi): \bigoplus_{\pi \in W(q)} \mathrm{Hom}(X(\pi), X) \longrightarrow \omega(X)$$

is an isomorphism for all motives X.

Let **T** be a Tannakian category over a field k, and let ω be a fibre functor over some extension field L. Then $\mathit{Aut}^{\otimes}(\omega)$ is an affine group scheme over L. In general, it only has the structure of a band over k, but when it is commutative, it is independent of the fibre functor and it is defined over k. (For an intrinsic way of looking at the group, see the subsection on the fundamental group below.)

An affine group scheme over a field k is said to be of *multiplicative type* if it becomes diagonalizable over k^{al}. For fields k of characteristic zero, the correspondence between diagonalizable groups and abstract abelian groups extends to a correspondence between group schemes of multiplicative type and discrete Γ-modules, $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$.

COROLLARY 2.25. *The category* $\mathbf{Mot}(\mathbb{F}_q)$ *has a fibre functor* ω *over* $\mathbb{Q}^{\mathrm{al}}$; *for any such* ω, $\mathit{Aut}^{\otimes}(\omega)$ *is the group scheme of multiplicative type* $P(q)$ *over* $\mathbb{Q}$ *such that* $X^*(P(q)) = W(q)$ *(as a* Γ*-module).*

PROOF. If ω is a fibre functor for $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$, then the composite

$$\mathbf{Mot}(\mathbb{F}_q) \longrightarrow \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}} \xrightarrow{\omega} \mathbf{Vec}_{\mathbb{Q}^{\mathrm{al}}}$$

is a fibre functor for $\mathbf{Mot}(\mathbb{F}_q)$. Clearly the associated affine group scheme is a group of multiplicative type P with character group W(q), and one verifies directly that the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on P agrees with its natural action on $W(q)$. □

REMARK 2.26. The same arguments show that $\mathbf{Mot}(\mathbb{F})$ has a fibre functor over $\mathbb{Q}^{\mathrm{al}}$, and that the associated affine group scheme is the pro-torus $P(p^\infty)$ with $X^*(P(p^\infty)) = W(p^\infty)$.

The group schemes $P(q)$ **and** $P(p^\infty)$**.** By definition $P(q)$ and $P(p^\infty)$ are the affine group schemes of multiplicative type over $\mathbb{Q}$ such that

$$X^*(P(q)) = W(q), \qquad X^*(P(p^\infty)) = W(p^\infty).$$

For a CM-field $L \subset \mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$, define $W^L(q)$ to be the subgroup of $W(q)$ of $\pi \in L$ such that

$$\|\pi\|_w \in q^{\mathbb{Z}}, \qquad \text{all primes } w \text{ of } L.$$

Note that this condition has to be checked only for the primes w of L lying over p or ∞ since for other primes $\|\pi\|_w = 1$. Let $W_0^L(q)$ be the subgroup of $\pi \in W^L(q)$ of weight 0. Define group schemes over $\mathbb{Q}$ by

$$X^*(P^L(q)) = W^L(q), \qquad X^*(P_0^L(q)) = W_0^L(q).$$

PROPOSITION 2.27. *Let F be the maximal totally real subfield of L.*

(a) *If p is a square in L, then $W^L(q) = W_0^L(q) \oplus q^{\frac{1}{2}\mathbb{Z}}$; if further q is a square in $\mathbb{Q}$, then $P^L(q) = P_0^L(q) \times \mathbb{G}_m$.*

(b) *Let $q = p^n$, for $n >> 1$; there is an exact sequence*

$$0 \longrightarrow W_0^L(q)/torsion \xrightarrow{\alpha} \bigoplus_{w|p} \mathbb{Z}w \xrightarrow{\beta} \bigoplus_{v|p} \mathbb{Z}v \longrightarrow 0,$$

where the sums are over the primes of L and F, respectively, dividing p, and α and β are defined as follows:

$$\alpha(\pi) = \sum n(w) \cdot w \quad \text{if } \|\pi\|_w = q^{n(w)};$$
$$\beta(\sum n(w) \cdot w) = \sum n(w) \cdot (w|F).$$

PROOF. (a) For any integer m and $w|p$,

$$\|q^{\frac{m}{2}}\|_w = q^{-[L_w : \mathbb{Q}_p]\frac{m}{2}},$$

and the hypothesis on L implies that $[L_w : \mathbb{Q}_p]$ is even. Obviously therefore, $q^{\frac{m}{2}} \in W^L(q)$, and an element π of $W^L(q)$ of weight m can be written $\pi = (\pi/q^{\frac{m}{2}}) \cdot q^{\frac{m}{2}}$ with $(\pi/q^{\frac{m}{2}}) \in W_0^L(q)$. If q is an even power of p, then $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts trivially on $q^{\frac{1}{2}\mathbb{Z}}$, and the corresponding group scheme is $\mathbb{G}_m$.

(b) The only serious difficulty is in showing that α maps onto the kernel of β. For this one has to be able to construct Weil numbers. We explain how to do this in (4.14). □

Define

$$W^L(p^\infty) = \varinjlim W^L(p^n), \qquad W_0^L(p^\infty) = \varinjlim W_0^L(p^n),$$

and let $P^L(p^\infty)$ and $P_0^L(p^\infty)$ be the groups of multiplicative type over $\mathbb{Q}$ with character groups $W^L(p^\infty)$ and $W_0^L(p^\infty)$. Sometimes we drop the p^∞ from the notation. For any $N \geq 1$, there is a commutative diagram:

$$\begin{array}{ccc} W_0^L(q) & \xrightarrow{\alpha} & \oplus_{w|p}\mathbb{Z} \\ \downarrow{\scriptstyle \pi \mapsto \pi^N} & & \| \\ W_0^L(q^N) & \xrightarrow{\alpha} & \oplus_{w|p}\mathbb{Z}. \end{array}$$

Therefore, on passing to the limit in (2.27), we obtain the following result.

COROLLARY 2.28. (a) *If p is a square in L, $W^L(p^\infty) = W_0^L(p^\infty) \oplus \mathbb{Z}$ and $P^L(p^\infty) = P_0^L(p^\infty) \times \mathbb{G}_m$.*

(b) *There is an exact sequence*

$$0 \longrightarrow W_0^L(p^\infty) \longrightarrow \bigoplus_{w|p} \mathbb{Z}w \longrightarrow \bigoplus_{v|p} \mathbb{Z}v \longrightarrow 0.$$

In particular, we see that $P^L(p^\infty)$ is an algebraic group.

REMARK 2.29. The group $W(p^\infty)$ is torsion-free, and the subgroup $W_0(p^\infty)$ is divisible: a Weil p^n-number π of weight zero is also a Weil p^{nN}-number of weight zero, and (π, nN) represents the N^{th} root of the class of (π, n) in $W_0(p^\infty)$. Thus, $W_0(p^\infty)$ is a $\mathbb{Q}$-vector space.

Fix a CM-field $L \subset \mathbb{Q}^{\text{al}}$ Galois over $\mathbb{Q}$. Let $P_0^{L,n}(p^\infty)$ be the torus with character group $n^{-1}W_0^L(p^\infty) \subset W(p^\infty)$. For all n and N there is a commutative diagram

$$\begin{array}{ccc} P_0^{L,nN}(p^\infty) & \xrightarrow{\approx} & P_0^L(p^\infty) \\ \downarrow & & \downarrow N \\ P_0^{L,n}(p^\infty) & \xrightarrow{\approx} & P_0^L(p^\infty) \end{array}$$

corresponding to

$$\begin{array}{ccc} (nN)^{-1}W_0(p^\infty) & \xleftarrow{(nN)^{-1}} & W_0^L(p^\infty) \\ \uparrow \text{inclusion} & & \uparrow N \\ n^{-1}W_0^L(p^\infty) & \xleftarrow{n^{-1}} & W_0^L(p^\infty), \end{array}$$

and so the projective system $(P_0^{L,n}(p^\infty))_n$ is the universal covering torus[4] of $P_0^L(p^\infty)$.

The fundamental group of Mot($\mathbb{F}$). Let $\mathbf{T}$ be a Tannakian category. Then Ind($\mathbf{T}$) also has a tensor structure, and we define a commutative ring in Ind($\mathbf{T}$) to be an object A of Ind($\mathbf{T}$) together with a commutative associative product $A \otimes A \to A$ admitting an identity $1 \to A$. In order to be able to use our geometric intuition, we define the *category of affine schemes* in $\mathbf{T}$ to be the opposite of the category of commutative rings in Ind($\mathbf{T}$), and we write Sp(A) for the affine scheme in $\mathbf{T}$ corresponding to A. (For more details, see [**7**, §5].)

For example, if $\mathbf{T}$ is the category of finite-dimensional vector spaces over k, then a commutative ring in Ind($\mathbf{T}$) is just a commutative k-algebra in the usual sense, and the category of affine schemes in $\mathbf{T}$ can be identified with the category of affine schemes over k.

Since tensor products exist in the category of commutative rings in Ind($\mathbf{T}$), fibre products exist in the category of affine schemes in $\mathbf{T}$. Therefore, we can define an *affine group scheme in* $\mathbf{T}$ to be a group in the category of affine schemes in $\mathbf{T}$. An *action* of an affine group scheme $G = \text{Sp}(A)$ in $\mathbf{T}$ on an object X of $\mathbf{T}$ is a morphism $X \to X \otimes A$ satisfying the usual axioms for a comodule (Waterhouse [**31**, 3.2]).

[4]For a torus T, the projective system $(T_n, T_{mn} \xrightarrow{m} T_n)$ with $T_n = T$ for all n is called the *universal covering torus* of T. It has character group $X^*(T) \otimes \mathbb{Q}$.

THEOREM 2.30. *Let* $\mathbf{T}$ *be a Tannakian category over a field* k. *There exists an affine group scheme* $\boldsymbol{\pi}(\mathbf{T})$ *in* $\mathbf{T}$ *together with an action of* $\boldsymbol{\pi}(\mathbf{T})$ *on every object* X *of* $\mathbf{T}$ *such that, for every fibre functor* ω *over a* k*-algebra* R, *the actions of the affine group scheme* $\omega(\boldsymbol{\pi}(\mathbf{T}))$ *on the* R*-modules* $\omega(X)$ *identify* $\omega(\boldsymbol{\pi}(\mathbf{T}))$ *with* $Aut_R^{\otimes}(\omega)$. *The affine group scheme* $\boldsymbol{\pi}(\mathbf{T})$ *and the actions of it on the objects of* $\mathbf{T}$ *are uniquely determined by this condition.*

PROOF. See [7, 8.13, 8.14]. □

EXAMPLE 2.31. Let $\mathbf{T} = \mathbf{Rep}_k(G)$ with $G = \operatorname{Spec} A$. Then $\boldsymbol{\pi}(\mathbf{T}) = G$. The action of $\boldsymbol{\pi}(\mathbf{T})$ on the objects of $\mathbf{T}$ extends to objects of $\operatorname{Ind}(\mathbf{T})$, and for $\mathbf{T} = \mathbf{Rep}_k(G)$, the action of G on A is induced by the action of G on itself by inner automorphisms (ibid. 8.14).

REMARK 2.32. An exact tensor functor $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ of Tannakian categories over a field k defines a morphism $\boldsymbol{\pi}(\mathbf{T}_2) \to \eta(\boldsymbol{\pi}(\mathbf{T}_1))$ of affine group schemes in $\mathbf{T}_2$. For each object X of $\mathbf{T}_1$, $\eta(\boldsymbol{\pi}(\mathbf{T}_1))$ acts on $\eta(X)$, and this action is compatible via $\boldsymbol{\pi}(\mathbf{T}_2) \to \eta(\boldsymbol{\pi}(\mathbf{T}_1))$ with the natural action of $\boldsymbol{\pi}(\mathbf{T}_2)$ on $\eta(X)$ (ibid. 8.15).

THEOREM 2.33. *Let* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ $(\neq 0)$ *be Tannakian categories over a field* k, *and let* $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ *be an exact tensor functor. Then* η *defines a tensor equivalence of* $\mathbf{T}_1$ *with the category of pairs* (Y, ρ) *consisting of an object* Y *of* $\mathbf{T}_2$ *and an action* ρ *of* $\eta(\boldsymbol{\pi}(\mathbf{T}_1))$ *on* Y *compatible with the action of* $\boldsymbol{\pi}(\mathbf{T}_2)$.

PROOF. See [7, 8.17]. □

REMARK 2.34. When $\mathbf{T}$ is a Tannakian category over k and η is a fibre functor over k, then (2.33) becomes the fundamental classification theorem for neutral Tannakian categories ([**3**, §1]; [**8**, 2.11]).

COROLLARY 2.35. *Let* $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ *be an exact tensor functor of Tannakian categories over a field* k. *If* $\boldsymbol{\pi}(\mathbf{T}_2) \to \eta(\boldsymbol{\pi}(\mathbf{T}_1))$ *is an isomorphism, then* $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ *is an equivalence of tensor categories.*

PROOF. This is an immediate consequence of the theorem. □

COROLLARY 2.36. *Let* $\mathbf{T}$ *be a Tannakian category over* k. *An object* X *in* $\mathbf{T}$ *is isomorphic to a direct sum of copies of* 1 *if and only if* $\boldsymbol{\pi}(\mathbf{T})$ *acts trivially on it.*

PROOF. Take $\mathbf{T}_1$ in (2.33) to be the category ($\approx \mathbf{Vec}_k$) of multiples of 1 in $\mathbf{T}$, and note that $\boldsymbol{\pi}(\mathbf{Vec}_k) = 1$. □

REMARK 2.37. It follows from (2.36) that we can identify $\mathbf{Vec}_k$ with the subcategory of $\mathbf{T}$ of objects on which $\boldsymbol{\pi}(\mathbf{T})$ acts trivially. If $\boldsymbol{\pi}(\mathbf{T}) = \operatorname{Sp}(A)$ is commutative, then the action of $\boldsymbol{\pi}(\mathbf{T})$ on A is trivial, and so $\boldsymbol{\pi}(\mathbf{T})$ is an affine group scheme in the Tannakian category $\mathbf{Vec}_k \subset \mathbf{T}$, i.e., it is an affine group scheme over k in the usual sense.

PROPOSITION 2.38. *Let* $\langle 1 \rangle$ ($\approx \mathbf{Vec}_{\mathbb{Q}}$) *be the subcategory of* $\mathbf{Mot}(\mathbb{F}_q)$ *on which* $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q))$ *acts trivially. Then* $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q))$ *is the affine group scheme* *in* $\langle 1 \rangle$ *of multiplicative type having character group* $W(q)$. *Similarly,* $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}))$ *is the affine group scheme in the subcategory* $\langle 1 \rangle$ *of* $\mathbf{Mot}(\mathbb{F})$ *of multiplicative type having character group* $W(p^{\infty})$.

PROOF. The affine group scheme $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q))$ in $\mathbf{Mot}(\mathbb{F}_q)$ is commutative because its image under one (hence, every) fibre functor is commutative. The remaining statements follow from (2.25) and (2.26). □

In (3.4) we make the result more precise by describing the action of $\boldsymbol{\pi}$ on each motive.

The decomposition of $\mathbf{Mot}(\mathbb{F})$ into a tensor product. We first recall from [7, §5], the notion of the tensor product of two Tannakian categories. We say that a k-bilinear functor is *left* (or *right*) *exact* if it is left (or right) exact in each variable.

THEOREM 2.39. *Let* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *be Tannakian categories over a field* k *which, for simplicity, we take to be of characteristic zero. There exists a category* $\mathbf{T}_1 \otimes \mathbf{T}_2$ *together with a right exact* k*-bilinear functor*

$$\otimes\colon \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbf{T}_1 \otimes \mathbf{T}_2$$

such that, for any abelian k*-linear category* $\mathbf{C}$, *the functor* $\otimes$ *defines an equivalence from the category of right exact* k*-linear functors* $\mathbf{T}_1 \otimes \mathbf{T}_2 \to \mathbf{C}$ *to the category of right exact* k*-bilinear functors* $\mathbf{T}_1 \times \mathbf{T}_2 \to \mathbf{C}$.

PROOF. See [7, 5.13]. □

PROPERTIES.

(2.40.1) The pair $(\mathbf{T}_1 \otimes \mathbf{T}_2, \otimes)$ is uniquely determined up to an equivalence which itself is unique up to a unique isomorphism (ibid. p. 143).

(2.40.2) The functor $\otimes$ is exact in each variable (ibid. 5.13).

(2.40.3) For objects X_1, Y_1 of $\mathbf{T}_1$ and X_2, Y_2 of $\mathbf{T}_2$,

$$\mathrm{Hom}(X_1, Y_1) \otimes_k \mathrm{Hom}(X_2, Y_2) \xrightarrow{\approx} \mathrm{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2).$$

(2.40.4) There is a unique tensor structure on $\mathbf{T}_1 \otimes \mathbf{T}_2$ such that

$$\otimes(X_1 \otimes Y_1, X_2 \otimes Y_2) = X_1 \otimes Y_1 \otimes X_2 \otimes Y_2,$$
$$X_1, Y_1 \in \mathrm{ob}(\mathbf{T}_1), \qquad X_2, Y_2 \in \mathrm{ob}(\mathbf{T}_2).$$

(The $\otimes$ on the left is the functor $\otimes\colon \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbf{T}_1 \otimes \mathbf{T}_2$). Relative to this tensor structure, $\mathbf{T}_1 \otimes \mathbf{T}_2$ is a Tannakian category (ibid. 5.17, 6.9).

(2.40.5) The functor

$$\mathrm{inj}_1\colon \mathbf{T}_1 = \mathbf{T}_1 \otimes \mathbf{Vec}_k \to \mathbf{T}_1 \otimes \mathbf{T}_2, \qquad X_1 \mapsto X_1 \otimes 1,$$

identifies $\mathbf{T}_1$ with a full subcategory of $\mathbf{T}_1 \otimes \mathbf{T}_2$ stable under passage to subquotients. A similar statement holds for $\mathbf{T}_2$, and

$$\otimes(X_1, X_2) = (X_1 \otimes 1) \otimes (1 \otimes X_2).$$

The canonical map

$$\boldsymbol{\pi}(\mathbf{T}_1 \otimes \mathbf{T}_2) \to \operatorname{inj}_1(\boldsymbol{\pi}(\mathbf{T}_1)) \times \operatorname{inj}_2(\boldsymbol{\pi}(\mathbf{T}_2))$$

is an isomorphism. If $\mathbf{T}_1$ and $\mathbf{T}_2$ are both semisimple, then so also is $\mathbf{T}_1 \otimes \mathbf{T}_2$, and every object of $\mathbf{T}_1 \otimes \mathbf{T}_2$ is a direct factor of an object of the form $X_1 \otimes X_2$, $X_1 \in \operatorname{ob}(\mathbf{T}_1)$, $X_2 \in \operatorname{ob}(\mathbf{T}_2)$. (Ibid. p. 183.)

Let $\mathbf{Mot}_0(\mathbb{F})$ be the subcategory of $\mathbf{Mot}(\mathbb{F})$ of motives of weight zero, and let $\mathbf{E}$ be the strictly full Tannakian subcategory of $\mathbf{Mot}(\mathbb{F})$ generated by a supersingular elliptic curve A over $\mathbb{F}$. Since any two such curves are isogenous, $\mathbf{E}$ is independent of the choice of A. Note that $\mathbf{E}$ is graded and contains the Tate object.

THEOREM 2.41. *The functor*

$$(X, Y) \mapsto X \otimes Y : \mathbf{Mot}_0(\mathbb{F}) \times \mathbf{E} \longrightarrow \mathbf{Mot}(\mathbb{F})$$

defines an equivalence of tensor categories

$$\eta : \mathbf{Mot}_0(\mathbb{F}) \otimes \mathbf{E} \to \mathbf{Mot}(\mathbb{F}).$$

PROOF. According to (2.35), it suffices to check that the homomorphism

$$\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F})) \longrightarrow \eta(\pi(\mathbf{Mot}_0(\mathbb{F}) \otimes \mathbf{E}))$$

induced by η is an isomorphism. But by (2.40.5),

$$\boldsymbol{\pi}(\mathbf{Mot}_0(\mathbb{F}) \otimes \mathbf{E}) = \boldsymbol{\pi}(\mathbf{Mot}_0(\mathbb{F})) \times \pi(\mathbf{E}),$$

and the homomorphism can be identified with the isomorphism

$$P(p^\infty) \longrightarrow P_0(p^\infty) \times \mathbb{G}_m$$

of (2.28a). □

Thus, the study of $\mathbf{Mot}(\mathbb{F})$ breaks down into the study of $\mathbf{Mot}_0(\mathbb{F})$ and $\mathbf{E}$.

The polarization on $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$. For any CM-field $L \subset \mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$ and any $n \geq 1$, let $\mathbf{Mot}_0^{L,n}(\mathbb{F})$ be the subcategory of $\mathbf{Mot}_0(\mathbb{F})$ containing those motives X such that $\pi_X \in n^{-1} W_0^L(p^\infty)$. The fundamental group of $\mathbf{Mot}_0^{L,n}(\mathbb{F})$ is $P_0^{L,n}(p^\infty)$ (see Remark 2.29 for this notation).

PROPOSITION 2.42. *For any CM-field $L \subset \mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$, $\mathbf{Mot}_0^{L,n}(\mathbb{F}) \otimes \mathbb{R}$ is neutral.*

PROOF. As we explained in the proof of (2.22), the obstruction to the existence of a fibre functor is an element of $H^2(\mathbb{R}, P_0^{L,n}(p^\infty))$. But $P_0^{L,n}(p^\infty)_{\mathbb{R}}$ is an anisotropic torus over $\mathbb{R}$, and hence is isomorphic to U^d, $d = \dim P_0^{L,n}(p^\infty)$, where U is the kernel of

$$1 \longrightarrow U \longrightarrow (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}} \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

Clearly, $H^2(\mathbb{R}, U) = H^1(\mathbb{R}, \mathbb{G}_m) = 0$, and so $H^2(\mathbb{R}, P_0^{L,n}(p^\infty)) = 0$. □

We shall need the notion of a polarization of a nongraded Tannakian category. Let $\mathbf{T}$ be a Tannakian category over $\mathbb{R}$, and let Z be the centre of $\boldsymbol{\pi}(\mathbf{T})$. We can regard Z as a commutative affine group scheme over $\mathbb{R}$ in the usual sense (cf. Proposition 2.38), and $Z(\mathbb{R}) = \mathrm{Aut}^{\otimes}(\mathrm{id}_{\mathbf{T}})$. Let $\varepsilon \in Z(\mathbb{R})$, and suppose there is given for each object X of $\mathbf{T}$ an equivalence class (for the relation of compatibility) $\Pi(X)$ of Weil forms of parity ε. We say that Π is a *polarization* on $\mathbf{T}$ if

(2.43.1) for all X and Y

$$\varphi \in \Pi(X),\ \psi \in \Pi(Y) \quad \Longrightarrow \quad \varphi \oplus \psi \in \Pi(X \oplus Y);$$

(2.43.2) for all X and Y

$$\varphi \in \Pi(X),\ \psi \in \Pi(Y) \quad \Longrightarrow \quad \varphi \otimes \psi \in \Pi(X \otimes Y).$$

THEOREM 2.44. *There are exactly two graded polarizations on* $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$.

PROOF. A graded polarization on $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$ restricts to a graded polarization on $\mathbf{E}$ and a polarization of parity 1 on $\mathbf{Mot}_0(\mathbb{F}) \otimes \mathbb{R}$, and so the theorem follows from the next two lemmas. □

LEMMA 2.45. *There are exactly two graded polarizations on* $\mathbf{E} \otimes \mathbb{R}$.

PROOF. The fundamental group of $\mathbf{E}$ is $\mathbb{G}_m$. Therefore, $\mathbf{E} \otimes \mathbb{R}$ is determined up to a tensor equivalence inducing the identity map on $\mathbb{G}_m$ by its cohomology class in $H^2(\mathbb{R}, \mathbb{G}_m) = \mathrm{Br}(\mathbb{R}) = 2^{-1}\mathbb{Z}/\mathbb{Z}$. This class can not be zero, because $\mathbf{E}$ does not have a fibre functor over $\mathbb{R}$. Therefore, $\mathbf{E} \otimes \mathbb{R}$ is $\mathbb{G}_m$-equivalent to $\mathbf{V}_\infty$, which, as we observed in (1.7), has exactly two graded polarizations. □

LEMMA 2.46. *There exists a unique polarization on* $\mathbf{Mot}_0(\mathbb{F}) \otimes \mathbb{R}$ *with parity* 1.

PROOF. We first recall the classification of polarizations on neutral Tannakian categories over $\mathbb{R}$ [**8**, pp. 179–183]. Let G be an algebraic group over $\mathbb{R}$ with centre Z, and let $C \in G(\mathbb{R})$. A G-invariant bilinear form $\psi: V \times V \to \mathbb{R}$ is said to be a *C-polarization* if

$$(x, y) \mapsto \psi(x, Cy)$$

is a positive-definite symmetric form on V. When every object of $\mathbf{Rep}_{\mathbb{R}}(G)$ has a C-polarization, then C is called a *Hodge element.* There is then a polarization Π_C of $\mathbf{Rep}_{\mathbb{R}}(G)$ with parity C^2 for which the positive forms are exactly the C-polarizations. Every polarization of $\mathbf{Rep}_{\mathbb{R}}(G)$ is of the form Π_C for some Hodge element. If C and C' are Hodge elements, then there exists a $g \in G(\mathbb{R})$ and a unique $z \in Z(\mathbb{R})$ such that $C' = zgCg^{-1}$; moreover, $\Pi_{C'} = z\Pi_C$, and so $\Pi_{C'} = \Pi_C$ if and only if C and C' are conjugate in $G(\mathbb{R})$. An element $C \in G(\mathbb{R})$ such that $C^2 \in Z(\mathbb{R})$ is a Hodge element if and only if $\operatorname{ad} C$ is a Cartan involution.

Fix a CM-field L, and consider the subcategory $\mathbf{Mot}_0^{L,n}(\mathbb{F})$ of $\mathbf{Mot}(\mathbb{F})$ described above. The polarizations of parity 1 of $\mathbf{Mot}_0^{L,n}(\mathbb{F}) \otimes \mathbb{R}$ are in one-to-one correspondence with the elements C of $P_0^{L,n}(\mathbb{R})$ of order 2. Consider one such polarization Π_C. If Π_C extends to a polarization of $\mathbf{Mot}_0^{L,2n}(\mathbb{F}) \otimes \mathbb{R}$, say to $\Pi_{C'}$ where C' is an element of order 2 in $P_0^{L,2n}(\mathbb{R})$, then C' maps to C under the canonical map (2.29) $P_0^{L,2n}(\mathbb{R}) \to P_0^{L,n}(\mathbb{R})$. But it is clear from the commutative diagram in (2.29) that this map kills all elements of order 2. Therefore, $C = 1$ and we have proved the uniqueness.

Because $(P_0^{L,n})_{\mathbb{R}}$ is compact, $\mathrm{id}_{P_0^{L,n}}$ is a Cartan involution, and so the element $C = 1$ defines a polarization on $\mathbf{Mot}_0^{L,n}(\mathbb{F})$. For varying n and L, these polarizations are compatible, and so they define a polarization on $\bigcup_{L,n} \mathbf{Mot}_0^{L,n}(\mathbb{F}) = \mathbf{Mot}_0(\mathbb{F})$. □

REMARK 2.47. We have shown that the Tate conjecture implies that $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$ is polarizable. Grothendieck's standard conjectures imply more, namely, that there is a polarization on $\mathbf{Mot}(\mathbb{F})$ whose Weil forms for the motive $h(V)$ of an algebraic variety V have a specific algebraic construction **[26**, VI.4.4]. In particular, it implies that there is a polarization Π on $\mathbf{Mot}(\mathbb{F})$ such that for any abelian variety A the Weil form defined by a polarization on A (in the usual sense of algebraic geometry) lies in $\Pi(h^1(A))$.

Alternative approach. In the above we have made use of Deligne's results on the Weil conjectures. Grothendieck originally envisaged that these results would be obtained as a consequence of his standard conjectures **[15]**. The standard conjectures imply directly that $\mathbf{Mot}(\mathbb{F}_q)$ is a polarizable (hence, semisimple) Tannakian category. Using only that, we have the following result.

PROPOSITION 2.48. *Let X be a motive of weight m over $\mathbb{F}_q$, and let $\alpha \mapsto \alpha^t$ be the involution of* $\operatorname{End}(X)$ *defined by a Weil form φ. The following statements hold for $\pi = \pi_X$:*

(a) $\pi \cdot \pi^t = q^m$; *hence $\mathbb{Q}[\pi]$ is stable under the involution $\alpha \mapsto \alpha^t$;*
(b) $\mathbb{Q}[\pi] \subset \operatorname{End}(X)$ *is a product of fields;*
(c) *for every homomorphism $\rho\colon \mathbb{Q}[\pi] \to \mathbb{C}$, $\rho(\pi^t) = \iota(\rho\pi)$, and $|\rho\pi| = q^{m/2}$.*

PROOF. (a) By definition, φ is a morphism $X \otimes X \to T^{\otimes(-m)}$. It is invariant under π, and so

$$\varphi(\pi x, \pi y) = \pi(\varphi(x, y)) = q^m \varphi(x, y) = \varphi(x, q^m y).$$

But $\varphi(\pi x, \pi y) = \varphi(x, \pi^t \pi y)$, and because φ is nondegenerate, this implies that $\pi^t \cdot \pi = q^m$. Therefore $\mathbb{Q}[\pi]$ is stable under $\alpha \mapsto \alpha^t$, and we obtain (a).

(b) Let R be a commutative subalgebra of $\operatorname{End}(X)$ stable under $\alpha \mapsto \alpha^{t}$, and let r be a nonzero element of R. Then $s = rr^{t} \neq 0$ because $\operatorname{Tr}(rr^{t}) > 0$. Since $s^{t} = s$, $\operatorname{Tr}(s^{2}) = \operatorname{Tr}(ss^{t}) > 0$, and so $s^{2} \neq 0$. Similarly $s^{4} \neq 0$, and so on, which implies that s is not nilpotent, and so neither is r. Thus R is a finite-dimensional commutative $\mathbb{Q}$-algebra without nonzero nilpotents, and the only such algebras are products of fields.

(c) In an abuse of notation, we set $\mathbb{R}[\pi] = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}[\pi]$. As in (b), this is a product of fields stable under $\alpha \mapsto \alpha^{t}$. This involution permutes the maximal ideals of $\mathbb{R}[\pi]$ and, correspondingly, the factors of $\mathbb{R}[\pi]$. If the permutation were not the identity, then $\alpha \mapsto \alpha^{t}$ would not be a positive involution. Therefore each factor of $\mathbb{R}[\pi]$ is stable under the involution. The only involution of $\mathbb{R}$ is the identity map (= complex conjugation), and the only positive involution of $\mathbb{C}$ is complex conjugation. Therefore we obtain the first statement of (c), and the second then follows from (a). □

This (conjectural) proof of the Riemann hypothesis for motives is very close to Weil's original proof for abelian varieties [32].

Mixed motives over a finite field.

THEOREM 2.49. *Every mixed motive over a finite field is a direct sum of pure motives.*

PROOF. If the category of mixed motives over a finite field does not exist, then there is nothing to prove. Otherwise, according to any reasonable definition, a mixed motive X over $\mathbb{F}_q$ will have an increasing weight filtration,

$$\cdots \subset W_{i-1}X \subset W_iX \subset \cdots$$

such that $W_iX/W_{i-1}X$ is a pure motive of weight i. Let π_X be the Frobenius endomorphism of X. The same argument as in §1 shows that there is a polynomial $P_i(X)$ with rational coefficients such that $P_i(\pi_X) \cdot X = W_iX/W_{i-1}$ X, and so $X = \bigoplus W_iX/W_{i-1}X$. □

REMARK 2.50. As S. Lichtenbaum pointed out to me, the theorem is *not* expected to be true for the category of mixed motives over a finite field with coefficients in $\mathbb{Z}$ (a $\mathbb{Z}$-linear Tannakian category). In fact, it is expected that $\operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}(1)) \approx k^{\times}$ in the category of mixed motives over a field k.

Notes. The results (2.41), (2.46), and (2.49) were explained to me by Deligne (who credits them to Grothendieck). For the rest, this section represents my attempt to extend the Weil-Tate-Honda theory of abelian varieties over finite fields to motives.

3. Characterizations of the category of motives over $\mathbb{F}$ and its fibre functors

Characterization of $P(q)$ **and** $P(p^{\infty})$. Let $P = P(q)$, and let $\Gamma = \operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. Then

$$P(\mathbb{Q}) = \operatorname{Hom}(X^{*}(P), \mathbb{Q}^{\mathrm{al}\times})^{\Gamma} = \operatorname{Hom}(W(q), \mathbb{Q}^{\mathrm{al}\times})^{\Gamma}.$$

The inclusion map $W(q) \hookrightarrow \mathbb{Q}^{\mathrm{al}\times}$ commutes with the action of Γ—that is how we define the Galois action on $W(q)$—and hence corresponds to an element $f \in P(\mathbb{Q})$, which we call the *Frobenius element.* It is characterized by the following condition: if χ_π is the character of P corresponding to the Weil q-number π, then $\chi_\pi(f) = \pi$.

PROPOSITION 3.1. *Let $P = P(q)$. For any algebraic group T over $\mathbb{Q}$ of multiplicative type and any element $a \in T(\mathbb{Q})$ such that $\chi(a) \in W(q)$ for every character χ of T defined over $\mathbb{Q}^{\mathrm{al}}$, there is a unique homomorphism $\alpha\colon P \to T$ carrying f to a.*

PROOF. If α exists, then for every character χ of T, we must have

$$(\chi \circ \alpha)(f) = \chi(a).$$

Define α to be the homomorphism corresponding to the map on characters

$$X^*(T) \to W(q), \qquad \chi \mapsto \chi(a). \quad \square$$

Obviously, the pair $(P(q), f)$ is uniquely determined by the condition in the proposition (up to a unique isomorphism).

For each L Galois over $\mathbb{Q}$, there is similarly a canonical element $f^L \in P^L(p^n)$ having the following universal property: for any algebraic group T of multiplicative type over $\mathbb{Q}$ and $a \in T(\mathbb{Q})$ such that $\chi(a) \in W^L(p^n)$ for all characters of T, there is a unique homomorphism $\alpha\colon P^L(p^n) \to T$ such that $\alpha(f^L) = a$.

There is a similar, but more complicated, characterization of $P(p^\infty)$, but first we compare $P(p^\infty)$ with $P(q)$.

PROPOSITION 3.2. *Let $L \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field Galois over $\mathbb{Q}$, and let m be the number of roots of 1 in L. For $n >> 1$, there is an exact sequence*

$$0 \longrightarrow P^L(p^\infty) \longrightarrow P^L(p^n) \xrightarrow{f^L \mapsto 1} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

identifying $P^L(p^\infty)$ with the identity component of $P^L(p^n)$. For any n, there is an exact sequence

$$P(p^\infty) \longrightarrow P(p^n) \xrightarrow{f \mapsto 1} \widehat{\mathbb{Z}} \longrightarrow 0.$$

PROOF. It follows from (2.27b) that, for $n >> 1$, the map

$$W^L(p^n)/\text{torsion} \longrightarrow W^L(p^\infty)$$

is bijective. Since the torsion subgroup of $W^L(p^n)$ is $\mu(L)$, the group of roots of 1 in L, this gives an exact sequence

$$0 \longrightarrow \mu(L) \longrightarrow W^L(p^n) \longrightarrow W^L(p^\infty) \longrightarrow 0.$$

This is the sequence of character groups of the first sequence in the proposition. Because $X^*(P^L(p^\infty))$ is the quotient of $X^*(P^L(p^n))$ by its torsion subgroup, $P^L(p^\infty)$ is the identity component of $P^L(p^n)$. The second exact sequence can be derived in the same way as the first. □

Consider $f^L \in P^L(p^n)(\mathbb{Q})$. Then $(f^L)^m \in P^L(p^\infty)(\mathbb{Q})$ if m is the number of roots of 1 in L, and we write f^L_{nm} for this element. In this way we obtain a family $(f^L_n)_{n>>1}$ of elements of $P^L(p^\infty)(\mathbb{Q})$ with the property that $(f^L_n)^N = f^L_{nN}$ for all $N > 1$.

If $L' \supset L$, then $f^{L'}_n \mapsto f^L_n$ under $P^{L'}(p^\infty)(\mathbb{Q}) \to P^L(p^\infty)(\mathbb{Q})$ whenever $f^{L'}_n$ is defined. Unfortunately, as L grows, the smallest n for which f^L_n is defined tends to infinity. Thus, for no n do we get an element $(f^L_n)_L \in P(p^\infty)(\mathbb{Q}) \underset{\text{df}}{=} \varprojlim P^L(p^\infty)(\mathbb{Q})$.

This suggests the following definition: let M be an affine group scheme over a field k, and write it as a projective limit, $M = \varprojlim M^L$, of its quotients of finite type; suppose that for each L and $n >> 1$ (depending on L) there is given an element $f^L_n \in M^L(k)$; if for each $L < L'$ and $n|n'$, the element $(f^L_n)^{n'/n}$ is the image of $f^{L'}_{n'}$ under the map $M^{L'}(k) \to M^L(k)$, then we call the family (f^L_n) a *germ of an element* of $M(k)$. Note that, for any homomorphism $\alpha\colon M \to G$ from M into an algebraic group G, there is a well-defined element $\alpha(f_n) \in G(k)$, $n >> 1$, since we can set $\alpha(f_n) = \alpha(f^L_n)$ for any choice of L such that α factors through M^L.

PROPOSITION 3.3. *There is a unique germ of an element $f = (f^L_n)$ in $P(p^\infty)(\mathbb{Q})$ having the following property: for any algebraic group T over $\mathbb{Q}$ of multiplicative type and element $a \in T(\mathbb{Q})$ such that $\chi(a) \in W(p^n)$ for every character of T defined over $\mathbb{Q}^{\text{al}}$, there is a unique homomorphism $\alpha\colon P \to T$ such that $\alpha(f_{nN}) = a^N$ for some $N \geq 1$.*

PROOF. The proof is straightforward from the above discussion. □

Applications. We apply the above results and Theorem 2.33 to obtain descriptions of $\mathbf{Mot}(\mathbb{F}_q)$, $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$, and $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell$.

Recall (2.38) that $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q)) = P(q)$, and that it acts on each object X of $\mathbf{Mot}(\mathbb{F}_q)$.

LEMMA 3.4. *The element f of $P(q)(\mathbb{Q})$ acts on a motive X over $\mathbb{F}_q$ as π_X; the germ of an element f of $P(p^\infty)(\mathbb{Q})$ acts on a motive X over $\mathbb{F}$ as π_X.*

PROOF. The first statement follows directly from the various definitions, and the second follows directly from the first. □

PROPOSITION 3.5. *Let $q = p^n$. The natural functor*

$$\mathbf{Mot}(\mathbb{F}_q) \longrightarrow \mathbf{Mot}(\mathbb{F})$$

identifies $\mathbf{Mot}(\mathbb{F}_q)$ with the category of pairs (X, π) consisting of a motive X over $\mathbb{F}$ and an endomorphism π of X such that (π, n) represents π_X.

PROOF. According to (2.33), the functor identifies $\mathbf{Mot}(\mathbb{F}_q)$ with the category of pairs (X, ρ) in which X is a motive over $\mathbb{F}$ and ρ is an action of $P(q)$ on X compatible with the action of $P(p^\infty)$. It follows from (3.1) that to give an action of $P(q)$ on X commuting with the action of the endomorphisms of X is to give an element $\pi \in (\mathbb{G}_m)_{\mathbb{Q}[\pi_X]/\mathbb{Q}}$ such that $\chi(\pi) \in W(q)$ for all characters χ. The action of $P(q)$ on X defined by π is compatible with the action of P if and only if (π, n) represents π_{X_n}. □

Since the proposition determines $\mathbf{Mot}(\mathbb{F}_q)$ in terms of $\mathbf{Mot}(\mathbb{F})$, we shall concentrate on characterizing $\mathbf{Mot}(\mathbb{F})$.

PROPOSITION 3.6. *The choice of a functor*

$$\omega_\infty : \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R} \longrightarrow \mathbf{V}_\infty$$

as in (1.10) *identifies* $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$ *with the category of pairs* (V, π) *consisting of an object* V *of* $\mathbf{V}_\infty$ *and a semisimple endomorphism* π *of* V *whose eigenvalues on the part of* V *of weight* m *are Weil* q*-numbers of weight* m. *Similarly, the choice of a functor*

$$\omega_\infty : \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R} \longrightarrow \mathbf{V}_\infty$$

identifies $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$ *with the category of pairs* $(V, (\pi_n))$, *where* (π_n) *is a germ of an endomorphism of* V *satisfying an analogous condition.*

PROOF. The fundamental group of $\mathbf{V}_\infty$ is $\mathbb{G}_m$, and the map $\mathbb{G}_m \to P(q)_\mathbb{R}$ induced by ω_∞ is the weight map w (corresponding to the map on characters $\pi \to \mathrm{wt}(\pi)$). According to (2.33), ω_∞ identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$ with the category of pairs (V, ρ) in which V is an object of $\mathbf{V}_\infty$ and ρ is an action of P on V compatible with the action of $\mathbb{G}_m$. To give such a ρ is the same as to give an endomorphism π as in the statement of the proposition. □

Let Frob_n be the geometric Frobenius element $x \mapsto x^{p^{-n}}$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^n})$.

PROPOSITION 3.7. *Let* $q = p^n$. *For* $\ell \neq p, \infty$, *the functor*

$$\omega_\ell : \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}_q)$$

identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$ *with the full subcategory of* $\mathbf{V}_\ell(\mathbb{F}_q)$ *consisting of semisimple representations* (V, ρ) *of* $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^n})$ *such that the eigenvalues of* $\rho(\mathit{Frob}_n)$ *are Weil* q*-numbers. The functor*

$$\omega_\ell : \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F})$$

identifies $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell$ *with the full subcategory of* $\mathbf{V}_\ell(\mathbb{F})$ *consisting of germs of semisimple representations* $(V, [\rho])$ *such that, for any* $\rho \in [\rho]$ *and any* n *for which it is defined,* $\rho(\mathit{Frob}_n)$ *has eigenvalues that are Weil* p^n*-numbers.*

PROOF. The functor ω_ℓ is fully faithful, and so (2.33) shows that it identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$ with the full subcategory of $\mathbf{V}_\ell(\mathbb{F}_q)$ of objects for which the action of $\boldsymbol{\pi}(\mathbf{V}_\ell(\mathbb{F}))$ factors through $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell)$. The fundamental group of $\mathbf{V}_\ell(\mathbb{F}_q)$ is the group of multiplicative type with character group U, the group of units in the ring of integers in $\mathbb{Q}_\ell^{\mathrm{al}}$, and the map on fundamental groups corresponds to the inclusion $W(q) \hookrightarrow U$ defined by some choice of an embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_\ell^{\mathrm{al}}$. The first statement is now clear, and the second is proved similarly. □

PROPOSITION 3.8. *The functor*

$$\omega_p\colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F}_q)$$

identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_p$ *with the full subcategory of objects* (M, F_M) *in* $\mathbf{V}_p(\mathbb{F}_q)$ *such that* π_M *acts semisimply on* M *with eigenvalues that are Weil* q*-numbers. The functor*

$$\omega_p\colon \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F}_q)$$

identifies $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_p$ *with the full subcategory of objects* (M, F_M) *in* $\mathbf{V}_p(\mathbb{F})$ *such that, for some model* $(M', F_{M'})$ *of* (M, F_M) *over a finite field* $\mathbb{F}_q$, $\pi_{M'}$ *acts semisimply on* M *with eigenvalues that are Weil* q*-numbers.*

PROOF. The proof is similar to that of (3.7). □

The cohomology of P. Choose a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ lying over p, and use the same symbol to denote its restriction to any subfield. Let $L \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field Galois over $\mathbb{Q}$, and let $D(w_0) \subset \mathrm{Gal}(L/\mathbb{Q})$ be the decomposition group of w_0. Define $E = L^{D(w_0)}$, and let F be the maximal totally real subfield of E. Thus, either $\iota \notin D(w_0)$ and E is a CM-field with F as its maximal totally real subfield, or $\iota \in D(w_0)$ and E and F are equal and totally real.

PROPOSITION 3.9. *There is an exact sequence*

$$0 \longrightarrow (\mathbb{G}_m)_{F/\mathbb{Q}} \longrightarrow (\mathbb{G}_m)_{E/\mathbb{Q}} \longrightarrow P_0^L(p^\infty) \longrightarrow 0.$$

PROOF. To verify that a sequence of tori is exact, it suffices to check that the corresponding sequence of character groups is exact. But on applying X^* to the sequence in the corollary, we obtain the sequence in (2.28b). □

PROPOSITION 3.10. *There are exact sequences*

$$0 \longrightarrow F^\times \longrightarrow E^\times \longrightarrow H^0(\mathbb{Q}, P_0^L(p^\infty)) \longrightarrow 0,$$

$$0 \longrightarrow H^1(\mathbb{Q}, P_0^L(p^\infty)) \longrightarrow Br(F) \longrightarrow Br(E) \longrightarrow H^2(\mathbb{Q}, P_0^L(p^\infty)) \longrightarrow 0.$$

PROOF. Except for the zero at the right of the second sequence, the statement follows directly from the preceding proposition and Hilbert's Theorem 90, but a theorem of Tate shows that $H^3(F, \mathbb{G}_m) \xrightarrow{\approx} \bigoplus_{v \text{ real}} H^3(F_v, \mathbb{G}_m)$ and $H^3(\mathbb{R}, \mathbb{G}_m) = H^1(\mathbb{R}, \mathbb{G}_m) = 0$ (see [**20**, I.4.10]). □

For an affine group scheme G over a field K, we define

$$H^r(K, G) = \varprojlim H^r(K, G') \qquad \text{(Galois cohomology)},$$

where the limit is over the quotients G' of G of finite type over K. When K is a number field, we set

$$\mathrm{Ker}^r(K, G) = \mathrm{Ker}(H^r(K, G) \longrightarrow \textstyle\prod_v H^r(K_v, G))$$

(product over all primes of K).

Proposition 3.11. *Let $L \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field Galois over $\mathbb{Q}$. Then*

(a) $\mathrm{Ker}^1(\mathbb{Q}, P_0^L(p^\infty)) = 0$;
(b) $H^1(\mathbb{Q}, P_0(p^\infty)) = 0 = H^1(\mathbb{Q}, P(p^\infty))$;
(c) $H^2(\mathbb{Q}, P_0^L(p^\infty)) \xrightarrow{\approx} \bigoplus_\ell H^2(\mathbb{Q}_\ell, P_0^L(p^\infty))$ *(sum over all primes of $\mathbb{Q}$)*;
(d) $\mathrm{Ker}^2(\mathbb{Q}, P^L(p^\infty)) = 0$ *when L contains $\sqrt{p}$.*

Proof. We drop p^∞ from the notation.

(a) The map

$$H^1(\mathbb{Q}, P_0^L) \longrightarrow \bigoplus_\ell H^1(\mathbb{Q}_\ell, P_0^L)$$

can be identified with the map

$$\mathrm{Br}(E/F) \longrightarrow \bigoplus_v \mathrm{Br}(E_w/F_v)$$

(sum over all primes of F; w is a prime of E lying over v), which class field theory shows to be injective.

(b) Let T be a torus, and let $\widetilde{T}$ be the universal covering torus of T. One sees immediately from the definition of $\widetilde{T}$ that, for any contravariant functor H from tori to abelian groups, $H(\widetilde{T}) = H(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular,

$$H^1(\mathbb{Q}, \widetilde{P}_0^L) = H^1(\mathbb{Q}, P_0^L) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Br}(E/F) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

But, as we noted in (2.29), the map $P_0 \to P_0^L$ factors through $\widetilde{P}_0^L \to P_0^L$, and so the map $H^1(\mathbb{Q}, P_0) \to H^1(\mathbb{Q}, P_0^L)$ is zero. Since $H^1(\mathbb{Q}, P_0) = \varprojlim_L H^1(\mathbb{Q}, P_0^L)$, this shows that it is zero. From the cohomology sequence of

$$0 \longrightarrow \mathbb{G}_m \longrightarrow P \longrightarrow P_0 \longrightarrow 0$$

we see that

$$H^1(\mathbb{Q}, P_0) = 0 \implies H^1(\mathbb{Q}, P) = 0.$$

(c) Consider the exact commutative diagram,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \mathrm{Br}(\mathrm{E}/\mathrm{F}) & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(\mathrm{E}) & \longrightarrow & H^2(\mathbb{Q}, P_0^L) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & \bigoplus_v \mathrm{Br}(E_w/F_v) & \longrightarrow & \bigoplus_v \mathrm{Br}(F_v) & \longrightarrow & \bigoplus_w \mathrm{Br}(E_w) & \longrightarrow & \bigoplus_v H^2(\mathbb{Q}_p, P_0^L) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \frac{1}{2}\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \xrightarrow{2} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the two middle columns are given by class field theory, and the top two rows are the cohomology sequences of the sequence in (3.9). A diagram chase shows that $H^2(\mathbb{Q}, P_0^L) \xrightarrow{\approx} \bigoplus_\ell H^2(\mathbb{Q}_\ell, P_0^L)$.

(d) If $\sqrt{p} \in L$, then $P^L = P_0^L \oplus \mathbb{G}_m$, and

$$\mathrm{Ker}^2(\mathbb{Q}, P^L) = \mathrm{Ker}^2(\mathbb{Q}, P_0^L) \oplus \mathrm{Ker}^2(\mathbb{Q}, \mathbb{G}_m) = 0. \quad \square$$

Characterization of Mot($\mathbb{F}$). As we noted in §2, the Frobenius endomorphisms of motives over $\mathbb{F}_q$ form a tensor endomorphism of the identity functor, i.e., $\alpha \circ \pi_X = \pi_Y \circ \alpha$ for any morphism $\alpha \colon X \to Y$, $\pi_1 = \mathrm{id}$, and $\pi_{X\otimes Y} = \pi_X \otimes \pi_Y$. In order to handle the Frobenius endomorphisms of motives over $\mathbb{F}$, we define (for any Tannakian category $\mathbf{T}$) a *germ of a tensor endomorphism* of $\mathrm{id}_{\mathbf{T}}$ to be a family π_X of germs of endomorphisms satisfying the same three conditions. For example, the Frobenius endomorphisms of the motives over $\mathbb{F}$ form a germ of a tensor endomorphism of $\mathrm{id}_{\mathbf{Mot}(\mathbb{F})}$.

Consider a Tannakian category $\mathbf{T}$ over $\mathbb{Q}$ and a germ π of a tensor endomorphism of $\mathrm{id}_{\mathbf{T}}$ such that

(3.12.1) For all objects X, $\mathrm{End}\, X$ is a semisimple algebra with centre $\mathbb{Q}[\pi_X]$ (hence, $\mathbf{T}$ is a semisimple category).

(3.12.2) For all simple objects X and representatives (π, n) for π_X, π is a Weil p^n-number. Moreover, the invariants of $E = \mathrm{End}(X)$ (as a central division algebra over $\mathbb{Q}[\pi_X]$) are given by the rule,

$$\|\pi\|_v = (p^n)^{\mathrm{inv}_v(E)},$$

and

$$\mathrm{rank}\, X = [E \colon \mathbb{Q}[\pi_X]]^{\frac{1}{2}} \cdot [\mathbb{Q}[\pi_X] \colon \mathbb{Q}].$$

(3.12.3) The map $X \mapsto [\pi_X]$ defines a bijection $\Sigma(\mathbf{T}) \longrightarrow \Gamma \backslash W(p^\infty)$.

For example, the pair $(\mathbf{Mot}(\mathbb{F}), (\pi_X))$ satisfies these conditions, and the next theorem shows that they determine it up to equivalence.

THEOREM 3.13. *Let* $(\mathbf{T}, \pi)$ *and* $(\mathbf{T}', \pi')$ *be two pairs satisfying the conditions* (3.12).

(a) *There is a tensor equivalence* $S: \mathbf{T} \to \mathbf{T}'$ *such that, for all objects* X *of* $\mathbf{T}$, $S(\pi_X) = \pi_{S(X)}$.
(b) *If* S_1 *and* S_2 *are two such tensor equivalences, then there is an isomorphism* $\alpha: S_1 \to S_2$ *of tensor functors; if* α' *is a second such isomorphism, then there is an* $a \in \mathbb{Q}^\times$ *such that* $\alpha' = w(a) \cdot \alpha$ *(i.e., such that* $\alpha'_X = a^m \alpha_X$ *if* X *is pure of weight* m*).*

PROOF. The proof will occupy the rest of this subsection.

Let $\mathbf{T}$ be a semisimple Tannakian category over a field K of characteristic zero, and consider $\mathbf{T} \otimes L$ where $L \supset K$ is a field. Let X be a simple object of $\mathbf{T}$, and let C be the centre of End(X). Then C is a field, and $X \otimes_K L$ decomposes into a sum of isotypic objects according as $C \otimes_K L$ decomposes into a product of fields (see the proof of 2.17). In more detail, if

$$C \otimes_K L = C_1 \times \cdots \times C_r,$$

then

$$\mathrm{End}(X \otimes L) = \prod \mathrm{End}(X) \otimes_C C_i$$

and $\mathrm{End}(X) \otimes_C C_i$ is a central simple algebra over C_i.

LEMMA 3.14. *In the above situation, there is a well-defined map*

$$\Sigma(\mathbf{T} \otimes L) \longrightarrow \Sigma(\mathbf{T})$$

sending the isomorphism class $[Y]$ *of a simple object* Y *of* $\mathbf{T} \otimes L$ *to* $[X]$ *if* Y *is a factor* $X \otimes L$. *The map is surjective, and when* L *is Galois over* K, *its fibres are the orbits of* $\mathrm{Gal}(L/K)$ *acting on* $\Sigma(\mathbf{T} \otimes L)$.

PROOF. From (1.3) we know that $\mathbf{T} \otimes L$ is a semisimple Tannakian category over L and every object of $\mathbf{T} \otimes L$ is a factor of an object of the form $X \otimes L$, $X \in \mathrm{ob}(\mathbf{T})$. Let Y be a simple object of $\mathbf{T} \otimes L$. Clearly it is a factor of $X \otimes L$ for some simple X. If it is also a factor of $X' \otimes L$ with X' simple, then

$$\mathrm{Hom}(X, X') \otimes L = \mathrm{Hom}(X \otimes L, X' \otimes L) \neq 0,$$

and so $X \approx X'$. Thus the map is well defined. It is obviously surjective.

Assume L is a Galois extension of K. The fibres of the map are invariant under the action of $\mathrm{Gal}(L/K)$, and hence are the unions of orbits. Let X be a simple object of $\mathbf{T}$, and let C be the centre of $\mathrm{End}(X)$. The elements of the fibre over [X] are indexed by the set of factors of $C \otimes_K L$, which equals $\mathrm{Hom}_K(C, L)$, and $\mathrm{Gal}(L/K)$ acts transitively on this set. □

When we apply the lemma to a pair $(\mathbf{T}, \pi)$ as in (3.12) and $L = \mathbb{Q}^{\mathrm{al}}$, we see that there is a canonical map

$$\Sigma(\mathbf{T} \otimes \mathbb{Q}^{\mathrm{al}}) \longrightarrow \Sigma(\mathbf{T}),$$

and for a simple X in $\mathbf{T}$ the fibre over $[X]$ is $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_X], \mathbb{Q}^{\mathrm{al}})$. But this set can be identified with $[\pi_X]$, and so there is a canonical map $\Sigma(\mathbf{T}\otimes\mathbb{Q}^{\mathrm{al}}) \to W(p^{\infty})$ making the following diagram commute:

$$\begin{array}{ccc} \Sigma(\mathbf{T} \otimes \mathbb{Q}^{\mathrm{al}}) & \longrightarrow & W(p^{\infty}) \\ \downarrow & & \downarrow \\ \Sigma(\mathbf{T}) & \longrightarrow & \Gamma \backslash W(p^{\infty}). \end{array}$$

Now the same arguments as in the proof of (2.22) show that there is a unique isomorphism $P \to \boldsymbol{\pi}(\mathbf{T})$ such that f acts on X as π_X, all X. Here $P = P(p^{\infty})$.

Let $(\mathbf{T}', \pi')$ be a second pair satisfying (3.12). A tensor equivalence $S: \mathbf{T} \to \mathbf{T}'$ maps π to π' if and only if it induces the identity map on P. Therefore there exists such an S if and only if $\mathbf{T}$ and $\mathbf{T}'$ define the same class[5] in $H^2(\mathbb{Q}, P)$. Thus we have to show that the conditions (3.12) determine this class.

Let X be a simple object of $\mathbf{T}$. The action of P on X defines a homomorphism $P \to (\mathbb{G}_m)_{\mathbb{Q}[\pi_X]/\mathbb{Q}}$, which is uniquely determined by the fact that it sends f to π_X.

LEMMA 3.15. *The map*

$$H^2(\mathbb{Q}, P) \longrightarrow H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\pi_X]/\mathbb{Q}}) = Br(\mathbb{Q}[\pi_X]),$$

sends the class of $\mathbf{T}$ *in* $H^2(\mathbb{Q}, P)$ *to the class of* $\mathrm{End}\, X$ *in* $Br(\mathbb{Q}[\pi_X])$.

PROOF. This can be proved by the same argument as [**26**, VI.3.5.3]. □

Thus we have to prove the following statement:

($*$) An element c of $H^2(\mathbb{Q}, P)$ is zero if its image in $H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})$ is zero for all $\pi \in W(p^{\infty})$.

The group $P = P_0 \times \mathbb{G}_m$, and the projection $P \to \mathbb{G}_m$ can be identified with the map $P \to (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}}$ where π is represented by $(p^{\frac{n}{2}}, n)$ for any even n (see 2.28). Therefore the component of c in $H^2(\mathbb{Q}, \mathbb{G}_m)$ is zero. Henceforth, we regard c as an element of $H^2(\mathbb{Q}, P_0)$.

[5]We are using that the gerb of fibre functors determines a Tannakian category up to a unique equivalence [**26**, III.3.2.3.2], that gerbs with band B are classified up to B-equivalence by $H^2(k, B)$ (this is how $H^2(k, B)$ is defined in [**14**], and that when B is the band defined by a smooth affine commutative group scheme P, $H^2(k, B)$ equals the group $H^2(k, P)$ defined above [**26**, III.3.1].

Let c^L be the image of c in $H^2(\mathbb{Q}, P_0^L)$. Because $\mathrm{Ker}^2(\mathbb{Q}, P_0^L) = 0$ (see 3.11), it suffices to show that the image of c^L in $H^2(\mathbb{Q}_\ell, P_0^L)$ is zero for all ℓ. This is automatic for $\ell = \infty$ because $H^2(\mathbb{R}, P_0^L) = 0$ (see the proof of 2.42).

Thus consider an $\ell \neq \infty$, and let $D(\ell)$ be the decomposition group of some prime of $\mathbb{Q}^{\mathrm{al}}$ lying over ℓ. Let $\pi \in W_0^L(p^\infty)$. A standard duality theorem [**22**, I.2.4] shows that the map

$$H^2(\mathbb{Q}_\ell, P_0^L) \longrightarrow H^2(\mathbb{Q}_\ell, (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})$$

is obtained from the map

$$X^*(P_0^L)^{D(\ell)} \leftarrow X^*((\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})^{D(\ell)}$$

by applying the functor $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$.

Thus we have to prove the following statement:

> (**) Every element of $W_0^L(p^\infty)^{D(\ell)}$ is the image of an element of $X^*((\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})^{D(\ell)}$ for some $\pi \in W_0^L(p^\infty)$.

We note that

$$X^*((\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}}) = \mathbb{Z}^{\mathrm{Hom}(\mathbb{Q}[\pi], \mathbb{Q}^{\mathrm{al}})}$$

and that the map

$$\mathbb{Z}^{\mathrm{Hom}(\mathbb{Q}[\pi], \mathbb{Q}^{\mathrm{al}})} \longrightarrow W_0^L(p^\infty)$$

is $\chi \mapsto \chi(\pi)$.

Let $\pi \in W_0^L(p^\infty)$ be represented by (π', n), $\pi' \in W_0^L(p^n)$. By definition, $\mathbb{Q}[\pi] = \mathbb{Q}[\pi'^N]$ for all $N >> 1$. If π is fixed by $D(\ell)$, then the elements of $D(\ell)$ multiply π' by roots of 1, and so π'^N is fixed by $D(\ell)$ for all $N >> 1$. Hence, $\mathbb{Q}[\pi]$ (as a subfield of $\mathbb{Q}^{\mathrm{al}}$) is fixed by $D(\ell)$, and if we denote the given inclusion $\mathbb{Q}[\pi] \hookrightarrow \mathbb{Q}^{\mathrm{al}}$ by σ_0, then π is the image of the element $\chi = \sigma_0 \in (\mathbb{Z}^{\mathrm{Hom}(\mathbb{Q}[\pi], \mathbb{Q}^{\mathrm{al}})})^{D(\ell)}$, which proves (**).

Thus we have a tensor equivalence $S_1 : \mathbf{T} \to \mathbf{T}'$ sending π to π'. If S_2 is a second such equivalence, then $\mathcal{H}om^{\otimes}(S_1, S_2)$ is a torsor for $\mathcal{A}ut^{\otimes}(S_1) = P$. But $H^1(\mathbb{Q}, P) = 0$, and so the torsor is trivial. Therefore, there exists a tensor isomorphism $\alpha : S_1 \to S_2$. A second such isomorphism α' is of the form $\alpha' = \alpha \circ \beta$, where β is a tensor automorphism of S_1. But this is an element of $P(\mathbb{Q})$. The next lemma implies that $P_0(\mathbb{Q}) = 0$, and so $P(\mathbb{Q}) = \mathbb{Q}^\times$. □

LEMMA 3.16. *For any torus T over $\mathbb{Q}$, $\widetilde{T}(\mathbb{Q}) = 0$.*

PROOF. An element of $\widetilde{T}(\mathbb{Q})$ is a family $(a_n)_{n \geq 1}$, $a_n \in T(\mathbb{Q})$, such that $a_n = (a_{mn})^m$. In particular, a_n is infinitely divisible. If $T = (\mathbb{G}_m)_{L/\mathbb{Q}}$, then $T(\mathbb{Q}) = L^\times$, and $\bigcap L^{\times m} = 1$. Every torus T can be embedded in a product of tori of the form $(\mathbb{G}_m)_{L/\mathbb{Q}}$, and so again $\bigcap T(\mathbb{Q})^m = 1$. □

REMARK 3.17. (a) We shall prove in (3.32) below that, without assuming any conjectures, there does exist a pair $(\mathbf{T}, \pi)$ satisfying (3.12).

(b) The same proof shows that the pair $(\mathbf{Mot}_0(\mathbb{F}), \pi)$ is characterized by the conditions (3.12) (with π required to be a Weil p^n-number of weight 0 in (3.12.2)) up to a tensor equivalence which itself is uniquely determined up to unique isomorphism.

(c) The category $\mathbf{Mot}(\mathbb{F})$ has a canonical Tate object T and a canonical isomorphism class of objects

$$\{h^1(A) \mid A \text{ a supersingular elliptic curve over } \mathbb{F}\}.$$

There is a unique polarization Π on $\mathbf{Mot}(\mathbb{F})$ such that, whenever A is a supersingular elliptic curve, $\Pi(h^1(A))$ is the set of Weil forms defined by a polarization of A. For $a \in \mathbb{Q}^\times$, $w(a)$ acts on T as a^{-2}, and $w(-1)$ maps Π to a different polarization. Consequently, the system $(\mathbf{Mot}(\mathbb{F}), \pi, T, \{h^1(A)\}, \Pi)$ is uniquely determined up to a tensor equivalence (preserving π, T, $\{h^1(A)\}$, and Π) which itself is uniquely determined up to a unique isomorphism.

Characterization of Mot($\mathbb{F}$) and its fibre functors. We now characterize $\mathbf{Mot}(\mathbb{F})$ together with its standard fibre functors. Consider a triple $(\mathbf{T}, \pi, \omega)$ where

(3.18.1) $\mathbf{T}$ is a semisimple Tannakian category over $\mathbb{Q}$ for which there exists a tensor functor $\omega_\infty : \mathbf{T} \to \mathbf{V}_\infty$ preserving weights;

(3.18.2) π is a germ of an endomorphism of $\mathrm{id}_\mathbf{T}$ for which there exists an isomorphism $\gamma : P \to \boldsymbol{\pi}(\mathbf{T})$ sending f to π;

(3.18.3) $\omega = (\omega^p, \omega_p)$ with ω^p a fibre functor over $\mathbb{A}_f^p$ and ω_p an exact tensor functor $\mathbf{T} \to \mathbf{V}_p(\mathbb{F})$ such that, for each object X of $\mathbf{T}$, $\omega_p(f_X) = \pi_{\omega_p(X)}$.

The system $(\mathbf{Mot}(\mathbb{F}), \pi, \omega)$ satisfies these conditions, and the next theorem shows that they determine it up to equivalence.

THEOREM 3.19. *Suppose* $(\mathbf{T}, \pi, \omega)$ *and* $(\mathbf{T}', \pi', \omega')$ *are two triples satisfying* (3.18). *There exists an equivalence of tensor categories* $S : \mathbf{T} \to \mathbf{T}'$ *carrying* π *into* π' *and isomorphisms* $s = (s^p, s_p)$ *of fibre functors on* $\mathbf{T}$,

$$s^p : \omega^p \longrightarrow \omega'^p \circ S,$$

$$s_p : \omega_p \longrightarrow \omega'_p \circ S.$$

PROOF. By assumption,

$$\boldsymbol{\pi}(\mathbf{T}) = P = \boldsymbol{\pi}(\mathbf{T}'),$$

and an equivalence $S : \mathbf{T} \to \mathbf{T}'$ of tensor categories will map f to f' if and only if it induces the identity map on P. There exists such an S if and only if $\mathbf{T}$ and $\mathbf{T}'$ have the same cohomology class in $H^2(\mathbb{Q}, P)$. Because

$\mathrm{Ker}^2(\mathbb{Q}, P) = 0$, it suffices to check this locally. By assumption, there is a functor $\omega_\infty\colon \mathbf{T}\otimes\mathbb{R} \to \mathbf{V}_\infty$ such that the map $\boldsymbol{\pi}(\mathbf{V}_\infty) \to \boldsymbol{\pi}(\mathbf{T})$ is the weight map $w\colon \mathbb{G}_m \to P$. Therefore the class of $\mathbf{T}\otimes\mathbb{R}$ in $H^2(\mathbb{R}, P)$ is the image of the class of $\mathbf{V}_\infty$ in $H^2(\mathbb{R}, \mathbb{G}_m)$ under the map defined by w. Similarly, the functor $\omega_p\colon \mathbf{T} \to \mathbf{V}_p(\mathbb{F})$ determines the class of $\mathbf{T}\otimes\mathbb{Q}_p$ in $H^2(\mathbb{Q}_p, P)$. Finally, the assumption that there is a fibre functor over $\mathbb{Q}_\ell$ for all $\ell \neq p, \infty$, implies that the class of $\mathbf{T}$ in $H^2(\mathbb{Q}_\ell, P)$ is zero. Hence S exists.

Because $H^1(\mathbb{Q}, P) = 0$, the functor S is unique up to isomorphism.

Choose one S. Then ω_p and $\omega'_p \circ S$ are both fibre functors on $\mathbf{T}$, and $Hom^{\otimes}(\omega_p, \omega'_p \circ S)$ is a torsor for P over $\mathbb{Q}_p$. Since $H^1(\mathbb{Q}_p, P) = 0$, we see that there is an isomorphism $s_p\colon \omega_p \to \omega'_p \circ S$. The proof that s^p exists is similar. □

For the subcategory $\mathbf{Mot}_0(\mathbb{F})$ of motives of weight zero, we can be a little more precise.

THEOREM 3.20. *Let* $(\mathbf{T}, \pi, \omega)$ *and* $(\mathbf{T}', \pi', \omega')$ *be two triples satisfying the conditions* (3.18) *with* P *replaced by* P_0. *There exists an equivalence of tensor categories* $S\colon \mathbf{T} \to \mathbf{T}'$ *carrying* π *into* π' *and isomorphisms* $s = (s^p, s_p)$ *of fibre functors on* $\mathbf{T}$,

$$s^p\colon \omega^p \longrightarrow \omega'^p \circ S,$$

$$s_p\colon \omega_p \longrightarrow \omega'_p \circ S.$$

Any two such pairs (S_1, s_1) *and* (S_2, s_2) *are isomorphic, i.e., there is an isomorphism of tensor functors* $\alpha\colon S_1 \to S_2$ *such that the following diagram commutes for all objects* X *of* $\mathbf{T}$:

$$\begin{array}{ccc} \omega(X) & = & \omega(X) \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ \omega'(S_1(X)) & \xrightarrow{\omega'(\alpha(X))} & \omega'(S_2(X)). \end{array}$$

PROOF. The same proof as for (3.19) shows that there exists a pair (S, s).

Consider two pairs (S_1, s_1) and (S_2, s_2). We know from (3.13) that there is an isomorphism $\alpha\colon S_1 \to S_2$ of tensor functors. Both $\omega'(\alpha)\circ s_1$ and s_2 are isomorphisms of fibre functors $\omega \to \omega' \circ S_2$, and hence they differ by an automorphism of ω, i.e., by an element of $P_0(\mathbb{A}_f)$. Thus, it remains to prove that $P_0(\mathbb{A}_f) = 1$. This is achieved by the next lemma. □

LEMMA 3.21. *Let* T *be a torus over* $\mathbb{Q}$ *such that* $T(\mathbb{R})$ *is compact. Then* $\widetilde{T}(\mathbb{A}_f) = 1$.

PROOF. Because $T(\mathbb{R})$ is compact, $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$, and the quotient $T(\mathbb{A}_f)/T(\mathbb{Q})$ is compact. Therefore, ignoring finite groups, the quotient is isomorphic to $T(\widehat{\mathbb{Z}})$, and $\bigcap T(\widehat{\mathbb{Z}})^N = 1$. □

For the much simpler category $\mathbf{E}$ we have only the following result. Consider pairs $(\mathbf{T}, \omega)$ where

(3.22.1) $\mathbf{T}$ is a polarizable Tate triple over $\mathbb{Q}$ having no fibre functor over $\mathbb{R}$ for which the weight map is an isomorphism $w: \mathbb{G} \to \boldsymbol{\pi}(\mathbf{T})$;

(3.22.2) $\omega = (\omega^p, \omega_p)$ with ω^p a fibre functor over $\mathbb{A}_f^p$ and ω_p an exact tensor functor $\mathbf{T} \to \mathbf{V}_p(\mathbb{F})$ such that if X has weight m, then $\omega_p(X)$ has slope $m/2$.

For example, $(\mathbf{E}, \omega)$ is such a pair.

PROPOSITION 3.23. *Suppose we have two pairs* $(\mathbf{T}, \omega)$ *and* $(\mathbf{T}', \omega')$ *satisfying* (3.22). *Then there exists an equivalence of Tate triples* $S: \mathbf{T} \to \mathbf{T}'$ *and an isomorphism* $s: \omega \to \omega' \circ S$ *of tensor functors.*

PROOF. Straightforward. □

Unfortunately, two such pairs (S_1, s_1) and (S_2, s_2) need not be isomorphic, because we can replace s_1 with its product by an element of $a \in \mathbb{A}_f^\times$, and the resulting pair will not be isomorphic to the original pair unless $a \in \mathbb{Q}^\times$.

The groupoid attached to Mot($\mathbb{F}$). We shall need the notion of a groupoid in schemes (see **[6**, §10]; [7, §3]; **[23**, Appendix A]; [3]).

Let $S_0 = \operatorname{Spec} k$, where k is a field of characteristic zero, and let $S = \operatorname{Spec} k^{\mathrm{al}}$. An *$S/S_0$-groupoid* is a scheme $\mathfrak{G}$ over S_0 together with two S_0-morphisms $s, t: \mathfrak{G} \to S$ and a law of composition (morphism of $S \times_{S_0} S$-schemes)

$$\circ: \mathfrak{G} \times_{s, S, t} \mathfrak{G} \longrightarrow \mathfrak{G}$$

such that, for all schemes T over S_0, $(S(T), \mathfrak{G}(T), (t, s), \circ)$ is a groupoid in sets, i.e., $S(T)$ is the set of objects and $\mathfrak{G}(T)$ the set of morphisms for a category whose morphisms are all isomorphisms (t and s map a morphism to its target and source, respectively, and $\circ$ gives the composition). A groupoid is said to be *affine* if it is an affine scheme, and it is said to be *transitive* if the map $(t, s): \mathfrak{G} \to S \times_{S_0} S$ makes $\mathfrak{G}$ into a faithfully flat $S \times_{S_0} S$-scheme. We refer to [7, 1.6], for the notion of a representation of a groupoid over S. The collection of such representations forms a Tannakian category $\mathbf{Rep}(S: \mathfrak{G})$ over k.

Henceforth, all groupoids will be affine and transitive.

The *kernel* of an S/S_0-groupoid is

$$G \underset{\mathrm{df}}{=} \mathfrak{G}^\Delta \underset{\mathrm{df}}{=} \Delta^* \mathfrak{G}, \qquad \Delta: S \longrightarrow S \times_{S_0} S \quad \text{(diagonal morphism)}.$$

Under our assumptions, it is a faithfully flat affine group scheme over S.

Let $\mathbf{T}$ be a Tannakian category over k, and let ω be a fibre functor over k^{al}. Write $Aut^{\otimes}(\omega)$ for the functor sending an $S \times_{S_0} S$-scheme $(b, a): T \to S \times_{S_0} S$ to the set of isomorphisms of tensor functors $a^* \omega \to b^* \omega$.

THEOREM 3.24. *Let* $\mathbf{T}$ *be a Tannakian category over* k, *and let* ω *be a fibre functor of* $\mathbf{T}$ *over* k^{al}; *then* $Aut^{\otimes}(\omega)$ *is represented by an* S/S_0-*groupoid, and* ω *defines an equivalence of tensor categories* $\mathbf{T} \to \mathbf{Rep}(S\colon \mathfrak{G})$. *Conversely, let* $\mathfrak{G}$ *be an* S/S_0-*groupoid, and let* ω *be the forgetful fibre functor of* $\mathbf{Rep}(S\colon \mathfrak{G})$; *then the natural map* $\mathfrak{G} \to Aut^{\otimes}(\omega)$ *is an isomorphism.*

PROOF. See [7, 1.12]. □

REMARK 3.25. (a) Let $\mathfrak{G}$ be the groupoid attached to $(\mathbf{T}, \omega)$. Then $G \underset{\text{df}}{=} \mathfrak{G}^{\Delta}$ is an affine group scheme over S with a canonical "descent datum up to inner automorphisms", i.e., it represents a band (see [23, p. 223]. In fact, it represents the band of the gerb of fibre functors of $\mathbf{T}$. In the case that the band is commutative, the descent datum defines an affine group scheme over k, which can be identified with $\boldsymbol{\pi}(\mathbf{T})$.

(b) Assume $\mathfrak{G}$ has a section over $S \times_{S_0} S$. Then the map

$$\mathfrak{G}(S) \xrightarrow{(t,s)} (S \times_{S_0} S)(S) = \mathrm{Gal}(k^{\text{al}}/k)$$

is surjective and the law of composition on $\mathfrak{G}$ defines a group structure on $\mathfrak{G}(S)$ for which following sequence is exact:

$$1 \longrightarrow G(S) \longrightarrow \mathfrak{G}(S) \longrightarrow \mathrm{Gal}(k^{\text{al}}/k) \longrightarrow 1.$$

EXAMPLE 3.26. The $\mathbb{C}/\mathbb{R}$-groupoid $\mathfrak{G}_{\infty}$ associated with $\mathbf{V}_{\infty}$ and the forgetful fibre functor has kernel $\mathbb{G}_m$, and the associated exact sequence

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow \mathfrak{G}_{\infty}(\mathbb{C}) \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

identifies $\mathfrak{G}_{\infty}(\mathbb{C})$ with the real Weil group.

EXAMPLE 3.27. Let G be a group scheme over k. The *neutral* S/S_0-groupoid defined by G is

$$\mathfrak{G}_G \underset{\text{df}}{=} G \times_{S_0} (S \times_{S_0} S).$$

The associated exact sequence is

$$1 \longrightarrow G(k^{\text{al}}) \longrightarrow G(k^{\text{al}}) \rtimes \mathrm{Gal}(k^{\text{al}}/k) \longrightarrow \mathrm{Gal}(k^{\text{al}}/k) \longrightarrow 1.$$

Let $\mathbf{T}$ be a Tannakian category over k with a fibre functor ω over k, and let $G = \mathrm{Aut}^{\otimes}(\omega)$; then the groupoid attached to $\mathbf{T}$ and $\omega \otimes k^{\text{al}}$ is $\mathfrak{G}_G$.

EXAMPLE 3.28. The $\mathbb{Q}_p^{\text{al}}/\mathbb{Q}_p$-groupoid $\mathfrak{G}_p$ attached to $\mathbf{V}_p(\mathbb{F})$ has kernel $\mathbb{G}$, the universal covering group of $\mathbb{G}_m$. If M is an isocrystal over $\mathbb{F}$ of slope λ, then $\mathbb{G}$ acts on M through the character $\lambda \in \mathbb{Q} = X^*(\mathbb{G})$.

Choose for each prime ℓ a commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}^{\text{al}} & \xrightarrow{\text{injective}} & \mathbb{Q}_\ell^{\text{al}} \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\text{injective}} & \mathbb{Q}_\ell \end{array}.$$

For $\ell = \infty$, $\mathbb{Q}_\ell = \mathbb{R}$ and $\mathbb{Q}_\ell^{\mathrm{al}} = \mathbb{C}$. On pulling back a $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid $\mathfrak{P}$ by the map

$$\mathrm{Spec}(\mathbb{Q}_\ell^{\mathrm{al}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}}) \to \mathrm{Spec}(\mathbb{Q}^{\mathrm{al}} \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{al}}),$$

we obtain a $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid $\mathfrak{P}(\ell)$.

Write z_∞ for the weight homomorphism $\mathfrak{G}_\infty^\Delta = \mathbb{G}_m \to P(p^\infty)_{\mathbb{R}}$ (corresponding to the map $W(p^\infty) \to \mathbb{Z}$ sending π to its weight).

Write z_p for the homomorphism $\mathfrak{G}_p^\Delta = \mathbb{G} \to P(p^\infty)_{\mathbb{Q}_p}$ corresponding to the map $\pi \mapsto \mathrm{ord}_p(\pi_n)/n\colon W(p^\infty) \to \mathbb{Q}$, where (π_n, n) represents π and ord_p is the extension of the p-adic valuation on $\mathbb{Q}$ corresponding to the chosen embedding of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{Q}_p^{\mathrm{al}}$.

For $\ell \neq p, \infty$, write $\mathfrak{G}_\ell$ for the trivial $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid $\mathrm{Spec}(\mathbb{Q}_\ell^{\mathrm{al}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}})$, and z_ℓ for the unique homomorphism $\mathfrak{G}_\ell^\Delta = 1 \to P(p^\infty)_{\mathbb{Q}_\ell}$.

THEOREM 3.29. *Let $\mathfrak{M}(\omega)$ be the $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid defined by a fibre functor ω of* **Mot**($\mathbb{F}$) *over $\mathbb{Q}^{\mathrm{al}}$. Then*

(a) *the kernel of $\mathfrak{M}(\omega)$ is $P(p^\infty)$;*
(b) *for each prime ℓ of $\mathbb{Q}$ (including p and ∞), there is a homomorphism $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{M}(\omega)(\ell)$, well defined up to isomorphism, whose restriction to the kernel is z_ℓ.*

If $\mathfrak{M}(\omega')$ is the groupoid attached to a second fibre functor over $\mathbb{Q}^{\mathrm{al}}$, then the choice of an isomorphism $\omega \approx \omega'$ determines an isomorphism $\alpha\colon \mathfrak{M}(\omega) \to \mathfrak{M}(\omega')$ whose restriction to the kernel is the identity map; moreover, $\alpha(\ell)\circ\zeta_\ell \approx \zeta'_\ell$, and changing the isomorphism between the fibre functors replaces α with an isomorphic isomorphism.

PROOF. That $\mathcal{M}(\omega)^\Delta = P(p^\infty)$ follows from (3.25a) and (2.38). The homomorphism $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathcal{M}(\omega)(\ell)$ is induced by the choice of an isomorphism $\omega \otimes_{\mathbb{Q}^{\mathrm{al}}} \mathbb{Q}_\ell^{\mathrm{al}} \to \omega_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}}$. The rest of the proof is a straightforward application of the theory of Tannakian categories using what has already been proved. □

REMARK 3.30. A fibre functor ω of **Mot**($\mathbb{F}$) over $\mathbb{Q}^{\mathrm{al}}$ defines by composition a fibre functor ω' of **Mot**($\mathbb{F}_q$) over $\mathbb{Q}^{\mathrm{al}}$. The groupoid $\mathcal{M}(\omega')$ attached to **Mot**($\mathbb{F}_q$) and ω' is obtained from $\mathcal{M}(\omega)$ by pushing out with respect to $P(p^\infty) \to P(q)$ (see [**6**, 10.8], for the "push-out" of a groupoid).

Existence results. Now drop the assumption of the Tate conjecture (1.14).

THEOREM 3.31. *There exists a system $(\mathfrak{P}, (\zeta_\ell))$ consisting of a $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid $\mathfrak{P}$ such that $\mathfrak{P}^\Delta = P(p^\infty)$ and a family of morphisms $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{P}(\ell)$ such that $\zeta_\ell^\Delta = z_\ell$. If $(\mathfrak{P}', (\zeta'_\ell))$ is a second such system, then there is an isomorphism $\alpha\colon \mathfrak{P} \to \mathfrak{P}'$ such that $\alpha^\Delta = \mathrm{id}$ and $\zeta'_\ell \approx \alpha\circ\zeta_\ell$; moreover, α is uniquely determined up to isomorphism.*

PROOF. Let c_ℓ be the cohomology class of the groupoid $\mathfrak{G}_\ell$ in $H^2(\mathbb{Q}_\ell, G_\ell)$. I claim that there is a unique class $c \in H^2(\mathbb{Q}, P)$ mapping to $z_\ell(c_\ell)$ for all ℓ. Since $P = P_0 \oplus \mathbb{G}_m$, it suffices to prove this for each factor. But

$$H^2(\mathbb{Q}, P_0) \xrightarrow{\approx} \bigoplus_\ell H^2(\mathbb{Q}_\ell, P_0),$$

and so this is obvious on the first factor. On the other hand, $z_\ell(c_\ell) = 0$ (in $H^2(\mathbb{Q}_\ell, \mathbb{G}_m)$) for $\ell \neq p, \infty$, and

$$\mathrm{inv}_p(z_p(c_p)) = \frac{1}{2} = \mathrm{inv}_\infty(z_\infty(c_\infty)),$$

and so it is also obvious for the second factor. Choose a groupoid $\mathfrak{P}$ corresponding to c.

If $(\mathfrak{P}', (\zeta'_\ell))$ is a second pair, then the existence of the maps ζ'_ℓ implies that the cohomology class of $\mathfrak{P}'$ is the same as that of $\mathfrak{P}$ locally, and hence (see 3.11d) globally. Therefore there is an isomorphism $\alpha: \mathfrak{P} \to \mathfrak{P}'$ that is the identity map on the kernel. The scheme $Hom^{\otimes}(\alpha \circ \zeta_\ell, \zeta'_\ell)$ is a torsor for $\mathfrak{P}_{\mathbb{Q}_\ell}$. Now (3.11b) shows that we can modify α by a global torsor (unique up to isomorphism) and force the local torsors to be trivial; then $\alpha \circ \zeta_\ell \approx \zeta'_\ell$. □

COROLLARY 3.32. *There exists a Tate triple* $(\mathbf{T}, w, T)$, *a germ of a tensor endomorphism* π *of* $\mathbf{T}$, *and a pair* $\omega = (\omega^p, \omega_p)$ *such that the system* $(\mathbf{T}, \pi, \omega)$ *satisfies the conditions* (3.18).

PROOF. Take $\mathbf{T} = \mathbf{Rep}(S: \mathfrak{P})$. The weight homomorphism $\mathbb{G}_m \to P$ defines a weight filtration on $\mathbf{T}$. The action of f defines π, and the homorphisms ζ_ℓ define ω. □

Notes. The form of the statement of Theorem 3.13 was suggested by a general remark of Grothendieck on the classification of Tannakian categories. Theorems 3.19 and 3.20 were explained to me by Deligne (who credits them to Grothendieck), and Theorem 3.31 is proved in Langlands and Rapoport [20].

4. The reduction of CM-motives to characteristic p

Hodge structures of CM-type. The *Mumford-Tate group* $\mathrm{MT}(H)$ of a polarizable rational Hodge structure $H = (V, h)$ is the algebraic group attached to the forgetful fibre functor on the Tannakian subcategory of $\mathbf{Hdg}_{\mathbb{Q}}$ generated by H and $\mathbb{Q}(1)$. It can also be described as the largest algebraic subgroup of $\mathrm{GL}(V) \times \mathbb{G}_m$ fixing the Hodge tensors of V, or the smallest algebraic subgroup G of $\mathrm{GL}(V) \times \mathbb{G}_m$ such that $G_{\mathbb{C}}$ contains the image of

$$z \mapsto (\mu_h(z), z): \mathbb{G}_m \longrightarrow \mathrm{GL}(V \otimes \mathbb{C}) \times \mathbb{G}_m.$$

Here $\mu_h\colon \mathbb{G}_m \to \mathrm{GL}(V \otimes \mathbb{C})$ is the homomorphism such that $\mu_h(z)$ acts on $V^{r,s}$ as multiplication by z^{-r}. The Mumford-Tate group is connected and reductive.

A polarizable rational Hodge structure (V, h) is said to be of *CM-type* if its Mumford-Tate group is commutative, and hence, is a torus T. We regard $z \mapsto (\mu_h(z), z)$ as a cocharacter μ of T.

PROPOSITION 4.1. *A pair (T, μ) arises as above from a rational Hodge structure of CM-type if and only if*

(a) *the weight $-\mu - \iota\mu$ of μ is defined over $\mathbb{Q}$;*
(b) *μ is defined over a CM-field; and*
(c) *μ generates T, i.e., there does not exist a proper subtorus T' of T such that $T'_{\mathbb{C}}$ contains the image of μ.*

PROOF. See [6, pp. 42–47]. □

For a CM-field $L \subset \mathbb{C}$, let S^L be the quotient of $(\mathbb{G}_m)_{L/\mathbb{Q}}$ having character group

$$X^*(S^L) = \{\lambda \in \mathbb{Z}^{\mathrm{Hom}(L,\mathbb{C})} \mid \lambda(\tau) + \lambda(\iota\tau) = \text{constant}\}.$$

Define μ^L to be the cocharacter of S^L such that

$$\langle \lambda, \mu^L \rangle = \lambda(\tau_0), \quad \text{all } \lambda \in X^*(S^L),$$

where τ_0 is the given embedding of L into $\mathbb{C}$. If $L \subset L' \subset \mathbb{C}$, the norm map defines a homomorphism $S^{L'} \to S^L$ carrying $\mu^{L'}$ to μ^L. We define

$$S = \varprojlim S^L, \qquad \mu_{\mathrm{can}} = \varprojlim \mu^L.$$

The pair (S, μ_{can}) is called the *Serre group*. If $\mathbb{Q}^{\mathrm{cm}}$ denotes the union of all CM-subfields of $\mathbb{Q}^{\mathrm{al}}$, then $X^*(S)$ can be identified with the set of all locally constant functions

$$\lambda\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}) \longrightarrow \mathbb{Z}$$

such that $\lambda(\tau) + \lambda(\iota\tau) = -m$ for some integer m (called the *weight* of λ).

PROPOSITION 4.2. *The rational Hodge structures of CM-type form a Tannakian subcategory $\mathbf{Hod}_{\mathbb{Q}}^{\mathrm{cm}}$ of $\mathbf{Hdg}_{\mathbb{Q}}$. The affine group scheme attached to the forgetful fibre functor is S.*

PROOF. Since $Aut^{\otimes}(\omega_{\mathrm{forget}}) = \varprojlim \mathrm{MT}(H)$ where H ranges over the Hodge structures of CM-type, this follows from (4.1) and the next lemma. □

LEMMA 4.3. *Let (T, μ) be a pair satisfying the conditions* (a) *and* (b) *of* (4.1). *Then there is a unique homomorphism $\rho_\mu\colon S \to T$ (defined over $\mathbb{Q}$) such that $(\rho_\mu)_{\mathbb{Q}} \circ \mu_{\mathrm{can}} = \mu$; moreover,*

$$(S, \mu_{\mathrm{can}}) = \varprojlim (T, \mu),$$

where the limit is over all pairs (T, μ) satisfying (4.1a,b,c).

PROOF. When restated in terms of character groups, the lemma becomes obvious. □

REMARK 4.4. Let T be a torus over a field k of characteristic zero. If k is algebraically closed, then each character χ of T defines a one-dimensional representation $V(\chi)$ of T over k, and every irreducible representation is isomorphic to $V(\chi)$ for exactly one χ; consequently

$$\Sigma(\mathbf{Rep}_k(T)) = X^*(T).$$

More generally, $\mathbf{Rep}_k(T)$ is a semisimple Tannakian category over k, and $\mathbf{Rep}_k(T) \otimes k^{\mathrm{al}} = \mathbf{Rep}_{k^{\mathrm{al}}}(T)$. Therefore (3.14) shows that there is a bijection

$$\Gamma\backslash X^*(T) \longrightarrow \Sigma(\mathbf{Rep}_k(T)), \qquad \Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k),$$

under which a simple representation V of T over k corresponds to the set of characters occurring in $V \otimes_k k^{\mathrm{al}}$.

Motives of CM-type. For an abelian variety (or motive) A over $\mathbb{C}$, the *Mumford-Tate group* of A is defined to be the Mumford-Tate group of $H_B(A) \underset{\mathrm{df}}{=} H_1(A, \mathbb{Q})$.

A simple abelian variety A over an algebraically closed field k is said to be of *CM-type* if $\mathrm{End}(A) \otimes \mathbb{Q}$ is a field of degree $2 \dim A$ over $\mathbb{Q}$, and a general abelian variety over k is said to be of *CM-type* if its simple (isogeny) factors are. An abelian variety over an arbitrary field k is of *CM-type*[6] if it becomes of CM-type over k^{al}.

PROPOSITION 4.5. *An abelian variety over $\mathbb{C}$ is of CM-type if and only if the rational Hodge structure $H_B(A)$ is of CM-type.*

PROOF. See [6, 5.1]. □

PROPOSITION 4.6. *The category $\mathbf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}}$ is generated by*

$$\{H_B(A) \mid A \text{ an abelian variety of CM-type over } \mathbb{C}\}.$$

PROOF. We have to show that $\mathbf{Rep}_{\mathbb{Q}}(S)$ is generated by the representations of S on $\{H_B(A)\}$. For this it suffices to show that $X^*(S)$ is generated by the set of characters arising from abelian varieties of CM-type over $\mathbb{C}$.

Let $L \subset \mathbb{Q}^{\mathrm{cm}}$ be Galois over $\mathbb{Q}$. A CM-*type* Φ for L is a function Φ: $\mathrm{Hom}(L, \mathbb{C}) \to \{0, 1\}$ such that $\Phi + \iota\Phi = \mathrm{id}$. An abelian variety A over $\mathbb{C}$ together with a homomorphism $L \to \mathrm{End}(A) \otimes \mathbb{Q}$ is said to be of *CM-type* (L, Φ) if $H_B(A)$ is a one-dimensional vector space over L and the representation of L on the tangent space to A at 0 is equivalent to $\sum \Phi(\varphi)\varphi$. An abelian variety of CM-type (L, Φ) always exists, and for such a variety A, Φ, when regarded as a character of S, occurs in the representation of S on $H_B(A) \otimes \mathbb{Q}^{\mathrm{al}}$.

[6]Some authors prefer to say "potentially of CM-type".

Thus it suffices to show that, for any CM-field L Galois over $\mathbb{Q}$, $X^*(S^L)$ is generated by CM-types. Choose a set of representatives $R = \{\varphi_1, \dots, \varphi_g\}$ for $\mathrm{Hom}(L, \mathbb{C})/\{1, \iota\}$, and let Φ_j be the CM-type with support $\{\varphi_1, \dots, \varphi_{j-1}, \iota\varphi_j, \varphi_{j+1}, \dots, \varphi_g\}$. For any $\lambda \in X^*(S)$, $\lambda - \sum_{i=1}^{g} \lambda(\iota\varphi_i)\Phi_i$ takes the value 0 on any element of ιR, and hence is a multiple of the CM-type Φ having support R. □

For any variety V over a field k of characteristic zero and integer r, Deligne has defined a space $A^r_{\mathrm{aH}}(V)$ of *absolute Hodge cycles of codimension* r on V [**6**, p. 36]. When $k = \mathbb{C}$ there are maps

$$A^r(V) \leftarrow Z^r(V) \rightarrow A^r_{\mathrm{aH}}(V) \subset A^r_{\mathrm{H}}(V)$$

where $A^r_{\mathrm{H}}(V)$ is the space of Hodge cycles of codimension r. The Hodge conjecture asserts that the map $Z^r(V) \rightarrow A^r_{\mathrm{H}}(V)$ is surjective, which implies that it has the same kernel as $Z^r(V) \rightarrow A^r(V)$, and hence induces isomorphisms

$$A^r(V) \xrightarrow{\approx} A^r_{\mathrm{aH}}(V) \xrightarrow{\approx} A^r_{\mathrm{H}}(V).$$

Fix a field k of characteristic zero. Analogously to $\mathbf{CV}^0(k)$ we can define a category having one object $h(V)$ for each smooth projective variety V over k, and having the absolute Hodge cycles as morphisms, i.e.,

$$\mathrm{Hom}(h(V), h(W)) = A^{\dim V}_{\mathrm{aH}}(V \times W).$$

On adding the images of projectors and inverting the Lefschetz motive, we obtain a $\mathbb{Q}$-linear tensor category. In this case, the Künneth components of the diagonal are automatically morphisms, and so we can define a gradation on the category and use it to modify the commutativity constraint. In this way we obtain the category $\mathbf{Mot}_{\mathrm{aH}}(k)$ of *motives over* k *for absolute Hodge cycles* (see [**8**, §6].

Define $\mathbf{CM}(k)$ to be the Tannakian subcategory of $\mathbf{Mot}_{\mathrm{aH}}(k)$ generated by the objects $h_1(A)$ for A an abelian variety of CM-type over k, the Tate motive, and the objects $h(V)$ for V a finite scheme over k. We refer to the objects of $\mathbf{CM}(k)$ as *CM-motives* over k.

PROPOSITION 4.7. *For any algebraically closed field* $k \subset \mathbb{C}$, *the functor*

$$X \mapsto H_B(X_{\mathbb{C}}) : \mathbf{CM}(k) \longrightarrow \mathbf{Hdg}^{\mathrm{cm}}_{\mathbb{Q}}$$

is an equivalence of Tannakian categories.

PROOF. Assume first that $k = \mathbb{C}$. The main theorem of [**6**] shows that for abelian varieties A and B over $\mathbb{C}$,

$$A^r_{\mathrm{aH}}(A \times B) = A^r_{\mathrm{H}}(A \times B),$$

and therefore,

$$\mathrm{Hom}(h_1(A), h_1(B)) = \mathrm{Hom}(H_B(A), H_B(B)).$$

That $X \mapsto H_B(X)$ is fully faithful is now obvious, and (4.6) shows that it is essentially surjective.

Now consider an arbitrary algebraically closed field $k \subset \mathbb{C}$. For any smooth projective varieties V and W over k,

$$A^r_{\mathrm{aH}}(V \times W) = A^r_{\mathrm{aH}}(V_{\mathbb{C}} \times W_{\mathbb{C}})$$

(ibid. 2.9a) and so the functor

$$X \mapsto X_{\mathbb{C}} \colon \mathbf{Mot}_{aH}(k) \longrightarrow \mathbf{Mot}_{aH}(\mathbb{C})$$

is fully faithful. Hence its restriction to $\mathbf{CM}(k)$ is also fully faithful, and because every abelian variety of CM-type over $\mathbb{C}$ has a model[7] over k, it is also essentially surjective. □

COROLLARY 4.8. *For any algebraically closed field $k \subset \mathbb{C}$, the affine group scheme attached to the fibre functor H_B on $\mathbf{CM}(k)$ is S. Hence,*

$$\boldsymbol{\pi}(\mathbf{CM}(k)) = S,$$

and

$$\Sigma(\mathbf{CM}(k)) = \Sigma(\mathbf{Rep}_{\mathbb{Q}}(S)) = \Gamma\backslash X^*(S).$$

PROOF. This is an immediate consequence of (4.7), (4.2), and (4.4). □

REMARK 4.9. In fact, for any algebraically closed field k of characteristic 0, $\mathbf{CM}(k)$ is a neutral Tannakian category over $\mathbb{Q}$, and the affine group scheme attached to any fibre functor ω over $\mathbb{Q}$ is canonically isomorphic to S. In more detail, each object of $\mathbf{CM}(k)$ has a (de Rham) filtration, and there is a unique isomorphism $\alpha\colon S \to \boldsymbol{\pi}(\mathbf{CM}(k))$ such that $\alpha \circ \mu_{\mathrm{can}}$ splits the de Rham filtration on each X.

Discussion of the problem of reducing CM-motives. For the rest of this section, we fix a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ lying over p and define $\mathbb{Q}_p^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}_p$ in the completion of $\mathbb{Q}^{\mathrm{al}}$ at w_0. We take $\mathbb{F}$ to be the residue field of $\mathbb{Q}_p^{\mathrm{al}}$.

Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type. Then A will be defined over a number field K, and it follows easily from Néron's criterion for good reduction that, after we pass to a finite extension L of K, A will acquire good reduction at w_0 (see Serre and Tate [**27**, Theorem 6]). We therefore obtain an abelian variety $A(w_0)$ over the residue field $k(w_0)$ of w_0 in L, and, by extension of scalars, we obtain an abelian variety $A(p)$ over $\mathbb{F}$.

LEMMA 4.10. *The abelian variety $A(p)$ is well defined by A (up to a canonical isomorphism).*

[7] An abelian variety A over $\mathbb{C}$ of CM-type will have a specialization over k that is of the same CM-type as A, and hence becomes isogenous to A over $\mathbb{C}$.

PROOF. Consider two models (A_1, φ_1) and (A_2, φ_2) of A over number fields K_1 and K_2. There will be a number field L containing both K_1 and K_2 and such that

(a) A_1 and A_2 both acquire good reduction over L at w_0;

(b) the map $\varphi \underset{\text{df}}{=} \varphi_2 \circ \varphi_1^{-1} \colon (A_1)_{\mathbb{Q}^{\text{al}}} \to (A_2)_{\mathbb{Q}^{\text{al}}}$ is defined over L.

Now the reduction of φ is an isomorphism $A_1(p) \to A_2(p)$. □

In this way, we obtain a functor $A \mapsto A(p)$ from the category of abelian varieties of CM-type over $\mathbb{Q}^{\text{al}}$ to the category of abelian varieties over $\mathbb{F}$.

Consider a CM-motive X over $\mathbb{Q}^{\text{al}}$. After replacing X with $X(m)$ for some m, there will exist a CM-motive Y and abelian varieties A_i of CM-type such that

$$X \oplus Y = \bigotimes h_1(A_i),$$

i.e., $X = (\bigotimes h_1(A_i), q)$ for some projector q. If q is algebraic, then we can define $X(p)$ to be $(\bigotimes h_1(A_i(p)), q(p))$. Consequently, if the Hodge conjecture holds for abelian varieties of CM-type, then there is a functor

$$R = (X \mapsto X(p)) \colon \mathbf{CM}(\mathbb{Q}^{\text{al}}) \to \mathbf{Mot}(\mathbb{F})$$

such that, for any abelian variety A of CM-type over $\mathbb{Q}^{\text{al}}$, $h(A)(p) = h(A(p))$. In particular, we will obtain the following:

(a) a map $\Sigma(\mathbf{CM}(\mathbb{Q}^{\text{al}})) \to \Sigma(\mathbf{Mot}(\mathbb{F}))$;

(b) a map $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F})) \to \boldsymbol{\pi}(\mathbf{CM})$;

(c) for all ℓ, a functor $\omega_\ell \circ R(\ell) \colon \mathbf{CM}(\mathbb{Q}^{\text{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell$.

Recall that, under the assumption of the Tate conjecture, we showed that $\Sigma(\mathbf{Mot}(\mathbb{F})) = \Gamma \backslash W(p^\infty)$ and $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F})) = P(p^\infty)$. We shall construct a canonical homomorphism $\gamma \colon P(p^\infty) \to S$, a canonical map $\Gamma \backslash X^*(S) \to \Gamma \backslash W(p^\infty)$, and canonical functors

$$\xi_\ell \colon \mathbf{Rep}_{\mathbb{Q}_\ell}(S) = \mathbf{CM}(\mathbb{Q}^{\text{al}}) \otimes \mathbb{Q}_\ell \longrightarrow \mathbf{V}_\ell.$$

Then we show that if $(\mathbf{T}, \pi, \omega)$ is a triple satisfying the conditions (3.18), there is a functor

$$R \colon \mathbf{CM}(\mathbb{Q}^{\text{al}}) \longrightarrow \mathbf{T}$$

such that $\boldsymbol{\pi}(R) = \gamma$ and $\omega_\ell \circ R(\ell) \approx \xi_\ell$.

The map on isomorphism classes.

LEMMA 4.11. *Let L be a CM-field that is Galois over $\mathbb{Q}$, and let w_0 be a prime of L lying over p. Let h be such that $\mathfrak{p}_{w_0}^h$ is principal; let $r = (U \colon U^+)$, where U is the group of units in L and U^+ is the subgroup of totally real units, and let f be the residue class degree $f(w_0/p)$. Let a be a generator of $\mathfrak{p}_{w_0}^h$. For any n divisible by $2hrf$ and $\chi \in X^*(S^L)$, $\chi(a^{-n/hf})$ is independent of the choice of a, and lies in $W^L(p^n)$.*

PROOF. The proof is straightforward. □

Thus we have a well-defined map

$$\chi \mapsto \pi_n^L(\chi) = \chi(a^{n/hf}) \colon X^*(S^L) \longrightarrow W(p^n).$$

For a fixed L, these maps define a homomorphism

$$\chi \mapsto \pi^L(\chi) \colon X^*(S^L) \longrightarrow W(p^\infty),$$

and when we let L vary over the CM-subfields of $\mathbb{Q}^{\mathrm{al}}$, they define a homomorphism

$$\chi \mapsto \pi(\chi) \colon X^*(S) \longrightarrow W(p^\infty).$$

This map is invariant under the action of $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and so we have proved the following result.

Proposition 4.12. (a) *There is a canonical homomorphism*

$$\gamma \colon P(p^\infty) \longrightarrow S.$$

(b) *There is a canonical homomorphism*

$$\Sigma(\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})) = \Gamma\backslash X^*(S) \xrightarrow{[\chi]\mapsto[\pi(\chi)]} \Gamma\backslash W(p^\infty) = \Gamma\backslash\Sigma(\mathbf{Mot}(\mathbb{F})).$$

Proposition 4.13. *The homomorphism in* (4.12) *is compatible with the reduction of abelian varieties of CM-type, i.e., if* χ *is the character of* S *associated with a simple abelian variety of CM-type* A *over* $\mathbb{Q}$, *then* $[\pi(\chi)]$ *is the Frobenius element of* $A(p)$.

Proof. This is a restatement of the theorem of Shimura and Taniyama [**28**, p. 110, Theorem 1]. □

Remark 4.14. Let $X^*(S^L)_0$ be the subset of $X^*(S^L)$ of elements of weight 0. For any n divisible by hrf, the composite

$$X^*(S^L)_0 \xrightarrow{\pi} W_0^L(q)/\text{torsion} \xrightarrow{\alpha} \bigoplus_{w|p} \mathbb{Z}w,$$

where α is as in (2.27b), is

$$\lambda \mapsto \sum \left(\sum_{\sigma w_0 = w} \lambda(\sigma) \right) w.$$

The image of this map is equal to the kernel of β, which completes the proof of (2.27b). This remark also proves that the map $X^*(S) \to W(p^\infty)$ is surjective. In conjunction with the Hodge and Tate conjectures, this implies that the reduction functor

$$\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \longrightarrow \mathbf{Mot}(\mathbb{F})$$

is surjective: every motive over $\mathbb{F}$ lifts to a motive of CM-type.

The functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{R} \to \mathbf{V}_\infty$. Let (V, ρ) be a real representation of S. Then $w(\rho) \underset{\mathrm{df}}{=} w_{\mathrm{can}} \circ \rho$ defines a gradation on $V \otimes \mathbb{C}$. Let F be the map

$$v \mapsto \rho(\mu(i)^{-1})\bar{v} : V \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}.$$

Clearly, F is semilinear and F^2 is multiplication by $\mu(i)\mu(i) = w(-1)$. Therefore it acts as $(-1)^m$ on the m^{th} graded piece, and so $(V(\rho) \otimes \mathbb{C}, \alpha)$ is an object of $\mathbf{V}_\infty$.

PROPOSITION 4.15. *The above construction defines a tensor functor*

$$\xi_\infty : \mathbf{Rep}_{\mathbb{R}}(S) \to \mathbf{V}_\infty.$$

PROOF. The proof is straightforward. □

The functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F})$, $\ell \neq p, \infty$. Let X be a CM-motive over $\mathbb{Q}^{\mathrm{al}}$. Then X will have a model over a finite extension L of $\mathbb{Q}$, and after replacing L with a finite extension, we may assume that the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/L)$ on $\omega_\ell(X)$ is unramified at w_0. Therefore, we obtain a representation of $D(w_0)/I(w_0) = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on $\omega_\ell(X)$.

PROPOSITION 4.16. *The germ of a representation of* $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ *on* $\omega_\ell(X)$ *given by the above construction is independent of the choices involved. In this way we obtain a canonical functor*

$$\xi_\ell : \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}).$$

PROOF. The proof is straightforward. □

REMARK 4.17. It is possible to give a direct construction (i.e., without mentioning CM-motives) of ξ_ℓ. The construction uses the Taniyama group and a result of Grothendieck [**27**, p. 515]).

The functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F})$. Let (V, ρ) be a representation of S over $\mathbb{Q}_p$. Then ρ will factor through S^L for some $L \subset \mathbb{Q}^{\mathrm{cm}}$. Choose a generator a for the maximal ideal in L_{w_0}, and let $b = Nm_{L_{w_0}/K}(\mu^L(a^{-1})) \in S^L(K)$, where K is the maximal unramified extension of $\mathbb{Q}_p$ contained in L_{w_0}. Define

$$M = V \otimes K(\mathbb{F}), \qquad F(x) = (1 \otimes \sigma)(bx).$$

PROPOSITION 4.18. *The above construction defines a tensor functor*

$$\xi_p : \mathbf{Rep}_{\mathbb{Q}_p}(S) \longrightarrow \mathbf{V}_p(\mathbb{F}).$$

PROOF. The proof is straightforward. □

REMARK 4.19. The functor ξ_p defines a homomorphism $\mathbb{G} \to S$ on the fundamental groups. The corresponding map on the character groups is

$$X^*(S^L) \longrightarrow \mathbb{Q}, \qquad \lambda \mapsto -[L_{w_0} : \mathbb{Q}_p]^{-1} \cdot \sum_{\sigma \in D(w_0)} \lambda(\sigma)$$

where $D(w_0) \subset \mathrm{Gal}(L/\mathbb{Q})$ is the decomposition group.

The cohomology of S. It is convenient at this point to compute the cohomology of S.

LEMMA 4.20. *Let L be a CM-field with largest totally real subfield F. There is a canonical exact sequence*

$$1 \longrightarrow (\mathbb{G}_m)_{F/\mathbb{Q}} \longrightarrow (\mathbb{G}_m)_{L/\mathbb{Q}} \times \mathbb{G}_m \longrightarrow S^L \longrightarrow 1.$$

PROOF. It suffices to check that the corresponding sequence of character groups is exact, but this follows from the fact that the map

$$\mathbb{Z}^{\operatorname{Hom}(L,\mathbb{C})} \times \mathbb{Z} \to \mathbb{Z}^{\operatorname{Hom}(F,\mathbb{C})},$$

$$\left(\sum_{\tau\in\operatorname{Hom}(L,\mathbb{C})} \lambda(\tau)\tau, m\right) \mapsto \sum_{\tau\in\operatorname{Hom}(L,\mathbb{C})} \lambda(\tau)\tau|F - m\left(\sum_{\tau\in\operatorname{Hom}(F,\mathbb{C})} \tau\right)$$

is surjective with kernel $X^*(S^L)$. □

PROPOSITION 4.21. *For any CM-field L,*

$$H^1(\mathbb{Q}, S^L) \xrightarrow{\approx} \bigoplus_\ell H^1(\mathbb{Q}_\ell, S^L),$$

$$H^2(\mathbb{Q}, S^L) \hookrightarrow \bigoplus_\ell H^2(\mathbb{Q}_\ell, S^L).$$

PROOF. Consider the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^1(\mathbb{Q}, S^L) & \longrightarrow & \bigoplus_\ell H^1(\mathbb{Q}_\ell, S^L) & & \\
 & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \operatorname{Br}(F) & \longrightarrow & \bigoplus_v \operatorname{Br}(F_v) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & \operatorname{Br}(L)\times\operatorname{Br}(\mathbb{Q}) & \longrightarrow & \bigoplus_w \operatorname{Br}(L_w) \times \bigoplus_\ell \operatorname{Br}(\mathbb{Q}_\ell) & \longrightarrow & \mathbb{Q}/\mathbb{Z}\times\mathbb{Q}/\mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & H^2(\mathbb{Q}, S^L) & \longrightarrow & \bigoplus_\ell H^2(\mathbb{Q}_\ell, S^L) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The columns are the cohomology sequences over $\mathbb{Q}$ and $\mathbb{Q}_\ell$ of the exact sequence in (4.20), and the two middle rows come from class field theory.

The vertical map on the right is the one that makes the following diagram commute:

$$\begin{array}{ccc} H^2(F, C) & \xrightarrow[\approx]{\text{inv}} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle (\text{res,cores})} & & \downarrow \\ H^2(L, C) \times H^2(\mathbb{Q}, C) & \xrightarrow[\approx]{(\text{inv,inv})} & \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}. \end{array}$$

Here C is the idèle class group and inv is the invariant map of class field theory. Let $m = [F : \mathbb{Q}]$. It is known that the restriction map

$$H^2(\mathbb{Q}, C) \longrightarrow H^2(F, C)$$

induces multiplication by m on $\mathbb{Q}/\mathbb{Z}$. Because

$$\text{cores} \circ \text{res} = m,$$

we see that cores must induce the identity map on $\mathbb{Q}/\mathbb{Z}$. Therefore the map at right is injective, and now the snake lemma completes the proof.

The functor $\mathbf{CM}(\mathbb{Q}^{\text{al}}) \to \mathbf{Mot}(\mathbb{F})$.

THEOREM 4.22. *Let $(\mathbf{T}, \pi, \omega)$ be a triple satisfying the conditions of* (3.18). *Then there exists a tensor functor*

$$R: \mathbf{CM}(\mathbb{Q}^{\text{al}}) \to \mathbf{T}$$

such that

(a) *the homomorphism $P \to S$ defined by R on the fundamental groups is equal to the map γ in* (4.12a).

(b) *for all ℓ, the composite*

$$\mathbf{CM}(\mathbb{Q}^{\text{al}}) \otimes \mathbb{Q}_\ell \xrightarrow{R} \mathbf{T} \otimes \mathbb{Q}_\ell \xrightarrow{\omega_\ell} \mathbf{V}_\ell$$

is isomorphic to the functor ξ_ℓ.

Any other tensor functor with these properties is isomorphic to R.

PROOF. We first should note that the two conditions are compatible, i.e., the map

$$G_\ell \xrightarrow{z_\ell} P_{\mathbb{Q}_\ell} \xrightarrow{\gamma} S_{\mathbb{Q}_\ell}$$

is equal to that induced by ξ_ℓ on the fundamental groups. Only the prime $\ell = p$ presents difficulties, but this case follows easily from the formula in (4.19).

There exists a tensor functor satisfying (a) if and only if the class of $\mathbf{T}$ in $H^2(\mathbb{Q}, P)$ maps to zero in $H^2(\mathbb{Q}, S)$. After (4.21), it suffices to check this in the local cohomology groups $H^2(\mathbb{Q}_\ell, S)$.

Consider

$$H^2(\mathbb{Q}_\ell, G_\ell) \xrightarrow{z_\ell} H^2(\mathbb{Q}_\ell, P) \longrightarrow H^2(\mathbb{Q}_\ell, S).$$

The existence of the functors ω_ℓ shows that the class of $\mathbf{T}$ in $H^2(\mathbb{Q}_\ell, P)$ is the image of the class of $\mathbf{V}_\ell$ in $H^2(\mathbb{Q}_\ell, G_\ell)$. But the existence of the functors ξ_ℓ show that this class maps to zero in $H^2(\mathbb{Q}_\ell, S)$.

Hence there exists a functor $R: \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{T}$ satisfying (a). Then $\omega_\ell \circ R(\ell)$ and ξ_ℓ are both tensor functors $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell$, and $Hom^{\otimes}(\omega_\ell \circ R(\ell), \xi_\ell)$ is a torsor for S over $\mathbb{Q}_\ell$. According to (4.21), the cohomology classes of these torsors arise from a unique element of $H^1(\mathbb{Q}, S)$, which we use to modify R. Then R satisfies (b) and is uniquely determined up to isomorphism. □

REMARK 4.23. Consider a pair $(R, (r_\ell))$ where R is a tensor functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{T}$ and r_ℓ is an isomorphism $\omega_\ell \circ R(\ell) \to \xi_\ell$. If $(R', (r'_\ell))$ is a second such pair, then the theorem tells us there exists an isomorphism $\alpha: R \to R'$, but it may not be possible to choose α to carry r_ℓ into r'_ℓ.

Notes. This section gives a geometric re-interpretation of the cocycle calculations in Langlands and Rapoport [**20**, pp. 118–152].

BIBLIOGRAPHY

1. P. Berthelot, *Cohomologie Cristalline des Schémas de Caractéristique* $p > 0$, Lecture Notes in Math., vol. 407, Springer-Verlag, Berlin and New York, 1974.
2. N. Bourbaki, *Algèbre*, Chapitre 8, Modules et Anneaux Semi-Simples, Hermann, Paris, 1958.
3. L. Breen, *Tannakian categories*, these Proceedings, vol. 1, pp. 337–376.
4. P. Deligne, *La conjecture de Weil*. I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.
5. ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, part 2, pp. 313–346.
6. P. Deligne (Notes by J. S. Milne), *Hodge cycles on abelian varieties*, in Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math., vol. 900, Springer-Verlag, Heidelberg, 1982, pp. 9–100.
7. P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over $\mathbb{Q}$, Springer-Verlag, New York, 1989, pp. 79–297.
8. ——, *Catégories tannakiennes*, in The Grothendieck Festschrift, vol. II, Progress in Math., vol. 87, Birkhäuser, Boston, 1990, pp. 111–195.
9. P. Deligne and J. Milne, *Tannakian categories*, in Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math, vol. 900, Springer-Verlag, Berlin and New York, 1982, pp. 101–228.
10. M. Demazure, *Lectures on p-Divisible Groups*, Lecture Notes in Math., vol. 302, Springer-Verlag, Berlin and New York, 1972.
11. ——, *Motifs des variétés algébriques*, Séminaire Bourbaki, 1969/70, no. 365.
12. M. Demazure and P. Gabriel, *Groupes Algébriques*, North-Holland, Amsterdam, 1970.
13. H. Gillet and W. Messing, *Cycle classes and Riemann-Roch for crystalline cohomology*, Duke Math. J. **55** (1987), 501–538.
14. J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, New York, 1971.
15. A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic Geometry, Oxford University Press, London and New York, 1969, pp. 193–199.
16. U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), 447–452.
17. N. Katz and W. Messing, *Some consequences of the Riemann hypothesis*, Invent. Math. **23** (1974), 73–77.

18. S. Kleiman, *Motives*, Algebraic Geometry, Oslo 1970 (F. Oort, ed.), Walters-Noordhoff, Groningen, 1972, 53–82.
19. R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444.
20. R. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378** (1987), 113–220.
21. J. Milne, *Étale Cohomology*, Princeton University Press, Princeton, NJ, 1980.
22. ——, *Arithmetic Duality Theorems*, Academic Press, New York, 1986.
23. ——, *The points on a Shimura variety modulo a prime of good reduction*, The Zeta Functions of Picard Modular Surfaces, Les Publications CRM, Montréal, 1992, pp. 153–255.
24. R. Pierce, *Associative Algebras*, Springer-Verlag, New York, 1982.
25. I. Reiner, *Maximal Orders*, Academic Press, London, 1975.
26. N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin and New York, 1972.
27. J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517.
28. G. Shimura and Y. Taniyama, *Complex multiplication of abelian varieties and its applications to number theory*, Publ. Math. Soc. Japan, Tokyo, 1961.
29. J. Tate, *Classe d'isogénie des variétés abéliennes sur un corps fini* (*d'après T. Honda*), Séminaire Bourbaki, 1968/69, no. 352.
30. ——, *Conjectures on algebraic cycles in ℓ-adic cohomology*, these Proceedings, vol. 1, pp. 71–83.
31. W. Waterhouse, *Introduction to Affine Group Schemes*, Springer-Verlag, New York, 1979.
32. A. Weil, *Sur les fonctions algébriques à corps de constantes fini*, C.R. Acad. Sci. Paris **210** (1940), 592–594.

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Motives for Absolute Hodge Cycles

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Each cohomology theory may be regarded as a kind of linearization of the theory of algebraic varieties. Grothendieck's theory of motives **[Ma, Kl1, SR]** is supposed to provide the universal linearization; that is, the category of motives is intended to be the target of a universal cohomology theory. However, many of the wished-for properties of this category (such as the existence of a good tensor product and of Tannakian duality) depend on some unknown properties of algebraic cycles. Over $\mathbf{C}$ these properties, viz., the standard conjectures **[Kl2]**, would follow from the validity of the stronger Hodge conjecture which asserts that every Hodge cycle is the class of an algebraic cycle.

Deligne defined an intermediate group of so-called absolute Hodge cycles and showed that these have very useful properties. His definition is based on the following observation. An algebraic cycle on a smooth algebraic variety X gives rise to not only a Hodge class γ_B on X, but also to a Hodge class $\sigma\gamma_B$ on σX, for every automorphism σ of $\mathbf{C}$; namely, the one coming from the conjugate algebraic cycle. This property can be described in purely cohomological terms. Of course, in general, σ is not continuous and does not act on the singular cohomology. But via the de Rham isomorphism one can map a topological cycle γ into the algebraic de Rham cohomology, apply σ there, and go back to the singular cohomology (with coefficients in $\mathbf{C}$) to obtain a cycle $\sigma_{\mathrm{dR}}\gamma$ on σX. Something similar can be done for the ℓ-adic étale cohomology instead, giving cycles $\sigma_\ell \gamma$ with coefficients in $\mathbf{Q}_\ell$. Then a Hodge cycle γ is called an absolute Hodge cycle, if $\sigma_{\mathrm{dR}}\gamma$ is a Hodge cycle again and agrees with $\sigma_\ell\gamma$, for all automorphisms σ.

As we review below, this definition has the merit of enabling the construction of a category of motives that is Tannakian and neutral. Instead of defining, in the manner of Grothendieck, an effective motive to be a pair

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(X, p) where p is an idempotent cycle class of codimension $\dim(X)$ on $X \times X$, one requires that p be an absolute Hodge class. Then the standard conjectures hold in this setting. The same would hold if one took Hodge classes instead, but absolute Hodge cycles behave "more algebraically" in that $\mathrm{Aut}(\mathbf{C})$ acts on them and they can, in fact, be defined for an arbitrary base field of characteristic zero. This makes them especially valuable for arithmetic applications. On the other hand, it is difficult to understand which cohomology classes are absolute Hodge on a given variety. Here Deligne has expressed the famous "espoir" that every Hodge class is absolute Hodge **[D3]**. He was able to prove this for abelian varieties **[DM0S, I]**, and this result has had fundamental consequences for the theory of abelian varieties, especially those of CM type, Shimura varieties, and the special values of L-functions.

Let us briefly mention certain of these applications. First recall that the Mumford–Tate group G_A of an abelian variety A is the group attached by Tannakian duality (with the evident fiber functor) to the tensor category of rational Hodge structures generated by $H^1(A)$. For a rational prime ℓ and finitely generated field of definition K for A, let G_ℓ be the connected component of the Zariski closure of the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{GL}(H^1_\ell(A))$. Then Deligne's result implies at once that G_ℓ is contained in $G_A \otimes \mathbf{Q}_\ell$, thus verifying a part of a well-known conjecture of Mumford which asserts simply that $G_A \otimes \mathbf{Q}_\ell = G_\ell$. This part already suffices to show the following: If the Tate conjecture is true for A over every finite extension L of K, i.e., if each Tate class in $H^{2j}(A \otimes L)(j)$ is in the $\mathbf{Q}_\ell$-span of the classes defined by algebraic cycles on A defined over L, then the Hodge conjecture is true for A.

The most significant applications to date are to the theory of abelian varieties with complex multiplication and related topics. In particular, Deligne's result enabled him to prove **[DMOS, IV]** a generalization to all automorphisms of $\mathbf{C}$ (not just for those fixing the reflex field) of the Shimura–Taniyama reciprocity law for abelian varieties ofCM type. The form of the generalization was conjectured by Langlands **[Lgl]** in the language of motives: he constructed an explicit group, called the Taniyama group, which he conjectured to be isomorphic to the group attached by Tannakian duality to the also conjectural category of motives generated by abelian varieties defined over $\mathbf{Q}$ which are of CM type. Deligne's result provided the means to construct such a category and to prove the mentioned isomorphism for it. This reciprocity law, which was formulated by Tate in a very explicit, noncategorical way **[La]**, is the starting point of the proof, due to Milne, of Langlands's general reciprocity law for Shimura varieties, and hence of the most general existence theorems for such varieties. See the papers of Milne **[Mi2]** and Schappacher **[Sch]** in these Proceedings for more details and references.

The applications to special values of Hecke L-series are as follows. These series give the Hasse–Weil L-functions of motives arising from abelian varieties with complex multiplication. For the case of such motives defined over CM-fields, Deligne's general conjecture [**D3**], which express a critical value in terms of the periods of algebraic integrals attached to the motive, has been proven [**Bl1**]. Similarly, Anderson [**A**] constructed an extension of the Taniyama group as well as a corresponding category of motives in order to clarify the relation between values of gamma-functions and the periods of motives attached to Jacobi sum Hecke characters. He thus proved a conjecture of Lichtenbaum which refined the formula of Chowla and Selberg. Both methods make essential use of Deligne's category of motives and of the following general principle: An isomorphism of motives for absolute Hodge cycles gives a period relation.

It should be mentioned that variants of the notion of absolute Hodge cycle have also been considered and applied. For example, several authors [**Og**, **Bl2**, **Wi**] have studied p-adic properties of Hodge cycles, which would follow from the Hodge conjecture. These results, together with Deligne's theorem, also have interesting applications, e.g., to points mod p on Shimura varieties or to p-adic periods and their relations. See [**Bl2**] for some discussion and references.

Finally, the use of absolute Hodge cycles allows for the treatment of arbitrary, not necessarily projective or smooth varieties, and the definition of a corresponding category of mixed motives [**J1**, **D4**] in a situation in which Grothendieck's definition via algebraic correspondences does not apply.

In this paper we essentially confine ourselves to a review of the definition and some examples of absolute Hodge cycles, and of the constructions and basic properties outlined in [**DMOS, II**] and [**J1**]. However, to illustrate some techniques, we recall the application to the Mumford–Tate groups mentioned above, and Jannsen's proof that the realizations attached to modular forms of higher weight belong to motives for absolute Hodge cycles. Of course, the stronger result that they even belong to Grothendieck motives was later proved by Scholl [**Scho**]. However, the proof for absolute Hodge cycles is much simpler and generalizes to many other Shimura varieties.

The paper is divided into the following sections:

1. Cohomology groups and absolute Hodge cycles
2. A construction of motives using absolute Hodge cycles
3. Motivic Galois groups
4. Motives and categories of realizations
5. Polarizations and semisimplicity
6. Motives of cusp forms.

1. Cohomology groups and absolute Hodge cycles

Let X be an object of the category $V = V_k$ of smooth projective varieties over a field k, which is embeddable in $\mathbf{C}$, so that for an embedding $\sigma : k \hookrightarrow \mathbf{C}$ we have a complex variety $\sigma X = (X \times_{k,\sigma} \mathbf{C})(\mathbf{C})$. Fix an algebraic closure k^{al} of k and put $G_k = \mathrm{Gal}(k^{\mathrm{al}}/k)$. We shall consider the following cohomology groups $H^r_\alpha(X)$ for $\alpha = \sigma, \ell, \mathrm{dR}$:

1. $H^r_\sigma(X) = H^r(\sigma X, \mathbf{Q})$, the singular cohomology of σX for each $\sigma : k \hookrightarrow \mathbf{C}$. This is a $\mathbf{Q}$-vector space with a $\mathbf{Q}$-rational Hodge structure $H^r_\sigma(X) \otimes \mathbf{C} = \bigoplus_{i+j=r} H^{i,j}_\sigma$, where the complex vector spaces $H^{i,j}_\sigma = H^{i,j}_\sigma(X)$ are given by Hodge theory for σX. If the embedding σ is real, then the complex conjugation ρ on σX induces a canonical $\mathbf{Q}$-rational involution $F_\infty = F^\sigma_\infty$ on $H^r_\sigma(X)$.

2. $H^r_\ell(x) = H^r_{\text{ét}}(X \otimes_k k^{\mathrm{al}}, \mathbf{Q}_\ell)$, the étale cohomology. This is a $\mathbf{Q}_\ell$-vector space on which G_k acts continuously via functoriality of étale cohomology.

3. $H^r_{\mathrm{dR}}(X) = \mathbf{H}^r(X, \Omega^\cdot_X)$, the de Rham cohomology, i.e., the hypercohomology of the complex $\Omega^\cdot_X$ of algebraic differential forms on X. This is a k-vector space with a decreasing k-linear filtration $F^i H_{\mathrm{dR}}(X)$.

The notion of an absolute Hodge cycle is based on nontrivial interrelations between these groups given by the comparison isomorphisms:

$$I_{\infty,\sigma} : H^r_\sigma(X) \otimes \mathbf{C} \xrightarrow{\sim} H^r_{\mathrm{dR}} \otimes_{k,\sigma} \mathbf{C} \cong H^r_{\mathrm{dR}}(\sigma X),$$
$$I_{\ell,\bar\sigma} : H^r_\sigma(X) \otimes \mathbf{Q}_\ell \xrightarrow{\sim} H^r_\ell(X) \cong H^r_{\text{ét}}(\bar\sigma(X \otimes_k k^{\mathrm{al}}), \mathbf{Q}_\ell)$$

for each $\sigma : k \hookrightarrow \mathbf{C}$ and each extension $\bar\sigma : k^{\mathrm{al}} \hookrightarrow \mathbf{C}$ of σ. The isomorphism $I_{\infty,\sigma}$ relates the Hodge structure on $H_\sigma(X)$ and the Hodge filtration $F^i H^r_{\mathrm{dR}}(X)$ in the following way:

$$I_{\infty,\sigma}\left(\bigoplus_{i'\geq i} H^{i',j}_\sigma\right) = F^i H^r_{\mathrm{dR}}(X) \otimes_{\sigma,k} \mathbf{C}.$$

Also, the $I_{\ell,\bar\sigma}$ are related by the action of G_k: for each $\tau \in G_k$ there is the commutative diagram

$$\begin{array}{ccc} & & H^r_\ell(X) \\ & \overset{I_{\ell,\bar\sigma}}{\nearrow} & \uparrow \\ H^r_\sigma(X) \otimes \mathbf{Q}_\ell & & \tau \\ & \underset{I_{\ell,\bar\sigma\tau}}{\searrow} & | \\ & & H^r_\ell(X) \end{array}$$

If $\rho = \rho_{\bar\sigma} \in G_k$ is the complex conjugation with respect to $\bar\sigma$ lying over a real embedding σ of k, then the involutions ρ_σ on $H^r_\ell(X)$ and F^σ_∞ on $H^r_\sigma(X)$ correspond under the comparison isomorphisms. The involution F^σ_∞ can therefore be regarded as a kind of Frobenius element at infinity.

We also need the twisted cohomology groups $H^r_\alpha(X)(m)$, defined as follows:

1. For $\alpha = \sigma$, the $\mathbf{Q}$-rational Hodge structure is multiplied by $(2\pi i)^m$:

$$H^r_\sigma(X)(m) = (2\pi i)^m H^r_\sigma(X) \subset H^r_\sigma(X) \otimes \mathbf{C},$$

and there is a shift $(i, j) \to (i-m, j-m)$ in the Hodge decomposition:

$$(H_\sigma^r(X)(m))^{i-m,j-m} = H_\sigma^{i,j}(X) \subset H_\sigma^r(X) \otimes \mathbf{C}.$$

2. For $\alpha = \ell$

$$H_\ell^r(X)(m) = H_\ell^r(X) \otimes \mathbf{Z}_\ell(1)^{\otimes m},$$

where $\mathbf{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}$ as a G_k-module, μ_{ℓ^n} being the groups of roots of unity of degree ℓ^n in k^{al}.

3. For $\alpha = \mathrm{dR}$, $H_{\mathrm{dR}}^r(X)(m) = H_{\mathrm{dR}}^r(X)$ as k-vector spaces, but the twist changes the index of the Hodge filtration:

$$F^{i-m} H_{\mathrm{dR}}^r(X)(m) = F^i H_{\mathrm{dR}}^r(X).$$

An important fact is that all three cohomology theories $H_\alpha^r(X)$ satisfy the axioms of a Weil cohomology (with Tate twists):

1. (Cycle maps). Let $Ch^r(X)$ be the Chow group of algebraic cycles of codimension r on X modulo linear equivalence. There are natural homomorphisms

$$cl_\alpha^r : Ch^r(X) \to H_\alpha^{2r}(X)(r),$$

which are compatible with multiplication in the Chow ring and in the cohomology rings.

2. (Poincaré duality). $H_\alpha^r(X) = 0$ for $r \notin [0, 2d]$, $d = \dim X$, and there is a natural nondegenerate pairing

$$H_\alpha^r(X) \times H_\alpha^{2d-r}(X) \to H_\alpha^{2d}(X) \xrightarrow{\mathrm{Tr}} \mathbf{Q}_\alpha(-d)$$

for $r = 0, 1, \ldots, 2d$, where Tr is defined by $H_\alpha^{2d}(X)(d) \to \mathbf{Q}_\alpha$, $cl^d(\mathrm{pt}) = 1$ ($\mathbf{Q}_\alpha = \mathbf{Q}$, $\mathbf{Q}_\ell$, or k, according to whether $\alpha = B$, ℓ, or dR);

3. (Künneth formula). The natural map

$$H_\alpha^r(X \times Y) = \sum_{r+s=n} H_\alpha^r(X) \otimes H_\alpha^s(Y)$$

is an isomorphism. If we put

$$H^r(X)(m) = H_{\mathrm{dR}}^r(X)(m) \times \prod_\ell H_\ell^r(X)(m) \times \prod_\sigma H_\sigma^r(X)(m),$$

then we have the map

$$cl^r : Ch^r(X) \otimes \mathbf{Q} \to H^{2r}(X)(r),$$

whose image is a finite-dimensional $\mathbf{Q}$-vector space.

For $\alpha = \sigma$, the image of cl_α^r is known to consist of rational elements of type $(0, 0)$ (the Hodge cycles), and according to the Hodge conjecture there are no other Hodge cycles if $k = \mathbf{C}$.

For $\alpha = \ell$, the above property of the Galois action implies that Im cl_ℓ^r is a $\mathbf{Q}$-subspace of $H_\ell^r(X)^{G_k}$ (the $\mathbf{Q}_\ell$-subspace of elements fixed by G_k), and according to the Tate conjecture we have that (for k finitely generated over $\mathbf{Q}$)

$$H^r_\ell(X)^{G_k} = \mathrm{Im}\ cl^r_\ell \otimes_{\mathbf{Q}} \mathbf{Q}_\ell.$$

Moreover, the cycle maps cl^r_α (for $\alpha = \sigma$, ℓ, dR) are compatible with the comparison isomorphisms. This motivates the following definition.

DEFINITION. An absolute Hodge cycle (AHC) x is an element of the (finite-dimensional) Q-vector space

$$C^r_{\mathrm{AH}}(X) = \{(x_{\mathrm{dR}}, x_\ell, x_\sigma)_{\ell,\sigma} \in H^{2r}(X)(r) \mid I_{\infty,\sigma}(x_\sigma) = x_{\mathrm{dR}}, I_{\ell,\bar\sigma}(x_\sigma) = x_\ell$$
$$\text{for all } \sigma : k \hookrightarrow \mathbf{C} \text{ and } \bar\sigma : k^{\mathrm{al}} \hookrightarrow \mathbf{C} \text{ restricting to } \sigma,\ x_{\mathrm{dR}} \in F^0 H^{2r}_{\mathrm{dR}}(X)(r)\}.$$

The above definition of AHC is contained in **[J1]**. In the original definition of Deligne **[DMOS, I]**, an AHC t is defined first for an algebraically closed $k = k^{\mathrm{al}}$ embeddable in $\mathbf{C}$, as an element of $H^{2r}_{\mathrm{dR}}(X)(r) \times \prod_\ell H^{2r}_\ell(X)(r)$ such that:

1. For each $\sigma : k \hookrightarrow \mathbf{C}$ the element t lies in the rational subspace $H^{2r}_\sigma(X)(r)$ given as the image of the comparison isomorphisms.
2. The first component t_{dR} of t lies in $F^0 H^{2r}(X)(r) = F^r H^{2r}(X)$.

Secondly, for an arbitrary k embeddable in $\mathbf{C}$ an AHC is defined as an AHC on $X \otimes_k k^{\mathrm{al}}$ that is fixed under the natural action by G_k. Since in the above definition an AHC $(x_{\mathrm{dR}}, x_\ell, x_\sigma)_{\ell,\sigma}$ necessarily has components $x_\ell \in H^{2r}_\ell(X)^{G_k}$ and since $H^{2r}_{\mathrm{dR}}(X \otimes_k k^{\mathrm{al}})^{G_k} = H^{2r}_{\mathrm{dR}}(X)$, both definitions are equivalent.

Here are some simple examples of absolute Hodge cycles:

1. For all $Z \in Ch^r(X)$ we have that $cl^r(Z) = (cl^r_{\mathrm{dR}}(Z), cl^r_\ell(Z), cl^r_\sigma(Z)) \in C^r_{\mathrm{AH}}(X)$ by the compatibility of the cl_α with the comparison isomorphisms.
2. For $d = \dim X$ and for the diagonal $\Delta \subset X \times X$ consider the Künneth decomposition

$$H^{2d}(X \times X)(d) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)(d),$$

and the corresponding decomposition $cl^d(\Delta) = \sum_{i=0}^{2d} \pi^i$. Then the classes π^i are absolute Hodge cycles: $\pi^i \in C^d_{\mathrm{AH}}(X \times X)$.

This fact is an example of the following, somewhat vague principle: if we start from AHC and use some "canonical operations" compatible with the comparison isomorphisms, then we always get AHC.

3. Let L be the class of a hyperplane section of X (given with a projective embedding), $L \in C^1_{\mathrm{AH}}(X)$. According to the Hard Lefschetz Theorem the map

$$H^{2r}(X)(r) \to H^{2d-2r}(X)(d-r), \qquad x \mapsto L^{d-2r} \cdot x,$$

is an isomorphism. Then $\alpha \in C^r_{\mathrm{AH}}$ is equivalent to $L^{d-2r} \cdot x \in C^{d-r}_{\mathrm{AH}}$.

The last two examples show that some conjectural properties of algebraic cycles are valid for AHC by a simple reasoning. These conjectural properties are given by the standard conjectures (see [Kl1, Kl2]): examples 2 and 3 are the analogues for AHC of the standard conjectures $A(X, L)$ and $C(X)$, respectively. Note that the AHC analogue of the standard conjecture of Hodge type is true by the embedding $C^r_{\mathrm{AH}}(X) \subset \prod_\sigma H^{2r}_\sigma(X)(r)$ and classical Hodge theory. Hence, all the standard conjectures are true in the AHC setting.

In general, it is not so easy to construct absolute Hodge cycles on an arbitrary smooth projective variety X. This would change significantly if Deligne's "hope" was true that every Hodge cycle is absolute Hodge [D3, 0.10]). Deligne was able to prove this for abelian varieties [DMOS, I]:

THEOREM. *For an abelian variety A over $\mathbf{C}$, every Hodge cycle is an AHC.*

Here a Hodge cycle is one in the classical sense, i.e., one in $H^{2r}_{\mathrm{id}}(A)(r)$, for some $r \geq 0$, in the notation used above. Deligne's theorem says that to such a Hodge cycle one has corresponding ones in $H^{2r}_\sigma(A)(r)$ for all automorphisms $\sigma : \mathbf{C} \to \mathbf{C}$, which is a highly nontrivial statement.

For a sketch of the proof we refer to the article by Blasius [Bl2]. Some important applications of Deligne's theorem will be mentioned in the next section.

Another interesting corollary of this theorem is that for abelian varieties the Tate conjecture implies the Hodge conjecture. Indeed, let A be an abelian variety over $\mathbf{C}$, and choose a model A_0 of A over a finitely generated field k imbedded in $\mathbf{C}$ via $\sigma : k \hookrightarrow \mathbf{C}$. By Deligne's theorem, the inclusion

$$C^r_{\mathrm{AH}}(A) \to H^{2r}(A(\mathbf{C}), \mathbf{Q}(2\pi i)^r) \cap H^{r,r}$$

is a bijection. On the other hand, it is known that

$$C^r_{\mathrm{AH}}(A_0 \otimes_k k^{\mathrm{al}}) \to C^r_{\mathrm{AH}}(A)$$

is an isomorphism [DMOS, I, 2.9] and that G_k acts on the left-hand group via a finite quotient (loc. cit. and [J1, 2.19]). Hence, we may assume that

$$C^r_{\mathrm{AH}}(A_0) = C^r_{\mathrm{AH}}(A_0 \otimes_k k^{\mathrm{al}}),$$

by enlarging k if necessary. Since the ℓ-adic cycle map factors as

$$Ch^r(A_0) \otimes \mathbf{Q} \xrightarrow{cl^r} C^r_{\mathrm{AH}}(A_0) \longrightarrow H^{2r}_\ell(A_0)^{G_k},$$

and since $C^r_{\mathrm{AH}}(A_0) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \hookrightarrow H^{2r}_\ell(A)^{G_k}$ is an injection,

$$\dim_{\mathbf{Q}} \operatorname{Im} cl^r \leq \dim_{\mathbf{Q}} C_{\mathrm{AH}}(A_0) \leq \dim_{\mathbf{Q}_\ell} H^{2r}_\ell(A)^{G_k};$$

hence equality holds if we assume the Tate conjecture. In this case, cl^r is surjective, and the Hodge conjecture follows.

2. A construction of motives using absolute Hodge cycles

Now we explain how AHC can be used for constructing motives. We wish to attach to V_k a $\mathbf{Q}$-linear Tannakian category (cf. [DMOS, II; Br]) $\mathscr{M} = \mathscr{M}_k$

and a "universal cohomology functor" $h : V_k \to \mathcal{M}$. This easily can be done if we restrict ourselves to the subcategory V_k^0 of varieties of dimension zero over k:

Let $\mathcal{A}rt_k = \mathrm{Rep}_{\mathbf{Q}}\, G_k$ be the Tannakian category of $\mathbf{Q}$-linear discrete G_k-modules, and set

$$h(X) = \mathrm{Hom}\,(X(k^{\mathrm{al}}), \mathbf{Q}) \in \mathrm{Rep}_{\mathbf{Q}}\, G_k$$

for $X \in \mathrm{Ob}\, V_k^0$. Then for $X = \mathrm{Spec}(R) \to \mathrm{Spec}(k)$ the zero cohomology groups of X can be constructed from $V = h(X)$ as follows:

$$H_\sigma(X) = \mathrm{Hom}\,(\sigma X, \mathbf{Q}) = V,$$

$$H_\ell(X) = \mathrm{Hom}\,(X(k^{\mathrm{al}}), \mathbf{Q}_\ell) = V \otimes \mathbf{Q}_\ell, \qquad H_{\mathrm{dR}}(X) = R = (V \otimes k^{\mathrm{al}})^{G_k};$$

This shows that H_σ, H_ℓ and H_{dR} can be extended to tensor functors on $\mathcal{A}rt_k$ in a unique way such that the composition with $h : V_k^0 \to \mathcal{A}rt_k$ gives the singular, ℓ-adic, and de Rham cohomology, respectively. Note that $\mathcal{A}rt_k$ is generated as an abelian category by the objects $h(\mathrm{Spec}(K))$, where K runs through the finite Galois extension of k. The category $\mathcal{A}rt_k$ is called the category of Artin motives over k **[DMOS, II** §6; **Mi1]** (cf. below for a different construction).

In the general case, the original idea of Grothendieck was to first build an additive $\mathbf{Q}$-linear category $\mathcal{M}'$ with $h : V_k \to \mathcal{M}'$ such that

$$\mathrm{Hom}\,(h(X), h(Y)) = Ch_R^d(X \times Y) \qquad (d = \dim X)$$

is the space of $\mathbf{Q}$-linear correspondences from X to Y modulo some equivalence relation R, and such that the map $\mathrm{Hom}\,(Y, X) \to \mathrm{Hom}\,(h(X), h(Y))$ is given by associating to a morphism the class of its graph. Then one embeds $\mathcal{M}'$ in a bigger category $\mathcal{M}$ by adjoining images of projectors and powers of the Tate object and defines a tensor product $\otimes : \mathcal{M}' \times \mathcal{M}' \to \mathcal{M}'$ in order to get the category of motives $\mathcal{M}_k$ (we discuss these constructions in more detail below in the AHC case). Unfortunately, not enough is known about algebraic cycles to construct a Tannakian category of motives. On the one hand, one does not know that $\mathcal{M}_k$ is abelian, if the equivalence relation R is homological equivalence (which is needed for the functors H_α ($\alpha = dR, l, \sigma$) to factor through $\mathcal{M}_k$). Another somewhat surprising difficulty lies in adjusting the commutativity constraints (the functorial isomorphism $h(X) \otimes h(Y) \cong h(Y) \otimes h(X)$, see **[Kl1, Ma]** and below).

If one uses Hodge cycles or AHC instead of algebraic cycles, then the construction works, and we now describe its four steps more precisely in the case of AHC.

1. First define CV_k to be the $\mathbf{Q}$-additive category with objects hX, one for each $X \in Ob(V_k)$, and $\mathrm{Hom}\,(h(X), h(Y)) = C_{\mathrm{AH}}^d(X \times Y)$ for $d = \dim X$. Then there is a graph map $\mathrm{Hom}\,(Y, X) \to \mathrm{Hom}\,(h(Y), h(X))$,

which makes h a contravariant functor. There is a natural **Q**-linear tensor law on CV_k for which $h(X)\otimes h(Y) = h(X\times Y)$, the commutativity and the associativity constraints are induced by the natural isomorphisms $X\times Y \cong Y\times X$, $(X\times Y)\times Z \cong X\times (Y\times Z)$, and $1 = h(\mathrm{pt})$ is the identity object.

2. The category $\dot{\mathcal{M}}_k^+$ (of false effective motives) is the pseudo-abelian envelope of the **Q**-linear, additive category CV_k, i.e., obtained by adjoining the images of projectors p in CV_k, cf. **[Kl1]**. More precisely, the objects of $\dot{\mathcal{M}}_k^+$ are pairs $(h(X), p)$, where $p\in \mathrm{End}(h(X))$ is a projector (i.e., $p^2 = p$) and

$$\begin{aligned}\mathrm{Hom}\,&((h(X), p),\ (h(Y), q))\\ &= \{f\in \mathrm{Hom}\,(h(X), h(Y)) \mid fp = qf\}\ /\ \{f \mid fp = 0 = qf\}\\ &\cong q\,\mathrm{Hom}\,(h(X), h(Y))p.\end{aligned}$$

The tensor law in CV_k extends in a canonical way to $\dot{\mathcal{M}}_k^+$.

3. The category $\dot{\mathcal{M}}_k$ of false motives is obtained by adjoining to $\dot{\mathcal{M}}_k^+$ all inverse powers of the Lefschetz object $L = h^2(\mathbf{P}^1)$, where for $X\in \mathrm{Ob}\ V_k$ the objects $h^i(X)$ in $\dot{\mathcal{M}}_k{}^+$ are defined as pairs $(h(X), \pi^i)$, with π^i as in Example 1 in §1. More precisely, the objects of $\dot{\mathcal{M}}_k^+$ are pairs (M, m), also written as "Tate twists" $M(m)$, where $M\in \mathrm{Ob}\dot{\mathcal{M}}_k^+$ and $n\in \mathbf{Z}$, defining

$$\mathrm{Hom}\,(M(m), N(n)) = \mathrm{Hom}\,(M\otimes L^{N-m}, N\otimes L^{N-n}),$$

provided $N\geq m, n$. This definition is independent of N since one has canonical morphisms

$$\mathrm{Hom}\,(M', N') \ \xrightarrow{\sim}\ \mathrm{Hom}\,(M'\otimes L, N'\otimes L)$$

for M', $N'\in \mathrm{Ob}\dot{\mathcal{M}}_k{}^+$.

In the following, we identify $h(X)$ with its images under the embeddings

$$\begin{gathered}CV_k \to \dot{\mathcal{M}}_k{}^+ \to \dot{\mathcal{M}}_k,\\ h(X)\mapsto (h(X), \mathrm{id}),\qquad M\mapsto M(0).\end{gathered}$$

The tensor law extends to $\dot{\mathcal{M}}_k$ in a canonical way, and the important fact is that in $\dot{\mathcal{M}}_k$ internal Homs exist. Indeed, for $\dim X = d$ one can define $\underline{\mathrm{Hom}}(h(X), h(Y))$ as $h(X\times Y)(d)$, and this extends canonically to all of $\dot{\mathcal{M}}_k$. As a result we get the category $\dot{\mathcal{M}}_k$ of "false motives", which is a **Q**-linear rigid abelian tensor category, but not a Tannakian category. To see this, note that the natural commutativity constraints

$$\dot{\Psi} : h(X)\otimes h(Y) \xrightarrow{\sim} h(Y)\otimes h(X)$$

are those which in the cohomology induce the isomorphisms

$$H_\alpha^r(X) \otimes H_\alpha^s(Y) \xrightarrow{\sim} H_\alpha^s(Y) \otimes H_\alpha^r(X), \qquad u \otimes v \mapsto (-1)^{rs} v \otimes u.$$

This implies that the H_α do not give tensor functors, and moreover, that there is no fiber functor at all over any field extension of $\mathbf{Q}$. In fact, the above shows that the canonical rank function, which is defined for any rigid tensor category (cf. **[DMOS, II,** 1.7.3]), has the value

$$\operatorname{rk} h(X) = \sum (-1)^r \dim H_\alpha^r(X).$$

This is not necessarily positive (for example, the Euler characteristic of X could be equal to zero), but in a Tannakian category it would coincide with the dimension of a vector space (the image of $h(X)$ under a fiber functor).

4. The category $\mathcal{M}_k$ of true motives (for AHC) is the same as $\dot{\mathcal{M}}_k$, except that the commutativity constraints are modified to remedy the mentioned problem. This is possible since one can write $\dot{\Psi} = \bigoplus \dot{\Psi}^{r,s}$, with

$$\dot{\Psi}^{r,s} : h^r(X) \otimes h^s(Y) \xrightarrow{\sim} h^s(Y) \otimes h^r(X),$$

so that one defines the new constraints as

$$\Psi = \bigoplus (-1)^{rs} \dot{\Psi}^{r,s}.$$

(We have used the result that the π^i of Example 1 in §1 are AHC; it would suffice to have cycles which separate the even from the odd part in the cohomology.)

First properties of the category $\mathcal{M}_k$ are given by the following

THEOREM (**[DMOS, II,** 6.1, 6.5, 6.7]). (a) *$\mathcal{M}_k$ is a neutral, semisimple Tannakian category.*

(b) *Every object M in $\mathcal{M}_k$ is a direct factor of $h(X)(m)$ for some $X \in \operatorname{Ob} V_k$ and $m \in \mathbf{Z}$.*

(c) *There exists a contravariant functor $h : V_k \to \mathcal{M}_k$ such that $h(X \coprod Y) = h(X) \oplus h(Y)$, $h(X \times Y) = h(X) \otimes h(Y)$; if $\dim(X) = d$, then $h(X)^\vee = h(X)(d)$, where $h(X)^\vee = \underline{\operatorname{Hom}}(h(X), 1)$.*

(d) *The functor H_α on V_k can be extended to a $\mathbf{Q}_\alpha$-linear fiber functor on $\mathcal{M}_k$, where $\mathbf{Q}_\alpha = \mathbf{Q}, \mathbf{Q}_\ell$, or k, according to whether $\alpha = \sigma, \ell$, or dR.*

(e) *for $M, N \in \operatorname{Ob} \mathcal{M}_k$ we have that $\operatorname{Hom}(M, N)$ coincides with the $\mathbf{Q}$-vector space of families of maps $f_\alpha : M_\alpha \to N_\alpha$ such that f_{dR} preserves the Hodge filtration, f_ℓ is a G_k-morphism, and such that the f_α are compatible under the comparison isomorphisms.*

We sketch the proof: claims (b) and (c) are clear from the definitions; the functor h sends X to the object $h(X)$ defined before. In (a) we first have to show that $\mathcal{M}_k$ is an abelian category. For this it suffices to prove that $\operatorname{End}(h(X))$ is a semisimple $\mathbf{Q}$-algebra for every $X \in \operatorname{Ob} V_k$; this will imply that $\mathcal{M}_k$ is semisimple abelian (cf. **[J2,** Lemma 2]; the proof in **[DMOS, II,** 6.5, 6.6] is not correct). But the semisimplicity of $\operatorname{End}(h(X))$ follows from the fact that the analogues of the standard conjectures are true for AHC, so that the reasoning of [**Kl2**, 3.12] applies (cf. also §4). Hence, $\mathcal{M}_k$ is a rigid **Q**-linear abelian tensor category.

If we put $H_\alpha(M) = \bigoplus_i pH_\alpha^{i+2m}(X)(m)$ for $M = (h(X), p)(m) \in \operatorname{Ob} \mathcal{M}_k$, it is easily seen that $M \mapsto H_\alpha(M)$ is an additive, $\mathbf{Q}_\alpha$-linear, faithful tensor functor with values in the category $\operatorname{Vec}_{\mathbf{Q}_\alpha}$ of finite-dimensional $\mathbf{Q}_\alpha$-vector spaces. Since $\mathcal{M}_k$ is semisimple, the additivity of the H_α means that they are also exact, i.e., fiber functors. In particular, $\mathcal{M}_k$ is neutral Tannakian. In fact, for every $\sigma : k \hookrightarrow \mathbf{C}$, $H_\sigma : \mathcal{M}_k \to \operatorname{Vec}_{\mathbf{Q}}$ is a **Q**-linear fiber functor.

The proof of (e) is based on the Weil cohomology property of the functors H_α. First consider the case that $M = h(X)$ and $N = h(Y)(n)$ for $X, Y \in \operatorname{Ob} V_k$, $n \in \mathbf{Z}$. Simplifying the notation, we have for $d = \dim X$

$$\begin{aligned}\operatorname{Hom}(h(X), h(Y)(n)) &\overset{\text{def}}{=} C_{AH}^{d+n}(X \times Y) \subset H^{2d+2n}(X \times Y)(d+n) \\ &= \sum_{s+t=2d} H^s(X) \otimes H^{t+2n}(Y)(d+n) \\ &= \sum_{s+t=2d} H^{2d-s}(X)^\vee \otimes H^{t+2n}(Y)(n) \\ &= \bigoplus_{t=0}^{2d} \operatorname{Hom}(H^t(X), H^{t+2n}(Y)(n)),\end{aligned}$$

establishing (e) in this case, since the subspace of AHC is given exactly by the condition that the corresponding f_α are as in (e). The general case follows from this by twisting with powers of L and applying projectors.

3. Motivic Galois group

One important fact about Tannakian categories is that they can be determined up to equivalence in terms of some pro-algebraic groups. To be more precise, let $\mathcal{T}$ be a neutral F-linear Tannakian category, for a field F of characteristic zero, with fiber functor $V : \mathcal{T} \to \operatorname{Vec}_F$. (The nonneutral case, which is more complicated [**Br**], is not needed here.) Let $G = \operatorname{Aut}^{\otimes} V$ be the automorphism group of V: for an F-algebra R one has

$$G(R) = \{\text{isomorphisms } \varphi : V \otimes R \xrightarrow{\sim} V \otimes R \text{ of tensor functors}\},$$

where $V \otimes R : \mathcal{T} \to R\text{-Modules}$ is the functor with $V \otimes R(T) = V(T) \otimes_F R$ for $\mathcal{T} \in \operatorname{Ob}\mathcal{T}$. Then G is a pro-algebraic group over F, and V defines an equivalence of tensor categories

$$\begin{aligned}\mathcal{T} &\xrightarrow{\sim} \operatorname{Rep}_F G, \\ T &\mapsto V(T),\end{aligned}$$

where $\operatorname{Rep}_F G$ is the category of F-linear algebraic representations of G. (Note that $G(T)$ acts F-rationally on $V(T)$ by definition.)

Let $\mathcal{T}'$ be another neutral F-linear Tannakian category, with fiber functor $V' : \mathcal{T}' \to \mathrm{Vec}_F$ and associated group $G' = \mathrm{Aut}^{\otimes} V'$. Then any morphism $\Phi : \mathcal{T}' \to \mathcal{T}$ of tensor categories for which

$$\begin{array}{ccc} \mathcal{T}' & \xrightarrow{\quad\Phi\quad} & \mathcal{T} \\ & {\scriptstyle V'}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathrm{Vec}_F & \end{array}$$

is commutative, obviously induces a morphism of pro-algebraic groups

$$\phi : G \to G'.$$

Properties of $\mathcal{T}$ and Φ are reflected in the group data in the following way.

Proposition **[DMOS, II,** pp. 138–144]. (a) *The category $\mathcal{T}$ is semisimple if and only if the identity connected component G° of G is pro-reductive.*

(b) *The group G is algebraic if and only if $\mathcal{T}$ is generated by one object $\mathcal{T}$ as a tensor category.*

(c) *Assume that $\mathcal{T}$ is semisimple. Then ϕ is faithfully flat if and only if Φ is fully faithful.*

(d) *The morphism ϕ is a closed immersion if and only if every object T in $\mathcal{T}$ is isomorphic to a subquotient of an object $\Phi(T')$, where $T' \in \mathrm{Ob}\,\mathcal{T}'$.*

Assume that $\mathcal{T}$ is semisimple. For an object T of $\mathcal{T}$, let $\mathcal{T}_T$ be the full Tannakian subcategory generated by T. It is generated by $T \oplus T^{\vee}$ as a tensor category. Hence, according to parts (b) and (c) of the proposition, $\mathcal{T}_T$ corresponds to the algebraic group $G_T = \mathrm{Aut}^{\otimes}(V\,|_{\mathcal{T}_T})$, which is a quotient of G, and G is the projective limit of the G_T for all $T \in \mathrm{Ob}\,\mathcal{T}$.

If we apply this to $\mathcal{M}_k$ and the fiber functor $H_\sigma : \mathcal{M}_k \to \mathrm{Vec}_{\mathbf{Q}}$ for a fixed $\sigma : k \hookrightarrow \mathbf{C}$, we obtain an equivalence of categories

$$H_\sigma : \mathcal{M}_k \xrightarrow{\sim} \mathrm{Rep}_{\mathbf{Q}}\, G(k, \sigma),$$

where $G(k, \sigma) = \mathrm{Aut}^{\otimes} H_\sigma$ is a pro-reductive group over $\mathbf{Q}$. A first structure result is

Theorem. *If $\bar{\sigma} : \bar{k} \hookrightarrow \mathbf{C}$ is an embedding restricting to σ, then there is an exact sequence of algebraic groups*

$$1 \to G(k^{\mathrm{al}}, \bar{\sigma}) \xrightarrow{i} G(k, \sigma) \xrightarrow{\pi} G_k \to 1,$$

where $G(k^{\mathrm{al}}, \bar{\sigma})$ is the automorphism group of $H_{\bar{\sigma}} : \mathcal{M}_{k^{\mathrm{al}}} \to \mathrm{Vec}_{\mathbf{Q}}$ and where G_k is regarded as a constant pro-algebraic group. Moreover, $G(k^{\mathrm{al}}, \bar{\sigma})$ is the identity component of $G(k, \sigma)$, and is pro-reductive.

For a proof cf. [**DMOS, II**, 6.23]; it relies on the proposition. The reductivity reflects the semisimplicity of $\mathcal{M}_k$, the map i is induced by the base extension functor $\mathcal{M}_k \to \mathcal{M}_{k^{\mathrm{al}}}$, and π is induced by the embedding

$$\mathcal{A}rt_k \hookrightarrow \mathcal{M}_k$$

which is obtained as follows. Let $\mathcal{M}_k^0$ be the Tannakian subcategory of $\mathcal{M}_k$ generated by the objects $h(X)$ where X is a variety of dimension zero over k. It is easily seen that this coincides with CV_k^0, i.e., is obtained by applying steps 1 and 2 of the above construction to V_k^0 (Grothendieck's construction—where R is homological equivalence—and the AHC construction give the same result here, since on zero-dimensional varieties every AHC is algebraic). Moreover, the functor

$$\mathcal{M}_k^0 \xrightarrow{\sim} \mathcal{A}rt_k, \qquad (h(X), p) \mapsto p \operatorname{Hom}(X(k^{\mathrm{al}}), \mathbf{Q})$$

is an equivalence of categories: it is fully faithful by part (e) of the previous theorem, and essentially surjective since $\mathcal{A}rt_k$ is generated by the $\operatorname{Hom}(X(k^{\mathrm{al}}), \mathbf{Q})$ as mentioned before. The embedding $\mathcal{A}rt_k \hookrightarrow \mathcal{M}_k$ induces π, since G_k is the automorphism group of the forgetful functor

$$\mathcal{A}rt_k = \operatorname{Rep}_k \mathbf{Q} \;\to\; \operatorname{Vec}_{\mathbf{Q}}.$$

Since $G(k, \sigma)$ does for motives what G_k does for zero-dimensional varieties, $G(k, \sigma)$ is sometimes called the "motivic Galois group". Little is known about its structure, cf. **[Se]**. However, Deligne's fundamental result recalled in §1 allows us to calculate certain quotients. First recall the following general fact. Let M be a motive over k, let $\mathcal{M}_M$ be the Tannakian subcategory of $\mathcal{M}_k$ generated by M, and let G_M be the corresponding quotient of $G(k, \sigma)$ (i.e., the group of tensor automorphisms of H_σ restricted to $\mathcal{M}_M$). Since morphisms in $\mathcal{M}_k$ are given by AHC, one easily obtains:

LEMMA. *The group G_M can be identified with the algebraic subgroup of $\mathrm{GL}(H_\sigma)(M)$ fixing all AHC on all tensor products*

$$M^{\otimes n_1} \otimes \check{M}^{\otimes n_2}$$

for all $n_1, n_2 \in \mathbf{N}$.

Here by an AHC on a motive N we mean an element in

$$H(N) = \prod_\alpha H_\alpha(N)$$

satisfying properties analogous to these in the definition of $C_{\mathrm{AH}}(X)$ in §1. By projection these can be regarded as elements in $H_\alpha(N)$, and the action of $\mathrm{GL}(H_\sigma(M))$ on $H_\sigma(M)^{\otimes n_1} \otimes H_\sigma(M^\vee)^{\otimes n_2}$ in the natural way. The AHC are elements fixed by G_M in this space, so the lemma follows from **[DMOS, I]**.

COROLLARY. (a) *The identity component G_M° of G_M contains the Mumford-Tate group $\mathrm{MT}(H_\sigma(M))$ of the Hodge structure $H_\sigma(M)$.*

(b) *If M is pure of weight w, then G_M is contained in an orthogonal or symplectic group of similitudes, depending on whether w is even or odd.*

(c) *Let* G_ℓ *be the image of the homomorphism* $\rho_\ell : G_k \to \mathrm{GL}(H_\ell(M))$ *given by the action of* G_k *on* $H_\ell(M)$. *Then* G_ℓ *is contained in* $G_M(\mathbf{Q}_\ell)$.

In fact, for a polarizable $\mathbf{Q}$-Hodge structure H its Mumford–Tate group $\mathrm{MT}(H)$ is, equivalently, the smallest algebraic subgroup of $\mathrm{GL}(H)$ whose $\mathbf{C}$-points contain the image of the character $\mu : \mathbf{C}^\times \to \mathrm{GL}(H_\mathbf{C})$ for which $\mu(\lambda) = \lambda^p v$ for $\lambda \in \mathbf{C}^\times$ and $v \in H^{p,q}$. The equivalence of both definitions follows as in the above lemma, since H is semisimple (cf. [**DMOS, I**, p. 43], where the normalizations are slightly different and correspond to our Mumford–Tate group for $H \oplus \mathbf{Q}(1)$). Since $\mathrm{MT}(H)$ is connected and every AHC is in particular a Hodge cycle, (a) follows from the first decomposition. Part (b) follows from the existence of polarizations, cf. §5. The weight of a motive M is equal to that of the associated Hodge structure $H_\sigma(M)$; it is $i - 2n$ if M is a subquotient of $h^i(X)(n)$. For (c) we regard $G_M(\mathbf{Q}_\ell)$ as a subgroup of $\mathrm{GL}(H_\ell(M))$ via any of the comparison isomorphisms $H_\sigma(M) \otimes \mathbf{Q}_\ell \cong H_\ell(M)$. Then the claim follows, since the ℓ-adic components of AHC are fixed by G_k.

Corollary. (a) *The identity component* G_A° *of* G_A *is equal to the Mumford–Tate group* $\mathrm{MT}(\sigma A)$ *of the abelian variety* σA *over* $\mathbf{C}$.

(b) *For any prime* ℓ, $\mathrm{MT}(\sigma A)(\mathbf{Q}_\ell)$ *contains an open subgroup of the image of*

$$\rho_\ell : G_k \to \mathrm{GL}(H_\ell(M)).$$

Indeed, $G_A^\circ = G_{A\otimes k^{\mathrm{al}}}$ as in the above proposition, and $G_{A\otimes k^{\mathrm{al}}} = G_{\sigma A}$ since the AHC are the same for $A \otimes k^{\mathrm{al}}$ and σA, cf. [**DMOS, I**, 2.9]. On the other hand,

$$\mathrm{MT}(\sigma A) = \mathrm{MT}(H^1_{\mathrm{id}}(\sigma A)) = \mathrm{MT}(H^1_\sigma(A))$$

by definition, and on

$$H^1_{\mathrm{id}}(\sigma A)^{\otimes n_1} \otimes H^1_{\mathrm{id}}(\sigma A)^\vee)^{\otimes n_2}$$

Hodge cycles and AHC are the same by Deligne's result. This shows (a), and (b) is an immediate consequence of (a) and the previous corollary. Part (b) was independently proved by Piatetski-Shapiro ([**PS**], under some conditions on k), and Borovoi ([**Bo**], in general), by different methods. By Deligne's approach this extends to other motives, in the following way. If Hodge cycles are absolute Hodge in $\mathscr{M}_{\sigma M}$, for a motive M and its base change σM to $\mathbf{C}$, then $G_M^\circ = \mathrm{MT}(H_\sigma(M))$. As above this equality implies that $\mathrm{MT}(H_\sigma(M))(\mathbf{Q}_\ell)$ contains an open subgroup of $\mathrm{Im}\,(\rho_\ell : G_k \to \mathrm{GL}(H_\ell(M))$ which was conjectured by Mumford [**Mu**]. By Deligne's theorem this applies to all motives (for AHC) in the Tannakian subcategory $\mathscr{M}_k^{\mathrm{av}}$ of $\mathscr{M}_k$ which is generated by motives $h^1(A)$ for abelian varieties A over k. In particular, the above conjecture is true for motives $h^r(X)$, where X is a curve, a Fermat hypersurface, or a K3-surface, or a product of such varieties, cf.

[DMOS, II, 6.26]. The most striking success, however, of the theory of absolute Hodge cycles lies in its application to abelian varieties with complex multiplication. Namely, for a number field k Deligne's theorem allows us to calculate the whole pro-algebraic group (not just its identity component) associated to the subcategory CM_k generated by the $h(A)$ for abelian varieties A over k which become of CM-type over an extension of k: it is the Taniyama group introduced by Langlands. For this and for applications to the general theory of complex multiplication we refer to **[Scha]**.

In spite of all these successes, there are some limitations of AHC. For example, the theory only applies to characteristic zero, and there is no notion of reduction $\bmod p$ of a motive. In particular, it is not known in general, if for an AHC motive M over $\mathbf{Q}$, the $H_\ell(M)$ form a compatible system of ℓ-adic representations. While questions on periods (like Deligne's conjecture) can adequately be treated in the AHC setting (cf. **[H]**), the latter is not suitable for Beilinson's conjectures, since AHC are not known to act on algebraic K-theory. Certainly a stronger theory is desirable.

4. Motives and categories of realizations

A different approach to defining motives for AHC (i.e., AH-motives) was developed by U. Jannsen **[J1]**. One first introduces a big category $\mathscr{R}_k$ of realizations in which the realizations of AH-motives live. Then $\mathscr{M}_k$ may be defined as a subcategory of $\mathscr{R}_k$ generated by the realizations of all $X \in V_k$. A precise definition of $\mathscr{R}_k$ is made by defining a bigger category $\mathscr{MR}_k$ of mixed realizations, in which also mixed structures are allowed. With the help of $\mathscr{MR}_k$, it is possible to treat nonproper and singular varieties and to define a corresponding category of mixed motives (for absolute Hodge cycles) **[J1]**: one proves that for a variety V over k, its ℓ-adic, de Rham, and Betti cohomology define an object $h(V)$ in $\mathscr{MR}_k$, and one defines $\mathscr{MM}_k$ as the subcategory of $\mathscr{MR}_k$ generated by these $h(V)$. The fact that the cohomology of arbitary varieties gives rise to mixed structures, i.e., to (nontrivial) extensions of pure structures of different weight, was proved by Deligne **[D2]**. To be more precise, we start with the following

DEFINITION. *The category $\mathscr{MR}_k$ of mixed realizations (for AHC) over k consists of families*

$$M = (M_{\mathrm{dR}}, M_\ell, M_\sigma, I_{\infty,\sigma} I_{\ell,\bar\sigma})_{\ell \text{ prime number}, \sigma : k \hookrightarrow \mathbf{C},\ \bar\sigma : k^{\mathrm{al}} \hookrightarrow \mathbf{C}},$$

where

1. M_{dR} *is a finite-dimensional k-vector space with a decreasing filtration $(F^n)_{n\in\mathbf{Z}}$ (the Hodge filtration) and an increasing filtration $(W_m)_{m\in\mathbf{Z}}$ (the weight filtration)*;

2. M_ℓ *is a finite-dimensional $\mathbf{Q}_\ell$-vector space with a continuous G_k-action and an increasing filtration $(W_m)_{m\in\mathbf{Z}}$ (the weight filtration), which is G_k-equivariant*;

3. M_σ *is a mixed* **Q**-*Hodge structure, i.e., there is an increasing filtration* $(W_m)_{m\in\mathbf{Z}}$ (*the weight filtration*) *on* M_σ *and a decreasing filtration* $(F^n)_{n\in\mathbf{Z}}$ (*the Hodge filtration*) *on* $M_\sigma\otimes\mathbf{C}$, *which induces a* **Q**-*Hodge structure of weight* m *on* $\mathrm{Gr}_m^W M_\sigma = W_m M_\sigma / W_{m-1} M_\sigma$, *that is* $\mathrm{Gr}_m^W M_\sigma \otimes \mathbf{C} = \bigoplus_{i+j=m} M_\sigma^{i,j}$ *with* $\overline{M_\sigma^{i,j}} = M_\sigma^{j,i}$, *and*

$$F^p\left(\mathrm{Gr}_m^W M_\sigma \otimes \mathbf{C}\right) = \bigoplus_{i'\geq i} M_\sigma^{i',j};$$

4. $I_{\infty,\sigma} : M_\sigma \otimes_{\mathbf{Q}} \mathbf{C} \tilde{\to} M_{\mathrm{dR}} \otimes_{k,\sigma} \mathbf{C}$ *is an isomorphism identifying the filtrations induced by the Hodge filtrations* (*respectively, the weight filtrations*) *on both sides*;

5. $I_{\ell,\bar\sigma} : M_\sigma \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \tilde{\to} M_\ell$, *for* $\sigma = \bar\sigma|_k$ *is an isomorphism transforming the weight filtration on* M_σ *into the weight filtration of* M_ℓ *such that* $I_{\ell,\bar\sigma} = \rho \circ I_{\ell,\bar\sigma\rho}$ *for* $\rho \in G_k$ (*as above, see* §1).

A morphism $f : M \to M'$ *of mixed realizations is a family*

$$(f_{\mathrm{dR}}, f_\ell, f_\sigma)_{\ell \text{ prime number},\ \sigma : k \hookrightarrow \mathbf{C}},$$

where

1. $f_{\mathrm{dR}} : M_{\mathrm{dR}} \to M'_{\mathrm{dR}}$ *is* k-*linear and respects the filtrations* W *and* F;
2. $f_\ell : M_\ell \to M'_\ell$ *is a* $\mathbf{Q}_\ell$-*linear* G_k-*morphism which respects the weight filtrations;*
3. $f_\sigma : M_\sigma \to M'_\sigma$ *is a morphism of mixed* **Q**-*Hodge structures, i.e., compatible with the filtrations* W *and* F;
4. $f_{\mathrm{dR}}, f_\ell, f_\sigma$ *correspond under the comparison isomorphisms.*

It is easily seen that $\mathcal{MR}_k$ is a neutral Tannakian category over **Q** with the obvious constraints and fiber functors $M \mapsto M_\sigma \in \mathrm{Ob}\,\mathrm{Vec}_{\mathbf{Q}}$ for every $\sigma : k \hookrightarrow \mathbf{C}$. To prove that $\mathcal{MR}_k$ is abelian one uses the fact that the category of mixed Hodge structures is abelian.

Definition. 1. *A mixed realization* $M \in \mathrm{Ob}\,\mathcal{MR}_k$ *is pure of weight* m, *if* $W_m M = M$ *and* $W_{m-1}M = 0$, *where*

$$W_m M = (W_m M_{\mathrm{dR}}, W_m M_\ell, W_m M_\sigma; I_{\infty,\sigma}|_{W_m}, I_{\ell,\bar\sigma}|_{W_m})_{\ell,\sigma,\bar\sigma}.$$

2. *The category of realizations* $\mathcal{R}_k$ *is the full subcategory of* $\mathcal{MR}_k$ *whose objects are direct sums of pure realizations.*

3. *Let* $h : V_k \to \mathcal{R}_k$ *be the contravariant functor with*

$$h(X) = (H_{\mathrm{dR}}(X), H_\ell(X), H_\sigma(X); I_{\infty,\sigma}, I_{\ell,\bar\sigma})_{\ell,\sigma,\bar\sigma},$$

where

$$H_\alpha(X) = \bigoplus_{r=0}^{2d} H_\alpha^r(X),$$

$$I_{\infty,\sigma} = \bigoplus_{r=0}^{2d} I_{\infty,\sigma}^r,$$

$$I_{\ell,\bar\sigma} = \bigoplus_{r=0}^{2d} I_{\ell,\bar\sigma}^r \quad (\text{for } d = \dim X,\ X \in V_k),$$

and $W_m H_\alpha(X) = \bigoplus_{r\le m} H_\alpha^r(X)$.

Notice that to each motive M one can naturally associate a realization

$$H(M) = (H_{\mathrm{dR}}(M),\, H_\ell(M),\, H_\sigma(M);\, I_{\infty,\sigma},\, I_{\ell,\bar\sigma})_{\ell,\sigma,\bar\sigma},$$

where the $H_\alpha(M)$ are as at the end of §2. In view of the theorem in §2, and by using polarizations, one then can show [**J1**, Theorem 4.4]: the association $M \mapsto H(M)$ identifies the category $\mathscr{M}_k$ defined in §2 with the full Tannakian subcategory of $\mathscr{R}_k$ that is generated by the $h(X)$ for $X \in \mathrm{Ob}\, V_k$. This approach can be generalized to the mixed situation (where the method of §2 does not apply). Thus, one has

THEOREM. *Let V_k' be the category of all varieties over k. There are contravariant functors $h^n : V_k' \to \mathscr{MR}_k$ for $n \in \mathbf{Z}$, such that $h = \bigoplus h^n$ extends the functor $h : V_k \to \mathscr{R}_k$ introduced above.*

For a proof see [**J1**, 6.11.1]; it relies heavily on techniques developed by Deligne [**D2**] and Beilinson. The essential point is to define the realization

$$H^n(U) = (H_{\mathrm{dR}}^n(U),\, H_\ell^n(U),\, H_\sigma^n(U);\, I_{\infty,\sigma},\, I_{\ell,\bar\sigma})_{\ell,\sigma,\bar\sigma}$$

for a smooth variety $U \in V_k'$, cf. [**J1**, §3]. For example, the filtrations F and W on $H_{\mathrm{dR}}^n(U) = H_{\mathrm{dR}}^n(U/k) = \mathbf{H}^n(U_{\mathrm{Zar}}, \Omega_{U/k}^\cdot)$ (Zariski hypercohomology of the de Rham complex) are constructed using the logarithmic de Rham complex $\Omega_X^\cdot\langle Y\rangle$:

$$O_X \xrightarrow{d} \Omega_X^1\langle Y\rangle \xrightarrow{d} \Omega_X^2\langle Y\rangle \to \cdots,$$

for a good compactification of U: X is smooth and proper, $j : U \hookrightarrow X$ is an open immersion, and $Y = X\backslash U$ is the union of smooth divisors with normal crossing. This exists by resolution of singularities. One has the isomorphism $\mathbf{H}^n(U_{\mathrm{Zar}}, \Omega_{U/k}^\cdot) \cong H^n(X_{\mathrm{Zar}}, \Omega_X^\cdot\langle Y\rangle)$, and F and W are induced by the "stupid" filtration

$$F^n\Omega_X^\cdot\langle Y\rangle = \Omega_X^{\ge n}\langle Y\rangle$$

and the filtration by the order of pole along Y, respectively, as in [**D2**]. One has $H_\ell^n(X) = H^n(X \otimes_k k^{\mathrm{al}}, \mathbf{Q}_\ell)$ and $H_\sigma^n(X) = H^n(\sigma X, \mathbf{Q})$ also for an arbitrary variety X over k.

A possible definition of $\mathscr{MM}_k$ is now

DEFINITION. *$\mathcal{MM}_k$ is the full Tannakian subcategory of $\mathcal{MR}_k$ generated by $h(X) = \bigoplus h^n(X)$ for $X \in \mathrm{Ob}\, V_k'$.*

This definition is tentative, since one may want to include less realizations (e.g., only for smooth varieties X as in the original definition [**J1**, 4.1]) or more realizations of "geometric origin" as in [**De4**, 1.11], where a similar definition of mixed motives was given. Cf. [**J1**, Appendix C2] for further discussions of this problem.

5. Polarizations and semisimplicity

Absolute Hodge cycles can be effectively used for constructing pairings and polarizations in the categories of motives.

PROPOSITION ([**DMOS, II**, pp. 197–198]). 1. *Let $X \in \mathrm{Ob}\, V_k$, $\dim X = d$. There is a one-to-one correspondence between AHC $\Psi \in C_{\mathrm{AH}}^{2d-r}(X \times X)$ and families of pairings*

$$\Psi_\alpha^s : H_\alpha^s(X) \times H_\alpha^{2r-s}(X) \to \mathbf{Q}_\alpha(-r), \qquad s = 0, 1, \dots;$$

such that the Ψ_l^s are G_k-equivariant, the Ψ_σ^s are pairings of Hodge structures, and the Ψ_α^s are compatible under the comparison isomorphisms.

2. *Moreover, for some $\Psi \in C_{AH}^{2d-r}$ the induced pairing*

$$\Psi_\sigma^r : H_\sigma^r(X, \mathbf{R}) \times H_\sigma^r(X, \mathbf{R}) \to \mathbf{R}(-r)$$

is a polarization of $\mathbf{R}$-Hodge structures for every

$$\sigma : k \hookrightarrow \mathbf{C}$$

(i.e., $(2\pi i)^r \Psi_\sigma(u, Cv) > 0$ is a positive-definite form, where C is the Weil operator: $C = i \in S(\mathbf{R}) = \mathbf{C}^\times$ acting on every $\mathbf{R}$-Hodge structure H by multiplying $v \in H^{p,q}$ with i^{-p}).

The proof of 1 is similar to that of the theorem in §2. We have that

$$\begin{aligned}\Psi \in C_{\mathrm{AH}}^{2d-r}(X \times X) &\subset H^{4d-2r}(X \times X)(2d - r)\\ &= \bigoplus_{s+t=4d-2r} H^s(X) \otimes H^t(X)(2d - r)\\ &= \bigoplus H^{2d-s}(X)^\vee \otimes H^{2d-t}(X)^\vee(-r).\end{aligned}$$

Writing s for $2d - s$ we get the pairings Ψ^s as elements of

$$H^s(X)^\vee \otimes H^{2r-s}(X)^\vee \otimes \mathbf{Q}(-r) = \mathrm{Hom}\,(H^s(X) \otimes H^{2r-s}(X), \mathbf{Q}(-r)),$$

and 1 follows, since the subspace of AHC in $H^{4d-2r}(X \times X)(2d - r)$ is given exactly by the conditions on the Ψ_α^s as in 1.

In order to obtain the polarization in 2 one constructs an AHC in $C_{\mathrm{AH}}^r(X \times X)$ corresponding to a morphism

$$\Phi : h^r(X) \to h^{2d-r}(X)(d - r)$$

which is the motivic version of the $*$-operator of the Hodge theory. Note that as above

$$C^r_{\mathrm{AH}}(X \times X) \subset \bigoplus_s \mathrm{Hom}\,(H^s(X),\, H^{2d-2r+s}(X)(d-r)),$$

and the desired homomorphism by construction is trivial for all s, except for $s = r$ in which case it coincides with the $*$-operator. In order to define this operator, we recall that $L^{d-r} : H^r(X) \xrightarrow{\sim} H^{2d-r}(X)(d-r)$ is the isomorphism of the hard Lefschetz theorem for $r \leq d = \dim X$. By definition, the subspace of primitive cohomology is defined by $H^r_{\mathrm{prim}}(X) = \mathrm{Ker}(L^{d-r+1} : H^r(X) \to H^{2d-r+2}(X)(d-r+1))$, so that there is the following decomposition

$$H^r(X) = \bigoplus_{s \geq r-d,\, s \geq 0} L^s H^{r-2s}(X)(-s)_{\mathrm{prim}}.$$

Thus, any $x \in H^r(X)$ can be written uniquely in the form

$$x = \sum L^s(x_s), \quad \text{with } x_s \in H^{r-2s}(X)(-s)_{\mathrm{prim}}.$$

Define

$${}^*x = \sum (-1)^{(r-2s)(r-2s+1)/2} L^{d-r+s} x_s \in H^{2d-r}(X)(d-r).$$

Then

$$x \mapsto {}^*x : H^r(X) \to H^{2d-r}(d-r)$$

respects the structures and it is compatible with the comparison isomorphisms. It follows from the theorem in §2 that this map, or rather the map $H(X) \to H(X)(d-r)$ that is $x \to {}^*x$ on H^r and zero otherwise, is defined by an AHC. We take Ψ^r to be

$$H^r(X) \otimes H^r(X) \xrightarrow{\mathrm{id} \otimes *} H^r(X) \otimes H^{2d-r}(X)(d-r) \to H^{2d}(X)(d-r) \xrightarrow{\mathrm{Tr}} \mathbf{Q}(-r).$$

Clearly, it is defined by an AHC, and the Hodge–Riemann bilinear relations show that it defines a polarization on the real Hodge structure $H^r_\sigma(X, \mathbf{R})$ for each $\sigma : k \hookrightarrow \mathbf{C}$ (**[DMOS, II**, p. 199] and **[We**, 5.3]).

An important corollary of this proposition is that the category $\mathcal{M}_k$ is semisimple (i.e., $\mathrm{End}(M)$ is a semisimple $\mathbf{Q}$-algebra for all $M \in \mathcal{M}_k$). This is similar to the fact that any substructure of a polarizable Hodge structure is a direct factor. More precisely, the $*$-operator

$$* : H^{2r}(X)(r) \to H^{2d-2r}(X)(d-r)$$

carries $C^r_{\mathrm{AH}}(X)$ to $C^{d-r}_{\mathrm{AH}}(X)$, and

$$\Psi : H^{2r}(X)(r) \times H^{2r}(X)(r) \to \mathbf{Q}$$

is positive definite on $C^r_{\mathrm{AH}}(X)$. Applied to $X \times X$ this implies that $C^d_{\mathrm{AH}}(X \times X) = \mathrm{End}\,(h(X))$ is a semisimple $\mathbf{Q}$-algebra, cf. **[DMOS, II**, 6.3] or the formalism of the standard conjectures mentioned in the proof of the theorem in §3.

The polarizations also allow the following description of the possible summands M of $h(X)(m)$ which is very useful for the construction of motives.

LEMMA ([**J1**, 1.1]). *Let* $M \in \operatorname{Ob}\mathcal{M}_k$. *Suppose given*

1. *a* K*-subspace* $U_{\mathrm{dR}} \subset H_{\mathrm{dR}}(M)$,
2. *for each* ℓ *a* $\mathbf{Q}_\ell$*-subspace* $U_\ell \subset H_\ell(M)$, *which is a* G_k*-submodule, and*
3. *for each* $\sigma : k \to \mathbf{C}$ *a* $\mathbf{Q}$*-subspace* $U_\sigma \subset H_\sigma(M)$, *which is a sub-*$\mathbf{Q}$*-Hodge structure,*

such that these subspaces correspond under the comparison isomorphisms. Then there is a decomposition $M = M_1 \oplus M_2$ *in motives such that* $U_\alpha = H_\alpha(M_1)$ *where* α *runs through* dR, ℓ, *and* σ.

Indeed, one can construct orthogonal complements V_α of the U_α with respect to polarization (as in part (2)) of the above proposition (by Tate twist we may assume that M is $h^r(X)$ for $X \in \operatorname{Ob} V_k$). Therefore, the projectors $p_\alpha : H_\alpha(M) \to U_\alpha \to H_\alpha(M)$ are given by a projector $p \in \operatorname{End}(M)$, and we may define $M_1 = \operatorname{Im} p$, $M_2 = \operatorname{Ker} p$. Here is another interesting corollary.

COROLLARY. *If* X, Y *are varieties over* k *with* X *smooth and projective, then for any morphisms* $f : Y \to X$, $g : X \to Y$ *the kernel of* $f_\alpha^* : H_\alpha^r(X) \to H_\alpha^r(Y)$ *is defined by a motive* $\operatorname{Ker} f^* \subset h^r(X)$ *and the image of* $g_\alpha^* : H_\alpha^r(Y) \to H_\alpha^r(X)$ *is defined by a motive* $\operatorname{Im}\, g^* \subset h^r(X)$, *and these are direct factors of* $h^r(X)$.

6. Motives of cusp forms

Let $f(z) = \sum_{n=q}^{\infty} a_n q^n$ $(q = \exp(2\pi i z))$ be a primitive cusp form on the upper half-plane $\mathcal{H}$ of weight $k+2$ $(k \geq 0)$, conductor N and character ε. There is a smooth projective curve $X_1(N)$ over $\mathbf{Q}$ and an open subvariety $j : Y_1(N) \hookrightarrow X_1(N)$ such that the $\mathbf{C}$-valued points can be identified with

$$\Gamma_1(N)\backslash\mathcal{H} \hookrightarrow \overline{\Gamma_1(N)\backslash\mathcal{H}} = \text{compactification by adding the cusps.}$$

Using the universal elliptic curve $g : E \to Y_1(N)$ $(N \geq 3)$, Deligne described some realizations $M(f)_\alpha$ [**D6**, §7] $(\alpha = B, \ell, dR)$ attached to f (over $k = \mathbf{Q}$), such that the L-function of the system of the ℓ-adic representations $r_\ell : \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \operatorname{GL}(M(f)_\ell)$ coincides with the Mellin transform $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$ of f. This L-function is defined as the Euler product

$$L(M(f)_\ell, s) = \prod_p L_p(M(f)_\ell, p^{-s}),$$
$$L_p(M(f)_\ell, X)^{-1} = \det(1 - X r_\ell(Fr_p) | M(f)_\ell^{I_p}) \quad (\ell \neq p),$$

where $M(f)_\ell^{I_p}$ is the $\mathbf{Q}_\ell$-subspace of elements fixed by the inertia group I_p.

We suppose for simplicity that $a_n \in \mathbf{Q}$; otherwise, one needs realizations with coefficients in the number field $T = \mathbf{Q}(\langle a_n \rangle)$. In Deligne's construction the $M(f)_\alpha$ are defined as parts of the cohomology

$$H_\alpha^1(X_1(N), j_* \operatorname{Sym}^k(R^1 g_* \mathbf{Q})),$$

namely as kernel of $T_n - a_n$, where T_n is the nth Hecke operator, for all n prime to N. Consider $g_k : E_k \to Y_1(N)$ the k-fold fiber product of g, where $E_0 = Y_1(N)$. Then one can show that the realizations $M(f)_\alpha$ are direct factors of the parabolic cohomology

$$H^{k+1}_{p,\alpha}(E_k) = \operatorname{Im}(H^{k+1}_{c,\alpha}(E_k) \to H^{k+1}_\alpha(E_k))$$

($H^{k+1}_{c,\alpha}$ denotes the cohomology with compact support) which are defined by the behaviour of several algebraic correspondences acting on it: the Hecke operators T_n (which are also defined as correspondences of E, and thus, of E_k) act by multiplication with a_n, the morphism $m_1 \operatorname{id}_E \times \cdots \times m_k \operatorname{id}_E$ ($m_i \in \mathbf{Z}$) induces the multiplication by $m_1 \cdots m_k$, and the symmetric group $S_k \subset \operatorname{Aut} E_k$ acts by the sign character.

THEOREM ([**J1**, p. 5]). *The realizations attached to an elliptic modular form f by Deligne belong to a motive $M(f)$.*

PROOF. This is easily proved with the tools of §4 and §5. Let $j : E_k \hookrightarrow \bar{E}_k$ be an open immersion into a smooth projective variety $\bar{E}_k$. Then the maps defining parabolic cohomology factor as

$$H^{k+1}_{c,\alpha}(E_k) \xrightarrow{j_!} H^{k+1}_\alpha(\bar{E}_k) \xrightarrow{j^*} H^{k+1}_\alpha(E_k).$$

These maps come from morphisms in $\mathcal{MR}_k$

$$h^{k+1}_{c,\alpha}(E_k) \xrightarrow{j_!} h^{k+1}_\alpha(\bar{E}_k) \xrightarrow{j^*} h^{k+1}_\alpha(E_k).$$

Cohomology with compact support was not mentioned in §4, but $j_!$ could be defined as the morphism

$$h^{k+1}(E_k)^\vee(-k-1) \xrightarrow{(j^*)^\vee} h^{k+1}(\bar{E}_k)^\vee(-k-1) \cong h^{k+1}(\bar{E}_k),$$

by the commutative diagram from Poincaré duality

$$\begin{array}{ccccccc} H^{k+1}_\alpha(\overline{E}_k) & \times & H^{k+1}_\alpha(\overline{E}_k) & \to & H^{2(k+1)}_\alpha(\overline{E}_k) & \overset{\mathrm{Tr}}{\to} & \mathbf{Q}_\alpha(-k-1) \\ j_! \uparrow & & \downarrow j^* & & \uparrow j_! & & \\ H^{k+1}_{c,\alpha}(E_k) & \times & H^{k+1}_\alpha(E_k) & \to & H^{2(k+1)}_{c,\alpha}(E_k). & & \end{array}$$

The mentioned algebraic correspondences define a subrealization $\tilde{M}$ of $h^{k+1}(E_k)$, and the realization $M(f)$ in question is $\operatorname{Im} j_! \cap (j^*)^{-1}(\tilde{M})$, which is a motive by the lemma in §5 (Note that

$$\operatorname{Im} j_! \cap \operatorname{Ker} j^* = (\operatorname{Ker} j^*)^\perp \cap Ker j^* = 0,$$

since the Poincaré pairing is nondegenerate on $\operatorname{Ker} j^*$, again by the polarization).

For $N = 1, 2$ one gets the motive $M(f)$ from a motive with bigger N' via taking the fixed part under a finite subgroup of $\mathrm{SL}_2(\mathbf{Z}/N'\mathbf{Z})$. This again gives a motive [**DMOS**, p. 206].

Note that a purely algebraic construction of $M(f)$ in the category of motives defined by algebraic correspondences, is given by A. Scholl **[Scho]**. However, the proof of this stronger result is considerably more difficult.

REFERENCES

[A] G. Anderson, *Cyclotomy and an extension of the Taniyama group*, Compositio Math. **57** (1986), 153–217.

[Bl1] D. Blasius, *On the critical values of Hecke L-series*, Ann. of Math. **124** (1986), 23–63.

[Bl2] ——, *A p-adic property of Hodge classes on abelian varieties*, these Proceedings, vol. 2, pp. 293–308.

[Bo] M. Borovoĭ, *The action of the Galois group on the rational cohomology classes of type (p, p) of abelian varieties*, Mat. Sb. **94** (1974), 649–652, 656. (Russian)

[Br] L. Breen, *Tannakian categories*, these Proceedings, vol. 1, pp. 337–376.

[D1] P. Deligne, *Formes modulaires et représentations ℓ-adiques*, Sem. Bourbaki 1968/69, exp 355, Lecture Notes in Math., vol. 179, Springer-Verlag, Berlin and New York, 1971, pp. 139–172.

[D2] ——, *Théorie de Hodge* III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–78.

[D3] ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 313–342.

[D4] ——, *Le groupe fondamental de la droit projective moins trois points*, Galois Groups over $\mathbf{Q}$, Math. Sci. Res. Inst. Publ. Proceedings, Springer-Verlag, New York, 1989.

[DMOS] P. Deligne, J. Milne, A. Ogus, and K. Shih, *Hodge cycles, motives and Shimura varieties*, Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982.

[H] M. Harris, *Hodge-de Rham structures and periods of automorphic forms*, these Proceedings, vol. 2, pp. 573–624.

[J1] U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, Berlin and New York, 1990.

[J2] ——, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), 447–452.

[Kl1] S. Kleiman, *Motives*; *Algebraic geometry Oslo* 1970, Wolters-Noordhoff, Gröningen 1972, pp. 53–82.

[Kl2] ——, *Algebraic cycles and the Weil conjectures*, Dix Exposés sur la Cohomologie des Schémas (J. Giraud et al., eds.), North-Holland, Amsterdam, 1968, pp. 359–386.

[La] S. Lang, *Complex multiplication*, Grundlehren Math. Wiss, vol. 255, Springer-Verlag, New York, 1983.

[Lgl] R. Langlands, *Automorphic representations, Shimura varieties, and motives*, Ein Märchen, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 205–246.

[Ma] Yu. I. Manin, *Correspondences, motives, and monoidal transformations*, Mat. Sb. 77 (1968), 457–507. (Russian)

[Mi1] J. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Proc. Conf., Automorphic forms, Shimura varieties and L-functions, Ann Arbor, July 6–16 (L. Clozel and J. S. Milne, eds.), Academic Press, New York, 1990.

[Mi2] ——, *Shimura varieties and motives*, these Proceedings, vol. 2, pp. 447–523.

[Mu] D. Mumford, *Families of abelian varieties*, Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, RI, 1966, pp. 347–351.

[Og] A. Ogus, *A p-adic analogue of the Chowla–Selberg formula*, Lecture Notes in Math., vol. 1454, Springer-Verlag, Berlin and New York, 1991, pp. 319–341.

[PS] I. I. Piatetski-Shapiro, *Interrelations between the Hodge and the Tate hypotheses for abelian varieties* (in Russian), Mat. Sb. **85** (1971), 610–620; English transl. in Math. USSR - Sb. **14** (1971), 615–625.

[SR] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin and New York, 1972.

[Scha] N. Schappacher, *CM motives and the Taniyama group*, these Proceedings, vol. 1, pp. 485–508.

[Scho] A. Scholl, *Motives for modular forms*, Invent Math. **100** (1990), 419–430.

[Se] J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques*, these Proceedings, vol. 1, pp. 377–400.

[We] R. Wells, *Differential analysis on complex manifolds*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[Wi] J. P. Wintenberger, *Théorèmes de comparison p-adique pour les schémas abéliens*, publ. Paris - Sud, Paris, 91 - 26.

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CM Motives and the Taniyama Group

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0. Introduction

0.1. The ring of endomorphisms End(A) of an elliptic curve A over **C** is either isomorphic to **Z** or to an order $\mathscr{O}$ in an imaginary quadratic field K. In the latter case, A is said to have complex multiplication by $\mathscr{O}$. The classical theory of complex multiplication describes explicitly the action of Aut($\mathbf{C}/K$) on A—i.e., on its j-invariant—and on the torsion points of A, i.e., on its Tate module. In particular, one shows that A always has a model over the so-called ring class field of $\mathscr{O}$, a certain abelian extension of K, and the action on torsion points (the so-called "reciprocity law") is given in terms of the class field theory of K. Furthermore, the L-function of an elliptic curve with complex multiplication defined over a number field k is seen to be a product of two L-functions of Hecke characters of k with values in K.[1]

The generalisation of this theory of complex multiplication to abelian varieties of higher dimension is due to [**Shimura and Taniyama**, 1961]. An abelian variety $A/\mathbf{C}$ of dimension n (say, A simple) is said to have complex multiplication if its ring of endomorphisms has maximal possible rank over **Z**, i.e., if it is isomorphic to an order in a field E of degree $[E:\mathbf{Q}] = 2n$. Then E is necessarily a so-called CM-field, i.e., a totally imaginary quadratic extension of a totally real field. In this case, A (together with its endomorphisms) has a model over some number field k, and the action of End(A) on the tangent space at 0 of A defines a k-linear representation of E which diagonalises over some smallest CM subfield K of k, called the reflex field of E (with respect to A). The Shimura-Taniyama reciprocity law—see [**Shimura**, 1971, Theorem 5.15]—describes the action of Aut($\mathbf{C}/K$) on A and its torsion

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[1] These notions go back to [**Hecke**, 1918, 1920]. But the link with CM elliptic curves is due to [**Deuring**, 1953ff].

points in terms of the class field theory of K. Furthermore, the L-function of A over k is expressed as a product of L-functions of Hecke characters of k with values in E.

In the elliptic curve case K is imaginary quadratic, and since it is easy to analyse the action of the continuous automorphism of complex conjugation, one obtains the complete action of Aut $\mathbf{C}$ on A. Also, if A (but not all its endomorphisms) is defined over a real field k_0 (for instance, $k_0 = \mathbf{Q}$), then it is not difficult to identify the L-function of A over k_0 as one of the two Hecke L-functions occurring as factors of the L-function of A over $k_0 \cdot K$. See [**Deuring** 1953ff].

In the general case however, the action of automorphisms on the abelian variety A which are not trivial on the reflex field K was not known before 1980. And in the case where A (not its complex multiplication) descends to a number field k_0 not containing K, it was not clear in general whether and how the L-function of A over k_0 could be expressed by Hecke L-functions.

0.2. This incompleteness of the "classical" theory of complex multiplication of abelian varieties was not just an esthetic blunder, but represented a serious desideratum in view of the applications to (special points of) Shimura varieties. According to Deligne, Shimura varieties should parametrise motives. This prompted Langlands [**Langlands**, 1979] to look for a motivic formulation of the problem of conjugation of Shimura varieties.

In doing so, Langlands reduces to Shimura varieties of tori and uses Serre's results from [**Serre**, 1968], where a group-theoretic treatment of algebraic Hecke characters and abelian ℓ-adic representations had been given. In particular, Serre had defined the (connected) "Serre group" $\mathcal{S}$, the representations of which may be thought of as "CM-motives over $\overline{\mathbf{Q}}$"—see 1.3 below.[2]

The group scheme corresponding to a (hypothetical) category of CM motives over $\mathbf{Q}$ should be an extension of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) by $\mathcal{S}$—see 1.2, 1.4 below. In [**Langlands**, 1979], Langlands wrote down explicit cocycles characterising a certain extension of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) by $\mathcal{S}$ which he called the Taniyama group and which he conjectured to be the group scheme corresponding to "the" category of CM motives over $\mathbf{Q}$.

It seems that it was Milne and Shih who shortly hereafter observed how Langlands's conjectural description of CM motives implied a generalisation of the Shimura-Taniyama reciprocity law to all automorphisms of $\mathbf{C}$—cf. [**Milne, Shih**, 1981 and 1982] and [**Milne**, 1981]. (For the reverse implication, see Milne's contribution "Motives and Shimura varieties" to the present Proceedings.)

0.3. In Langlands's *Märchen*, the category of CM motives was hypothetical and the word "motive" was not given a precise meaning. Soon after the

[2] Serre himself mentions Grothendieck's hypothetical theory of motives as an inspirational background in the introduction to [**Serre**, 1968].

Corvallis conference Deligne made progress on absolute Hodge cycles and the category of motives defined with them—see [**Deligne** 1979, 0.9–0.11], cf. [**Panchishkin** 1993].

More precisely, Deligne was able to show that, on an abelian variety, every Hodge cycle is an absolute Hodge cycle: [**Deligne** (**Milne**), 1982]. Thus, any category of motives for absolute Hodge cycles which is generated by a class of abelian varieties is not only well defined, but in it, we actually have at our disposal essentially all homomorphisms and objects that are expected to exist.

0.3.1. Using this manageable category of motives, Deligne was then able to prove, not later than the summer of 1981 [**Deligne**, 1982], that *Langlands's Taniyama group is isomorphic to the motivic Galois group of the category* $\mathcal{CM}_{\mathbf{Q}}$ of absolute Hodge cycle CM motives over $\mathbf{Q}$ (see §1 below for the precise definition of $\mathcal{CM}_{\mathbf{Q}}$).

This theorem of Deligne solves the two problems mentioned in §0.1 above, to wit: (a) it generalises the Shimura-Taniyama reciprocity law to all automorphisms of $\mathbf{C}$, and (b) it settles the problem about the L-function of a potentially CM abelian variety. See [**Deligne**, 1982, pp. 262/3: Remarques 4 and 2].

0.4. To appreciate (b), recall that, *a priori*, absolute Hodge cycle motives might have some undesirable properties. Specifically, absolute Hodge cycles cannot in general be demonstrated to behave well under reduction. Therefore, we cannot be sure at first that our motives have strictly compatible systems of ℓ-adic representations. So their L-functions are not under control. However, the Taniyama group is related to the Weil group (of $\mathbf{Q}$). Thus, via Deligne's result, the objects of $\mathcal{CM}_{\mathbf{Q}}$ do give finite-dimensional representations of the Weil group. See §5 below for details.

In fact, (b) had essentially been established independently slightly earlier, by H. Yoshida [**Yoshida**, 1981].

0.5. As for the general reciprocity law (a), John Tate (who spent the academic year 1980–81 in Paris) worked out, and partially proved, in the Spring of 1981 a conjecture giving a completely explicit class field theoretic generalisation of the Shimura-Taniyama reciprocity law: see [**Tate**, 1981], a manuscript which, among others, is projected to appear in the Collected Papers of Tate. In our notation of 4.4 below, Tate's conjecture says that $f_E(s, \lambda) = g_E(s, \lambda)$ for all $s \in G_{\mathbf{Q}}$ and for all "CM-types" λ, i.e., for all characters λ of the Serre group associated to abelian varieties of CM type by E.

Everything that Tate formulated and proved was (motivated by, but) logically independent of Langlands's construction and Deligne's work on Langlands's conjecture. In fact, Tate's conjecture and his partial proof of it could have in principle been obtained by Shimura and Taniyama back in the fifties.

However, in the fall of 1981 Deligne gave a complete proof of Tate's statement using the theory of absolute Hodge cycles. More precisely, Deligne showed that all one had to know to derive the full conjecture of Tate from Tate's own partial result was the following: the quantitites $f_E(s, \lambda)$ and $g_E(s, \lambda)$ satisfy a relation $\prod f_E(s, \lambda_i)^{a_i} = 1$ and $\prod g_E(s, \lambda_i)^{a_i} = 1$ ($a_i \in \mathbf{Z}$), whenever the corresponding linear combination of the CM-types λ_i is the trivial character of the Serre group. This follows trivially if we can consistently define $f_E(s, \lambda)$ and $g_E(s, \lambda)$ for any character of the Serre group, not just for CM-types of abelian varieties. [3]

0.6. In the following presentation we do the following: we generalise Tate's formalism from CM abelian varieties to arbitrary CM motives. Here the word "motive" will always refer to the absolute Hodge cycle theory, and "CM motives" are defined as arising from abelian varieties which, over $\mathbf{C}$, have complex multiplication. Note that this implies that we have the corresponding quantitites f and g, for instance for the motives of Fermat hypersurfaces, as well as for motives of CM type obtained from K3 surfaces—see [**Deligne and Milne**, 1982, p. 217].

Now, by generalising Tate's formalism in this way we also retrieve Langlands's construction of the Taniyama group and Deligne's theorem (0.3.1). In fact, Tate's quantities f_E, g_E (generalised to arbitrary CM motives) each characterise a certain extension of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by $\mathscr{S}$, and it is elementary to show (see §§3 and 4 below) that g_E gives the motivic Galois group of Deligne's category $\mathscr{CM}_{\mathbf{Q}}$, and that f_E defines Langlands's Taniyama group! In this way, the proof of Tate's (generalised) conjecture gives, by the same token, the proof of Deligne's theorem on the Taniyama group—see Theorem 4.4 below.

0.7. Deligne's theorems mentioned in 0.3 were first published in [**Deligne et al.**, 1982], where the reader can also find a survey of Langlands's definition of the Taniyama group [**Milne and Shih**, 1981, (I)]. Later treatments can be found in [**Schappacher**, 1988, I.6] and [**Milne**, 1989, chapter I]. See also the brief accounts contained in [**Anderson**, 1986] and [**Blasius**, 1986].

The presentation given here is somewhat analogous to Milne's unpublished manuscript [**Milne**, 1981]. I saw this manuscript for the first time after the first version of this paper was written. It was in fact G. Anderson who had explained these things to me in 1984.

0.8. Thanks are due to the referee and the editor for helpful suggestions, and especially for prodding me to put in more details and explanations. All

[3] Note here the perfect analogy with Deligne's motivic proof of Shimura's monomial relations between periods of CM abelian varieties: see [**Deligne** (**Brylinski**), 1980, **Schappacher**, 1988, Chapter IV].

mistakes or shortcomings still are only mine, of course; but they may have become more easily detectable in this way.

1. The category $\mathscr{CM}_{\mathbf{Q}}$

Let k be a field embeddable into $\mathbf{C}$. Fix an algebraic closure $\overline{k}$ of k, and write $G_k = \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k. The letter σ denotes embeddings $\sigma : k \hookrightarrow \mathbf{C}$.

Throughout this article, the word "motive over k" will refer to the semi-simple Tannakian category $\mathscr{M}_k$ of motives over k for absolute Hodge cycles, cf. [**Deligne**, 1979, 0.9 – 0.11, **Deligne et al.**, 1982, **Panchishkin**, 1993]. Recall the realisation fibre functors built into the theory of absolute Hodge cycles, Betti realisation H_σ, for each embedding $\sigma : k \to \mathbf{C}$, the de Rham realisation H_{dR}, and for each prime number ℓ, the ℓ-adic realisation H_ℓ. If X is a nonsingular projective algebraic variety defined over k, then X defines an object $h(X)$ of $\mathscr{M}_k$, and for each $i \in \mathbf{Z}$ there exists a motive $h^i(X)$, an object of $\mathscr{M}_k$, such that $h(X) = \sum_i h^i(X)$. The various realisations of $h^i(X)$ are, respectively, singular cohomology $H^i([X \times_{k,\sigma} \mathbf{C}](\mathbf{C}), \mathbf{Q})$, algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X/k)$, and ℓ-adic cohomology $H^i_{\mathrm{\acute{e}t}}(X \times_k \overline{k}, \mathbf{Q}_\ell)$.

A simple abelian variety A over k is said to be of *CM-type* if $\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}_{/k} A$ is a CM-field (i.e., a purely imaginary quadratic extension of a totally real algebraic number field) of degree $2 \cdot \dim A$. An arbitrary abelian variety A over k is said to admit *complex multiplication*, or "*to be CM*", if every simple isogeny factor is of CM-type. Finally, A/k is said to admit *potential complex multiplication*, or for short, "*to be potentially CM*", if $A \times_k L$ is CM for some finite extension L of k.

Define $\mathscr{CM}_k$ to be the smallest full ($\mathbf{Q}$-linear neutralised) Tannakian subcategory of $\mathscr{M}_k$ containing the motives $h^1(A)$ for all abelian varieties A defined over k which are potentially CM.

1.1. Lemma. *$\mathscr{CM}_k$ contains arbitrary Tate twists $\mathbf{Q}(m)$, $m \in \mathbf{Z}$, and the category $\mathscr{M}^0_k$ of Artin motives over k is a full Tannakian subcategory of $\mathscr{CM}_k$.*

This latter category $\mathscr{M}^0_k$ is by definition the Tannakian subcategory of motives generated by the $h^0(X)$, where X is a variety of dimension zero defined over k—cf. [**Deligne and Milne**, 1982, p. 211, **Panchishkin**, 1993]. In particular, the automorphism group scheme of the fibre functor H_σ (for *any* embedding σ) is, $\underline{Aut}^{\otimes}(\mathscr{M}^0_k, H_\sigma) \cong \mathscr{G}_k$, where the Galois group is viewed as a constant group scheme. This follows from the fact that for X of dimension zero, $X(\overline{\mathbf{Q}})$ is just a finite collection of points with Galois action.

Proof. First, the cohomology ring of an abelian variety is well known to be the exterior algebra on $H^1(A)$. This holds for every realisation of $h(A)$ and carries over to absolute Hodge cycle motives. Thus, the category $\mathscr{CM}_k$ contains all motives $h^i(A)$ for CM abelian varieties A defined over k.

The claim of the lemma concerning Tate twists follows from the isomorphism of motives $h^2(E) = \mathbf{Q}(-1)$ for any elliptic curve E (with or without CM) which is geometrically irreducible over k. In fact, this follows from the nondegenerate alternating pairing in every realisation, $H^1(E) \times H^1(E) \to H(\mathbf{Q}(-1))$.

To see that every object of $\mathscr{M}_k^0$ occurs in $\mathscr{CM}_k$, note that for every geometrically irreducible abelian variety A/k, one has $h^0(A) = h^0(\mathrm{Spec}\, k)$. Thus, for any extension L/k, $h^0(A \times_k L) = h^0(\mathrm{Spec}\, L)$. These latter motives generate $\mathscr{M}_k^0$ as a Tannakian category.

The central object of this survey is the $\mathbf{Q}$ group scheme of tensor automorphisms of the fibre functor H_B on $\mathscr{CM}_{\mathbf{Q}}$:

$$\mathscr{U} = \underline{Aut}^{\otimes}(\mathscr{CM}_{\mathbf{Q}}, H_B).$$

Here we write "B" (for *Betti*) instead of σ because there is only one embedding of $\mathbf{Q}$ into $\mathbf{C}$.

This pro-algebraic group, which determines the category $\mathscr{CM}_{\mathbf{Q}}$ up to equivalence, is the motivic Galois group mentioned in the introduction. It will eventually be identified with the Taniyama group. But before defining the Taniyama group (in §4 below) we will introduce and formalise the finer structures with which $\mathscr{U}$ is naturally equipped.

By the Tannakian formalism, the fully faithful inclusion functor $\mathscr{M}_{\mathbf{Q}}^0 \to \mathscr{CM}_{\mathbf{Q}}$ induces a faithfully flat homomorphism of $\mathbf{Q}$ group schemes $\mathscr{U} \to G_{\mathbf{Q}}$,[4] and the essentially surjective base change functor $\mathscr{CM}_{\mathbf{Q}} \to \mathscr{CM}_{\overline{\mathbf{Q}}}$ induces a closed immersion $\mathscr{U}^{\circ} \to \mathscr{U}$, where $\mathscr{U}^{\circ} = \underline{Aut}^{\otimes}(\mathscr{CM}_{\overline{\mathbf{Q}}}, H_\sigma)$ for some fixed embedding $\sigma : \overline{\mathbf{Q}} \to \mathbf{C}$.

These homorphisms, in fact, form an *exact sequence of* $\mathbf{Q}$ *group schemes*

$$1 \to \mathscr{U}^{\circ} \to \mathscr{U} \to G_{\mathbf{Q}} \to 1. \tag{1.2}$$

PROOF. More generally, for any $s \in G_{\mathbf{Q}}$ the fibre over s in $\mathscr{U}$ may be written $\underline{Isom}^{\otimes}(H_\sigma, H_{\sigma\circ s})$, where we view H_B on $\mathscr{CM}_{\overline{\mathbf{Q}}}$ via σ. The point is that for any $\mathbf{Q}$-algebra R, any automorphism $g \in \mathscr{U}(R) = \underline{Isom}^{\otimes}(H_B \otimes R, H_B \otimes R)$ in the fibre above s and any object M of $\mathscr{CM}_{\mathbf{Q}}$, the identifications

$$\begin{aligned} H_\sigma(M \times \overline{\mathbf{Q}}) \otimes R = H_B(M) \otimes R &\xrightarrow{g_M} H_B(M) \otimes R \\ &= H_{\sigma\circ s}(M \times \overline{\mathbf{Q}}) \otimes R \end{aligned}$$

define homomorphisms in a way functorial in $M \times \overline{\mathbf{Q}}$ and R, and compatible with tensor products. But any object N of $\mathscr{CM}_{\overline{\mathbf{Q}}}$ is a direct factor of an object of the form $M \times \overline{\mathbf{Q}}$ for M defined over $\mathbf{Q}$: in fact, N is defined

[4] Here, as before, $G_{\mathbf{Q}}$ is considered as a constant group scheme.

over a number field, so take the restriction of scalars to $\mathbf{Q}$ and extend back up to $\overline{\mathbf{Q}}$. Therefore, g defines an element of $\mathrm{Isom}(H_\sigma \otimes R, H_{\sigma\circ s} \otimes R)$. This establishes a bijection between the fibre above s and $\underline{Isom}^{\otimes}(H_\sigma, H_{\sigma\circ s})$. Cf. [**Jannsen**, 1990, pp. 52–54, proof of 4.7.e].

1.2.1. Remark. 1.2 shows that $\mathscr{U}^\circ$ may be identified with the connected component of the identity in $\mathscr{U}$.

1.3. Proposition. *$\mathscr{U}^\circ$ is isomorphic to the (connected) Serre group $\mathscr{S}$.*

We will sketch the proof, at the same time introducing $\mathscr{S}$. For $k = \overline{\mathbf{Q}}$, Deligne's absolute Hodge cycle theorem implies that for all $\sigma : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, the Betti realisation functor H_σ defines an equivalence of Tannakian categories between $\mathscr{CM}_{\overline{\mathbf{Q}}}$ and the subcategory of rational Hodge structures generated by those coming from CM abelian varieties. The identification of $\mathscr{U}^\circ = \underline{Aut}^{\otimes}(\mathscr{CM}_{\overline{\mathbf{Q}}}, H_\sigma)$ with $\mathscr{S}$ comes from an explicit description of those Hodge structures.

1.3.1. *CM-types.* Let $k \subset \mathbf{C}$ be a number field, i.e., a finite extension of $\mathbf{Q}$, and let A be an *abelian variety of CM-type* defined over k. So A is equipped with an isomorphism $E \to \mathbf{Q} \otimes \mathrm{End}_{/k} A$ for some CM-field E of degree $[E:\mathbf{Q}] = 2 \cdot \dim A$. Then every element of E defines an endomorphism of the rational Hodge structure $H^1_B(A)$.[5] But

$$H^1_B(A) \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{\tau:E \hookrightarrow \mathbf{C}} H^1_B(A) \otimes_{E,\tau} \mathbf{C}.$$

Thus, since $\dim_E H^1_B(A) = 1$, for each $\tau : E \hookrightarrow \mathbf{C}$ there exists $n_\tau \in \{0, 1\}$ such that

$$H_B(A) \otimes_{E,\tau} \mathbf{C} \subseteq H^{n_\tau, 1-n_\tau}.$$

This system of Hodge numbers (n_τ) determines the *CM-type of A* which, classically,[6] is simply the half-system of embeddings $T = \{\tau \mid n_\tau = 1\}$. Note that for $c =$ complex conjugation, one has $T \cap cT = \varnothing$, $T \cup cT = \mathrm{Hom}(E, \mathbf{C})$.

Conversely, every such half-system T of embeddings of E satisfying these conditions occurs as the CM-type of an abelian variety of CM-type with CM by E which is defined over some finite extension of k. In fact, over $\mathbf{C}$, if $g = [E:\mathbf{Q}]$, then take $\mathbf{C}^g/\mathscr{O}_E$ with the ring of integers $\mathscr{O}_E$ embedded into $\mathbf{C}^g$ via the direct sum of the elements of T. Like any CM abelian variety, this has a model over $\overline{\mathbf{Q}}$—see footnote 7 for a more precise statement.

1.3.2. Lemma. *The $\mathbf{Z}$-span in $\mathbf{Z}[\mathrm{Hom}(E, \mathbf{C})]$ of all CM-types (n_τ) is equal to the set of all $(m_\tau) \in \mathbf{Z}[\mathrm{Hom}(E, \mathbf{C})]$ such that $w = m_{c\tau} + m_\tau$ is independent of τ.*

[5] That is the Betti cohomology $H^1(A \times_\sigma \mathbf{C})$, for the fixed inclusion $\sigma : k \hookrightarrow \mathbf{C}$.

[6] See [**Shimura and Taniyama**, 1961, §5.2] or [**Shimura**, 1971, §5.5 B]. There is, however, a difference in normalisation: we work with $H^1(A)$ rather than the classical $H_1(A)$. Recall that $H^{1,0} = H^0(A, \Omega^1) \otimes_k \mathbf{C}$.

PROOF. It is clear that all elements in the **Z**-span of CM-types have the desired property. Conversely, let the system (m_τ) satisfy the condition of the lemma, and choose $\tau_1 \in \mathrm{Hom}(E, \mathbf{C})$ as well as some CM-type T_1 containing τ_1; then $(m_\tau)-m_1\cdot T_1$ has coefficient 0 at τ_1. Next, for $\tau_1 \neq \tau_2 \in T_1$, choose a CM-type T_2 containing τ_2, but $c\tau_1$. Then $(m_\tau)-m_1 T_1-(m_2+m_1)\cdot T_2$ has coefficient 0 at τ_2. We can continue like this for all elements $\tau_1, \dots, \tau_n \in T_1$, where $n = [E:\mathbf{Q}]$. So we finally correct the given (m_τ), modulo an element from the **Z**-span of CM-types, to a system (m'_τ) which involves only $c\tau_1, \dots, c\tau_n$ and still satisfies the condition that $w' = m'_{c\tau} + m'_\tau$ is independent of τ. It follows that $(m'_\tau) = w' \cdot cT_1$, and the lemma is proved.

1.3.3. Another way to conceive of CM-types, which paves the way to the Serre group, is to view them as algebraic homomorphisms between k^* and E^*—cf. [**Deligne**, 1977, §5.1].

Consider the $k \otimes_{\mathbf{Q}} E$ module of holomorphic differentials $\Omega^1_{/k}(A)$. The homomorphism

$$\det_k(1 \otimes \cdot\,; \Omega^1_{/k}(A)) : E^* \to k^*$$

is in fact a homomorphism of **Q**-algebraic groups

$$R_{E/\mathbf{Q}}\mathbf{G}_m \to R_{k/\mathbf{Q}}\mathbf{G}_m.$$

The link with the system of Hodge numbers (n_τ) is simply given by $\det_k(1 \otimes x\,; \Omega^1_{/k}(A)) = \prod_{\tau:E\to\mathbf{C}} \tau(x)^{n_\tau}$ (remember that $k \subset \mathbf{C}$).

The analogous algebraic homomorphism in the opposite direction

$$(1.3.4) \qquad \det_E(\cdot \otimes 1\,; \Omega^1_{/k}(A)) : R_{k/\mathbf{Q}}\mathbf{G}_m \to R_{E/\mathbf{Q}}\mathbf{G}_m$$

corresponds to what is classically called *the dual* (or *reflex*) *CM-type* of the one given by the (n_τ). More precisely, this dual homomorphism will in general be extended up from proper CM subfields of k. The smallest CM subfield K of k to which this dual CM-type descends is called the *reflex field* of the initial CM-type of E.

If we write the dual type 1.3.4 in the form $(n^\#_\sigma)$, where σ ranges over the embeddings of k into $\mathbf{C}$, then K is the smallest CM subfield of k which admits a CM-type (n^*_ϕ) such that $n^\#_\sigma = 1 \Leftrightarrow n^*_\phi = 1$ for $\phi = \sigma_{|K}$. With this notation, it is the CM-type (n^*_ϕ) on the reflex field that is classically called the dual of (n_τ).

Alternatively, the reflex field $K \subset \overline{\mathbf{Q}}$ is simply the fixed field of the stabiliser in $G_{\mathbf{Q}}$ of the initial CM-type (n_τ). Cf. [**Shimura and Taniyama**, 1961, §8.3].

1.3.5. For any CM field E, let $\mathcal{S}_E$ be the biggest **Q**-algebraic quotient of $R_{E/\mathbf{Q}}\mathbf{G}_m$ through which all CM-types factorise. In other words, the character group $X(\mathcal{S}_E)$ is the group of all mappings $x \mapsto \prod_{\tau:E\to\mathbf{C}} \tau(x)^{m_\tau}$ for (m_τ) such that $w = m_{c\circ\tau} + m_\tau$ is independent of τ. This last condition makes sense for any number field L instead of E and defines $\mathcal{S}_L$ in general. The characters of $\mathscr{S}_L$ are necessarily extended up from CM subfields of L, via the norm.

1.3.6. DEFINITION. $\mathscr{S} = \varprojlim_L \mathscr{S}_L = \varprojlim_{E \text{ CM}} \mathscr{S}_E$ (with inverse limits taken with respect to the norm maps) is called the (connected) Serre group.

By 1.3.2, all characters of $\mathscr{S}_E$ defined over $\overline{\mathbf{Q}}$ are generated by the types of abelian varieties A defined over $\overline{\mathbf{Q}}$ of CM-type by E. The CM-type clearly characterises the rational Hodge structure $H^1_B(A)$ of A. By Deligne's theorem on absolute Hodge cycles on abelian varieties [**Deligne** (**Milne**), 1982], the type therefore characterises the motive $h^1(A)$ over $\overline{\mathbf{Q}}$. An integral linear combination of types corresponds to the corresponding E-linear tensor product of motives M in $\mathscr{CM}_{\overline{\mathbf{Q}}}$ (which admit coefficients in E such that $H_B(M)$ is a one-dimensional E vector space).

Furthermore, the norm map $N_{E'/E}$ corresponds to the extension of coefficients $\otimes_E E'$ on motives with an E action. So passing from characters of $\mathscr{S}_E$ to those of $\mathscr{S}$ has the effect of not having to worry about the fact that, in general, a product of two characters λ_1, λ_2 with fields of values E may have values generating a proper subfield E_0 of E, in which case $\lambda_1 \otimes \lambda_2$ corresponds to a motive with coefficients in E which is obtained from another motive, with coefficients in E_0, by extension of the field of coefficients.

Summing up we conclude that via $M \mapsto H_B(M)$, *the category* $\mathscr{CM}_{\overline{\mathbf{Q}}}$ *is equivalent to the category of finite-dimensional representations of* $\mathscr{S}$. This shows 1.3.

1.3.7. An alternative description of the Serre group is as follows.

DEFINITION. (i) A rational Hodge structure is CM if it is polarisable and its Mumford-Tate group is abelian.

(ii) $\mathscr{S}$ is the affine group scheme corresponding to the Tannakian category of CM Hodge structures, with the forgetful fibre functor.

EXAMPLE. If A is an abelian variety of CM-type (1.3.1), then $H^1_B(A)$ is a CM Hodge structure.

We do not go into this point of view here. See for instance [**Schappacher**, 1988, I 6.1]; cf. [**Milne**, 1989, p. 294ff]. The fact that this approach gives the same pro-torus $\mathscr{S}$ as described above is one way to see that the CM Hodge structures are precisely those obtained as Betti realisations of objects in $\mathscr{CM}_{\overline{\mathbf{Q}}}$.

We now describe some finer properties of the extension 1.2.

1.4. Just as for the Serre group $\mathscr{S}$, the whole of $\mathscr{U}$ is also an inverse limit of algebraic groups coming from finite levels, and in fact, the whole exact sequence 1.2 is the inverse limit of the following:

$$1 \to \mathscr{S}_E \to \mathscr{U}_E \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to 1. \tag{1.4.1}$$

Here E runs over finite CM fields Galois over $\mathbf{Q}$, E^{ab} is the maximal abelian extension of E in $\overline{\mathbf{Q}}$, and $\mathscr{U}_E$ is the affine group scheme corresponding to the full Tannakian subcategory of $\mathcal{CM}_{\mathbf{Q}}$ generated by the objects that admit coefficients in E. Equivalently[7] $\mathcal{U}_E$ is generated (as a Tannakian category) by $R_{L/\mathbf{Q}}h^1(A)$, with A an abelian variety of CM type by E, defined over some algebraic number field $L \subset E^{\mathrm{ab}}$. For $E \hookrightarrow E'$ the map $\mathcal{U}_{E'} \to \mathcal{U}_E$ is induced by extension of coefficients $\otimes_E E'$ on objects with coefficients in E.

1.5. Since $\mathcal{S}$ is a (pro-)torus, the left action of $\mathcal{U}$ on $\mathcal{S}$ by conjugation (i.e., $x \mapsto uxu^{-1}$) factors through $G_{\mathbf{Q}}$. This defines a left Galois action on $\mathcal{S}$ which, on characters $\lambda \in X^*(\mathcal{S})$, transports to the left action of $G_{\mathbf{Q}}$ by left translation: $\lambda^s(x) = \lambda(s^{-1}(x))$—see [**Deligne**, 1982, §(B)].

If λ factors through $\mathcal{S}_E$ with E Galois over $\mathbf{Q}$ and λ corresponds to $(n_\tau)_\tau$, then λ^s corresponds to $(n_{\tau s})_\tau$.

1.5*. There is also the "usual" left Galois action on $\mathcal{S}$ and its characters $\lambda \mapsto s \circ \lambda$. In terms of the other notation, $s \in G_{\mathbf{Q}}$ takes $(n_\tau)_\tau$ to $(n_{s^{-1}\circ\tau})_\tau$.

For example, if λ is the CM-type of an abelian variety A/k as in 1.3.1, then the conjugate abelian variety A^s, together with the conjugate isomorphism $E \to \mathrm{End}(A^s)$, has CM-type $s \circ \lambda$. Writing λ as (n_τ), the set $T = \{\tau \mid n_\tau = 1\}$ is transformed into sT.

1.6. For each prime ℓ, the absolute Galois group $G_{\mathbf{Q}}$ acts on the ℓ-adic realisation $H_\ell(M)$ of an object M of $\mathcal{CM}_{\mathbf{Q}}$. Choose an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$. Then in view of the comparison isomorphism $H_\ell(M) \cong H_B(M) \otimes \mathbf{Q}_\ell$, each $s \in G_{\mathbf{Q}}$ gives an automorphism of the fibre functor $H_B \otimes \mathbf{Q}_\ell$, and therefore, by definition, an element of $\mathcal{U}(\mathbf{Q}_\ell)$. This defines a continuous homomorphism

$$G_{\mathbf{Q}} \xrightarrow{\varepsilon_\ell} \mathcal{U}(\mathbf{Q}_\ell).$$

Artin motives carry a rational $G_{\mathbf{Q}}$ action on their Betti realisation. This implies that ε_ℓ is a continuous splitting on the $\mathbf{Q}_\ell$-rational points of the map $\mathcal{U} \to G_{\mathbf{Q}}$—but note that it is not a homomorphism of $\mathbf{Q}_\ell$-algebraic groups.

Putting together all finite places ℓ (and writing the finite adèles of a number field k as $k_{\mathbf{A}_f}$), we obtain a splitting

$$G_{\mathbf{Q}} \xrightarrow{\varepsilon} \mathcal{U}(\mathbf{Q}_{\mathbf{A}_f}). \tag{1.6.1}$$

Analogous splittings exist for the sequences 1.4.1 as well; they will be written

$$\mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \xrightarrow{\varepsilon_E} \mathcal{U}_E(\mathbf{Q}_{\mathbf{A}_f}).$$

The splitting ε is the limit of the splittings ε_E.

[7] Here we need to know that abelian varieties of CM type by E, defined initially, say over $\overline{E}$, admit a model over E^{ab}. This follows from E being Galois over $\mathbf{Q}$: it contains the reflex field of any of its CM-types, and the result follows, for instance, from the consideration of the moduli fields of suitable structures without automorphisms; see [**Shimura**, 1971, p. 130, p. 216].

We will briefly denote the exact sequence 1.2, equipped with all the extra structures described above, by the following diagram:

$$(1.7)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{U} & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1 \\ & & & & & & \| & & \\ & & & & \mathcal{U}(\mathbf{Q}_{\mathbf{A}_f}) & \overset{\varepsilon}{\longleftarrow} & G_{\mathbf{Q}} . & & \end{array}$$

2. Taniyama extensions and their invariant

2.0. DEFINITION. A *Taniyama extension* is an affine group scheme $\mathcal{T}$ over $\mathbf{Q}$ that fits into an exact sequence

$$(2.0.1)\qquad 1 \to \mathcal{S} \to \mathcal{T} \to G_{\mathbf{Q}} \to 1,$$

which is the inverse limit of exact sequences indexed by CM fields E, as in the sequence (1.4.1), and such that the Galois action induced by the sequence on $\mathcal{S}$ is as described in §1.5. Furthermore, a Taniyama extension by definition admits ℓ-adic splittings ε_ℓ, for every ℓ, as in 1.6. As a shorthand notation for Taniyama extensions we will use diagrams such as 1.7.

The aim of this section is to characterise Taniyama extensions by a cocycle-like invariant.

Given a CM field $E \subset \mathbf{C}$ Galois over $\mathbf{Q}$, the map $\mathcal{T}_E(E) \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ from the analogue of (1.4.1) for $\mathcal{T}$ is surjective, because S_E splits over E, so that $H^1(E, S_E) = 0$ by Hilbert 90. Choose any set-theoretic splitting α_E of this surjection. Recall that ε_E is the splitting (as in (1.6.1)) of the surjection $\mathcal{T}_E(\mathbf{Q}_{\mathbf{A}_f}) \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ which is implicit in the Taniyama extension $\mathcal{T}$. Then for $s \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$, define

$$(2.1)\qquad \beta_E(s) = \alpha_E(s)^{-1} \cdot \varepsilon_E(s) \ (\mathrm{mod}\ S_E(E)) \in S_E(E_{\mathbf{A}_f})/S_E(E) .$$

2.2. PROPOSITION. (i) *The map β_E does not depend on the choice of the splitting α_E.*

(ii) *For all $s, t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$, we have $\beta_E(st) = t^{-1}(\beta_E(s)) \cdot \beta_E(t)$.*

(iii) *The system of maps β_E, for E ranging over CM algebraic number fields Galois over $\mathbf{Q}$, characterises the Taniyama extension $\mathcal{T}$ (i.e., the exact sequence* 2.0.1, *inverse limit of the sequences that serve to define β_E, with all additional structures) up to unique isomorphism.*

PROOF. (i) If α_E and α'_E are two E-rational splittings, then for all $s \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$, one has $\beta_E(s) \cdot \beta'_E(s)^{-1} = \alpha_E(s)^{-1} \cdot \alpha'_E(s) \in \mathcal{S}_E(E)$.

(ii) Dropping the subscript E from the notation momentarily, we find that

$$\begin{aligned} \beta(st) \cdot [t^{-1}(\beta(s)) \cdot \beta(t)]^{-1} &= \alpha(st)^{-1}\varepsilon(st)\varepsilon(t)^{-1}\alpha(t)\alpha(t)^{-1}\varepsilon(s)^{-1}\alpha(s)\alpha(t) \\ &= \alpha(st)^{-1}\alpha(s)\alpha(t) \in \mathcal{S}_E(E) . \end{aligned}$$

(iii) is straightforward—cf. the more detailed discussion in §2 (in particular, Proposition 2.7) of [**Milne and Shih**, 1982], where our β_E corresponds to $(\bar{b})^{-1}$.

2.2.1 Remark. In [**Milne and Shih**, 1982] it is also proved that maps

$$\beta_E : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to S_E(E_{\mathbf{A}_f})/S_E(E)$$

satisfying 2.2.(ii) come from an extension

$$1 \to \mathcal{S}_E \to \mathcal{V}_E \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to 1$$

if and only if (i) their values are invariant under $\mathrm{Gal}(E/\mathbf{Q})$ and (ii) β_E lifts to a map $b : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to S_E(E_{\mathbf{A}_f})$ such that $b(st)^{-1}t^{-1}(b(s))b(t)$ is locally constant.

Now for any character $\lambda : \mathcal{S}_E \to \mathbf{G}_m$ define a finite idèle class of E as follows:

$$\gamma_E(s, \lambda) = \lambda(\beta_E(s)) \in E^*_{\mathbf{A}_f}/E^*. \tag{2.3}$$

Then the following properties are easily checked.

2.4. Corollary. *For all automorphisms $s, t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ and all characters $\lambda, \lambda' \in X(\mathcal{S}_E)$, one has*

(i) $\gamma_E(s, \lambda)\gamma_E(s, \lambda') = \gamma_E(s, \lambda \cdot \lambda')$;

(ii) $\gamma_E(t, \lambda)^s = \gamma_E(t, s \circ \lambda)$;

(iii) $\gamma_E(st, \lambda) = \gamma_E(s, \lambda^t)\gamma_E(t, \lambda)$.[8]

(iv) *Let $E \subset E'$ be a finite extension of CM fields, both Galois over $\mathbf{Q}$.*[9] *Then $\gamma_{E'}(s, \lambda \circ N_{E'/E}) = \gamma_E(s, \lambda)$.*

(v) *The system of maps γ_E, for E ranging over CM algebraic number fields Galois over $\mathbf{Q}$, characterises the Taniyama extension $\mathcal{V}$ up to unique isomorphism.*

3. The invariant of $\mathcal{U}$

For the rest of the paper, $\overline{\mathbf{Q}}$ will denote the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and E will range over CM fields contained in $\overline{\mathbf{Q}}$ that are Galois extensions of $\mathbf{Q}$.

3.1. Definition. For every CM field E Galois over $\mathbf{Q}$, let $g_E(s, \lambda)$ be the invariant $\gamma_E(s, \lambda)$ of the previous section for the Taniyama extension $\mathcal{U}$ of 1.2.

To determine g_E explicitly, let M be a motive in $\mathcal{CM}_{\overline{\mathbf{Q}}}$ with coefficients in E and of rank 1 over E. Then for any $s \in G_{\mathbf{Q}}$, the conjugate motive

[8] Recall from 1.5, 1.5* the notations $t \circ \lambda$, resp., λ^t for the two actions of $t \in G_{\mathbf{Q}}$ on characters of $\mathcal{S}_E$.

[9] By Hilbert 90, $E^*_{\mathbf{A}_f}/E^* \hookrightarrow E'^*_{\mathbf{A}_f}/E'^*$.

M^s is well defined in $\mathscr{CM}_{\overline{\mathbf{Q}}}$ and is also a motive of rank 1 over E. Fix bases

$$\theta : E \xrightarrow{\simeq} H_B(M), \qquad \xi : E \xrightarrow{\simeq} H_B(M^s).$$

Then there exists a finite idèle $a \in E^*_{\mathbf{A}_f}$ such that the following diagram commutes:

$$\begin{array}{ccc} E_{\mathbf{A}_f} & \xrightarrow{\theta\otimes\mathbf{Q}_{\mathbf{A}_f}} & H_B(M)\otimes_E E_{\mathbf{A}_f} \\ \cdot a\downarrow & & \downarrow s \\ E_{\mathbf{A}_f} & \xrightarrow{\xi\otimes\mathbf{Q}_{\mathbf{A}_f}} & H_B(M^s)\otimes_E E_{\mathbf{A}_f}. \end{array} \tag{3.1.1}$$

Recall that $\mathscr{S}_E$ is the group scheme associated to CM motives defined over $\overline{\mathbf{Q}}$ with coefficients in E. So let $\lambda : \mathscr{S}_E \to \mathbf{G}_m$ be the character corresponding to the motive M. Then the finite idèle class $a \cdot E^* \in E^*_{\mathbf{A}_f}/E^*$ depends only on λ and on s.

3.2. Proposition. $g_E(s, \lambda) = a \cdot E^*$.

Proof. It suffices—possibly after enlarging E—to treat the case where the given character λ of $\mathscr{S}_E$ is the restriction of an E-rational representation $\rho : \mathscr{U}_E \to \mathrm{GL}_E(V)$ with $\dim_E V = 1$. Then $g_E(s, \lambda) = \lambda(\beta_E(s)) = \rho(\alpha(s))^{-1} \cdot \rho(\varepsilon_E(s))$. But $\rho(\alpha(s)) \in E^*$, and the proposition follows from the definition of ε_E.

As a consequence, these finite idèle classes $a{\cdot}E^*$ have the formal properties of the γ_E recorded in 2.4 above. They also satisfy the following lemma for all characters λ, where c denotes complex conjugation, w is the weight of the (CM) Hodge structure given by λ (cf. 1.3.2 ff above), and Ψ is the cyclotomic character. If $s \in \mathbf{G}_{\mathbf{Q}}$ and if $\zeta \in \overline{\mathbf{Q}}^*$ is a root of unity, then $\Psi(s) \in \hat{\mathbf{Z}}$ satisfies $\zeta^s = \zeta^{\Psi(s)}$.

3.3. Lemma. (i) $g_E(c, \lambda) = 1$.
(ii) $g_E(s, \lambda)^{1+c} = \Psi(s)^{-w} \cdot E^*$.

Proof. (i) Complex conjugation is defined rationally on $H_B(M)$. In other words, $\varepsilon_E(c) \in \mathscr{U}(\mathbf{Q}) \subset \mathscr{U}(\mathbf{Q}_{\mathbf{A}_f})$. So (i) follows from 2.2 and 2.3.

(ii) Use 2.4 (ii). By 1.3.2, $\lambda \cdot (c \circ \lambda) = {N_{E/\mathbf{Q}}}^w$. This norm corresponds to the Tate motive $E(-1)$ with coefficients in E because the Hodge structure of $\mathbf{Q}(-1)$ is pure of type $(1, 1)$. The ℓ-adic realisation of $E(-1)$ is the dual of $E \otimes \varprojlim \mu_{\ell^n}$—which explains the minus sign in the exponent of Ψ.

4. Tate's invariant and Langlands's construction

The commutative diagram preceding theorem 3.2 specialises to the situation of the Shimura-Taniyama reciprocity law when $M = h_1(A)$, with A as in §1.3.1, provided s fixes the reflex field of A. In this case, Shimura and

Taniyama provide a class field theoretic description of the finite idèle class $a \cdot E^*$. A generalisation of this reciprocity law to all s, with $M = h_1(A)$, was found by Tate in [**Tate**, 1981]. We will now present Tate's approach, generalising it to arbitrary rank 1 motives M, and explain how this vindicates Langlands's construction of the Taniyama group. When we specialise our formalism to abelian varieties, the formulas are not literally the same as in Tate's original paper—essentially because we represent an abelian variety A by the motive $h^1(A)$.

4.1. For any CM field $E \hookrightarrow \overline{\mathbf{Q}}$ that is Galois over $\mathbf{Q}$, choose some system of representatives

$$v : \mathrm{Gal}(E/\mathbf{Q}) \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}), \qquad v(\tau)_{|E} = \tau$$

in such a way that for all $\tau \in \mathrm{Gal}(E/\mathbf{Q})$ one has $v(c\tau) = v(\tau c) = v(\tau)c$, where c denotes complex conjugation.

Let λ be a character of S_E as above, lift it via $R_{E/\mathbf{Q}}\mathbf{G}_m \to S_E$ to write it in the form $\lambda = (n_\tau)_{\tau \in \mathrm{Gal}(E/\mathbf{Q})}$. Then given $s \in G_{\mathbf{Q}}$, the following formula defines an element of $\mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ which is independent of the choice of v:

$$V_E(s, \lambda) = \prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} \left(v(\tau) \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v(\tau s^{-1})^{-1}\right)^{n_\tau}. \tag{4.1.1}$$

4.1.2. *Notation.* Normalise the reciprocity map $r_E : E^*_{\mathbf{A}} \to \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ of global class field theory for E to be the reciprocal of the classical Artin map. Note that since E is totally imaginary, r_E factorises through $E^*_{\mathbf{A}_f}/E^*$. Our normalisation implies, in particular, that $r_{\mathbf{Q}}(\Psi(s)) = s_{|\mathbf{Q}^{\mathrm{ab}}}$ for all $s \in G_{\mathbf{Q}}$ with the cyclotomic character Ψ defined as in 3.3. Also, as in 3.3, write the weight of λ as w.

4.2. Proposition. *For s, λ as above there exists a unique idèle class $f_E(s, \lambda) \in E^*_{\mathbf{A}_f}/E^*$ satisfying the following two identities:*

(i) $r_E(f_E(s, \lambda)) = V_E(s, \lambda)$,

(ii) $f_E(s, \lambda)^{1+c} = \Psi(s)^{-w} \cdot E^*$.

The proof is a straightforward generalisation of Tate's proof for the same result in case λ is a CM-type. We need

4.2.1. Lemma. *The quotient group $\ker(r_E)/E^*$ is uniquely divisible and complex conjugation c acts trivially on it. In particular, $1 + c$ acts bijectively on $\ker(r_E)/E^*$.*

We quote the proof of the lemma from [**Tate**, 1981]—cf. [**Lang**, 1983, chapter 7, Lemma 2.1]:

If U is a subgroup of finite index in $\mathcal{O}_E^*$, then the group in question is isomorphic to $\overline{U}/U$, where $\overline{U}$ is the closure of U in $\widehat{\mathcal{O}_E}^*$. By a theorem of Chevalley [**Chevalley**, 1951; **Artin and Tate**, 1967], $\overline{U}$ is isomorphic to

$\varprojlim(U/U^n)$. On taking U in the real subfield E_0 of E and torsion free, the lemma follows because U is isomorphic to a product of $\mathbf{Z}$'s, and $\hat{\mathbf{Z}}/\mathbf{Z}$ is uniquely divisible.

PROOF OF 4.2. First, r_E is surjective, so there exists $a \in E^*_{\mathbf{A}_f}$ such that $r_E(a) = V_E(s, \lambda)$. Then

$$r_E(a^{1+c}) = r_E(a) \cdot cr_E(a)c^{-1} = V_E(s, \lambda) \cdot V_E(s, c \circ \lambda).$$

The latter identity follows from the fact that $c^2 = 1$, that c commutes with $\mathrm{Gal}(E/\mathbf{Q})$ (because E is a CM field), and by taking $v' = c \circ v \circ c$ instead of v to define V_E: $cv(\tau)c \cdot c(s^{-1}{}_{|E^{ab}})c \cdot cv(\tau s^{-1})^{-1}c = v'(\tau c) \cdot (s^{-1}{}_{|E^{ab}}) \cdot v'(\tau c s^{-1})$.

Next we check that $V_E(s, \lambda) \cdot V_E(s, c \circ \lambda) = \mathrm{Ver}_{E/\mathbf{Q}}(s^{-1})^w = \mathrm{Ver}_{E/\mathbf{Q}}(s)^{-w}$, where $\mathrm{Ver}_{E/\mathbf{Q}}$ is the transfer map from $G_\mathbf{Q}$ to G_E^{ab}. It is given by the formula $\mathrm{Ver}_{E/\mathbf{Q}}(s) = \prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} v(s\tau)^{-1} \cdot (s_{|E^{ab}}) \cdot v(\tau)$. So we have to show that

$$\prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} (v(\tau) \cdot (s^{-1}{}_{|E^{ab}}) \cdot v(\tau s^{-1})^{-1}) = \prod_{\tau \in \mathrm{Gal}(E/\mathbf{Q})} v(s^{-1}\tau)^{-1} \cdot (s^{-1}{}_{|E^{ab}}) \cdot v(\tau).$$

This is done by first relabelling τ as $\tau^{-1}s$, and then passing from v to $v'(\tau) = v(\tau^{-1})^{-1}$; the product is invariant under these substitutions.

By class field theory, $\mathrm{Ver}_{E/\mathbf{Q}} \circ r_\mathbf{Q} = r_E \circ i$, where i is the natural embedding of $\mathbf{Q}^*_\mathbf{A}$ into $E^*_\mathbf{A}$. Applying both sides to $\Psi(s)$, we see that $a^{1+c}\Psi(s)^w \in \ker(r_E)$.

But the structure of $\ker(r_E)/E^*$ has been studied in the lemma. It follows that one can correct a to satisfy conditions (i) and (ii) and that this correction is unique modulo E^*.

While the requirement 4.2 (ii) matches 3.3 (ii), the following list of properties of the f_E is reminiscent of 2.4. All are easily checked for V_E instead of f_E and follow from this using 4.2.

4.3. PROPOSITION. *For all automorphisms* $s, t \in \mathrm{Gal}(E^{ab}/\mathbf{Q})$ *and all characters* $\lambda, \lambda' \in X(\mathcal{S}_E)$, *one has*

(i) $f_E(s, \lambda) f_E(s, \lambda') = f_E(s, \lambda \cdot \lambda')$.

(ii) $f_E(t, \lambda)^s = f_E(t, s \circ \lambda)$.

(iii) $f_E(st, \lambda) = f_E(s, \lambda^t) f_E(t, \lambda)$.

(iv) *Let* $E \subset E'$ *be a finite extension of CM fields. Then* $f_{E'}(s, \lambda \circ N_{E'/E}) = f_E(s, \lambda)$.

(v) $f_E(c, \lambda) = 1$.

We can now formulate the central result presented in this chapter.

4.4. THEOREM (Tate-Deligne). $f_E = g_E$. *In words, the class field theoretic invariant* f_E *also comes from a Taniyama extension and in fact from one that is isomorphic to the motivic Galois group* $\mathcal{U}$ *of the category* $\mathcal{CM}_\mathbf{Q}$.

For the proof, define $e_E(s, \lambda) = g_E(s, \lambda) \cdot f_E(s, \lambda)^{-1}$.

4.4.1. PROPOSITION. (i) $e_E(s, \lambda)e_E(s, \lambda') = e_E(s, \lambda \cdot \lambda')$.

(ii) $e_E(t, \lambda)^s = e_E(t, s \circ \lambda)$.

(iii) $e_E(st, \lambda) = e_E(s, \lambda^t)e_E(t, \lambda)$.

(iv) *Let* $E \subset E'$ *be a finite extension of CM fields. Then* $e_{E'}(s, \lambda \circ N_{E'/E}) = e_E(s, \lambda)$.

(v) *If* λ *is the trivial character, then* $e_E(s, \lambda) = 1$.

(vi) $e_E(c, \lambda) = 1$.

(vii) *If* λ *is the CM-type of an abelian variety as in* 1.3.1 *and if* $s \circ \lambda = \lambda$ (*i.e., if* s *fixes the reflex field of this CM-type: see* 1.5*), *then* $e_E(s, \lambda) = 1$.

All these properties, except the last one, are immediate consequences of what we know about f_E, g_E. As for (vii), it is but a reformulation of the Shimura-Taniyama reciprocity law: [**Shimura**, 1971, Theorem 5.15], cf. [**Shimura and Taniyama**, 1961, §13]. Here is how one checks (vii):

Let $\{\phi_1^*, \dots, \phi_n^*\}$ be the reflex type of λ on the reflex field K (see (1.3.4)). The Shimura-Taniyama reciprocity law says precisely that for a finite idèle $x \in K^*_{\mathbf{A}_f}$ such that $r_K(x) = s$, one has

$$\begin{aligned} g_E(s, \lambda) &= \det_E(x \otimes 1, \Omega^1(A))^{-1} \\ &= \prod_i \phi_i^*(x)^{-1} \in E^*_{\mathbf{A}_f}/E^*. \end{aligned}$$

(The inverse comes in because we not only have the opposite sign convention for the reciprocity map from Shimura, but also work with $h^1(A)$ instead of its dual.)

To check that $f_E(s, \lambda)$ gives the same value, Tate decomposes the original type T of E given by λ into its orbits under the left action of G_K. This gives a disjoint union $T = \bigcup T_j$. Then $V_E(s, \lambda) = \prod F_j(s)$, where $F_j(s) = \prod_{\tau \in T_j} \left(v(\tau) \cdot (s^{-1}{}_{|E^{\mathrm{ab}}}) \cdot v(\tau s^{-1})^{-1}\right)$.

Following Tate, fix j temporarily, choose $w_j \in G_{\mathbf{Q}}$ such that $w_{j|E} \in T_j$, and let $L = K^{w_j^{-1}} \cdot E$ so that $G_L = w_j^{-1} G_K w_j \cap G_E$. By the basic functorial properties of Artin's reciprocity law, we see that the following diagram is commutative, where the vertical arrows are the respective Artin maps.

$$\begin{array}{ccccccc} K^*_{\mathbf{A}_f} & \xrightarrow{\text{incl}} & L^{w_j *}_{\mathbf{A}_f} & \xrightarrow{w_j^{-1}} & L^*_{\mathbf{A}_f} & \xrightarrow{\text{norm}} & E^*_{\mathbf{A}_f} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_K^{\mathrm{ab}} & \xrightarrow{\text{Ver}} & w_j G_L^{\mathrm{ab}} w_j^{-1} & \xrightarrow{w_j^{-1} \cdot w_j} & G_L^{\mathrm{ab}} & \xrightarrow{\text{incl}_*} & G_E^{\mathrm{ab}}. \end{array}$$

If we denote by G_j the composite of the maps in the top row, we see that $\prod_i \phi_i^*(x) = \prod_j G_j(x)$. In order to prove (vii), i.e., $r_E(g_E(s, \lambda)) = V_E(s, \lambda)$, it therefore suffices to show that the composite of the maps in the bottom row of our diagram is just $s \mapsto F_j(s^{-1})$. This follows in the same way as we checked 4.2 (ii).

Theorem 4.4.1 is proved. Theorem 4.4 now follows from

4.4.2. PROPOSITION (Deligne). *Any family of maps* e_E *satisfying all the properties listed in Proposition* 4.4.1 *is trivial:* $e_E(s, \lambda)$ *is a principal idèle for all* s, λ.

We start to prove this proposition by showing that *the idèle classes* $e_E(s, \lambda)$ *all have square* 1. This is the part of the theorem already proved by Tate in [**Tate**, 1981].

4.4.3. LEMMA. (i) *If* λ *is the CM-type of an abelian variety as in* 1.3.1 *and if* $\lambda^s = \lambda^t$ *and* $s \circ \lambda = t \circ \lambda$ *(in other words, if* s *and* t *act alike on the reflex field of* λ*), then* $e_E(s, \lambda) = e_E(t, \lambda)$.

(ii) *For all* s *and all characters* λ, $e_E(s, \lambda)^c = e_E(s, \lambda)$.

(iii) *For all* s *and all* λ, $e_E(s, \lambda)^2 = 1$.

(iv) *For all* s *and all* λ, *there exists a representative* $e \in e_E(s, \lambda)$, $e \in E^*_{\mathbf{A}_f}$, *such that* $e^c = e$ *and* $e^2 = 1$.

PROOF. (i) We use 4.4.1 (iii) and (vii):

$$\begin{aligned} e_E(s, \lambda) &= e_E(tt^{-1}s, \lambda) = e_E(t, \lambda^{t^{-1}s})e_E(t^{-1}s, \lambda) \\ &= e_E(t, \lambda) \cdot 1. \end{aligned}$$

(ii) First, we assume that λ is as in part (i). By 4.4.1 (ii), $e_E(s, \lambda)^c = e_E(s, c \circ \lambda) = e_E(s, \lambda^c)$, because c commutes with every automorphism of a CM field. Now $e_E(s, \lambda^c) = e_E(sc, \lambda)$ by 4.4.1 (iii) and (vi), and $e_E(sc, \lambda) = e_E(cs, \lambda)$ by 4.4.3 (i), again because c commutes with every automorphism of a CM field. Thus, applying 4.4.1 (iii) and (vi) once more, we find that $e_E(s, \lambda)^c = e_E(cs, \lambda) = e_E(s, \lambda)$. Finally, for general λ, the claim follows from 4.4.1 (i) (recall Lemma 1.3.2).

The statements 3.3 (ii) and 4.2 (ii) imply that $e_E(s, \lambda)^{1+c} = 1$. So (ii) implies (iii).

As for (iv), start with some representative $e \in e_E(s, \lambda)$, $e \in E^*_{\mathbf{A}_f}$. By part (ii), $e^{1-c} \in E^*$. By Hilbert 90, we may therefore correct e by an element of E^* to achieve $e^{1-c} = 1$. In other words, we may (and do) assume that e is an idèle of the totally real subfield F of E. By part (iii), $e^2 \in E^* \cap F_{\mathbf{A}}{}^* = F^*$ — an element that is locally a square at all finite places. By class field theory, it is the square of an element of F^*. Therefore, we can adjust e to satisfy (iv).

This proves Lemma 4.4.3.

We now conclude the proof of Proposition 4.4.2, by following Deligne's argument contained in a letter from Deligne to Tate dated 8 October 1981. The substance of this letter is also recorded in [**Lang**, 1983, chapter 7, §4].

4.4.4. There exists some CM field E Galois over $\mathbf{Q}$ and some CM-type λ on E such that $e_E(s, \lambda) = 1$ for all $s \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$. Indeed, take for instance $E = \mathbf{Q}(\sqrt{-1})$, and let λ be the CM-type of the abelian variety of dimension 1 given by $y^2 = x^3 - x$. Then $e_E(s, \lambda) = 1$, for s fixing E (which in this case is equal to its own reflex field), is a special case of the Shimura-Taniyama reciprocity law — so special a case in fact, that it was essentially established already in **[Eisenstein**, 1850]. On the other hand, $e_E(c, \lambda) = 1$ by 4.4.1 (vi). So $e_E(s, \lambda) = 1$ for all s by 4.4.3 (i).

4.4.5. We now reduce Proposition 4.4.2 to the case $\overline{\lambda^c} = \overline{\lambda}$, where $\lambda \mapsto \overline{\lambda}$ denotes reduction modulo 2 on the character group: $X^*(\mathcal{S}_E) \to X^*(\mathcal{S}_E) \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z}$.

Indeed, by 4.4.1 (i) it suffices to prove that $e_E(s, \lambda) = 1$ for all s and all CM-types λ on E. Now given E_1, E_2 two CM fields Galois over $\mathbf{Q}$, and for $i = 1, 2$ CM-types λ_i on E_i, then by 4.4.1 (iv), we may compare the $e_{E_i}(s, \lambda_i)$ on a suitable common overfield; in other words, we may (and do) assume that $E_1 = E_2$. Then $\lambda = \lambda_1\lambda_2^{-1}$ has weight $w = 0$, and therefore satisfies $\overline{\lambda^c} = \overline{\lambda}$. Then $e_E(s, \lambda) = 1$ implies that $e_{E_1}(s, \lambda_1) = e_{E_2}(s, \lambda_2)$. This proves 4.4.2 in view of 4.4.4.

4.4.6. It now remains to show $e_E(s, \lambda) = 1$ for all λ such that $\overline{\lambda^c} = \overline{\lambda}$.

Note that $e_E(s, \lambda)$ depends only on $\overline{\lambda}$ because we know from 4.4.3 that $e_E(s, \lambda^2) = e_E(s, \lambda)^2 = 1$.

It is convenient to switch to additive notation: write λ as $(n_\tau)_{\tau\in\mathrm{Hom}(E, \overline{\mathbf{Q}})}$ or as $\sum n_\tau \tau$. Considering $\overline{\lambda}$ amounts to reading the n_τ modulo 2, and our hypothesis on λ says that for all τ, we have $n_\tau \equiv n_{c\tau} \pmod 2$. It follows that over $\mathbf{Z}/2\mathbf{Z}$, $\overline{\lambda}$ is a linear combination of characters of the form $\tau - c\tau = \tau - \tau c$ for $\tau \in \mathrm{Hom}(E, \overline{\mathbf{Q}})$.

Given such a $\tau - \tau c$, choose $t \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$ such that $\tau = t \circ \mathrm{id}_E$. Then the formula $e_E(st, (\mathrm{id}_E - c)) = e_E(s, (\mathrm{id}_E - c)^t) \cdot e_E(t, (\mathrm{id}_E - c)) = e_E(s, \tau - \tau c) \cdot e_E(t, \mathrm{id}_E - c)$ shows that it suffices to verify that $e_E(s, \mathrm{id}_E - c) = 1$ for all s.

To do this, choose a representing idèle $e \in F^*_{\mathbf{A}_f}$ for $e_E(s, \mathrm{id}_E - c)$ as in 4.4.3 (iv). Everywhere locally it is ± 1, and we have to show that the sign is everywhere the same—say $+1$ after multiplying globally with $\pm 1 \in F^*$. For this it is enough to prove that, given any two distinct finite places v_1, v_2 of F, the signs of e at v_1 and v_2 agree. Let F' be a totally real quadratic extension of F that is inert at v_1 and v_2, and let E' be the composite of F' and E. Then $e_E(s, \mathrm{id}_E - c) = e_{E'}(s, (\mathrm{id}_E - c) \circ N_{E'/E}) = e_{E'}(s, N_{E'/E} \circ (\mathrm{id}_{E'} - c)) = N_{E'/E}(e_{E'}(s, \mathrm{id}_{E'} - c))$. Representing $e_{E'}(s, \mathrm{id}_{E'} - c)$ by a finite idèle e' of F' as in 4.4.3 (iv), we see that, up to a global sign, $e = N_{F'/F}e'$. But $e'_{v_i} = \pm 1$, and the norm takes this to the square, so $e_{v_i} = +1$, which is what we wanted to show.

This completes the proof of Proposition 4.4.2 and thereby of Theorem 4.4.

4.4.7. Remark. We saw that 4.4.1 (vii) is equivalent to the Shimura-Taniyama reciprocity law. Thus, Proposition 4.4.2 implies a generalisation of this law which describes the action of all of $G_{\mathbf{Q}}$ on any CM abelian variety. This was mentioned in the introduction. Some details concerning the Galois action on additional data of the abelian variety (polarisation type) are worked out explicitly in [**Tate**, 1981] and [**Lang**, 1983, chapter 7].

In view of §2, Theorem 4.4 implies that for all E and s as above, there exists a class $f_E(s) \in S_E(E_{\mathbf{A}_f})/S_E(E)$ satisfying $f_E(s, \lambda) = \lambda(f_E(s))$ for all λ. Following Milne [**Milne**, 1989, pp. 305–307], we now give a direct construction of the elements $f_E(s)$ from a slight reformulation of Langlands's construction of the Taniyama group. This provides additional insight into Tate's invariant $f_E(s, \lambda)$, and via 4.4, a new perspective on the Taniyama extension $\mathscr{U}$ associated with CM motives. It is actually this point of view that is going to give us control over the L-functions of CM motives—see §5 below.

4.5. THEOREM. *For every CM field E and every $s \in G_{\mathbf{Q}}$, there exists a unique, explicit class $f_E(s) \in S_E(E_{\mathbf{A}_f})/S_E(E)$ such that for all λ, one has $f_E(s, \lambda) = \lambda(f_E(s))$.*

Here is Langlands's approach in a nutshell (cf. [**Milne**, 1989, loc. cit.]):

Let E be a CM field Galois over $\mathbf{Q}$, and let $W^f_{E/\mathbf{Q}}$ be the quotient of the Weil group of $E/\mathbf{Q}$ by the image of the kernel of r_E under the inclusion $E^*_{\mathbf{A}}/E^* \hookrightarrow W_{E/\mathbf{Q}}$. Then we have the following commutative diagram with surjective vertical arrows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^*_{\mathbf{A}_f}/E^* & \longrightarrow & W^f_{E/\mathbf{Q}} & \longrightarrow & \mathrm{Gal}(E/\mathbf{Q}) & \longrightarrow & 1 \\ & & r_E\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/E) & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) & \longrightarrow & \mathrm{Gal}(E/\mathbf{Q}) & \longrightarrow & 1. \end{array}$$

Given $s \in G_{\mathbf{Q}}$, let $\hat{s} \in W^f_{E/\mathbf{Q}}$ be any element mapping to $s_{|E^{\mathrm{ab}}} \in \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q})$. Furthermore, for all $\tau \in \mathrm{Gal}(E/\mathbf{Q})$ choose representatives $w(\tau) \in W^f_{E/\mathbf{Q}}$. Let λ and (n_τ) be as in §4.1.

4.5.1. LEMMA. $f_E(s, \lambda) = \prod_{\tau\in\mathrm{Gal}(E/\mathbf{Q})} (w(\tau)\hat{s}^{-1}w(\tau s^{-1})^{-1})^{n_\tau}$.

For the proof, one has to check that the formula gives elements satisfying the two conditions of Proposition 4.2. This is easy. (If $\lambda = N_{E/\mathbf{Q}}$, then the formula defines the transfer map from $W^f_{E/\mathbf{Q}}$ to the subgroup $E^*_{\mathbf{A}_f}/E^*$.)

Now in order to get the class $f_E(s)$ from $f_E(s, \lambda)$, use the canonical cocharacter $\mu_E : \mathbf{G}_m \to \mathscr{S}_E$. It is canonical over $\mathbf{C}$ and defined over E which is considered as a subfield of $\mathbf{C}$. Its dual map $\check{\mu}$ on the character groups sends a character of $\mathscr{S}_E$ given by $(n_\tau)_{\tau\in\mathrm{Hom}(E,\mathbf{C})}$ to the character of $\mathbf{G}_m$ given by n_{id}. Let $\varphi \in \mathrm{Gal}(E/\mathbf{Q})$ operate on μ_E in such a way that $(\mu_E^\varphi)\check{}\,((n_\tau)_\tau) = \check{\mu}_E\,((n_{\varphi\circ\tau})_\tau)$.

Define

$$f_E(s) = \prod_{\tau\in\mathrm{Gal}(E/\mathbf{Q})} \mu_E^\tau(w(\tau)\hat{s}^{-1}w(\tau s^{-1})^{-1}). \tag{4.5.2}$$

Then Theorem 4.5 follows immediately from Lemma 4.5.1.

At this point it is an exercise to check independently of Theorem 4.4 that $f_E(s, \lambda)$ is indeed the invariant of a Taniyama extension—cf. end of proof of Proposition 2.2. Following Langlands, we call the Taniyama extension

$$\text{(4.5.3)}\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{T} & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1 \\ & & & & & & \| & & \\ & & & & \mathcal{T}(\mathbf{Q}_{\mathbf{A}_f}) & \overset{\varepsilon}{\longleftarrow} & G_{\mathbf{Q}} & & \end{array}$$

charcterised by the invariants $f_E(s, \lambda)$ **The Taniyama Group**. Theorem 4.4 then says that the Taniyama extensions $\mathcal{U}$ and $\mathcal{T}$ are uniquely isomorphic.

To end this section we show the relationship between the Taniyama group and the Weil group. It was suggested to Langlands by Casselman and will be used in §5 to control the L-functions of motives in $\mathcal{CM}_{\mathbf{Q}}$.

4.5.4. Proposition (Langlands). *For every CM field E Galois over $\mathbf{Q}$, there is a homomorphism $\phi_E : W_{E/\mathbf{Q}} \to \mathcal{T}_E(\mathbf{C})$ making the diagram below commutative:*

$$\begin{array}{ccccccccc} & & & & W_{E/\mathbf{Q}} & = & W_{E/\mathbf{Q}} & & \\ & & & & \phi_E \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{S}_E(\mathbf{C}) & \longrightarrow & \mathcal{T}_E(\mathbf{C}) & \longrightarrow & \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) & \longrightarrow & 1. \end{array}$$

4.5.5. Corollary. *There exists a commutative diagram*

$$\begin{array}{ccccccccc} & & & & W_{\mathbf{Q}} & = & W_{\mathbf{Q}} & & \\ & & & & \phi \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{S}(\mathbf{C}) & \longrightarrow & \mathcal{T}(\mathbf{C}) & \longrightarrow & G_{\mathbf{Q}} & \longrightarrow & 1. \end{array}$$

Sketch of the proof. The corollary follows by passing to the limit. To prove the proposition, Langlands simply checks (with slightly different normalisations) that the 2-cocycle[10] describing the extension $1 \to \mathcal{S}_E(E) \to \mathcal{T}_E(E) \to \mathrm{Gal}(E^{ab}/\mathbf{Q}) \to 1$,

$$d_{s,t} = \tilde{f}_E(st) \cdot t^{-1}(\tilde{f}_E(s))^{-1} \cdot \tilde{f}_E(t)^{-1},$$

becomes trivial after inflation to $W_{E/\mathbf{Q}}$ with values in $\mathcal{S}_E(\mathbf{C})$. Here we denote by $\tilde{f}_E(s)$ some representative in $\mathcal{S}_E(E_{\mathbf{A}_f})$ of the class $f_E(s)$. So in the notation of 2.1, we are working with some $\alpha_E(s)^{-1} \cdot \varepsilon_E(s)$ instead of $\beta_E(s)$. We may arrange these representatives such that the resulting map $\tilde{f}_E : \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) \to \mathcal{S}_E(E_{\mathbf{A}_f})$ is locally constant.

[10] Actually, in our somewhat unusual normalisation, we do not really get a 2-cocycle. In fact, in the notation of 2.1, we have $d_{s,t} = \alpha(st)^{-1}\alpha(s)\alpha(t)$, rather than the usual $\alpha(s)\alpha(t)\alpha(st)^{-1}$.

Lift $\hat{f}_E$ to the Weil group to get the following commutative diagram. (Note that the composite of the maps in the bottom row is f_E.)

$$\begin{array}{ccccc} W_{E/\mathbf{Q}} & \xrightarrow{\hat{f}_E} & \mathcal{S}_E(E_{\mathbf{A}}) & \longrightarrow & \mathcal{S}_E(E_{\mathbf{A}})/\mathcal{S}_E(E) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Gal}(E^{\mathrm{ab}}/\mathbf{Q}) & \xrightarrow{\hat{f}_E} & \mathcal{S}_E(E_{\mathbf{A}_f}) & \longrightarrow & \mathcal{S}_E(E_{\mathbf{A}_f})/\mathcal{S}_E(E). \end{array}$$

Then $\hat{f}_E(\hat{s}\hat{t}) \cdot t^{-1}(\hat{f}_E(\hat{s}))^{-1} \cdot \hat{f}_E(\hat{t})^{-1} \in \mathcal{S}_E(E) \subset \mathcal{S}_E(E_{\mathbf{A}})$ lifts $d_{s,t}$. For any infinite place v of E, let $\hat{f}_v$ be the local component at v. Then $\hat{s} \mapsto \hat{f}_v(\hat{s})$ trivialises $d_{s,t}$ over $E_v = \mathbf{C}$.

4.5.6. REMARK. The map ϕ of Proposition 4.5.4 is unique up to (reversed) 1-cocycles of $W_{E/\mathbf{Q}}$ with values in $\mathcal{S}_E(\mathbf{C})$.

5. CM motives and L-functions

One of the main applications that A. Weil drew from his newly defined "Weil groups" in [**Weil**, 1951b, section VI] was to the L-functions of their representations: via a generalisation of R. Brauer's induction theorem, they may be decomposed into Hecke L-functions "mit Grössencharakteren". In other words, the Grothendieck group of finite-dimensional continuous complex representations of $W_{\mathbf{Q}}$ is generated by representations of the form $\mathrm{Ind}_{L/\mathbf{Q}}\chi$, for algebraic number fields L and quasi-characters χ of $L^*_{\mathbf{A}_f}/L^*$.

Not all such quasi-characters will factor through $W_{\mathbf{Q}} \xrightarrow{\phi} \mathcal{T}(\mathbf{C})$ (where $\mathcal{T}$ is the Taniyama group). An obvious necessary condition that they do is that they define a (CM) Hodge structure (which would be their restriction to $\mathcal{S}(\mathbf{C})$). In other words, at most the *algebraic Hecke characters* (Weil's "quasi-characters of type A_0") will be visible in $\mathcal{CM}_{\mathbf{Q}}$—and in fact they all are.

5.1. THEOREM. *For an algebraic number field L, let ${}_L\mathcal{T}$ be the preimage of $G_L \subset G_{\mathbf{Q}}$ in $\mathcal{T}$ with respect to* 4.5.3. *Then the characters* $\mathrm{Hom}({}_L\mathcal{T}, \mathbf{C}^*)$ *are naturally identified with the algebraic Hecke characters of L.*

There are several ways to prove this theorem. We may, for instance, use 4.4, and prove 5.1 by constructing explicitly, for every algebraic Hecke character χ of L with values in an algebraic number field E, a motive $M(\chi)$ in $\mathcal{CM}_{\mathcal{L}}$ with coefficients in E whose λ-adic realisations—for finite places λ of E—are just the one-dimensional λ-adic G_k-representations attached to χ. We refer to [**Schappacher**, 1988, Chapter I, §4] for a detailed explanation of how to build up such motives $M(\chi)$ from abelian varieties of CM type.

Sticking with $\mathcal{T}$ rather than $\mathcal{U}$ one can show that the group $\tilde{\mathcal{S}}_L = \varprojlim \mathcal{S}_m$, constructed by Serre for every number field L in order to accomodate all algebraic Hecke characters of L [**Serre**, 1968], is isomorphic to the subquotient ${}_L\mathcal{T}_L$ of the Taniyama group, i.e., to the group at level L in the inverse system of quotients ${}_L\mathcal{T}_E$ the limit of which is

$$1 \to \mathcal{S} \to {}_L\mathcal{T} \to G_L \to 1.$$

See [**Langlands**, 1979, p. 224]; cf. [**Deligne**, 1982, §(E)]. We do not give the details here.

There are a few important consequences of Theorem 5.1 (and of Brauer's induction theorem applied to $\mathcal{T} \otimes \mathbf{C}$)[11] which are worth stressing, cf. also [**Anderson**, 1986, p. 181].

5.2. Corollary. *For every object M of $\mathscr{CM}_{\mathbf{Q}}$ with coefficients in some number field E, the system of λ-adic Galois representations $(H_\lambda(M))_\lambda$ (where λ ranges over the finite places of E) is strictly compatible.*

5.3. Corollary. *For each object M of $\mathscr{CM}_{\mathbf{Q}}$, there exist algebraic number fields $L_1, \dots, L_r$, integers $m_1, \dots, m_r$, and for each $i = 1, \dots, r$, an algebraic Hecke character χ_i of L_i such that*

$$L(M, s) = \prod_{i=1}^{r} L_{L_i}(\chi_i, s)^{m_i}.$$

5.5. Theorem. *If M, M' are two objects of $\mathscr{CM}_{\mathbf{Q}}$, if ℓ is a prime number and $H_\ell(M)$, $H_\ell(M')$ the ℓ-adic representations of the motives, then the natural map*

$$\mathrm{Hom}_{\mathscr{CM}_{\mathbf{Q}}}(M, M') \otimes \mathbf{Q}_\ell \hookrightarrow \mathrm{Hom}_{G_{\mathbf{Q}}}(H_\ell(M), H_\ell(M'))$$

is surjective.

Proof. Let $\mathcal{L}$ be the affine group scheme over $\mathbf{Q}_\ell$ corresponding to the $\mathbf{Q}_\ell$-linear neutral Tannakian category of ℓ-adic representations π of $G_{\mathbf{Q}}$ which are *potentially locally algebraic*, in the sense of [**Serre**, 1968, III.3.3], i.e., possibly after restricting to a subgroup G_L of finite index in $G_{\mathbf{Q}}$, π factors through $\mathrm{Gal}(L^{\mathrm{ab}}/L)$, admits a conductor, and is given by a certain character π_{alg} of $\mathcal{S}_L$ on principal ideals generated by numbers congruent to 1 modulo the conductor. Mapping π to π_{alg} (over some sufficiently large number field L) induces a morphism $\mathcal{S} \otimes \mathbf{Q}_\ell \to \mathcal{L}$. On the other hand, if π_{alg} is trivial, then π is of finite order. This defines $\mathcal{L}$ as an extension of $G_{\mathbf{Q}}$ by $\mathcal{S} \otimes \mathbf{Q}_\ell$, and 5.5 follows from

5.5.1. Lemma. *The extension*

$$1 \to \mathcal{S} \otimes \mathbf{Q}_\ell \to \mathcal{L} \to G_{\mathbf{Q}} \to 1$$

is isomorphic to

$$1 \to \mathcal{S} \otimes \mathbf{Q}_\ell \to \mathcal{U} \otimes \mathbf{Q}_\ell \to G_{\mathbf{Q}} \to 1.$$

[11] It applies because $\mathcal{T} \otimes \mathbf{C}$ is the inverse limit of $\mathbf{C}$-algebraic groups $\mathcal{T}_E \otimes \mathbf{C}$ whose connected components are tori; to wit $\mathcal{S}_E \otimes \mathbf{C}$.

PROOF. See [**Deligne**, 1982, §(D)]. The morphism $\mathscr{S} \to \mathscr{U} \otimes \mathbf{Q}_\ell$ is induced by mapping an E_λ-rational (with $\mathbf{Q}_\ell \subseteq E_\lambda$) representation ρ of $\mathscr{U} \otimes E_\lambda$ to $G_\mathbf{Q} \xrightarrow{\varepsilon_\ell} \mathscr{U}(\mathbf{Q}_\ell) \hookrightarrow \mathscr{U}(E_\lambda) \xrightarrow{\rho} \mathrm{GL}(V)$. One checks that $\rho_{|\mathscr{S}\otimes\mathbf{Q}_\ell}$ and $(\rho \circ \varepsilon_\ell)_{\mathrm{alg}}$ give the same representation of $\mathscr{S}_L$ by reducing to the case of a CM type (CM types generate all possible representations by Lemma 1.3.2 above). Then the morphism $\mathscr{S} \to \mathscr{U} \otimes \mathbf{Q}_\ell$ is trapped in a mapping between both extensions with identity on the left $(\mathscr{S} \otimes \mathbf{Q}_\ell)$ and on the right $(G_\mathbf{Q})$.

5.6. COROLLARY. *If two objects M, M' of $\mathscr{CM}_\mathbf{Q}$ have the same L-function, then they are isomorphic.*

This follows from 5.5 and the semisimplicity of our category.

REFERENCES

1. E. Artin and J. Tate, *Class field theory*, Benjamin, New York, 1967.
2. G. W. Anderson, *Cyclotomy and an extension of the Taniyama group*, Compositio Math. **57** (1986), 153–217.
3. D. Blasius, *On the critical values of Hecke L-series*, Ann. of Math. **124** (1986), 23–63.
4. C. Chevalley, *Deux théorèmes d'arithmétique*, J. Math. Soc. Japan **3** (1951), 36–44.
5. P. Deligne, *Applications de la formule des traces aux sommes trigonométriques*, Séminaire de Géometrie Algébrique $4\frac{1}{2}$, Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin and New York, 1977, pp. 168–232.
6. ——, *Valeurs de fonctions L*, Automorphic Forms, Representations, and L-Functions, Proc. Sympos. Pure Math., vol. **XXXIII**, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 313–346.
7. ——, *Motifs et groupe de Taniyama*, Hodge Cycles, Motives and Shimura Varieties (P. Deligne, J. S. Milne, A. Ogus, and K. Shih, eds.), Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982, pp. 261–279.
8. ——, (texte rédigé par J. L. Brylinski), *Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes*, Soc. Math. France, Mémoire no. 2 (2ème série) 1980, pp. 23–33.
9. P. Deligne and J. S. Milne, *Tannakian categories*, Hodge Cycles, Motives and Shimura Varieties (P. Deligne, J. S. Milne, A. Ogus, and K. Shih, eds.), Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982, pp. 101–228.
10. P. Deligne (notes by J. S. Milne), *Hodge cycles on abelian varieties*, Hodge Cycles, Motives and Shimura Varieties (P. Deligne, J. S. Milne, A. Ogus, and K. Shih, eds.), Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982, pp. 9–100.
11. P. Deligne, J. S. Milne, A. Ogus, and K. Shih, eds., *Hodge cycles, motives and Shimura varieties*, Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982.
12. M. Deuring, *Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins*, I–IV, Göttinger Nachr. 1953, 1955–1957.
13. G. Eisenstein, *Über die Irreductibilität und einige andere Eigenschaften der Gleichung, von welcher die Theilung der ganzen Lemniscate abhängt....*; J. Reine Angew. Math. **39** (1850), 160–179; 224–287.
14. U. Jannsen, *Mixed motives and algebraic K-Theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, Berlin and New York, 1990.
15. E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. Erste Mitteilung*, Math. Z. **1** (1918), 357–376.
16. ——, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. Zweite Mitteilung*, Math. Z. **6** (1920), 11–51.
17. S. Lang, *Complex multiplication*, Grundlehren Math. Wiss., vol. 255, Springer, New York, 1983.
18. R. P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*,Automorphic Forms, Representations, and L-Functions, Proc. Sympos. Pure Math., vol. **XXXIII**, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 205–246.

19. J. S. Milne, *Abelian varieties with complex multiplication* (for pedestrians), unpublished handwritten manuscript, 1981.
20. ——, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic Forms, Shimura Varieties, and L-functions (L. Clozel and J. S. Milne, eds.), vol. I, Perspect. Math., vol. 10, Academic Press, New York, 1989, pp. 283–414.
21. J. S. Milne and K. Shih, *The action of complex conjugation on a Shimura variety*, Ann. of Math. **113** (1981), 569–599.
22. ——, [1982 (I)], *Langlands's construction of the Taniyama group*, Hodge Cycles, Motives and Shimura Varieties (P. Deligne, J. S. Milne, A. Ogus, and K. Shih, eds.), Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982, pp. 229–260.
23. ——, [1982 (II)], *Conjugates of Shimura varieties*, Hodge Cycles, Motives and Shimura Varieties (P. Deligne, J. S. Milne, A. Ogus, and K. Shih, eds.), Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin and New York, 1982, pp. 280–356.
24. A. Panchishkin, *Motives for absolute Hodge cycles*, these Proceedings, vol. 1, pp. 461–483.
25. N. Schappacher, *Periods of Hecke characters*, Lecture Notes in Math., vol. 1301, Springer-Verlag, Berlin and New York, 1988.
26. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publ. Math. Soc. Japan, vol. 11, Iwanami Shoten/Princeton Univ. Press, Tokyo/Princeton, NJ, 1971.
27. G. Shimura and Y. Taniyama, *Complex multiplication of abelian varieties*, Publ. Math. Soc. Japan, vol. 6, Math. Soc. Japan, 1961.
28. J.-P. Serre, *Abelian l-adic representations and elliptic curves*, Benjamin, New York, 1968.
29. J. Tate, *Number theoretic background*, Automorphic Forms, Representations, and L-Functions, Proc. Sympos. Pure Math., vol. **XXXIII**, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 3–26.
30. ——, *On conjugation of abelian varieties of CM type*, manuscript, 1981.
31. A. Weil [1949b], *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. **55** (1949), 497–508.
32. ——, [1951b], *Sur la théorie du corps de classes*, J. Math Soc. Japan **3** (1951), 1–35.
33. H. Yoshida, *Abelian varieties with complex multiplication and representations of the Weil groups*, Ann. of Math. **114** (1981), 87–102.

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Structures de Hodge mixtes réelles

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Les structures de Hodge mixtes réelles (§2) forment une $\otimes$-catégorie abélienne rigide et le foncteur "espace vectoriel réel sous-jacent" est un foncteur fibre. Le foncteur Gr^W est un autre. Nous nous proposons de calculer ce que donne la théorie générale des catégories tannakiennes dans ce cas particulier.

1. Trois filtrations opposées

1.1. Ce numéro est une variation sur le numéro 1.2 de P. Deligne, *Théorie de Hodge* II, Publ. Math. IHES **40**, pp. 5–58. Soient $\mathfrak{A}$ une catégorie abélienne, et A un objet de $\mathfrak{A}$ muni de trois filtrations W, F, $\overline{F}$ opposées. "Filtration" signifie ici "filtration finie" (loc. cit. 1.1.4), la filtration W est supposée croissante, et les filtrations F, $\overline{F}$ décroissantes. La condition d'opposition s'écrit $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q \mathrm{Gr}_n^W A = 0$ pour $n \neq p+q$ (loc. cit. 1.2.7, 1.2.13).

Pour tout objet bigradué $B = \bigoplus B^{p,q}$, nous noterons W, F_W, et $\overline{F}_W$ les trois filtrations opposées:

$$W_n = \bigoplus_{p+q\leq n} B^{p,q},$$
$$F_W^i = \bigoplus_{p\geq i} B^{p,q},$$
$$\overline{F}_W^i = \bigoplus_{q\geq i} B^{p,q}.$$

Avec cette notation, que trois filtrations W, F, et $\overline{F}$ de A soient opposées signifie encore (loc. cit. 1.2.6) que chaque $\mathrm{Gr}_n^W A$ admet une (unique)

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This paper is in final form and no version of it will be submitted for publication elsewhere.

décomposition

$$\mathrm{Gr}_n^W A = \bigoplus_{p+q=n} A_W^{p,q}$$

telle que les filtrations de $\mathrm{Gr}_n^W A$ induites par F et $\overline{F}$ soient données par

$$F^i \mathrm{Gr}_n^W A = \bigoplus_{p+q=n,\, p\geq i} A_W^{p,q},$$

$$\overline{F}^i \mathrm{Gr}_n^W A = \bigoplus_{p+q=n,\, q\geq i} A_W^{p,q}.$$

En d'autres termes, F et $\overline{F}$ induisent sur $\mathrm{Gr}^W A$ les filtrations F_W et $\overline{F}_W$.

Posons

$$A_F^{p,q} := (W_{p+q} \cap F^p) \cap ((W_{p+q} \cap \overline{F}^q) + \sum_{i\geq 0} (W_{p+q-i} \cap \overline{F}^{q-i+1}))$$

et, échangeant les rôles de F et $\overline{F}$, et de p et q,

$$A_{\overline{F}}^{p,q} := (W_{p+q} \cap \overline{F}^q) \cap ((W_{p+q} \cap F^p) + \sum_{i\geq 0} (W_{p+q-i} \cap F^{p-i+1})).$$

Pour $p+q=n$, la projection de W_n sur $\mathrm{Gr}_n^W A$ induit un isomorphisme de A_F^{pq} avec A_W^{pq} : loc. cit. 1.2.8, restreint au cas particulier 1.2.11 (où $A_F^{p,q}$ est noté A_0^{pq} ; noter que dans la formule définissant A_F^{pq}, on peut remplacer la somme sur $i \geq 0$ par la même somme sur $i \geq 2$). On en déduit (loc. cit.) que les A_F^{pq} (resp. $A_{\overline{F}}^{pq}$) forment une bigraduation de A, et que la somme a_F (resp. $a_{\overline{F}}$) des isomorphismes $A_F^{pq} \xrightarrow{\sim} A_W^{pq}$ (resp. $A_{\overline{F}}^{pq} \xrightarrow{\sim} A_W^{pq}$) est un isomorphisme bifiltré de $(A; W, F)$ avec $(\mathrm{Gr}^W A; W, F_W)$ (resp. de $(A; W, \overline{F})$ avec $(\mathrm{Gr}^W A; W, \overline{F}_W)$).

LEMME. *L'automorphisme* $d = a_{\overline{F}} a_F^{-1}$ *de* $\mathrm{Gr}^W A$ *vérifie*

$$(1.1.1) \qquad (d-1)(A_W^{pq}) \subset \bigoplus_{r<p,\, s<q} A_W^{rs}.$$

Par passage au gradué, les isomorphismes a_F et $a_{\overline{F}}$ induisent l'identité de $\mathrm{Gr}^W A$. On a donc $\mathrm{Gr}^W(d) = 1$.

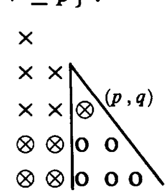
Pour chaque partie I de $\mathbb{Z}^2$, et pour B un objet bigradué, posons $B^I = \bigoplus_{(r,s)\in I} B^{rs}$. Posons

$$I(p,q) = \{(r,s) \mid r+s \leq p+q \text{ et } (r,s) = (p,q) \text{ ou } r < p\},$$

$$J(p,q) = \{(r,s) \mid r+s \leq p+q \text{ et } (r,s) = (p,q) \text{ ou } s < q\},$$

$$K(p,q) = \{(r,s) \mid r+s \leq p+q \text{ et } r \geq p\} :$$

$I(p,q)$: ×
$J(p,q)$: o
$K(p,q)$: entouré



Dans la définition de A_F^{pq} comme intersection de deux sous-objets de A, le premier sous-objet est $A_F^{K(p,q)}$ et le second est $A_{\overline{F}}^{I(p,q)}$. Cela résulte de ce que a_F (resp. $a_{\overline{F}}$) est un isomorphisme bifiltré. On a donc $A_F^{pq} \subset A_{\overline{F}}^{I(p,q)}$, d'où $dA_W^{pq} = a_{\overline{F}} A_F^{pq} \subset A_W^{I(p,q)}$, et $dA_W^{I(p,q)} \subset A_W^{I(p,q)}$. Parce que $\mathrm{Gr}^W(d) = 1$, on a égalité: $dA_W^{I(p,q)} = A_W^{I(p,q)}$. Echangeant F et $\overline{F}$, on trouve de même que $d^{-1}A_W^{J(p,q)} = A_W^{J(p,q)}$, donc que $dA_W^{J(p,q)} = A_W^{J(p,q)}$, et

$$dA_W^{pq} \subset A_W^{I(p,q)} \cap A_W^{J(p,q)} = A_W^{I(p,q)\cap J(p,q)} = A_W^{p,q} \oplus \bigoplus_{r<p\,,\,s<q} A_W^{p,q}.$$

On conclut en utilisant que $\mathrm{Gr}_W(d) = 1$:

Proposition 1.2. *Le foncteur $A \to \mathrm{Gr}^W A$ est une équivalence de la catégorie des objets de $\mathfrak{A}$, munis de trois filtrations opposées W, F, $\overline{F}$, avec la catégorie des objets de $\mathfrak{A}$, munis d'une bigraduation* (*sous-entendu: finie*) *et d'un automorphisme d vérifiant* (1.1.1).

L'isomorphisme $a_F : A \to \mathrm{Gr}^W A$ respecte la filtration W, et transforme F en F_W, et $\overline{F}$ en $d^{-1}(\overline{F}_W)$: on a $a_F = d^{-1} a_{\overline{F}}$. Le foncteur Gr^W admet donc pour inverse à gauche le foncteur $s : (B = \bigoplus B_W^{pq}, d) \mapsto (B, W, F_W, d^{-1}(\overline{F}_W))$ muni de l'isomorphisme fonctoriel $a_F : A \xrightarrow{\sim} s\mathrm{Gr}^W A$.

Réciproquement, partons d'un objet bigradué B, muni d'un automorphisme d, et définissons W, F, et $\overline{F}$ comme ci-dessus. Ces filtrations sont opposées dès que $\mathrm{Gr}^W(d) = 1$. Pour ces filtrations, $B_F^{pq} = B^{K(p,q)} \cap d^{-1}(B^{I(p,q)})$. Si d vérifie (1.1.1), on a $dB^{pq} \subset B^{I(p,q)}$, et $B^{pq} \subset B_F^{pq}$. Puisque (B^{pq}) et (B_F^{pq}) sont des bigraduations de B, on a égalité: $B^{pq} = B_F^{pq}$. Le même argument, appliqué à d^{-1} et à la bigraduation $\overline{B}^{pq} = d^{-1}B^{pq}$, montre que $d^{-1}B^{pq} = B_{\overline{F}}^{pq}$. On en déduit que l'isomorphisme évident de B avec $\mathrm{Gr}^W B$ est un isomorphisme de foncteurs $\mathrm{Id} \xrightarrow{\sim} \mathrm{Gr}^W \circ s$.

Remarque 1.3. Si 2 est inversible, d admet une unique racine carré $d^{1/2}$ vérifiant encore (1.1.1). Posons $a := d^{1/2} a_F = d^{-1/2} a_{\overline{F}}$. On a $a_{\overline{F}} = d^{1/2} a$ et $a_F = d^{-1/2} a$, de sorte que a transforme F (resp. $\overline{F}$) en la filtration $d^{1/2}(F_W)$ (resp. $d^{-1/2}(\overline{F}_W)$) de $\mathrm{Gr}^W A$. L'isomorphisme a est le même pour A muni de W, F, et $\overline{F}$ ou de W, $\overline{F}$, et F.

1.4. Ces constructions sont compatibles au produit tensoriel: si $\otimes : \mathfrak{A}_1 \times \mathfrak{A}_2 \to \mathfrak{A}$ est un foncteur biadditif exact, on l'étend aux objets filtrés en posant

$$F^n(A_1 \otimes A_2) = \sum_{a+b=n} F^a(A_1) \otimes F^b(A_2) \subset A_1 \otimes A_2.$$

Le foncteur "gradué associé" est compatible à $\otimes$ et, pour les objets bifiltrés, l'isomorphisme $\mathrm{Gr}^{W_1 \otimes W_2}(A_1 \otimes A_2) \xrightarrow{\sim} \mathrm{Gr}^{W_1}(A_1) \otimes \mathrm{Gr}^{W_2}(A_2)$ transforme la filtration induite par $F_1 \otimes F_2$ en le $\otimes$ de celles induites par F_1 et F_2.

Pour les espaces vectoriels sur un corps, cela peut se déduire de ce qu'une bifiltration est toujours sous-jacente à une bigraduation. On en déduit que si W_i, F_i, et $\overline{F}_i$ sont trois filtrations opposées sur $A_i \in \operatorname{Ob}\mathfrak{A}_i$ $(i = 1, 2)$, alors les filtrations $W_1 \otimes W_2$, $F_1 \otimes F_2$, et $\overline{F}_1 \otimes \overline{F}_2$ de $A_1 \otimes A_2$ sont encore opposées. La formation des bigraduations A_F^{pq} et $A_{\overline{F}}^{pq}$ de A et A_W^{pq} de $\operatorname{Gr}^W(A)$, ainsi que celle des isomorphismes a_F, $a_{\overline{F}} : A \xrightarrow{\sim} \operatorname{Gr}^W(A)$ et de l'automorphisme d de $\operatorname{Gr}^W(A)$ est compatible à $\otimes$.

1.5. Soient k un corps commutatif, $\mathfrak{A}$ la catégorie des espaces vectoriels (sous-entendu: de dimension finie) sur k et $\mathscr{O}$ la catégorie des espaces vectoriels munis de trois filtrations opposées W, F, et $\overline{F}$. C'est une $\otimes$-catégorie abélienne rigide, avec $\operatorname{End}(1) = k$. Elle admet le foncteur fibre ω_0: "espace vectoriel sous-jacent", donc est tannakienne et neutre. On dispose aussi du foncteur fibre $\operatorname{Gr}^W =: \omega_W$. Les isomorphismes a_F et $a_{\overline{F}}$ sont des isomorphismes de foncteurs fibres: $\omega_0 \to \omega_W$. Posons $\mathfrak{G} = \operatorname{Aut}(\omega_W)$. L'automorphisme fonctoriel d définit un point du groupe pro-algébrique $\mathfrak{G}$ sur k.

A la bigraduation fonctorielle de $\operatorname{Gr}^W(V)$ par les $V_W^{p,q}$ correspond un homomorphisme

$$\alpha : \mathbb{G}_m \times \mathbb{G}_m \to \mathfrak{G}.$$

C'est une section de la projection

$$\mathrm{pr} : \mathfrak{G} \to \mathbb{G}_m \times \mathbb{G}_m$$

correspondant à l'inclusion dans $\mathscr{O}$ de la catégorie des espaces vectoriels bigradués (par $B \mapsto (B, W, F_W, \overline{F}_W))$. Je désire normaliser α de sorte que $\alpha(\lambda, \mu) \cdot v = \lambda^{-p} \mu^{-q} v$ pour $v \in V_W^{p,q}$.

D'après la proposition 1.2, $\mathfrak{G}$ est aussi le groupe algébrique des automorphismes du foncteur fibre "espace vectoriel sous-jacent" sur la catégorie des espaces vectoriels bigradués munis d'un automorphisme d vérifiant (1.1.1). Nous allons le "calculer" lorsque k est de caractéristique 0.

Soit $\mathfrak{L}$ l'algèbre de Lie sur k, librement engendrée par des éléments $D^{i,j}$ $(i, j < 0)$, et munissons-la de l'unique bigraduation pour laquelle $D^{i,j}$ est de bidegré (i, j). Les $W_n(\mathfrak{L})$ sont des idéaux de $\mathfrak{L}$, et les quotients $\mathfrak{L}/W_n(\mathfrak{L})$ sont des algèbres de Lie nilpotentes. Nous noterons $\mathfrak{U}$ le groupe pro-algébrique limite projective des groupes algébriques unipotents $\exp(\mathfrak{L}/W_n(\mathfrak{L}))$ d'algèbre de Lie les $\mathfrak{L}/W_n(\mathfrak{L})$. Faisons agir le groupe $\mathbb{G}_m \times \mathbb{G}_m$ sur $\mathfrak{L}$, de sorte que (λ, μ) agisse sur l de degré (i, j) par multiplication par $\lambda^{-i}\mu^{-j}$. Cette action respecte la filtration W, et fournit donc une action de $\mathbb{G}_m \times \mathbb{G}_m$ sur $\mathfrak{U}$.

CONSTRUCTION 1.6. *$\mathfrak{G}$ est le produit semi-direct* $(\mathbb{G}_m \times \mathbb{G}_m) \ltimes \mathfrak{U}$.

Faire agir le produit semi-direct sur un espace vectoriel V de dimension finie revient à faire agir $\mathbb{G}_m \times \mathbb{G}_m$, et $\mathfrak{U}$, avec une condition de compatibilité reliant ces deux actions. L'action de $\mathbb{G}_m \times \mathbb{G}_m$ correspond à une bigraduation (celle pour laquelle $(\lambda, \mu) * v^{p,q} = \lambda^{-p}\mu^{-q}v^{p,q})$. Celle de $\mathfrak{U}$ à une action de $\mathfrak{L}$, nulle sur un $W_n\mathfrak{L}$ et nilpotente, i.e., à la donnée d'endomorphismes $D^{i,j}$, pour $i, j < 0$, tels que tout composé de $D^{i,j}$ de poids assez grand en valeur absolue soit nul. La compatibilité est que $D^{i,j}$ soit de bidegré (i, j), et cette propriété entraîne les annulations requises ci-dessus. La donnée des $D^{i,j}$ équivaut à celle de $D := \sum D_{ij}$: au total, l'action du produit semi-direct sur V s'identifie à la donnée d'une bigraduation, et d'un endomorphisme vérifiant $DV^{p,q} \subset \bigoplus_{i<p,\, j<q} V^{i,j}$. Posant $d = \exp(D)$, on l'identifie encore à la donnée d'une bigraduation, et d'un automorphisme d vérifiant (1.1.1). Cette construction est compatible au produit tensoriel, d'où l'isomorphisme voulu $(\mathbb{G}_m \times \mathbb{G}_m) \ltimes \mathfrak{U} \xrightarrow{\sim} \mathfrak{G} = \mathrm{Aut}(\omega_W)$. Cet isomorphisme est bien sûr compatible à α et pr ci-dessus.

2. Structures de Hodge mixtes

Rappelons qu'une structure de Hodge mixte réelle est un espace vectoriel réel (sous-entendu: de dimension finie) muni d'une filtration croissante W et dont le complexifié $V_{\mathbb{C}}$ est muni d'une filtration décroissante F. On exige que sur $V_{\mathbb{C}}$ la filtration $W_{\mathbb{C}}$—souvent notée simplement W—déduite de W par extension des scalaires, la filtration F et sa complexe conjuguée $\overline{F}$ forment un système de trois filtrations opposées.

Il revient au même de se donner un espace vectoriel réel ou un espace vectoriel complexe muni d'une structure réelle, i.e., d'une conjugaison complexe (= une involution antilinéaire). Via ce dictionnaire, il revient au même de se donner une structure de Hodge mixte réelle, ou un espace vectoriel complexe muni de trois filtrations opposées W, F, et $\overline{F}$, et d'une conjugaison complexe qui respecte W et échange F et $\overline{F}$. D'après la proposition 1.2, le foncteur Gr^W induit une équivalence de la catégorie de ces objets avec celle des espaces vectoriels complexes bigradués $V = \bigoplus V^{p,q}$, munis d'un automorphisme d vérifiant (1.1.1) et d'une conjugaison complexe σ qui échange $V^{p,q}$ et $V^{q,p}$, et pour laquelle le complexe conjugué $\bar{d}$ de d est $d^{-1} : \sigma d \sigma^{-1} = d^{-1}$.

La catégorie $\mathscr{M}$ des structures de Hodge mixtes réelles est une $\otimes$-catégorie abélienne rigide, avec $\mathrm{End}(1) = \mathbb{R}$. Elle admet le foncteur fibre ω_0: "espace vectoriel sous-jacent", donc est tannakienne et neutre. On dispose aussi du foncteur fibre $\mathrm{Gr}^W =: \omega_W$. Avec les notations de la remarque 1.3, le morphisme fonctoriel

$$a : V_{\mathbb{C}} \to \mathrm{Gr}^W(V_{\mathbb{C}})$$

respecte la structure réelle des deux membres (car il dépend symétriquement de F et $\overline{F}$). Il induit un isomorphisme de foncteurs fibres, encore noté $a : \omega_0 \to \omega_W$.

Avec les notations du numéro 1, faisons $k = \mathbb{C}$, et soit $\mathfrak{M}$ la forme réelle de $\mathfrak{G}$ définie par la conjugaison complexe suivante:

a) Sur $\mathbb{G}_m \times \mathbb{G}_m$, c'est $(\lambda, \mu) \to (\bar{\lambda}, \bar{\mu})$.

b) Sur $\mathfrak{U}$, c'est celle qui est compatible à la conjugaison complexe de $\mathfrak{L}$ donnée par

$$D^{i,j} \mapsto -D^{j,i}.$$

Comme en 1.1, on voit qu'il revient au même de faire agir $\mathfrak{M}$ sur un espace vectoriel réel V, ou de se donner

a) un bigraduation $V_{\mathbb{C}} = \bigoplus V^{p,q}$ de $V_{\mathbb{C}}$, telle que $V^{p,q}$ et $V^{q,p}$ soient complexes conjugués l'un de l'autre, et

b) un automorphisme d de $V_{\mathbb{C}}$, tel que $\bar{d} = d^{-1}$, et que

$$(d-1)V^{p,q} \subset \bigoplus_{i<p,\, j<q} V^{i,j}.$$

Dès lors,

PROPOSITION 2.1. *Le foncteur fibre ω_W induit une équivalence de $\mathcal{M}$ avec la catégorie des représentations de $\mathfrak{M}$: on a $\mathfrak{M} = \mathrm{Aut}(\omega_W)$.*

Le foncteur inverse ϕ est

(V, muni de l'action de $\mathfrak{M}$) $\leftrightarrow$

(V, muni d'une bigraduation de $V_{\mathbb{C}}$, et de d) $\leftrightarrow$

(V, muni de la filtration W de complexifiée la filtration W attachée à la bigraduation de $V_{\mathbb{C}}$, et de la filtration $F := d^{-1/2}(F_W)$ de son complexifié), avec pour isomorphisme $\phi \circ \mathrm{Gr}^W \xrightarrow{\sim} \mathrm{Id}$ l'isomorphisme fonctoriel a.

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L-functions

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L-Functions of Mixed Motives

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0. Introduction

L-functions of pure motives have been considered for a long time, and there are quite a number of introductions to this topic. We refer to **[Se2, D3, T]**, for example. More generally, L-functions of mixed motives have also proved to be a useful concept, for example, in Scholl's reformulation of the Beilinson conjectures in **[Scho]**. Recently Fontaine and Perrin-Riou have also defined the appropriate ε- and L-factors at infinity in the mixed case **[F-PR]**. This allows them to extend the usual conjecture on the functional equation of a completed L-series to any mixed motive.

In this note, starting from Hasse-Weil zeta functions, we explain how one is led to the notion of the L-series of a mixed motive. We then formulate the conjectures on the analytic properties of such L-series and mention some of the evidence. A new point of view on these questions is described in **[De2]**.

1. Some motivation

The Hasse-Weil zeta function of an algebraic scheme $\mathscr{X}$ over $\operatorname{spec}\mathbb{Z}$ is defined by the Euler product

$$\zeta_{\mathscr{X}}(s) = \prod_{x\in|\mathscr{X}|} (1 - Nx^{-s})^{-1} \quad \text{for } \operatorname{Re} s > \dim \mathscr{X}$$

where $|\mathscr{X}|$ denotes the set of closed points of $\mathscr{X}$ and Nx is the order of the residue class field $\kappa(x)$ of x. In $\operatorname{Re} s > \dim\mathscr{X}$ the function $\zeta_{\mathscr{X}}(s)$ is holomorphic **[Se1]**, and one expects that it has a meromorphic continuation to the whole plane. If $\mathscr{O}_k$ is the ring of integers in a number field k, the function $\zeta_{\operatorname{spec}\mathscr{O}_k}(s)$ is just the Dedekind zeta function of k. Questions on the asymptotic distribution of closed points on $\mathscr{X}$ can be translated into analytic questions about $\zeta_{\mathscr{X}}(s)$. For $\mathscr{X} = \operatorname{spec}\mathscr{O}_k$ this is the classical approach of analytical number theory to the study of the distribution of prime ideals.

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If $\mathscr{X}$ is in fact a scheme over $\operatorname{spec}\mathbb{F}_p$, then by an elementary argument (e.g., **[D2]**) one obtains

$$\zeta_{\mathscr{X}}(s) = \exp\left(\sum_{\nu=1}^{\infty} \frac{\#\mathscr{X}(\mathbb{F}_{p^\nu})}{\nu} p^{-\nu s}\right). \tag{1.1}$$

Thus $\zeta_{\mathscr{X}}(s)$ also carries the diophantine information $\#\mathscr{X}(\mathbb{F}_{p^\nu})$ on the "numbers of solutions" of $\mathscr{X}$ in the finite fields $\mathbb{F}_{p^\nu}$. Viewing $\zeta_{\mathscr{X}}(s)$ as a formal power series in p^{-s} via (1.1), the Lefschetz trace formula in ℓ-adic cohomology $\ell \neq p$ implies the equality

$$\zeta_{\mathscr{X}}(s) = \prod_{w=0}^{2\dim\mathscr{X}} \det_{\mathbb{Q}_\ell}(1 - p^{-s}\mathrm{Fr}_p^*|H_c^w(\mathscr{X}\otimes\overline{\mathbb{F}}_p, \mathbb{Q}_\ell))^{(-1)^{w+1}} \tag{1.2}$$

in $\mathbb{Q}_\ell[|p^{-s}|]$. We have to resort to formal power series because it is not known in general, although expected, that the characteristic polynomials of Frobenius on the ℓ-adic cohomology groups have coefficients in $\mathbb{Q}$ and are independent of ℓ. If $\mathscr{X}$ is smooth and proper over $\mathbb{F}_p$, however, this is known **[D2]**, and (1.2) is then true as an identity of meromorphic functions.

For an arbitrary algebraic scheme $\mathscr{X}$ over $\operatorname{spec}\mathbb{Z}$ setting $\mathscr{X}_p = \mathscr{X}\otimes\mathbb{F}_p$ we have

$$\zeta_{\mathscr{X}}(s) = \prod_p \zeta_{\mathscr{X}_p}(s) \quad \text{for } \operatorname{Re}s > \dim\mathscr{X}. \tag{1.3}$$

If $\mathscr{X}$ is proper and flat over $\operatorname{spec}\mathbb{Z}$ with smooth generic fibre $X = \mathscr{X}\otimes\mathbb{Q}$, then almost all p are good in the sense that $\mathscr{X}_p$ is smooth, and for these p we find

$$\zeta_{\mathscr{X}_p}(s) = \prod_{w=0}^{2\dim X} \det_{\mathbb{Q}_\ell}(1 - p^{-s}\mathrm{Fr}_p^*|H^w(X\otimes\overline{\mathbb{Q}}_p, \mathbb{Q}_\ell))^{(-1)^{w+1}}, \qquad \ell \neq p, \tag{1.4}$$

by (1.2) and the smooth and proper base change theorems in étale cohomology. Here Fr_p denotes a geometric Frobenius automorphism in the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, i.e., the inverse of a lift to $G_{\mathbb{Q}_p}$ of the Frobenius automorphism $x \mapsto x^p$ in $G_{\mathbb{F}_p}$. Combining (1.3) and (1.4) we get that with $\ell = \ell_p \neq p$

$$\zeta_{\mathscr{X}}(s) = \prod_{w=0}^{2\dim\mathscr{X}} \left(\prod_{p\text{ good}} \det_{\mathbb{Q}_\ell}(1 - p^{-s}\mathrm{Fr}_p^*|H^w(X\otimes\overline{\mathbb{Q}}_p, \mathbb{Q}_\ell))^{-1}\right)^{(-1)^w}$$

up to finitely many Euler factors $\zeta_{\mathscr{X}_p}(s)$.

Now it has turned out in a number of cases, for example, for abelian varieties with complex multiplication, that already the individual factors

$$\prod_{p\text{ good}} \det_{\mathbb{Q}_\ell}(1 - p^{-s}\mathrm{Fr}_p^*|H^w(X\otimes\overline{\mathbb{Q}}_p, \mathbb{Q}_\ell))^{-1} \tag{1.5}$$

have a meromorphic continuation to all of $\mathbb{C}$—and even a functional equation if suitably completed by factors at the bad primes, i.e., those where the inertia group of $G_{\mathbb{Q}_p}$ acts nontrivially on $H^w(X\otimes\overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$ and at the infinite prime. One is therefore led to study Euler products such as (1.5) for their own sake: they are viewed as being attached to the motive $H^w(X)$. The correct definition of the local Euler factors of $H^w(X)$ at the bad places is suggested by analogy with the function field case to which we now turn for a moment:

Let Y be a smooth projective curve over $\mathbb{F}_p$ with function field K, and consider a smooth and proper variety X/K:

$$\begin{array}{ccc} X & & \\ \pi\downarrow & & \\ \operatorname{spec} K & \overset{j}{\hookrightarrow} & Y \end{array}$$

The ℓ-adic sheaf on Y

$$\mathscr{F} = j_*R^w\pi_*\mathbb{Q}_\ell = j_*H^w(X\otimes\overline{K}, \mathbb{Q}_\ell), \qquad \ell\neq p,$$

has stalks

$$\mathscr{F}_{\overline{y}} = H^w(X\otimes\overline{K}, \mathbb{Q}_\ell)^{I_y} \cong H^w(X\otimes\overline{K}_y, \mathbb{Q}_\ell)^{I_y}$$

in the geometric points $\overline{y}\to y$ for $y\in|Y|$. Here K_y is the completion of K in y, and I_y is the inertia group of G_{K_y}. It follows from the Lefschetz trace formula that for the L-function

$$(1.6)\qquad L(H^w(X), s) = \prod_{y\in|Y|}\det{}_{\mathbb{Q}_\ell}(1 - Ny^{-s}\mathrm{Fr}_y^*|H^w(X\otimes\overline{K}_y, \mathbb{Q}_\ell)^{I_y})^{-1}$$

in $\mathbb{Q}_\ell[|p^{-s}|]$ we have

$$(1.7)\qquad L(H^w(X), s) = \prod_{i=0}^{2}\det{}_{\mathbb{Q}_\ell}(1 - p^{-s}\mathrm{Fr}_p^*|H^i(Y\otimes\overline{\mathbb{F}}_p, \mathscr{F}))^{(-1)^{i+1}}.$$

This expression shows that $L(H^w(X), s)$ is in fact rational in p^{-s} and by Poincaré duality [**M**, V, Proposition 2.2.c] that it has a functional equation.

By analogy it is now clear how to define the local Euler factors of $H^w(X)$ in the number field case in general and not only for the good primes. We have to take

$$(1.8)\ L_p(H^w(X), s) := \det{}_{\mathbb{Q}_\ell}(1 - p^{-s}\mathrm{Fr}_p^*|H^w(X\otimes\overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)^{I_p})^{-1}, \qquad \ell\neq p,$$

as the local factor at p assuming it has coefficients in $\mathbb{Q}$ and is independent of ℓ.

2. The L-series of a mixed motive

Before going on we generalize the setting at the end of §1 as follows:

- As ground field we allow any algebraic number field k of finite degree over $\mathbb{Q}$.

- We do not only consider the particular motives $H^w(X)$ but also any mixed motive M over k as in **[D4]** or **[J]**.
- We allow for coefficients in a number field E in order to include, for example, Artin L-series in the discussion as well.

For a $\mathbb{Q}$-linear category $\mathscr{A}$ and a number field E of finite degree let $\mathscr{A}(E)$ denote the E-linear category of objects A in $\mathscr{A}$ with multiplication by E:

$$E \to \operatorname{End} A, \qquad 1 \mapsto \mathrm{id}_A$$

and with the evident morphisms. According to **[D3**, 2.1**]** the category $\mathscr{A}(E)$ is equivalent to the pseudo-abelian completion of $\mathscr{A} \otimes E$, i.e., to $\mathscr{A} \otimes E$ with images of projectors added.

Let $\mathscr{M}\mathscr{M}_K$ denote the $\mathbb{Q}$-linear category of mixed motives over a field K of characteristic zero in the sense of **[D4]** or **[J]**.

(2.1). If K is a finite extension of $\mathbb{Q}_p$ with prime ideal $\mathfrak{p}$, let I be the inertia group of G_K, and define the geometric Frobenius F to be the inverse of the canonical generator of G_K/I that maps x to $x^{N\mathfrak{p}}$ in the residue field of K. We fix a prime $\ell \neq p$ and an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Denoting by

$$M_\ell = H^\bullet(M \otimes \overline{K}, \mathbb{Q}_\ell)$$

the ℓ-adic realization of a motive M in $\mathscr{M}\mathscr{M}_K(E)$, we get an E-linear functor

$$M \mapsto M^I_{\ell,\iota} = M^I_\ell \otimes_{\mathbb{Q}_{\ell,\iota}} \mathbb{C}$$

from $\mathscr{M}\mathscr{M}_K(E)$ to the category of $(E \otimes \mathbb{C})[F]$-modules of finite rank over $E \otimes \mathbb{C}$. It is hoped that these functors or at least their semisimplifications are isomorphic for different ℓ and ι. The latter is true by the work of Deligne **[D2]** if $M = H^w(X)$ for some smooth projective variety X over K with good reduction and $E = \mathbb{Q}$. For cases of bad reduction where some information is available see **[K]**.

Since

$$E \otimes \mathbb{C} \cong \mathbb{C}^{\operatorname{Hom}(E,\mathbb{C})} \quad \text{via } e \otimes \lambda \mapsto (\sigma \mapsto \lambda\sigma(e)),$$

we can view $M^I_{\ell,\iota}$ as the array of $\mathbb{C}$-vector spaces

$$M^I_{\ell,\iota} = (M^I_{\ell,\iota,\sigma})_{\sigma \in \operatorname{Hom}(E,\mathbb{C})} \quad \text{where } M^I_{\ell,\iota,\sigma} = M^I_{\ell,\iota} \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}$$

and define the $(E \otimes \mathbb{C})$-valued L-factor of M by

$$L_K(M, s) = (\det_{\mathbb{C}}(1 - FN\mathfrak{p}^{-s} | M^I_{\ell,\iota,\sigma})^{-1})_{\sigma \in \operatorname{Hom}(E,\mathbb{C})}.$$

By the above remark it should not depend on ℓ and ι. For a construction of $L_K(M, s)$ using p-adic cohomology see **[F-PR]**.

(2.2). We can now define the L-function of a mixed motive M in $\mathscr{M}\mathscr{M}_K(E)$ over a number field k by

$$L(M, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M, s)$$

where $\mathfrak{p}$ runs over the finite primes of k and

$$L_{\mathfrak{p}}(M, s) = L_{k_{\mathfrak{p}}}(M \otimes k_{\mathfrak{p}}, s).$$

Here choices of ℓ's and $\imath$'s are suppressed from the notation since the resulting function should be independent of them anyhow. Let $w_{\max}$ be the largest weight of M. By the definition of weights it follows that for $\operatorname{Re} s > w_{\max}/2 + 1$ the Euler product over the good primes converges and defines an array of holomorphic nowhere-vanishing functions. It is expected that $L(M, s)$ has a meromorphic continuation to $\mathbb{C}$ of the form

$$L(M, s) = \frac{L_1(M, s)}{L_2(M, s)} \tag{2.2.1}$$

where each $L_1(M, s)_\sigma$ is entire of genus one [**A**, V, 2.3] and $L_2(M, s)_\sigma$ is a polynomial in s whose zeros are integers.

In all cases where this can be proved one can in fact show a functional equation for the L-function of M completed by factors at infinity. So let us turn to these generalizing [**F-PR**] to coefficients $E \supset \mathbb{Q}$.

(2.3). We denote by $\mathcal{MH}_{\mathbb{C}}$ the category of real mixed Hodge structures $(H, W_\bullet H, F^\bullet H_{\mathbb{C}})$. By $\mathcal{MH}_{\mathbb{R}}$ we denote the category of real mixed Hodge structures equipped with an involution F_∞ that respects the weight filtration and whose $\mathbb{C}$-linear extension F_∞ to $H_{\mathbb{C}}$ maps $F^\bullet H_{\mathbb{C}}$ into $\overline{F}^\bullet H_{\mathbb{C}}$. We call these mixed Hodge structures over $\mathbb{R}$. We set

$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s) \quad \text{and} \quad \Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2).$$

For H in $\mathcal{MH}_K(E)$ where $K = \mathbb{C}$ or $\mathbb{R}$ consider the $(E \otimes \mathbb{C})$-invariant filtration

$$\gamma^\nu H_{\mathbb{C}} = F^\nu H_{\mathbb{C}} \cap \overline{F}^\nu H_{\mathbb{C}}$$

of $H_{\mathbb{C}}$. In case $K = \mathbb{R}$ it is F_∞-invariant. Note that

$$\gamma^\nu H_{\mathbb{C}} = \gamma^\nu H \otimes_{\mathbb{R}} \mathbb{C} \quad \text{where } \gamma^\nu H = F^\nu H_{\mathbb{C}} \cap H.$$

The filtration $\gamma^\nu H$ is used in [**F-PR**, III]. However, because we are dealing with coefficients in E, we need the γ-filtration of $H_{\mathbb{C}}$ in the following.

The $(E \otimes \mathbb{C})$-valued L-factor of H

$$L(H, s) = (L(H, s)_\sigma)_{\sigma \in \operatorname{Hom}(E, \mathbb{C})}$$

is defined as follows. For any σ let e_σ denote the corresponding idempotent in $E \otimes \mathbb{C}$ and set

$$n_{\nu,\sigma} = \dim_{\mathbb{C}} e_\sigma(\operatorname{Gr}_\gamma^\nu H_{\mathbb{C}}).$$

If $K = \mathbb{C}$, we define

$$L(H, s)_\sigma = \prod_\nu \Gamma_{\mathbb{C}}(s - \nu)^{n_{\nu,\sigma}}.$$

If $K = \mathbb{R}$, set

$$n_{\nu,\sigma}^{\pm} = \dim_{\mathbb{C}} e_\sigma(\operatorname{Gr}_\gamma^\nu H_{\mathbb{C}})^{\pm}$$

where $\pm$ denotes the ± 1 eigenspace of F_∞ and define

$$L(H, s)_\sigma = \prod_\nu \Gamma_{\mathbf{R}}(s+\varepsilon_\nu-\nu)^{n^+_{\nu,\sigma}}\Gamma_{\mathbf{R}}(s+1-\varepsilon_\nu-\nu)^{n^-_{\nu,\sigma}}$$

with $\varepsilon_\nu \in \{0, 1\}$ such that $\varepsilon_\nu \equiv \nu \bmod 2$.

REMARK. The definition of $\mathbb{D}(H)$ in [**De1**, §3] makes sense for mixed Hodge structures as well and Theorem (4.1) of loc. cit. generalizes easily to give

$$L(H, s)_\sigma = \det_\infty\left(\frac{1}{2\pi}(s-\Theta)|e_\sigma(\mathbb{D}(H)\otimes_{\mathbf{R}}\mathbb{C})\right)^{-1}$$

for any H in $\mathcal{MH}_K$. Since the functor $\mathbb{D}$ is left exact but not exact on $\mathcal{MH}_K(E)$, this formula makes it quite clear that in general $L(H, s)$ is not equal to the L-factor of the semisimplification of H. For more on L-factors of Hodge structures and regularized determinants we refer to [**De2**, §§5 and 6].

(2.4). The Betti realization with coefficients in $\mathbb{R}$ of a motive

$$\mathcal{MM}_K(E) \to \mathcal{MH}_K(E), \qquad M \mapsto M_B,$$

allows us to define

$$L_K(M, s) = L(M_B, s).$$

If finally k is a finite extension of $\mathbb{Q}$ and $\mathfrak{p}$ an infinite prime of k, we set for any M in $\mathcal{MM}_k(E)$:

$$L_\mathfrak{p}(M, s) = L_{k_\mathfrak{p}}(M\otimes k_\mathfrak{p}, s).$$

If M is a pure motive one gets back the usual definition of the L-factor at infinity [**Se2**; **D3**, 5.2] up to a slight difference in the normalization of $\Gamma_\mathbb{R}$ and $\Gamma_\mathbb{C}$.

REMARK. A motive M in $\mathcal{MM}_\mathbb{Q}$ is critical in the sense of [**Scho**, II] if and only if $L_\infty(M, s)$ and $L_\infty(M^*(1), s)$ do not have a pole at $s=0$. This is an immediate consequence of the considerations in [**F-PR**, III, 1.2.1–1.2.5].

3. The functional equation

The completed L-function of a motive M over a number field k with coefficients in E is defined as

$$\widehat{L}(M, s) = L(M, s)\prod_{\mathfrak{p}|\infty} L_\mathfrak{p}(M, s)$$

where $\mathfrak{p}$ runs over the infinite primes of k. It is therefore a holomorphic and nonvanishing function in $\operatorname{Re} s > w_{\max}/2+1$.

(3.1). CONJECTURE. (1) *$\widehat{L}(M, s)$ has a meromorphic continuation to $\mathbb{C}$. The completed L-series of M and its dual motive M^* are related by a functional equation of the form*

$$\widehat{L}(M, s) = \varepsilon(M, s)\widehat{L}(M^*, 1-s)$$

where $\varepsilon(M, s) = ae^{bs}$ *for some* $a \in (E \otimes \mathbb{C})^*$, $b \in E \otimes \mathbb{C}$.

(2) *We have*

$$\widehat{L}(M, s) = \frac{\widehat{L}_1(M, s)}{\widehat{L}_{02}(M, s)}$$

where $\widehat{L}_1(M, s)_\sigma$ *is entire of genus one and* $\widehat{L}_{02}(M, s)_\sigma$ *is a polynomial in* s *whose zeros are integers.*

In all cases where this conjecture is known one first relates the motivic L-series to automorphic L-series and then uses their theory. It will be difficult to prove (3.1) in general by this approach: recall that conjecturally all elliptic curves over $\mathbb{Q}$ are modular. Conversely this would follow from (3.1) and (3.5) applied to twists of elliptic curves by a theorem of Weil.

In the function field case an analogue of (3.1)(1) can be proved using ℓ-adic cohomology, for example, **[D1**, §10]. Some evidence that a cohomological approach might be possible in the number field case too is contained in **[De2]**.

It suffices to prove (3.1) for pure motives **[F-PR]**. Note that if M is a pure motive in $\mathcal{M}_k$ of the form $M = H^w(X)$ for a smooth and proper variety X/k then because of Poincaré duality and strong Lefschetz there is an isomorphism $M^* \cong M(w)$. Hence the functional equation can be rewritten in the form

$$\widehat{L}(M, s) = \varepsilon(M, s)\widehat{L}(M, w + 1 - s).$$

Do all the zeros of $\widehat{L}(M, s)$ lie on the line $\operatorname{Re} s = (w+1)/2$?

Conjecture (3.1) can be made more precise: the factor $\varepsilon(M, s)$ should have a description as a product of local ε-factors which are obtained from the restrictions $M \otimes k_\mathfrak{p}$ of the motive M to $\operatorname{spec} k_\mathfrak{p}$. We proceed with the details **[Se2**, D3, F-PR].

(3.2). Consider $K/\mathbb{Q}_p$ finite, $I, F \in G_K/I$, a prime $\ell \neq p$, and an embedding $\iota: \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ as in (2.1). The Weil group of K sits in an exact sequence

$$1 \to I \to W_K \overset{\|\ \|}{\to} q^{\mathbb{Z}} \to 1$$

where q is the order of the residue class field κ of K and $\|w\|$ is defined to be the power of q to which w raises elements of κ.

A complex representation (ρ, N, V) of the Weil-Deligne group W_K' consists of a continuous representation ρ of W_K in a finite-dimensional $\mathbb{C}$-vector space V with the discrete topology and of a nilpotent endomorphism N of V such that

$$\rho(w)N\rho(w)^{-1} = \|w\|N \quad \text{for all } w \text{ in } W_K.$$

Set $V^I = V^{\rho(I)}$ and $V_N^I = (\operatorname{Ker} N)^{\rho(I)}$. Denoting by $a(\rho)$ the Artin conductor of ρ, the conductor of V is defined by

$$a(V) = a(\rho) + \dim V^I - \dim V_N^I.$$

Given choices of a Haar measure dx and a nontrivial character ψ of K the ε-factor of V is defined by

$$\varepsilon(V, \psi, dx, s) = \varepsilon(\rho, \psi, dx)\det(-F|V^I/V_N^I)q^{-sa(V)}$$

where $\varepsilon(\rho, \psi, dx)$ is the local nonabelian ε-factor of Langlands and Deligne [**D1**, 4.1; **T**, (3.4.1)].

Let us now consider a motive M in $\mathcal{MM}_K(E)$. Given an embedding $\sigma\colon E \hookrightarrow \mathbb{C}$ choose a prime $\lambda|\ell$ of E and an embedding $\iota_\lambda\colon E_\lambda \hookrightarrow \mathbb{C}$ which prolongs σ and ι. From the λ-adic representation of W_K on $M_\lambda = M_\ell \otimes_{E\otimes\mathbb{Q}_\ell} E_\lambda$ we get in a standard way [**D1**, §8] or [**T**, Theorem (4.2.1)] a representation of W_K' on $M_\lambda \otimes_{E_\lambda, \iota_\lambda} \mathbb{C}$. Its isomorphism class $V_\sigma(M)$ is expected to depend only on σ and not on ℓ, λ and the auxiliary embeddings ι and ι_λ. Compare [**F-PR**] for a p-adic construction of it. We now set

$$\varepsilon_K(M, s, \psi, dx)_\sigma = \varepsilon(V_\sigma(M), \psi, dx, s).$$

(3.3). We now turn to the ε-factor at infinity. For $K = \mathbb{R}$ or $\mathbb{C}$ let ψ_0 be the character of K given by $\psi_0(x) = \exp(2\pi i \operatorname{Tr}_{K/\mathbb{R}}(x))$, and let d_0x denote $[K:\mathbb{R}]$ times the Lebesgue measure of K. For a Hodge structure H in $\mathcal{MH}_K(E)$ recall the numbers $n_{\nu,\sigma}$, $n^{\pm}_{\nu,\sigma}$ of (2.3) and set

$$h_{\nu,\sigma} = \dim_{\mathbb{C}} e_\sigma(\operatorname{Gr}^\nu_F H_{\mathbb{C}}).$$

Following [**F-PR**] we define

$$\varepsilon(H, \psi_0, d_0x)_\sigma = i^{\sum_\nu \nu(h_{\nu,\sigma}-n_{\nu,\sigma})+\varepsilon_\nu n^+_{\nu,\sigma}+(1-\varepsilon_\nu)n^-_{\nu,\sigma}} \quad \text{if } K = \mathbb{R}$$

and

$$\varepsilon(H, \psi_0, d_0x)_\sigma = (-1)^{\sum_\nu \nu(h_{\nu,\sigma}-n_{\nu,\sigma})} \quad \text{if } K = \mathbb{C}.$$

For a mixed motive M in $\mathcal{MM}_K(E)$ we set

$$\varepsilon_K(M, s, \psi_0, d_0x)_\sigma = \varepsilon(M_B, \psi_0, d_0x)_\sigma, \quad \text{a constant}.$$

(3.4). Let k be a finite extension of $\mathbb{Q}$, and choose a nontrivial character of $\mathbb{A}_k/k$

$$\psi = \bigotimes_{\mathfrak{p}} \psi_{\mathfrak{p}}$$

where $\mathfrak{p}$ runs over the finite and infinite primes of k such that $\psi_{\mathfrak{p}}$ is the character of (3.3) if $\mathfrak{p}|\infty$. Let $dx = \bigotimes_{\mathfrak{p}} dx_{\mathfrak{p}}$ be a decomposition of the Tamagawa measure of $\mathbb{A}_k/k$ into Haar measures $dx_{\mathfrak{p}}$ of $k_{\mathfrak{p}}$ that agree with the measures d_0x of (3.3) if $\mathfrak{p}|\infty$. For a motive M in $\mathcal{MM}_k(E)$ one then sets

(3.4.1) $$\varepsilon(M, s)_\sigma = \prod_{\mathfrak{p}} \varepsilon_{\mathfrak{p}}(M, s, \psi_{\mathfrak{p}}, dx_{\mathfrak{p}})_\sigma$$

where

$$\varepsilon_{\mathfrak{p}}(M, s, \psi_{\mathfrak{p}}, dx_{\mathfrak{p}})_\sigma = \varepsilon_{k_{\mathfrak{p}}}(M \otimes k_{\mathfrak{p}}, s, \psi_{\mathfrak{p}}, dx_{\mathfrak{p}})_\sigma.$$

The product makes sense because almost all factors are equal to one. It is independent of the choice of ψ and the decomposition of dx into local Haar measures $dx_{\mathfrak{p}}$.

One can now refine Conjecture (3.1) as follows.

(3.5). CONJECTURE. *The factor* $\varepsilon(M, s)$ *in the functional equation for* $\hat{L}(M, s)$ *is given by* (3.4.1).

In **[F-PR]** it is shown that if (3.5) holds for two motives in a short exact sequence then it holds for the third one as well. One is thus reduced to pure motives.

In the function field analogue a proof of (3.5) was given in special cases by Deligne in **[D1]** and in full generality by Laumon **[L]** using the ℓ-adic Fourier transform. The latter proof also gives a cohomological interpretation of the local ε-factors in the function field case expressing them as "determinants of local Fourier transforms" **[L**, Theorem (3.5.1.1)].

REFERENCES

[A] L. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1966.

[D1] P. Deligne, *Les constantes des équations fonctionelles des fonctions* L, Antwerp II, Lecture Notes in Math., vol. 349, Springer-Verlag, New York, 1973, pp. 501–595.

[D2] ——, *La conjecture de Weil.* I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

[D3] ——, *Valeurs de fonctions* L *et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 313–342.

[D4] ——, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over $\mathbb{Q}$, Math. Sci. Res. Inst. Publ., Springer, New York, 1989, pp. 79–297.

[De1] C. Deninger, *On the* Γ*-factors attached to motives*, Invent. Math. **104** (1991), 245–261.

[De2] ——, *Motivic L-functions and regularized determinants*, these Proceedings, vol. 1, pp. 707–743.

[F] J.-M. Fontaine, *Valeurs spéciales des fonctions* L *des motifs*, Sém. Bourbaki, exp. 751 (fevr. 1992).

[F-PR] J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato, cohomologie galoisienne et valeurs de fonctions* L, these Proceedings, vol. 1, pp. 599–706.

[J] U. Jannsen, *Mixed motives and algebraic* K*-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, New York, 1990.

[K] N. Katz, *Wild ramification and some problems of "Independence of* l", Amer. J. Math. **105** (1983), 201–227.

[L] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 131–210.

[M] J. S. Milne, *Étale Cohomology*, Princeton Univ. Press, Princeton, NJ, 1980.

[Scho] A. J. Scholl, *Remarks on special values of L-functions*, L-functions and Arithmetic, London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge and New York, 1991, pp. 373–392.

[Se1] J.-P. Serre, *Zeta and* $\mathscr{L}$*-functions*, Arithmetic Algebraic Geometry, Harper & Row, New York, 1965, pp. 82–92.

[Se2] ——, *Facteurs locaux des fonctions zêta des variétés algébriques* (*définitions et conjectures*), Sém. Delange-Pisot-Poitou, exp. 19 (1969/70).

[T] J. Tate, *Number theoretic background*, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 3–26.

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L-Functions at the Central Critical Point

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The famous formulae

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4};$$

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{(n)^{2k}}=\pi^{2k}\cdot\text{rational number},\quad k\geq 1,\quad \pi=\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}};$$

$$\sum_{\substack{n,m\in\mathbb{Z}\\ (n,m)\neq(0,0)}}\frac{1}{(n+mi)^{4k}}=\Omega^{4k}\cdot\text{rational number},\quad k\geq 1,\quad \Omega=\int_{-1}^{1}\frac{dx}{\sqrt{1-x^4}},$$

discovered by Leibniz, Euler, and Gauss are all special cases of Deligne's general conjecture on the special values of L-functions. The first gives the value of the Dirichlet L-function $L(\chi, s)$ at $s=1$, where χ is the quadratic character of conductor 4 for $\mathbb{Q}$; here Deligne's conjecture predicts that the infinite sum is a rational multiple of π. The second gives twice the value of the Riemann zeta function $\zeta(s)$ at $s=2k\geq 2$, up to a rational factor; Euler also gave an explicit expression for this rational factor. The third gives four times the value of the Hecke L-function $L(\chi^k, s)$ at $s=4k\geq 4$, up to a rational factor, where χ is the unramified Hecke character for $\mathbb{Q}(i)$ which maps an ideal to the 4th power of any generator. Euler's formula was generalized by Siegel to the special values of Hecke L-functions of totally real fields **[Si]**, and Gauss's formula was generalized by Shimura to the special values (up to an algebraic factor) of Hecke L-functions of CM-fields **[S, §3]**.

In this paper, we will briefly recall Deligne's general conjecture on special values of motivic L-functions **[D1]**. For the details, the reader is urged to consult the original article, where the exposition is unsurpassed. We will also describe an extension proposed by Beilinson **[Bel]** and Bloch **[B]**, on the leading term in the Taylor series of the L-function at the center of its

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critical strip. We shall make these conjectures more explicit for the L-series of abelian varieties and review some of the recent progress that has been made on the conjecture of Birch and Swinnerton-Dyer. Finally, we have included a general section on symplectic local root numbers, which play an important role in the study of central critical behavior.

1. The general conjectures

Let M be a pure motive of weight $w(M)$ over $\mathbb{Q}$, and let $L(M, s)$ denote the L-function of M. This is defined in the half-plane $\mathrm{Re}(s) > 1 + \frac{w(M)}{2}$ by an absolutely convergent Euler product; we assume that $L(M, s)$ has a meromorphic continuation to the entire complex plane. Deligne calls the motive M critical if the Archimedean L-factors $L_\infty(M, s)$ and $L_\infty(M^\vee, 1-s)$ are both regular at $s = 0$. In this case, his conjecture relates the value $L(M) = L(M, 0)$ to certain periods of the dual motive $M^\vee$. For simplicity, we only state this conjecture in the case when $w(M) < 0$. Note that when $w(M) < -2$, the point $s = 0$ is in the half-plane of absolute convergence for the Euler product.

Let $I : H_B(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{\mathrm{DR}}(M) \otimes \mathbb{C}$ be the comparison isomorphism coming from de Rham's theorem, let $H_B(M)^+$ be the fixed space of the involution F_∞ acting on $H_B(M)$, and let $F^\cdot$ be the Hodge filtration of $H_{\mathrm{DR}}(M)$. When $w(M) < 0$, Beilinson has observed that the motive M is critical if and only if the composite map

$$(1.1)\quad I^+ : H_B(M)^+ \otimes \mathbb{C} \hookrightarrow H_B(M) \otimes \mathbb{C} \underset{I}{\xrightarrow{\sim}} H_{\mathrm{DR}}(M) \otimes \mathbb{C} \twoheadrightarrow H_{\mathrm{DR}}(M)/F^0 \otimes \mathbb{C}$$

is an isomorphism. In this case, we may define the period invariant $c^+(M) = \det(I^+)$ in $\mathbb{C}^*/\mathbb{Q}^*$ by taking the determinant of I^+ with respect to rational bases of $H_B(M)^+$ and $H_{\mathrm{DR}}(M)/F^0$. Then $c^+(M)$ lies in $\mathbb{R}^*/\mathbb{Q}^*$, and Deligne's conjecture **[D1;** 1.8] is that the ratio $L(M)/c^+(M)$ is a rational number.

For example, if $M = \mathbb{Q}(2k)$ with $k \geq 1$, we have $w(M) = -4k \leq -4$ and $L(M, 0) = \zeta(2k)$. The rational vector spaces $H_B(M)^+ = H_B(M)$ and $H_{\mathrm{DR}}(M) = H_{\mathrm{DR}}/F^0$ each have dimension 1, and $c^+(M) = (2\pi i)^{2k}$ **[D1,** 3.1.3]. In this case, Deligne's conjecture reduces to the second formula of the introduction.

If M is critical and $w(M) \neq -1$, Deligne conjectures **[D1,** 1.3] that $L(M) \neq 0$. If $w(M) = -1$, the point $s = 0$ is the center of the critical strip for $L(M, s)$, and the value $L(M)$ can be zero. For example, if $M = H_1(A)$ for an abelian variety A over $\mathbb{Q}$, Birch and Swinnerton-Dyer **[T1]** have conjectured that the order of $L(M, s)$ at $s = 0$ is equal to the rank of the finitely generated group $A(\mathbb{Q})$. This rank is equal to the dimension of the Ext-group $\mathrm{Ext}^1(\mathbb{Q}, M)$ in the category of 1-motives over $\mathbb{Q}$ **[D4,** §10].

More generally, when $w(M) = -1$, Deligne has suggested that

$$\operatorname{ord}_{s=0} L(M, s) \stackrel{?}{=} \dim \operatorname{Ext}^1(\mathbb{Q}, M) \tag{1.2}$$

where the Ext-group is taken in a category of mixed motives over $\mathbb{Q}$ **[D1**, 4.3]. When $M = H^{2n-1}(X)(n)$ for X projective and smooth over $\mathbb{Q}$, the group $\operatorname{Ext}^1(\mathbb{Q}, M)$ is related to the Chow group $CH^n(X)^0_{\mathbb{Q}}$ of $\mathbb{Q}$-linear combinations of codimension n cycles on X over $\mathbb{Q}$ with trivial class in $H^{2n}(X)(n)$, modulo rational equivalence **[D1**, 4.3].

Beilinson and Bloch have independently pursued this point of view. Assume that $M = H^{2n-1}(X)(n)$; they have (conditionally) defined a $\mathbb{Q}$-bilinear height pairing **[Be 1**, 40.2], **[B]**:

$$\langle\ ,\ \rangle_M : CH^n(X)^0_{\mathbb{Q}} \times CH_{n-1}(X)^0_{\mathbb{Q}} \longrightarrow \mathbb{R}, \tag{1.3}$$

which is equal to the Néron-Tate height when $n = 1$. They conjecture that the real vector spaces $CH^n(X)^0_{\mathbb{R}} = CH^n(X)^0_{\mathbb{Q}} \otimes \mathbb{R}$ and $CH_{n-1}(X)^0_{\mathbb{R}} = CH_{n-1}(X)^0_{\mathbb{Q}} \otimes \mathbb{R}$ are both finite dimensional, of dimension equal to the order r of the function $L(M, s)$ at $s = 0$, and that the pairing $\langle\ ,\ \rangle_M$ of (1.3) is nondegenerate over $\mathbb{R}$. If this is true, one can define the discriminant (or regulator) of the pairing $\det\langle\ ,\ \rangle_M$ in $\mathbb{R}^*/\mathbb{Q}^*$ by taking the determinant of the $(r \times r)$-matrix of its values on rational basis elements.

Let $L^*(M)$ be the first nonzero term in the Taylor expansion of $L(M, s)$ at $s = 0$, so

$$L(M, s) = L^*(M) \cdot s^r + O(s^{r+1}) \quad \text{as } s \to 0. \tag{1.4}$$

Then Bloch and Beilinson conjecture that the ratio $L^*(M)/c^+(M) \cdot \det\langle\ ,\ \rangle_M$ is a rational number. When $r = 0$, $\det\langle\ ,\ \rangle_M = 1$, and this reduces to Deligne's conjecture.

2. Abelian varieties

We consider the conjectures of Deligne, Beilinson, and Bloch in the case when $M = H_1(A)$, for an abelian variety A over $\mathbb{Q}$. In this case, they reduce to the conjecture of Birch and Swinnerton-Dyer **[T1]**. Let $g = \dim A$, so $2g = \operatorname{rank} M$. By Faltings's proof of the Tate conjecture **[F]**, the motive M determines the abelian variety A up to isogeny over $\mathbb{Q}$. Let ${}^tA$ be the dual abelian variety, and fix a polarization: $\varphi : A \to {}^tA$. This isogeny induces an isomorphism of motives $H_1(A) \xrightarrow{\sim} H_1({}^tA)$ and an isomorphism of rational Mordell-Weil groups $A(\mathbb{Q}) \otimes \mathbb{Q} \xrightarrow{\sim} {}^tA(\mathbb{Q}) \otimes \mathbb{Q}$.

The Weil pairing: $H_1(A) \times H_1({}^tA) \longrightarrow H_1(\mathbb{G}_m)$ shows that $H_1(A) \xrightarrow{\sim} H_1({}^tA)^\vee(1)$. Hence $M \simeq M^\vee(1) \simeq H^1(A)(1)$, and the L-function of the motive $H^1(A)$ is given by

$$L(H^1(A), s) = L(M, s - 1). \tag{2.1}$$

The study of $L(H^1(A), s)$ at $s = 1$ is therefore equivalent to the study of $L(M, s)$ at the central critical point $s = 0$.

Let $\langle c_1, \ldots, c_g\rangle$ be a basis for $H_B(M)^+ = H_1(A(\mathbb{C}), \mathbb{Q})^+$, and let $\langle \omega_1, \ldots, \omega_g\rangle$ be a basis for $(H_{\mathrm{DR}}(M)/F^0)^\vee = F^0H_{\mathrm{DR}}(M^\vee) = H^0(A, \Omega^1)$. Then the period determinant is defined by

$$c^+(M) = \det\left(\int_{c_i} \omega_j\right) \quad \text{in } \mathbb{R}^*/\mathbb{Q}^*. \tag{2.2}$$

The Néron-Tate pairing $A(\mathbb{Q}) \times {}^tA(\mathbb{Q}) \longrightarrow \mathbb{R}$ is the canonical height associated to the Poincaré divisor on $A \times {}^tA$ (cf. **[L]**). When combined with the polarization, it induces a symmetric, positive definite bilinear form $\langle\ ,\ \rangle$ on $A(\mathbb{Q})\otimes\mathbb{R}$. Let $\langle e_1, \ldots, e_r\rangle$ be a basis for $A(\mathbb{Q})\otimes\mathbb{Q} = \mathrm{Ext}^1(\mathbb{Q}, M)$; then the regulator is defined by

$$R(M) = \det(\langle e_i, e_j\rangle) \quad \text{in } \mathbb{R}^*/\mathbb{Q}^*. \tag{2.3}$$

The basic conjecture is that $L(M, s)$ has a zero of order $r = \operatorname{rank} A(\mathbb{Q})$ at $s = 0$ and that

$$L^*(M) \equiv c^+(M)\cdot R(M) \pmod{\mathbb{Q}^*}. \tag{2.4}$$

To make any progress on (2.4), one first needs the analytic continuation of the function $L(M, s)$ to a neighborhood of $s = 0$, since the Euler product that defines the L-function is only absolutely convergent when $\mathrm{Re}(s) > \frac{1}{2}$. More precisely, one expects that the function $\Lambda(M, s) = \Gamma_{\mathbb{C}}(s+1)^g L(M, s)$ is entire and satisfies the functional equation

$$\Lambda(M, s) = \pm f(M)^{-s}\cdot\Lambda(M, -s) \tag{2.5}$$

where $f(M) \geq 1$ is the conductor of A **[S-T, §3]**. Fontaine has shown that $f(M) > 1$ whenever $M \neq 0$: there are no abelian varieties of dimension $g \geq 1$ over $\mathbb{Q}$ with everywhere good reduction **[Fo]**.

At the present moment, the analytic continuation and functional equation of $\Lambda(M, s)$ is known only for a very restrictive class of abelian varieties A. For example, the functional equation is known when A either has complex multiplication (over $\overline{\mathbb{Q}}$) or is isogenous to a factor of the Jacobian of a Shimura curve X over $\mathbb{Q}$. We review what is known in the special case when $X = X_0(N)$, the modular curve that classifies elliptic curves with a cyclic N-isogeny.

Let $J_0(N)$ be the Jacobian of $X_0(N)$, and let $J_0(N)'$ be the "new" quotient of $J_0(N)$, whose cotangent space corresponds to the newforms of weight 2 for $\Gamma_0(N)$ **[B-SD, §4]**. We will henceforth *assume* that there is a surjective homomorphism of abelian varieties over $\mathbb{Q}$:

$$J_0(N)' \longrightarrow A. \tag{2.6}$$

This implies that A has "real multiplication" over $\mathbb{Q}$: the semisimple $\mathbb{Q}$-algebra $E = \mathrm{End}_{\mathbb{Q}}(A)\otimes\mathbb{Q} = \mathrm{End}(M)$ is commutative of dimension $g = \dim A$, and $E\otimes\mathbb{R} \simeq \mathbb{R}^g$ is split over $\mathbb{R}$. The abelian variety A is simple over $\mathbb{Q}$ if and only if E is a field. Conversely, a natural generalization of the conjecture of Taniyama and Weil predicts that *every* simple abelian variety B with real multiplication over $\mathbb{Q}$ is a quotient of some $J_0(N)$ (cf. **[Se,** 4.7]).

PROPOSITION 2.7. *Assume that A is a simple quotient of $J_0(N)'$ of dimension g, and let $M = H_1(A)$.*

(1) *The function $L(M, s)$ is entire, and $\Lambda(M, s)$ satisfies the functional equation* (2.5), *with $f(M) = N^g$.*
(2) *If $L(M, 0) \neq 0$, then $A(\mathbb{Q})$ is finite and $L(M) \equiv c^+(M) \pmod{\mathbb{Q}^*}$.*
(3) *If $L(M, 0) = 0$, then $\operatorname{ord}_{s=0} L(M, s) \geq g$.*
(4) *If $\operatorname{ord}_{s=0} L(M, s) = g$, then $A(\mathbb{Q})$ is infinite, the $\mathbb{Q}$-vector space $A(\mathbb{Q}) \otimes \mathbb{Q}$ has dimension g, and $L^*(M) \equiv c^+(M) \cdot R(M) \pmod{\mathbb{Q}^*}$.*
(5) *If $\operatorname{ord}_{s=0} L(M, s) > g$, then $\operatorname{ord}_{s=0} L(M, s) \geq 2g$.*

PROOF. The analytic behavior of $L(M, s)$ follows from the factorization: $L(H^1(A), s) = \prod_{\alpha \in \operatorname{Hom}(E, \mathbb{R})} L(f^\alpha, s)$, where $f = \Sigma a_n q^n$ is the normalized newform with coefficients in $E = \operatorname{End}(A) \otimes \mathbb{Q}$ associated to the factor A of $J_0(N)'$ **[B-SD,** §4]. The fact that $f(A) = N^g$ is due to Carayol **[C]** in the most general case. Most of the other statements were proved in **[GZ,** Chapter V, §§1–2], with some analytic assistance from **[W]** and **[B-F-H]**. However, in part (4), Zagier and I were only able to show that the dimension of $A(\mathbb{Q}) \otimes \mathbb{Q}$ was $\geq g$. The upper bound ($\dim \leq g$) in part (4) and the finiteness of $A(\mathbb{Q})$ in part (2) are due to Kolyvagin (cf. **[G]**, **[K-L]**).

We do not review the progress which has been made on the more precise conjectures of Birch and Swinnerton-Dyer here. Suffice it to say that in the situations of (2) and (4) of Proposition 2.7, Kolyvagin has shown that the Tate-Shafarevich group Ш(A) is finite, with a good estimate on its order **[K]**. When A is an elliptic curve with complex multiplication (over $\overline{\mathbb{Q}}$), Rubin has established more precise results on this order **[R]**, especially when $L(M) \neq 0$.

3. Symplectic root numbers

The functional equation (2.5) conjecturally satisfied by the L-series of the motive $M = H_1(A)$ has a rather simple form. More generally, if M is a motive of weight -1 over $\mathbb{Q}$ with a nondegenerate alternating form

$$M \otimes M \longrightarrow \mathbb{Q}(1), \tag{3.1}$$

the complete L-function $\Lambda(M, s) = L_\infty(M, s) L(M, s)$ should be entire and is conjectured to satisfy a functional equation

$$\Lambda(M, s) = \pm f(M)^{-s} \Lambda(M, -s), \tag{3.2}$$

where $f(M) \geq 1$ is the conductor of M (an integer divisible only by primes that ramify in M).

The Archimedean L-factor $L_\infty(M, s)$ is (in this case) determined simply by the Hodge numbers $h^{pq}(M) = \dim H^{pq}(M)$ occurring in the decomposition $H_B(M) \otimes \mathbb{C} = \oplus H^{pq}(M)$. Using **[D1,** 5.3], we find that

$$L_\infty(M, s) = \prod_{p<0} \Gamma_{\mathbb{C}}(s-p)^{h^{pq}(M)}. \tag{3.3}$$

Since $L_\infty(M, s)$ is regular and nonzero at $s = 0$, the sign $\varepsilon(M) = \pm 1$, which occurs in the conjectural functional equation (3.2) of $\Lambda(M, s)$, is given by the analytic formula

$$\varepsilon(M) = (-1)^{\operatorname{ord}_{s=0} L(M, s)}. \tag{3.4}$$

In this section, we show how the global sign $\varepsilon(M)$ has a canonical decomposition as the product of local symplectic root numbers $\varepsilon_v(M) = \pm 1$, attached to M over the completions $\mathbb{Q}_v$ of $\mathbb{Q}$. More precisely, assuming some compatibility between the ℓ-adic realizations M_ℓ of M over $\mathbb{Q}_v$, we will define a local sign $\varepsilon_v(M)$. Then $\varepsilon_v(M) = +1$ for almost all v, and

$$\varepsilon(M) = \prod_v \varepsilon_v(M) \tag{3.5}$$

is the global sign in the conjectural functional equation for $L(M, s)$ **[D2,** 9.2], **[T2,** 4.5].

For example, at the infinite place we find **[D1,** 5.3]:

$$\varepsilon_\infty(M) = \prod_{p<0} (-1)^{p \cdot h^{pq}(M)}. \tag{3.6}$$

At a finite prime p that does not divide the conductor $f(M)$, we have $\varepsilon_p(M) = +1$. It would be highly desirable to have an interpretation of $\varepsilon_p(M)$ at those finite primes that divide $f(M)$, analogous to **[D3]** for orthogonal root numbers. For some work in this direction, see **[G-P]**.

To define $\varepsilon_p(M)$, we will work in a slightly more general setting. Let k be a non-Archimedean local field and let M_λ be a continuous, λ-adic representation of the Weil group $W(k)$ over E. Here E is a finite extension of $\mathbb{Q}_\ell$, where ℓ is a prime distinct from the residual characteristic of k. Assume that there is a nondegenerate alternating form

$$\langle\ ,\ \rangle : M_\lambda \otimes M_\lambda \longrightarrow E(1) \tag{3.7}$$

that is $W(k)$-equivariant. Following Deligne **[D2]**, we will define a symplectic local root number $\varepsilon(M_\lambda) = \pm 1$. The local constant $\varepsilon_p(M)$ is then defined as the root number of any of the ℓ-adic realizations of M over $k = \mathbb{Q}_p$ (assuming that the resulting sign is independent of ℓ).

We now give the definition of $\varepsilon(M_\lambda)$. The continuous λ-adic representation M_λ of $W(k)$ determines a representation $\rho' = (\rho, N)$ of the Weil-Deligne group $W(k)'$ over E **[T2,** 4.2.1]. Recall that $\rho : W(k) \to \mathrm{GL}(V)$ is a representation of the Weil group on a vector space V over E, whose kernel contains an open subgroup of the inertia group I in $W(k)$, and N is a nilpotent endomorphism of V that satisfies [**T2**, 4.1.2]

$$\rho(w)N\rho(w)^{-1} = \|w\| \cdot N. \tag{3.8}$$

In particular, N commutes with the action of I on V. The alternating form $\langle\ ,\ \rangle$ on M_λ gives a $W(k)$-equivariant nondegenerate alternating form

$$\langle\ ,\ \rangle : V \otimes V \longrightarrow E(1). \tag{3.9}$$

Since N lies in the Lie algebra of the symplectic group of V, it satisfies

$$\langle Nv\, ,\, w\rangle + \langle v\, ,\, Nw\rangle = 0 \tag{3.10}$$

for all $v\, ,\, w \in V$.

Let V^I be the invariants of the inertial group acting on V, and let $V^I_{N=0}$ denote the kernel of N on V^I. Let F be a geometric Frobenius element in $W(k)$; then F generates the quotient $W(k)/I \simeq \mathbb{Z}$ and operates unambiguously on V^I. Since $\|F\| = q^{-1}$, where q is the cardinality of the residue field of k, we have

$$NF = q \cdot FN \quad \text{in } \operatorname{End}(V^I). \tag{3.11}$$

In particular, F stabilizes $V^I_{N=0}$ and acts on the quotient $V^I/V^I_{N=0}$. Finally, we have the formula

$$\langle Fv\, ,\, Fw\rangle = q^{-1}\langle v\, ,\, w\rangle \tag{3.12}$$

for $v\, ,\, w \in V^I$.

Let ψ be a nontrivial additive character of k, and let dx be the unique Haar measure for k that is self-dual for Fourier transform with respect to the duality furnished by ψ. We define the nonzero complex number

$$\varepsilon(M_\lambda) = \varepsilon(V\, ,\, \psi\, ,\, dx) \cdot \det(-F \mid V^I/V^I_{N=0}). \tag{3.13}$$

Here $\varepsilon(V\, ,\, \psi\, ,\, dx)$ is the local constant which Deligne attaches to a representation V of $W(k)$ [**D2**, Theorem 4.1]. A priori, this definition depends on the choice of ψ (which determines dx), but we have the following.

PROPOSITION 3.14. *The local constant $\varepsilon(M_\lambda)$ is independent of the choice of ψ and satisfies $\varepsilon(M_\lambda)^2 = 1$.*

PROOF. If $a \in k^*$, we let ψ_a be the additive character $\psi_a(x) = \psi(ax)$. The associated Haar measure of k is given by $dx_a = \|a\|^{1/2}dx$. Since all nontrivial additive characters are given in this fashion, to show independence we must check that

$$\varepsilon(V\, ,\, \psi_a\, ,\, dx_a) = \varepsilon(V\, ,\, \psi\, ,\, dx).$$

But from [**T2**, 3.4.2–3.4.4] we find that

$$\frac{\varepsilon(V\, ,\, \psi_a\, ,\, dx_a)}{\varepsilon(V\, ,\, \psi\, ,\, dx)} = \|a\|^{(\dim V)/2} \cdot \det V(a) \cdot \|a\|^{-\dim V}.$$

Since (3.9) shows that $\det V = \| \ \|^{(\dim V)/2}$, $\varepsilon(M_\lambda)$ is independent of the choice of ψ.

To see that $\varepsilon(M_\lambda)^2 = 1$, we observe that (3.9) gives an isomorphism from V to $V^\vee(1)$. On the other hand, by **[T2, 3.4.7]** we have the general formula

$$\varepsilon(V, \psi, dx) \cdot \varepsilon(V^\vee(1), \psi_{-1}, dx) = 1.$$

Hence $\varepsilon(V, \psi, dx)^2 = 1$, and we are reduced to showing that $\det(-F \mid V^I/V^I_{N=0}) = \pm 1$. By (3.10), the bilinear form $B(v, w) = \langle Nv, w\rangle$ on V is symmetric; it is nondegenerate on the quotient space $V^I/V^I_{N=0}$. By (3.11) and (3.12) we find that $B(Fv, Fw) = B(v, w)$ on V^I. Hence F is an orthogonal transformation of $V^I/V^I_{N=0}$ and has determinant equal to ± 1. (This argument was suggested by the corresponding argument for orthogonal root numbers in **[D3, 5.5–5.6]**.)

We say that the λ-adic representation M_λ of $W(k)$ is unramified if the subgroup I acts trivially. This implies that I acts trivially on V and that $N = 0$ in $\operatorname{End}(V)$.

PROPOSITION 3.15. *If M_λ is unramified, then $\varepsilon(M_\lambda) = +1$.*

If $M_\lambda = P_\lambda \oplus P_\lambda^\vee(1)$ with $\dim P_\lambda = \frac{1}{2}\dim M_\lambda$, then $\varepsilon(M_\lambda) = \det P_\lambda(-1)$, where $\det P_\lambda$ is viewed as a character of k^* by local class field theory.

PROOF. When M_λ is unramified, V is the direct sum of unramified characters χ of $W(k)$. Since

$$\varepsilon(V_1 \oplus V_2, \psi, dx) = \varepsilon(V_1, \psi, dx) \cdot \varepsilon(V_2, \psi, dx)$$

and $\varepsilon(\chi, \psi, dx) = +1$ when both χ and ψ are unramified **[T2, 3.2.6.1]**, we have $\varepsilon(M_\lambda) = +1$.

Let U be the representation of $W(k)'$ associated to the λ-adic representation P_λ. Then by **[T2, 3.4.4 and 3.4.7]** we have

$$\begin{aligned}\varepsilon(V, \psi, dx) &= \varepsilon(U, \psi, dx) \cdot \varepsilon(U^\vee(1), \psi, dx)\\ &= \det U(-1).\end{aligned}$$

It suffices to show that

$$\det(-F \mid U^I/U^I_{N=0}) \cdot \det(-F \mid U^\vee(1)^I/U^\vee(1)^I_{N=0}) = +1.$$

To do this, one can imitate the proof of Proposition 3.14 to construct a nondegenerate F-invariant duality between the quotient spaces $U^I/U^I_{N=0}$ and $U^\vee(1)^I/U^\vee(1)^I_{N=0}$. We leave the details to the reader.

BIBLIOGRAPHY

[Be1] A. Beilinson, *Height pairing between algebraic cycles*, in Current Trends in Arithmetic Algebraic Geometry, Contemp. Math., vol. 67, 1987, pp. 1–24.

[Be2] ——, *Higher regulators and values of L-functions*, Soviet Math. **30** (1985), 2036–2070.

[B] S. Bloch, *Height pairings for algebraic cycles*, J. Proc. Appl. Algebra **34** (1984), 119–145.

[B-F-H] D. Bump, S. Friedburg, and J. Hoffstein, *Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives*, Ann. of Math. (2) **131** (1990), 53–127.

[B-SD] B. J. Birch and H. P. F. Swinnerton-Dyer, *Elliptic curves and modular functions*, Modular Functions of One Variable IV, Lecture Notes in Math., vol. 476, Springer, New York, 1975, pp. 2–32.

[C] H. Carayol, *Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup (4) **19** (1986), 409–468.

[D1] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 313–346.

[D2] ——, *Les constantes des équations fonctionnelles des fonctions L*, Modular Functions of One Variable II, Lecture Notes in Math., vol. 349, Springer, New York, 1973, pp. 501–595.

[D3] ——, *Les constantes locales de l'équation fonctionnelle de la fonction L d'Artin d'une représentation orthogonale*, Invent. Math. **35** (1976), 299–316.

[D4] ——, *Théorie de Hodge*. III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.

[F] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366; Erratum ibid. **75** (1984), 381.

[Fo] J.-M. Fontaine, *Il n'y a pas de variété abélienne sur $\mathbb{Z}$*, Invent. Math. **81** (1985), 515–538.

[G] B. Gross, *Kolyvagin's work on modular elliptic curves*, L-functions and arithmetic, London Math. Soc. Lecture Notes Ser., vol. 153, Cambridge Univ. Press, London and New York, 1991, pp. 235–256.

[G-P] B. Gross and D. Prasad, *On the decomposition of a representation of $SO(n)$ when restricted to $SO(n-1)$*, Canad. Math. J. **44** (5) (1992), 974–1002.

[G-Z] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.

[K] V. A. Kolyvagin, *Finiteness of $E(\mathbb{Q})$ and $Ш(E/\mathbb{Q})$ for a class of Weil curves*, Izv. Akad. Nauk SSSR **52** (1988).

[K-L] V. A. Kolyvagin and D. Y. Logachev, *Finiteness of the Shafarevich-Tate group and group of rational points for some modular abelian varieties*, Algebra and Analysis USSR **5** (1989).

[L] S. Lang, *Les formes bilinéaires de Néron et Tate*, Sém. Bourbaki 274 (1964).

[R] K. Rubin, *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, Invent. Math. **103** (1991), 25–68.

[Se] J.-P. Serre, *Sur les représentations modulaires de degré 2 de Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$*, Duke Math. J. **54** (1987), 179–230.

[S-T] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

[S] G. Shimura, *On some arithmetic properties of modular forms of one and several complex variables*, Ann. of Math. (2) **102** (1975), 491–515.

[Si] C. L. Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nach. Akad. Wiss. Göttingen **10** (1969), 87–102.

[T1] J. Tate, *On the conjecture of Birch and Swinnerton-Dyer and a geometric analogue*, Sém. Bourbaki, Exposé 306, in Dix Exposés sur la Théorie des Schémas, North-Holland, Amsterdam, 1968.

[T2] ——, *Number theoretic background*, Proc. Sympos. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 3–26.

[W] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), 173–242.

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Beilinson's Conjectures

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ABSTRACT. We give a survey of Beilinson's conjectures about special values of L-functions, with emphasis on the underlying philosophy of mixed motives and motivic cohomology.

Introduction

In his seminal paper [1], Beilinson formulated far-reaching conjectures about values of motivic L-functions at integers and produced a compelling body of evidence in their favour by making ingenious calculations in several special cases. The main gist of [1] was a construction of "higher regulators", expected to explain these L-values in the same spirit as the (slightly modified) classical Dirichlet regulator

$$r : \mathcal{O}_F^* \oplus \mathbf{Z} \to \mathbf{R}^{r_1+r_2}$$

does for the zeta function of a number field F at $s = 0$ (resp. $s = 1$). In this case, $\zeta_F(s)$ satisfies a functional equation relating its values at s and $1-s$. It has a simple pole at $s = 1$ and a zero of order $r_1 + r_2 - 1$ at $s = 0$. Its leading Taylor coefficient at $s = 0$ is equal to

$$\lim_{s\to 0} \zeta_F(s) s^{-(r_1+r_2-1)} = -\frac{\sharp\mathrm{Pic}(\mathcal{O}_F)\cdot R}{\sharp(\mathcal{O}_F^*)_{\mathrm{tors}}}, \tag{0.1}$$

where R denotes the covolume of the lattice $\mathrm{Im}(r)$ in $\mathbf{R}^{r_1+r_2}$.

The quest for higher regulators, extending (0.1) to other values of $\zeta_F(s)$, has been initiated by Lichtenbaum [47]. He observed that for $m > 1$, the order d_m of vanishing of $\zeta_F(s)$ at $s = 1 - m$ is equal to the dimension of the higher K-group $K_{2m-1}(F) \otimes \mathbf{Q}$. This led to a conjecture that the leading coefficient

$$\lim_{s\to 1-m} \zeta_F(s)(s+m-1)^{-d_m}$$

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should be equal, up to a rational factor, to the covolume of $\mathrm{Im}(r_m)$ for a certain map

$$r_m : K_{2m-1}(F) \to \mathbf{R}^{d_m}.$$

This conjecture has been proved by Borel **[14, 15]**, for a slightly modified version of the regulator map r_m.

The next step was taken by Bloch **[6, 7]**, who defined a regulator

$$r : K_2(E) \to H^1(E(\mathbf{C}), \mathbf{R})$$

for elliptic curves $E_{/\mathbf{C}}$ and verified that r computes the value $L(E, 2)$ for curves with complex multiplication defined over $\mathbf{Q}$. Later, Bloch **[8]** defined a regulator

$$K_2(X) \to H^1(X(\mathbf{C}), \mathbf{C}^*)$$

for any curve $X_{/\mathbf{C}}$.

Beilinson defines, for a given quasi-projective variety $X_{/\mathbf{Q}}$, its motivic cohomology $H^{\cdot}_{\mathscr{M}}(X, \mathbf{Q}(*))$ as a suitable piece of K-theory of X. Let X be a smooth projective variety over $\mathbf{Q}$. The L-function $L(h^i(X), s)$, associated to the i^{th} cohomology of X, is expected to satisfy a functional equation relating its values at s and $i+1-s$. Suppose that n is an integer greater than $1 + i/2$. Beilinson defines a regulator map

$$r : H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n)) \to H^{i+1}_{\mathscr{D}}(X_{/\mathbf{R}}, \mathbf{R}(n)) \tag{0.2}$$

from the motivic cohomology of X into Deligne cohomology. The significance of the latter lies in the fact that its dimension is equal to the order of vanishing of $L(h^i(X), s)$ at $i+1-n$. Deligne cohomology also admits a natural $\mathbf{Q}$-structure, and Beilinson conjectures that $\det(r) \in \mathbf{R}^*/\mathbf{Q}^*$ with respect to this $\mathbf{Q}$-structure is equal to the leading coefficient of the L-function at $i+1-n$. At the central point $n = (i+1)/2$, resp. near the central point $n = i/2 + 1$, the conjecture has to be modified.

In a letter to Soulé **[22]**, Deligne suggested a motivic formulation of the regulator 0.2, writing it as a "Hodge realization"

$$r : \mathrm{Ext}^1(\mathbf{Q}(0), M) \to \mathrm{Ext}^1(\mathbf{R}(0), M_{\mathrm{Hodge}}),$$

where the first Ext is the group of "motivic extensions" of the trivial motive by $M = h^i(X)(n)$ and the second one is the group of extensions in a suitable category of Hodge structures. Deligne also introduced another $\mathbf{Q}$-structure on the target of r, related to the value of $L(h^i(X), s)$ at $s = n$.

In **[3, 4]**, Beilinson followed this suggestion and put his conjectures about special values of L-functions into a general perspective of mixed motives and motivic sheaves. In this context, $H^i_{\mathscr{M}}(X, \mathbf{Q}(j))$ are expected to form a universal "absolute", or "arithmetic", cohomology theory of X, as opposed to Grothendieck's $h^i(X)$, which should provide only "geometric" information about $X \times_{\mathbf{Q}} \overline{\mathbf{Q}}$.

Following this motivic thread, Scholl **[63]** proposed a unified formulation of Beilinson's conjectures at all integers (including the central and near central points) as Deligne's conjecture **[21]** for critical mixed motives.

In a separate development, Bloch and Kato **[12]** formulated a conjecture about the precise value of $L(h^i(X), n)$, eliminating the undetermined rational factor in Beilinson's approach. Recently, Fontaine and Perrin-Riou **[32, 33]** found a common generalization of the conjectures of Bloch-Kato and Scholl.

Apart from Beilinson's original papers **[1–4]**, there exist excellent surveys of his conjectures **[29, 53, 56, 61, 67]**. For geometric aspects of the conjectures, **[43]** is indispensable.

This survey attempts to explain not only the K-theoretic formulation of the conjectures but also the underlying motivic intuition. The reader will have no trouble in distinguishing real mathematical statements from a mere wishful thinking (which prevails) simply by counting frequency of expressions "should", "is expected", and the like.

1. Pure motives and realizations

(1.1) We first recall basic notions of (pure) motives. To a smooth projective variety X, defined over a number field K, and integers $i \geq 0$, $n \in \mathbf{Z}$, one hopes to associate a "motive" $M = h^i(X)(n)$ (pure of weight $w = i - 2n$), being a universal cohomology group of X. In §§1–2, M will intervene only through its realizations, namely:

- étale realizations (for each prime number ℓ)

$$M_\ell = H^i((X \times_K \overline{K})_{\text{ét}}, \mathbf{Q}_\ell)(n),$$

a finite-dimensional ℓ-adic (continuous) representation of $G(\overline{K}/K)$, pure of weight w. The last condition has the following meaning: there is a finite set S of places at which X has bad reduction. If $v \notin S$ is non-Archimedean and prime to ℓ, then M_ℓ is unramified at v and all eigenvalues of the geometric Frobenius Fr_v on M_ℓ are algebraic numbers with absolute value $(Nv)^{w/2}$ (by **[20]**).

- Betti realizations (for each embedding $\sigma : K \hookrightarrow \mathbf{C}$)

$$M_{\sigma,\mathrm{B}} = H^i((X \times_{K,\sigma} \mathbf{C})(\mathbf{C}), \mathbf{Q}(n)),$$

a pure $\mathbf{Q}$-Hodge structure of weight w. If σ is a real embedding, then the action of the complex conjugation $c \in G(\mathbf{C}/\mathbf{R})$ on both $(X \times_{K,\sigma} \mathbf{C})(\mathbf{C})$ and $\mathbf{Q}(n) = (2\pi i)^n\mathbf{Q}$ induces an involution ϕ_σ on $M_{\sigma,\mathrm{B}}$ such that $\phi_\sigma \otimes c$ preserves the Hodge decomposition

$$M_{\sigma,\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=w} H^{p,q}.$$

- de Rham realization

$$M_{\mathrm{dR}} = H^i(X_{\mathrm{Zar}}, \Omega^{\cdot}_{X/K})(n),$$

a finite-dimensional K-vector space with a decreasing filtration

$$F^k M_{\mathrm{dR}} = H^i(X_{\mathrm{Zar}}, \Omega^{\geq k+n}_{X/K}).$$

(1.2) There are standard comparison isomorphisms between different realizations:

- $$I_\sigma : M_{\sigma,\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{K,\sigma} \mathbf{C}$$

("de Rham theorem"), under which $(F^k M_{\mathrm{dR}}) \otimes \mathbf{C}$ corresponds to $\bigoplus_{p \geq k} H^{p,q}$. If σ is a real embedding, then $\phi_\sigma \otimes c$ corresponds to the action of $1 \otimes c$ on the right-hand side.

- $$I_{\ell,\overline{\sigma}} : M_{\sigma,\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \xrightarrow{\sim} M_\ell$$

(depending on the choice of an extension $\overline{\sigma} : \overline{K} \hookrightarrow \mathbf{C}$ of σ). For a real embedding σ, the action of $\phi_\sigma \otimes 1$ corresponds to the automorphism $\overline{\sigma}^*(c) \in G(\overline{K}/K)$ induced on $\overline{K}$ by c (via $\overline{\sigma}$).

The common dimension

$$\mathrm{rk}(M) := \dim_{\mathbf{Q}}(M_{\sigma,\mathrm{B}}) = \dim_K(M_{\mathrm{dR}}) = \dim_{\mathbf{Q}_\ell}(M_\ell)$$

is called the rank of M (over K).

(1.3) All constructions of linear algebra (duals, tensor products, ...) apply to M, at least on the level of realizations. In particular,

$$M = h^i(X) \otimes \mathbf{Q}(1)^{\otimes n},$$

where the Tate motive $\mathbf{Q}(1) = h^2(\mathbf{P}^1_K)^\vee$ has realizations

$$\begin{aligned}
\mathbf{Q}(1)_\ell &= \varprojlim_k (\mu_{\ell^k}(\overline{K})) \otimes \mathbf{Q}; \\
\mathbf{Q}(1)_{\mathrm{B}} &= (2\pi i)\mathbf{Q}, \quad \text{Hodge type } (-1,-1), \ \phi_\sigma = -1; \\
\mathbf{Q}(1)_{\mathrm{dR}} &= K, \quad F^0 = 0, \ F^{-1} = K;
\end{aligned}$$

By Poincaré duality and the hard Lefschetz theorem (true in all realizations), we have

$$\begin{aligned}
M^\vee(1) &= h^i(X)^\vee(1-n) = h^{2d-i}(X)(d+1-n) \\
&= h^i(X)(i+1-n) = M(w+1),
\end{aligned} \tag{1.3.1}$$

where d is the dimension of X and $w = i - 2n$ the weight of M.

There is also an operation of restriction of scalars for motives:

$$R_{K/\mathbf{Q}}(M) = h^i(X_{/\mathbf{Q}})(n),$$

where $X_{/\mathbf{Q}}$ is X viewed as a $\mathrm{Spec}(\mathbf{Q})$-scheme.

(1.4) The local L-factor of M at a non-Archimedean place v of K is defined as

$$L_v(M, s) = \det(1 - Fr_v(Nv)^{-s} \mid M_\ell^{I_v})^{-1} \qquad (\ell \nmid Nv)$$

(where I_v is the inertia group of v), conjecturally independent of ℓ. For $v \notin S$, $L_v(M, s)$ is indeed independent of ℓ by **[20]** and all its poles have real part equal to $\mathrm{Re}(s) = w/2$. For $v \in S$, independence of ℓ is not known; if true, purity conjecture for monodromy filtration then predicts that all poles of $L_v(M, s)$ have real part $\mathrm{Re}(s) = w/2, (w-1)/2, \ldots, (w-i)/2$ (see **[40]**).

Local L-factors satisfy the following relations:

$$\begin{aligned} L_v(M(m), s) &= L_v(M, s+m), \\ L_p(R_{K/\mathbf{Q}}(M), s) &= \prod_{v|p} L_v(M, s), \qquad (1.4.1) \\ L_v(M_1 \oplus M_2) &= L_v(M_1, s) L_v(M_2, s). \end{aligned}$$

For an Archimedean place v, corresponding to an embedding $\sigma : K \hookrightarrow \mathbf{C}$, $L_v(M, s)$ depends only on the real Hodge structure $H = M_{\sigma, \mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{R}$. The relations (1.4.1) are satisfied for Archimedean places as well, and they determine $L_v(M, s)$ for all M, once they are known for three basic Hodge structures (for a real place v):

For M of rank 2 with H of Hodge type $(k, 0) + (0, k)$ with $k > 0$, $L_v(M, s) = \Gamma_{\mathbf{C}}(s)$; for M of rank 1 with H of Hodge type $(0, 0)$, $L_v(M, s) = \Gamma_{\mathbf{R}}(s)$ (resp. $\Gamma_{\mathbf{R}}(s+1)$), if ϕ_σ acts on H by $+1$ (resp. -1). Here we use the standard notation

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

The total L-factor at infinity will be denoted by

$$L_\infty(M, s) = \prod_{v|\infty} L_v(M, s).$$

See **[26–28]** for a unified definition of L-factors at all places.

(1.5) Assuming that L-factors at places of bad reduction are defined and behave as expected (i.e., the location of their poles is that predicted by the purity conjecture), then the L-function

$$L(M, s) = \prod_{v \nmid \infty} L_v(M, s),$$

a priori only a formal Dirichlet series with rational coefficients, is absolutely convergent (and without zeroes) for $\mathrm{Re}(s) > 1 + w/2$.

The full L-function

$$\Lambda(M, s) = L_\infty(M, s) L(M, s)$$

conjecturally satisfies a functional equation

$$\Lambda(M, s) = \varepsilon(M, s)\Lambda(M^\vee(1), -s)$$

with an ε-factor of the form $\varepsilon(M, s) = a \cdot b^s$ (see **[28]**). We have, of course,

$$\Lambda(M^\vee(1), -s) = \Lambda(M, w+1-s)$$

by (1.3.1) and (1.4.1), so the functional equation becomes

$$\Lambda(M, s) = \varepsilon(M, s)\Lambda(w + 1 - s). \tag{1.4.2}$$

Suppose that we are interested in the behaviour of $L(h^i(X), s)$ at an integer $s = n$. Restricting the scalars to $\mathbf{Q}$, applying a Tate twist by $\mathbf{Q}(n)$ and using (1.4.1), we may assume that $K = \mathbf{Q}$ and that the point of our interest is $s = 0$. Using the functional equation (1.4.2) (which we *assume* to hold), it is sufficient to treat only the case when $s = 0$ lies to the right of the central point $(w+1)/2$ (or coincides with it), which happens iff $w \le -1$. In fact,

$$\begin{aligned} w = -1 &\Leftrightarrow s = 0 \text{ is the central value } (w+1)/2, \\ w = -2 &\Leftrightarrow s = 0 \text{ is the near central value } w/2 + 1, \\ w \le -3 &\Leftrightarrow s = 0 \text{ is in the convergence region}. \end{aligned}$$

From now on, we shall assume that X is a smooth projective variety over $\mathbf{Q}$ and that $M = h^i(X)(n)$ is of weight $w = i - 2n \le -1$. Since $\mathbf{Q}$ admits only one embedding $\infty : \mathbf{Q} \hookrightarrow \mathbf{C}$, we shall often drop it from the notation.

(1.6) An easy calculation, based on basic properties of the Gamma function, shows that

$$\begin{aligned} -\operatorname{ord}_{s=0} L_\infty(M^\vee(1), s) &= \sum_{0>p>q} \dim(H^{p,q})(+\dim(H^{w/2,w/2})^-) \\ &= \dim_K(M_{\mathrm{dR}}/F^0) - \dim_{\mathbf{Q}}(M_{\mathrm{B}}^+), \\ \operatorname{ord}_{s=0} L_\infty(M, s) &= 0 \end{aligned} \tag{1.6.1}$$

(where the $\pm$ superscript denotes the (± 1)-eigenspace for ϕ_∞) and that leading terms of the Taylor expansions of L_∞ at $s = 0$ satisfy

$$\frac{L_\infty^*(M, 0)}{L_\infty^*(M^\vee(1), 0)} \in (2\pi)^{w\cdot \mathrm{rk}(M)/2 + \dim(M_{\mathrm{B}}^-)}\mathbf{Q}^*. \tag{1.6.2}$$

Here we define $f^*(a) = \lim_{z\to a}(z-a)^{-r} f(z) = b$, where $r = \operatorname{ord}_{z=a} f(z)$ (for a function f meromorphic in a neighbourhood of $a \in \mathbf{C}$).

2. Deligne's period map

(2.1) Under the comparison map $I_\infty : M_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{\mathbf{Q}} \mathbf{C}$, $M_{\mathrm{dR}} \otimes \mathbf{R}$ corresponds to $(M_{\mathrm{B}}^+ \otimes \mathbf{R}) \oplus (M_{\mathrm{B}}^- \otimes \mathbf{R}(-1))$. Deligne's period map is the induced map

$$\alpha_M : M_{\mathrm{B}}^+ \otimes \mathbf{R} \to (M_{\mathrm{dR}}/F^0) \otimes \mathbf{R},$$

first introduced in **[21]**. As M has negative weight by assumption, we have

$$\operatorname{Ker}(\alpha_M) \subseteq (F^0 \cap \overline{F}^0)(M_{\mathrm{B}} \otimes \mathbf{C}) = 0.$$

We may, therefore, reformulate (1.6.1) as

$$\begin{aligned} \operatorname{ord}_{s=0} L_\infty(M, s) &= \dim_{\mathbf{R}} \operatorname{Ker}(\alpha_M) = 0, \\ -\operatorname{ord}_{s=0} L_\infty(M^\vee(1), s) &= \dim_{\mathbf{R}} \operatorname{Coker}(\alpha_M). \end{aligned} \tag{2.1.1}$$

The isomorphism $M(-1)^+_{\mathrm{B}} \otimes \mathbf{R} = M^-_{\mathrm{B}} \otimes \mathbf{R}(-1) \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbf{R}/I_\infty(M^+_{\mathrm{B}} \otimes \mathbf{R})$ also induces a map

$$\beta_M : \mathrm{Ker}(\alpha_{M(-1)}) \to \mathrm{Coker}(\alpha_M).$$

For $w \neq -2$, the domain of β_M vanishes; for $w = -2$, β_M is injective for the same reason as α_M is.

(2.2) The **Q**-structures M^+_{B} and M_{dR}/F^0, on the domain and target of α_M respectively, define a natural **Q**-structure $\mathscr{D}(M)$ on the real vector space $\det(\mathrm{Coker}(\alpha_M))$ (where by $\det(V)$ we denote the highest exterior power of a vector space V).

Deligne **[21]** calls the motive M *critical*, if the period map α_M is an isomorphism. If this is the case, he defines the period of M as

$$c^+(M) = \det(\alpha_M) \in \mathbf{R}^*/\mathbf{Q}^*,$$

the determinant being taken with respect to the **Q**-structures M^+_{B}, M_{dR}/F^0. Of course, $\mathrm{Coker}(\alpha_M)$ vanishes for such M, so its determinant is canonically isomorphic to $\mathbf{R}$, and the **Q**-structure $\mathscr{D}(M)$ is equal to $c^+(M)^{-1}\cdot\mathbf{Q}$.

In general, there is a natural commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to (F^0M_{\mathrm{dR}} \oplus M^+_{\mathrm{B}}) \otimes \mathbf{R} & \to & (M^-_{\mathrm{B}}(-1) \oplus M^+_{\mathrm{B}}) \otimes \mathbf{R} & \to & \mathrm{Ker}(\alpha_{M^\vee(1)})^\vee & \to 0 \\ \| & & \downarrow \wr I_\infty & & \downarrow \wr & \\ 0 \to (F^0M_{\mathrm{dR}} \oplus M^+_{\mathrm{B}}) \otimes \mathbf{R} & \to & M_{\mathrm{dR}} \otimes \mathbf{R} & \to & \mathrm{Coker}(\alpha_M) & \to 0, \end{array}$$

which defines a canonical isomorphism

$$\mathrm{Coker}(\alpha_M)^\vee \xrightarrow{\sim} \mathrm{Ker}(\alpha_{M^\vee(1)}). \tag{2.2.1}$$

Similarly, $\mathrm{Coker}(\alpha_{M^\vee(1)}) \xrightarrow{\sim} \mathrm{Ker}(\alpha_M)^\vee = 0$, and thus $\det(\mathrm{Ker}(\alpha_{M^\vee(1)}))$ also has a natural **Q**-structure, inherited from the **Q**-structures $M^\vee(1)^+_{\mathrm{B}}$ and $M^\vee(1)_{\mathrm{dR}}/F^0$ on the domain and target of $\alpha_{M^\vee(1)}$. The above diagram shows that this **Q**-structure on $\det(\mathrm{Ker}(\alpha_{M^\vee(1)}))$ corresponds, via (2.2.1), to the **Q**-structure

$$\mathscr{B}(M) = (2\pi i)^{-\dim(M^-_{\mathrm{B}})}\delta(M)\mathscr{D}(M)$$

on $\det(\mathrm{Coker}(\alpha_M))$. Here $\delta(M) \in \mathbf{C}^*/\mathbf{Q}^*$ denotes the determinant of I_∞ with respect to the **Q**-structures M_{B}, M_{dR}.

The calculation in **[21,** 5.6], based on a conjectural description of motives of rank 1 over **Q**, shows that

$$(2\pi i)^{-\dim(M^-_{\mathrm{B}})}\delta(M) \in (2\pi)^{-\dim(M^-_{\mathrm{B}})-w\cdot\mathrm{rk}(M)/2}\varepsilon(M, 0)\mathbf{Q}^*;$$

hence, in view of (1.6.2),

$$L^*(M^\vee(1), 0)\mathscr{B}(M) = L^*(M, 0)\mathscr{D}(M). \tag{2.2.2}$$

(2.3) We have seen that (by 1.4.2 and 2.1.1)

$$\dim_{\mathbf{R}}(\mathrm{Coker}(\alpha_M)) = \mathrm{ord}_{s=0}\, L(M^\vee(1), s) - \mathrm{ord}_{s=0}\, L(M, s)$$

(where the last term vanishes if $w \leq -3$). This makes $\mathrm{Coker}(\alpha_M)$ a natural candidate for the target of a regulator map, which should "explain" the values $L^*(M, 0)$ and $L^*(M^\vee(1), 0)$. What we need is a **Q**-vector space A_M "of arithmetic nature" (a generalization of $\mathbf{Q} \oplus \mathscr{O}_F^* \otimes \mathbf{Q}$ for $\zeta_F(s)$ at $s = 0$) and regulator map $r : A_M \to \mathrm{Coker}(\alpha_M)$, inducing an isomorphism after tensoring with $\mathbf{R}$. The value $L^*(M, 0) \in \mathbf{R}^*/\mathbf{Q}^*$ (resp. $L^*(M^\vee(1), 0)$) should then be equal to the determinant of r with respect to the **Q**-structure $\mathscr{D}(M)$ (resp. $\mathscr{B}(M)$) on $\det(\mathrm{Coker}(\alpha_M))$.

(2.4) As a first step toward the construction of a regulator map we give an interpretation of $\mathrm{Coker}(\alpha_M)$ in terms of Hodge theory.

For a subring $A \subset \mathbf{R}$, denote by $\mathscr{MH}_A$ (resp. $\mathscr{MH}_A^+$) the category of mixed A-Hodge structures (resp. mixed A-Hodge structures with infinite Frobenius, i.e., an involution ϕ_∞ compatible with the weight filtration and such that $\phi_\infty \otimes c$ preserves the Hodge filtration). Both $\mathscr{MH}_A$ and $\mathscr{MH}_A^+$ are tensor categories with a unit object $\mathbf{1} = A(0)$.

For $H \in \mathrm{Ob}\,\mathscr{MH}_\mathbf{R}$, the complex (in degrees 0 and 1)

$$W_0H \oplus F^0(W_0H)_\mathbf{C} \xrightarrow{i_W - i_F} (W_0H)_\mathbf{C} \tag{2.4.1}$$

(where i_W and i_F denote the obvious inclusions) represents $R\,\mathrm{Hom}(\mathbf{R}(0), H)$, i.e., there are (natural) isomorphisms

$$\mathrm{Hom}_{\mathscr{MH}_\mathbf{R}}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0H \cap F^0H_\mathbf{C}, \tag{2.4.2}$$

$$\mathrm{Ext}^1_{\mathscr{MH}_\mathbf{R}}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0H\backslash(W_0H)_\mathbf{C}/F^0(W_0H)_\mathbf{C} \tag{2.4.3}$$

and higher Ext^i vanish for $i > 1$. For a proof of this basic fact of life in $\mathscr{MH}_\mathbf{R}$ we refer the reader to **[3, 16, 17]**. Note that (2.4.2) is obvious: we associate to a morphism $f : \mathbf{R}(0) \to H$ in $\mathscr{MH}_\mathbf{R}$ the value $f(1)$. Let us indicate how the morphism in (2.4.3) is defined: given an extension

$$0 \to H \to E \to \mathbf{R}(0) \to 0$$

in $\mathscr{MH}_\mathbf{R}$, we can lift $1 \in \mathbf{R}(0)$ to $1_W \in W_0E$ (defined modulo W_0H) and to $1_F \in F^0(W_0E)_\mathbf{C}$ (defined modulo $F^0(W_0H)_\mathbf{C}$). The class of $1_F - 1_W$ in

$$W_0H\backslash(W_0H)_\mathbf{C}/F^0(W_0H)_\mathbf{C}$$

is then well defined, depends only on the extension class of E and is additive in E.

(2.5) Similarly, for $H \in \mathrm{Ob}\,\mathscr{MH}_\mathbf{R}^+$, we just take $\phi_\infty \otimes c$-invariants: $R\,\mathrm{Hom}(\mathbf{R}(0), H)$ is represented by

$$W_0H^+ \oplus F^0(W_0H_{\mathrm{dR}}) \to W_0H_{\mathrm{dR}}; \tag{2.5.1}$$

hence,

$$\mathrm{Hom}_{\mathscr{MH}_\mathbf{R}^+}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0H^+ \cap F^0H_\mathbf{C}, \tag{2.5.2}$$

$$\mathrm{Ext}^1_{\mathscr{MH}_\mathbf{R}^+}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0H^+\backslash W_0H_{\mathrm{dR}}/F^0(W_0H_{\mathrm{dR}}), \tag{2.5.3}$$

where $H_{\mathrm{dR}} = H_{\mathbf{C}}^{\phi_\infty \otimes c = 1}$, $H^{\pm} = H^{\phi_\infty = \pm 1}$.

In particular, for $H = M_{\mathrm{B}} \otimes \mathbf{R} \in \mathrm{Ob}\mathscr{MH}_{\mathbf{R}}^{+}$, we have $H = W_0 H$ and $H_{\mathrm{dR}} = M_{\mathrm{dR}} \otimes \mathbf{R}$; hence,

$$Ext^1_{\mathscr{MH}_{\mathbf{R}}^+}(\mathbf{R}(0), H) \xrightarrow{\sim} M_B^+ \otimes \mathbf{R} \backslash M_{dR} \otimes \mathbf{R}/F^0 M_{dR} \otimes \mathbf{R} = Coker(\alpha_M).$$

(2.6) Having interpreted $\mathrm{Coker}(\alpha_M)$ as an Ext-group in the category of mixed Hodge structures, it is quite tempting to make a guess as to what the regulator map should be: simply the canonical map ("Hodge realization")

$$\mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), M) \to \mathrm{Ext}^1_{\mathscr{MH}_{\mathbf{R}}^+}(\mathbf{R}(0), M_{\mathrm{B}} \otimes \mathbf{R}), \tag{2.6.1}$$

where the first Ext-group is computed in a suitable category of "mixed motives", which extends Grothendieck's category of pure motives over $\mathbf{Q}$. In the following two sections we shall try to make this idea more precise.

3. Arithmetic vs. geometric cohomology

(3.1) The group $\mathrm{Coker}(\alpha_M)$ (where, as before, $M = h^i(X)(n)$ for a smooth projective variety $X_{/\mathbf{Q}}$ and $w = i - 2n \leq -1$) can be obtained as a composition of two cohomological functors: $H_B^i(X(\mathbf{C}), -)$ (applied to $\mathbf{R}(n)$) and $\mathrm{Ext}^1_{\mathscr{MH}_{\mathbf{R}}^+}(\mathbf{R}(0), -)$. This suggests that $\mathrm{Coker}(\alpha_M)$ is, in fact, equal to $H_?^{i+1}(X, \mathbf{R}(n))$ in some fancy cohomology theory $H_?^{\cdot}$, and that the isomorphism

$$H_?^{i+1}(X, \mathbf{R}(n)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathscr{MH}_{\mathbf{R}}^+}(\mathbf{R}(0), H_B^i(X(\mathbf{C}), \mathbf{R}(n)))$$

comes from the standard spectral sequence for composition of derived functors.

(3.2) This is indeed the case and the corresponding cohomology theory fits into the following general framework:

- Let $\mathscr{T}$ be a tensor category with a unit object $\mathbf{1}$ and Tate twists $- \mapsto -(j)$. Set $A = \mathrm{End}_{\mathscr{T}}(\mathbf{1})$,

$$\Gamma(\mathscr{T}, -) = \mathrm{Hom}_{\mathscr{T}}(\mathbf{1}, -) : \mathscr{T} \to (A - \mathrm{mod}).$$

- Let $\mathscr{V}$ be a sufficiently large subcategory of the category of schemes of finite type over a given field F (e.g., containing all smooth quasi-projective varieties over F).
- Suppose that for each $X \in \mathrm{Ob}\mathscr{V}$ and $j \in \mathbf{Z}$, there is a complex (contravariant in X)

$$\underline{R\Gamma}(X, j) = \underline{R\Gamma}(X, 0)(j) \in \mathrm{Ob}\, D^b(\mathscr{T}),$$

whose cohomology

$$\underline{H}^p(X, j) = \underline{H}^p(X, 0)(j) \in \mathrm{Ob}\mathscr{T}$$

are "geometric cohomology groups" of X.

Applying the functor $R\Gamma(\mathcal{T}, -) : D^b(\mathcal{T}) \to D^+(A - \text{mod})$, we get a complex

$$R\Gamma_{\mathcal{T}}(X, j) = R\Gamma(\mathcal{T}, \underline{R\Gamma}(X, j)) \in \mathrm{Ob}\, D^+(A - \text{mod})$$

with cohomology groups

$$H^p_{\mathcal{T}}(X, j) \in \mathrm{Ob}\,(A - \text{mod}),$$

"arithmetic", or "absolute" cohomology groups of X.

The spectral sequence, referred to in (3.1), is then

$$\mathrm{Ext}^p_{\mathcal{T}}(\mathbf{1}, \underline{H}^q(X, j)) \Longrightarrow H^{p+q}_{\mathcal{T}}(X, j). \tag{3.2.1}$$

Reasonable geometric cohomology theories are usually equipped with additional structure: cohomology with supports, cup products, dual homology theory (in the sense of **[13]**). See (3.5) for more details.

(3.3) As a basic example, consider étale cohomology. For a fixed integer n prime to the characteristic of F, let $\mathcal{T}$ be the category of finite $\mathbf{Z}/n\mathbf{Z}[G]$-modules, where $G = G(F^{\mathrm{sep}}/F)$. Then $\mathbf{1} = \mathbf{Z}/n\mathbf{Z}(0)$, $A = \mathbf{Z}/n\mathbf{Z}$, $R^p\Gamma_{\mathcal{T}}(-) = H^p(G, -)$ and the geometric, resp. arithmetic, cohomology groups are étale cohomology of X over F^{sep} resp. F:

$$\underline{H}^p(X, j) = H^p((X \times_F F^{\mathrm{sep}})_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})(j),$$
$$H^p_{\mathcal{T}}(X, j) = H^p(X_{\text{ét}}, (\mathbf{Z}/n\mathbf{Z})(j)).$$

They are related by the Hochschild-Serre spectral sequence.

If F is a finitely generated extension of $\mathbf{Q}$, then $\underline{R\Gamma}(X, j)$ exist also for ℓ-adic cohomology. In this case, $\mathcal{T}$ is the category of $\mathbf{Q}_\ell$-vector spaces of finite dimension equipped with a continuous action of G, $\mathbf{1} = \mathbf{Q}_\ell(0)$, $A = \mathbf{Q}_\ell$,

$$\underline{H}^p(X, j) = H^p((X \times_F \overline{F})_{\text{ét}}, \mathbf{Q}_\ell)(j),$$
$$H^p_{\mathcal{T}}(X, j) = H^p(X_{\text{ét}}, \mathbf{Q}_\ell(j)),$$

where the ℓ-adic cohomology over F is the *continuous* étale cohomology in the sense of **[41]**. Similarly, (3.2.1) becomes the Hochschild-Serre spectral sequence

$$H^p(G, H^q((X \times_F \overline{F})_{\text{ét}}, \mathbf{Q}_\ell(j)) \Longrightarrow H^{p+q}(X_{\text{ét}}, \mathbf{Q}_\ell(j))$$

for continuous Galois cohomology (**[41]**).

(3.4) Another example is provided by what Beilinson calls "absolute Hodge cohomology". For any separated scheme X of finite type over $\mathbf{C}$, Beilinson constructs in **[3]** a complex $\underline{R\Gamma}(X, 0) \in \mathrm{Ob}\, D^b(\mathcal{MH}_{\mathbf{R}})$, whose cohomology objects are $H^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{R})$ with Deligne's Hodge structures **[18, 19]**. The formalism of (3.2) (with $\mathcal{T} = \mathcal{MH}_{\mathbf{R}}$) then produces "absolute Hodge cohomology" of X, sitting in an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{MH}_{\mathbf{R}}}(\mathbf{R}(0), H^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{R}(n))) \to H^{i+1}_{\mathcal{MH}_{\mathbf{R}}}(X, n)$$
$$\to \mathrm{Hom}_{\mathcal{MH}_{\mathbf{R}}}(\mathbf{R}(0), H^{i+1}_{\mathrm{B}}(X(\mathbf{C}), \mathbf{R}(n))) \to 0.$$

For X over $\mathbf{R}$, we just replace $\mathscr{MH}_{\mathbf{R}}$ by $\mathscr{MH}_{\mathbf{R}}^{+}$ and get a similar sequence for $H^{i+1}_{\mathscr{MH}^{+}_{\mathbf{R}}}(X, n)$. If $X_{/\mathbf{R}}$ is proper and smooth, then (writing $w = i - 2n$, as usual) we get from (2.5.2–2.5.3)

$$(3.4.1)\quad H^{i+1}_{\mathscr{MH}^{+}_{\mathbf{R}}}(X, n) = \begin{cases} \operatorname{Ext}^1_{\mathscr{MH}^{+}_{\mathbf{R}}}(\mathbf{R}(0), H^i_{\mathbf{B}}(X(\mathbf{C}), \mathbf{R}(n))), & w \le -2, \\ H^{2n}_{\mathbf{B}}(X(\mathbf{C}), \mathbf{R}(n))^{+} \cap F^0, & w = -1, \\ 0, & w \ge 0. \end{cases}$$

(3.5) Grothendieck's philosophy of motives stipulates existence of functors $X \mapsto h^i(X)$, which are universal among cohomology groups for *smooth projective* varieties over a given field F. The point of view advocated by Beilinson and Deligne is that the category $\mathscr{M}_F$ of motives with respect to homological equivalence embeds into a larger category of "mixed motives" $\mathscr{MM}_F$, which should be a universal target for "geometric cohomology theories" in the sense of (3.2). A fundamental object associated to a quasi-projective variety $X_{/F}$ should then be the complex

$$\underline{R\Gamma}(X, 0) \in \operatorname{Ob} D^b(\mathscr{MM}_F),$$

rather than its cohomology groups

$$h^i(X) = H^i(\underline{R\Gamma}(X, 0)) \in \operatorname{Ob} \mathscr{MM}_F$$

(which would coincide with Grothendieck's $h^i(X)$ for smooth projective X).

There should be versions with support $\underline{R\Gamma}_Y(X, j)$ (for $Y \subseteq X$ closed), cup product $\underline{R\Gamma}(X, i) \otimes^{\mathbf{L}} \underline{R\Gamma}(X, j) \to \underline{R\Gamma}(X, i + j))$, and homology complexes $\underline{R\Gamma'}(X, j)$ satisfying several axioms (see [**1, 2.3.2**]), which ensure that

$$h^i_Y(X)(j) = H^i(\underline{R\Gamma}_Y(X, j)),$$
$$h_i(X, j) = H^{-i}(\underline{R\Gamma'}(X, -j))$$

form a twisted Poincaré duality theory in the sense of Bloch and Ogus [**13**]. The most important axiom is the duality isomorphism

$$\underline{R\Gamma'}(Y, j) \xrightarrow{\sim} \underline{R\Gamma}_Y(X, j + d)[2d],$$

valid for $Y \subseteq X$ closed in a smooth X of dimension d. If Y has pure codimension i, then its cycle class is a map

$$cl(Y) : \mathbf{1} \to h_{2d-2i}(Y)(d - i) \xrightarrow{\sim} h^{2i}_Y(X)(i).$$

Motivic cohomology and homology are then defined by

$$H^i_{\mathscr{MM}_F}(X, j) = \operatorname{Ext}^i_{\mathscr{MM}_F}(\mathbf{1}, \underline{R\Gamma}(X, j)) = \operatorname{Hom}_{D^b(\mathscr{MM}_F)}(\mathbf{1}, \underline{R\Gamma}(X, j)[i]),$$
$$H_i^{\mathscr{MM}_F}(X, j) = \operatorname{Ext}^{-i}_{\mathscr{MM}_F}(\mathbf{1}, \underline{R\Gamma'}(X, -j)) = \operatorname{Hom}_{D^b(\mathscr{MM}_F)}(\mathbf{1}, \underline{R\Gamma'}(X, -j)[-i]).$$

Of course, $\mathbf{1} = \mathbf{Q}(0) = h^0(\operatorname{Spec}(F))$.

(3.6) Beilinson conjectures that the spectral sequence (3.2.1)

$$(3.6.1)\qquad \operatorname{Ext}^i_{\mathscr{MM}_F}(\mathbf{1}, h^j(X)(n)) \Rightarrow H^{i+j}_{\mathscr{MM}_F}(X, n)$$

degenerates for smooth projective X and that $\mathrm{Ext}^i_{\mathcal{MM}_F}$ vanishes for i greater than the Kronecker dimension of F (equal to the transcendence degree $\mathrm{tr\,deg}(F/\mathbf{F}_p)$ in characteristic p, resp. to $1+\mathrm{tr\,deg}(F/\mathbf{Q})$ in characteristic zero). In particular, if X is a smooth projective variety over $\mathbf{Q}$, then the spectral sequence (3.6.1) should degenerate into exact sequences

$$0 \to \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{1}, h^i(X)(n)) \to H^{i+1}_{\mathcal{MM}_{\mathbf{Q}}}(X, n) \to \mathrm{Hom}_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{1}, h^{i+1}(X)(n)) \to 0,$$

with the third group vanishing, unless $w = i-2n = -1$ (for weight reasons). This suggests another description of the regulator map (2.6.1) for $w < -1$: the Hodge realization

$$H^{i+1}_{\mathcal{MM}_{\mathbf{Q}}}(X, n) \to H^{i+1}_{\mathcal{MH}^+_{\mathbf{R}}}(X, n). \tag{3.6.2}$$

For $i+1 = 2n$, the above sequence should be isomorphic to

$$0 \to CH^n(X)_0 \otimes \mathbf{Q} \to CH^n(X) \otimes \mathbf{Q} \xrightarrow{cl} \mathrm{Hom}_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), h^{2n}(X)(n)) \to 0,$$

where $CH^n(X)$ is the Chow group of codimension n cycles on X modulo rational equivalence, and $CH^n(X)_0$ the subgroup of homologically trivial cycles (cf. 4.2).

(3.7) Furthermore, there should be a relative version of motivic cohomology for morphisms $f : X \to Y$, and there should be a notion of "motivic sheaves" on every variety $X_{/F}$, together with the standard formalism of Grothendieck's six functors $(f^*, Rf_*, Rf_!, Rf^!, R\mathrm{Hom}, \otimes^{\mathbf{L}})$ between corresponding derived categories. Denoting the category of motivic sheaves on X by $\mathcal{M}(X)$, $\mathcal{MM}_F$ should be identified with $\mathcal{M}(\mathrm{Spec}(F))$. Writing $a : X \to \mathrm{Spec}(F)$ for the structural morphism, we should have

$$\underline{R\Gamma}(X, j) = Ra_*a^*\mathbf{1}(j),$$

$$H^i_{\mathcal{MM}_F}(X, j) = \mathrm{Ext}^i_{\mathcal{M}(\mathrm{Spec}(F))}(\mathbf{1}, Ra_*a^*\mathbf{1}(j)) = \mathrm{Ext}^i_{\mathcal{M}(X)}(a^*\mathbf{1}, a^*\mathbf{1}(j)).$$

Note that such a relative theory exists in both étale cohomology (3.3) and Hodge theory (3.4) (see **[58]**).

4. Mixed motives

(4.1) The category $\mathcal{MM}_{\mathbf{Q}}$ of mixed motives over $\mathbf{Q}$ is expected to enjoy (at least) the following four properties:

- The category of semisimple objects of $\mathcal{MM}_{\mathbf{Q}}$ is equivalent to the category $\mathcal{M}_{\mathbf{Q}}$ of motives with respect to homological equivalence (this makes sense only if homological and numerical equivalences of cycles coincide, which is one of Grothendieck's Standard Conjectures **[39]**; otherwise $\mathcal{M}_{\mathbf{Q}}$ itself would not be semisimple, by **[44]**).
- Each mixed motive $E \in \mathrm{Ob}\,\mathcal{MM}_{\mathbf{Q}}$ admits a functorial weight filtration $W_\cdot E$ (increasing) with graded factors $\mathrm{Gr}^W_i(E) \in \mathrm{Ob}\,\mathcal{M}_{\mathbf{Q}}$ pure of weight i.

- Each quasi-projective variety (more generally, simplicial variety) X has cohomology $h^i(X) \in \mathrm{Ob}\,\mathcal{MM}_{\mathbf{Q}}$.
- $\mathrm{Ext}^i_{\mathcal{MM}_{\mathbf{Q}}} = 0$ for $i > 1$.

There is no Grothendieck style definition of $\mathcal{MM}_{\mathbf{Q}}$ as yet. All definitions proposed so far **[23, 32, 43]** are based on the same principle: one constructs first a suitable Tannakian category of mixed realizations $\mathcal{MR}_{\mathbf{Q}}$ and then defines $\mathcal{MM}_{\mathbf{Q}}$ as a full subgategory of $\mathcal{MR}_{\mathbf{Q}}$ consisting of objects of "geometric origin", e.g., the smallest Tannakian subcategory of $\mathcal{MR}_{\mathbf{Q}}$ containing cohomology realizations of all quasi-projective varieties. This is based on a tacit assumption that the realization functor $\mathcal{MM}_{\mathbf{Q}} \to \mathcal{MR}_{\mathbf{Q}}$ is fully faithful.

All realizations discussed in (1.1) in the context of pure motives have analogues for mixed motives. For arbitrary quasi-projective variety $X_{/\mathbf{Q}}$, the mixed motive $E = h^i(X)$ has realizations

$$\begin{aligned} E_\ell &= H^i((X \times_{\mathbf{Q}} \overline{\mathbf{Q}})_{\text{ét}}, \mathbf{Q}_\ell), \\ E_{\mathrm{B}} &= H^i(X(\mathbf{C}), \mathbf{Q}), \\ E_{\mathrm{dR}} &= H^i((Y_\cdot)_{\mathrm{Zar}}, \Omega^\cdot_{Y_\cdot/\mathbf{Q}}), \end{aligned}$$

where $Y_\cdot \to X$ is a smooth hypercovering of X (for Zariski topology). In this situation E_{B} is a *mixed* Hodge structure with infinite Frobenius ϕ_∞ and the weight filtration on E_{B} corresponds, under comparison isomorphisms

$$I_{\ell,\overline{\sigma}} : E_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \xrightarrow{\sim} E_\ell, \qquad I_\infty : E_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} E_{\mathrm{dR}} \otimes_{\mathbf{Q}} \mathbf{C},$$

to natural filtrations $W_\cdot E_\ell$ (by subrepresentations of $G(\overline{\mathbf{Q}}/\mathbf{Q})$) and $W_\cdot E_{\mathrm{dR}}$. The graded objects $\mathrm{Gr}^W_j(E_\ell)$ and $\mathrm{Gr}^W_j(E_{\mathrm{B}})$ are pure of weight j (as representations of $G(\overline{\mathbf{Q}}/\mathbf{Q})$ in the sense of (1.1), resp. as $\mathbf{Q}$-Hodge structures).

(4.2) Let us give a few examples of mixed motives over an arbitrary (say, finitely generated) field F. We shall confidently use cohomology with supports $h^i_Y(X)$, resp. relative cohomology $h^i(X, Y)$ for $Y \subseteq X$ a closed subvariety of X (see **[64]** for a realization of the relative cohomology).

The first example gives a motivic interpretation of Abel-Jacobi maps. Let $X_{/F}$ be a smooth projective variety, $Y \subset X$ a cycle of (pure) codimension i, homologically trivial. Consider the long exact cohomology sequence

$$0 \to h^{2i-1}(X)(i) \to h^{2i-1}(X - Y)(i) \to h^{2i}_Y(X)(i) \xrightarrow{\beta} h^{2i}(X)(i) \to \cdots.$$

Since Y is homologically trivial, the composition of β with the cycle class of Y

$$cl(Y) : \mathbf{Q}(0) \to h^{2i}_Y(X)(i)$$

vanishes. Taking pull-back of the above exact sequence via β, we get an extension of motives

$$0 \to h^{2i-1}(X)(i) \to E \to \mathbf{Q}(0) \to 0.$$

Note that E is a motive with two weights, namely, -1 and 0. The extension class of E depends, in fact, only on the rational equivalence class of Y, and the map $Y \mapsto E$ induces a homomorphism

$$CH^i(X)_0 \to \mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), h^{2i-1}(X)(i)).$$

This is a "motivic" Abel-Jacobi map; if $F \subseteq \mathbf{C}$, we may do the same with singular cohomology with *integral* coefficients, getting the usual Abel-Jacobi map with values in Griffiths's Jacobian

$$\mathrm{Ext}^1_{\mathcal{MH}_{\mathbf{Z}}}(\mathbf{Z}(0), H_{\mathbf{Z}}) = H_{\mathbf{Z}} \backslash H_{\mathbf{C}} / F^0,$$

where $H_{\mathbf{Z}} = H^{2i-1}(X(\mathbf{C}), \mathbf{Z}(i))$ (cf. (2.4.3)).

(4.3) The second example is related to a motivic construction of height pairings (see [**64**]). Let $X_{/F}$ be a smooth projective variety, equidimensional of dimension d, let $Y, Z \subset X$ be homologically trivial cycles of pure codimensions $i, j = d + 1 - i$, with disjoint supports. The relative cohomology $H = h^{2i-1}(X - Y, Z)(i)$ appears in exact sequences

$$0 \to h^{2i-1}(X, Z)(i) \to h^{2i}(X - Y, Z)(i) \to h^{2i}_Y(X)(i) \to h^{2i}(X, Z)(i),$$

$$h^{2i-2}(X-Y)(i) \to h^{2i-2}(Z)(i) \to h^{2i-1}(X-Y, Z)(i) \to h^{2i-1}(X-Y)(i) \to 0.$$

Taking pull-back of H via the cycle class $cl(Y) : \mathbf{Q}(0) \to h^{2i}_Y(X)(i)$ and push-out via the trace map $\mathrm{Tr}(Z) : h^{2i-2}(Z)(i) \to \mathbf{Q}(1)$, we get a mixed motive E sitting in a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{Q}(1) & \to & E_2 & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{Q}(1) & \to & E & \to & E_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{Q}(0) & = & \mathbf{Q}(0) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $M = h^{2i-1}(X)(i)$ and E_1 (resp. $E_2^\vee(1)$) is associated to the cycle Y (resp. $-Z$) as in the previous example. The weight filtration of E is given by

$$W_{-3}E = 0, \qquad W_{-2}E = \mathbf{Q}(1), \qquad W_{-1}E = E_2, \qquad W_0E = E.$$

An important special case of this construction, when $X = \mathbf{P}^1$, $Y = (0) - (\infty)$, $Z = (1) - (u)$ $(u \in F^* - \{1\})$ gives a "Kummer motive" $E = h^1(\mathbf{G}_m, \{1, u\})(1)$, which is an extension

$$0 \to \mathbf{Q}(1) \to E \to \mathbf{Q}(0) \to 0.$$

It is believed that Kummer motives exhaust all motivic extensions of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$, in other words, that

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), \mathbf{Q}(1)) = F^* \otimes \mathbf{Q}. \tag{4.3.1}$$

Note that the extension class of E in various realizations is given by

$$[E_\ell] = u \otimes 1 \in H^1(F, \mathbf{Q}_\ell(1)) = F^* \hat{\otimes} \mathbf{Q}_\ell,$$
$$[E_\mathrm{B}] = \log(u) \in \mathrm{Ext}^1_{\mathcal{MH}^+_\mathbf{R}}(\mathbf{R}(0), \mathbf{R}(1)) \xrightarrow{\sim} \mathbf{R}$$

(for ℓ different from the characteristic of F, resp. for an embedding $F \hookrightarrow \mathbf{R}$), after a suitable normalization of signs **[64]**.

In general, we get an extension of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$ whenever $M = 0$. It is by no means clear that extensions obtained in this way are Kummer (I am grateful to the referee for this remark).

(4.4) The last example is borrowed from the appendix to **[29]**. Let

$$\Delta_n = \mathrm{Spec}\left(F[T_0, \dots, T_n]/\left(\sum T_i - 1\right)\right)$$

be a simplex of dimension n, and let $\partial_i : \Delta_n \hookrightarrow \Delta_{n+1}$ (for $0 \le i \le n+1$) be the i^{th} face map (sending T_i to 0 and renumbering $T_j \mapsto T_{j-1}$ for $j > i$).

Let $X_{/F}$ be an equidimensional smooth projective variety. Fix $n > 0$, and suppose that Y is a cycle of codimension i on $X \times \Delta_n$, meeting all faces of $X \times \Delta_n$ properly. By a face we mean the image of $X \times \Delta_m$ by any composition of face maps ∂_i (for $m < n$). We also assume that $\partial_j^*(Y) = 0$ for all $0 \le j \le n$.

We shall write $\Delta^n_X = X \times \Delta_n$, $\partial\Delta^n_X$ for the union of all faces of codimension one $\partial_i(X \times \Delta_{n-1})$, $|\partial Y| = |Y| \cap \partial\Delta^n_X$, $U = \Delta^n_X - Y$, $\partial U = U \cap \partial\Delta^n_X$.

There is a natural exact sequence **[29**, (A.3)**]**

$$0 \to h^{2i-n-1}(X)(i) \to h^{2i-1}(U, \partial U)(i) \to \mathrm{Ker}(\beta) \to h^{2i-n}(X)(i),$$

where

$$\beta : \mathrm{Ker}[h^{2i}_Y(\Delta^n_X)(i) \to h^{2i}(\Delta^n_X)(i)] \to \mathrm{Ker}[h^{2i}_{\partial Y}(\partial\Delta^n_X)(i) \to h^{2i}(\partial\Delta^n_X)(i)]$$

is induced by taking intersection with $\partial\Delta^n_X$.

The class of Y is a map $cl(Y) : \mathbf{Q}(0) \to \mathrm{Ker}(\beta)$ and its composite with $\mathrm{Ker}(\beta) \to h^{2i-n}(X)(i)$ vanishes. Taking pull-back by $cl(Y)$, gives, finally, an extension

$$0 \to h^{2i-n-1}(X)(i) \to E \to \mathbf{Q}(0) \to 0.$$

(4.5) The example of Kummer motives in (4.3) shows that the motivic regulator (2.6.1) still needs a minor adjustment: for $M = R_{F/\mathbf{Q}}\mathbf{Q}(1)$, we have

$$\mathrm{Ext}^1_{\mathcal{MM}_\mathbf{Q}}(\mathbf{Q}(0), M) = \mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), \mathbf{Q}(1)) \overset{?}{=} F^* \otimes \mathbf{Q},$$

but the classical regulator is made up only of units $\mathcal{O}^*$ (where $\mathcal{O}$ is the ring of integers of F). We need, therefore, a motivic interpretation of $\mathcal{O}^* \otimes \mathbf{Q}$.

We say, after Scholl, that a mixed motive $E \in \mathrm{Ob}\,\mathcal{MM}_F$ is defined over $\mathcal{O}$, if the weight filtration of E_ℓ splits as a representation of the inertia group I_v, for all ℓ and $v \nmid \ell$. Mixed motives defined over $\mathcal{O}$ form a full subcategory $\mathcal{MM}_\mathcal{O}$ of $\mathcal{MM}_F$, containing $\mathcal{M}_F$.

For $v \nmid \ell$, the valuation v induces isomorphisms

$$H^1(F_v, \mathbf{Q}_\ell(1)) = F_v^* \hat{\otimes} \mathbf{Q}_\ell \xrightarrow{\sim} H^1(I_v, \mathbf{Q}_\ell(1)) = F_{v,ur}^* \hat{\otimes} \mathbf{Q}_\ell \xrightarrow{\sim} \mathbf{Q}_\ell,$$

which shows that the Kummer motive corresponding to $u \in F^*$ is defined over $\mathscr{O}$ iff $u \in \mathscr{O}^*$. Modulo (4.3.1), this gives the desired motivic formula for $\mathscr{O}^* \otimes \mathbf{Q}$:

$$\mathscr{O}^* \otimes \mathbf{Q} = \mathrm{Ext}^1_{\mathscr{M}\mathscr{M}_{\mathscr{O}}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathrm{Ext}^1_{\mathscr{M}\mathscr{M}_{\mathbf{Z}}}(\mathbf{Q}(0), R_{F/\mathbf{Q}}\mathbf{Q}(1)).$$

Similarly, we denote by $H^{i+1}_{\mathscr{M}\mathscr{M}_{\mathbf{Z}}}(X, n)$ (for smooth projective $X_{/\mathbf{Q}}$ and $w = i - 2n \leq -2$) the image of

$$\mathrm{Ext}^1_{\mathscr{M}\mathscr{M}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \hookrightarrow \mathrm{Ext}^1_{\mathscr{M}\mathscr{M}_{\mathbf{Q}}}(\mathbf{Q}(0), M) \xrightarrow{\sim} H^{i+1}_{\mathscr{M}\mathscr{M}_{\mathbf{Q}}}(X, n).$$

The final version of the motivic regulator should then be given by restricting (3.6.1) to

$$(4.5.1) \qquad H^{i+1}_{\mathscr{M}\mathscr{M}_{\mathbf{Z}}}(X, n) \hookrightarrow H^{i+1}_{\mathscr{M}\mathscr{M}_{\mathbf{Q}}}(X, n) \to H^{i+1}_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n).$$

5. Motivic cohomology

(5.1) One of the most disturbing features of the motivic regulator map (4.5.1) is how the motivic cohomology $H^{i+1}_{\mathscr{M}\mathscr{M}_{\mathbf{Z}}}$ is "defined", in terms of homological algebra in $\mathscr{M}\mathscr{M}_{\mathbf{Q}}$ and $\mathscr{M}\mathscr{M}_{\mathbf{Z}}$, two Tannakian categories of rather dubious status. What we need is a direct description of motivic cohomology and the regulator map.

Beilinson **[4]** and Lichtenbaum **[48]** conjectured that motivic cohomology of X (even its version with integral coefficients) can be computed as hypercohomology of suitable complexes on Zariski, resp. étale site of X. We shall not discuss this approach to motivic cohomology and instead refer the reader to the articles of S. Bloch, A. B. Gončarov, and S. Lichtenbaum in these Proceedings.

There exist two candidates for motivic cohomology. Beilinson **[1]** defines, for a quasi-projective variety $X_{/F}$,

$$(5.1.1) \qquad H^i_{\mathscr{M}}(X, \mathbf{Q}(j)) = (K_{2j-i}(X) \otimes \mathbf{Q})^{(j)}$$

as the subspace of weight j for Adams operations of a suitable K-group of X. Here K-theory enters the picture for two reasons. First, Beilinson was guided by the relationship between K-theory and singular cohomology in the topological situation (see the discussion of the Atiyah-Hirzebruch spectral sequence in **[37]**). The second hint came from arithmetic, through the works of Lichtenbaum, Borel, and Bloch discussed in the introduction. In fact, in **[1]**, Beilinson states that his work "owes its origin to an attempt to understand Bloch's ideas and computations".

The second construction is due to Bloch **[9]**. In the notation of (4.4), let $Z^q(X)_n$ be the group of cycles of codimension q on $X \times \Delta_n$, meeting all faces transversally. Then $\underline{\underline{n}} \mapsto Z^q(X)_n$ is a simplicial abelian group $Z^q(X)$. One defines the higher Chow group $CH^q(X, p)$ as the homotopy group $\pi_p(|Z^q(X)|)$ of the geometric realization of $Z^q(X)$ (or as homology of the corresponding normalized chain complex; see **[37]**). Then

$$H^i(X, \mathbf{Z}(j)) = CH^j(X, 2j - i)$$

is a candidate for motivic cohomology with integral coefficients.

At present, it is not even known if both recipes give the same result, i.e., if there exist canonical isomorphisms

$$(K_p(X) \otimes \mathbf{Q})^{(j)} \xrightarrow{\sim} CH^j(X, p) \otimes \mathbf{Q} \tag{5.1.2}$$

for all smooth varieties $X_{/F}$. This is certainly true for $p = 0$. In this case, $CH^j(X, 0) = CH^j(X)$ is the standard Chow group of codimension j cycles on X modulo rational equivalence, and the isomorphism $CH^j(X) \otimes \mathbf{Q} \xrightarrow{\sim} (K_0(X) \otimes \mathbf{Q})^{(j)}$ is a classical result of Grothendieck **[66]**. For $p > 0$, (5.1.2) still remains open (the argument in **[9]** runs into difficulties when applying various moving lemmas).

An account of Beilinson's conjectures in terms of higher Chow groups is presented in **[29]**.

(5.2) It is possible to extend (5.1.1) and define K-theoretic cohomology with supports, homology theory (using $K'(X)$), cup products

$$\cup : H^i_{\mathscr{M}}(X, \mathbf{Q}(m)) \otimes H^j_{\mathscr{M}}(X, \mathbf{Q}(n)) \to H^{i+j}_{\mathscr{M}}(X, \mathbf{Q}(m+n))$$

and show that they satisfy Galois descent and almost all axioms of Bloch-Ogus **[13]** (see **[68, 69]**).

Next we need a regulator map

$$r_{\mathscr{H}} : H^i_{\mathscr{M}}(X, \mathbf{Q}(j)) \to H^i_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X, j) \tag{5.2.1}$$

to replace (3.6.2) (say, for a quasi-projective variety $X_{/\mathbf{Q}}$). Beilinson constructs $r_{\mathscr{H}}$ as a Chern class on higher K-theory, using the general machinery of characteristic classes due to Gillet **[34]**. We shall present a more direct construction of $r_{\mathscr{H}}$, which works for X smooth and quasi-projective.

Fix an integer N and denote by $B.\mathrm{GL}_{N/\mathbf{R}}$ the classifying space of the algebraic group $\mathrm{GL}_{N/\mathbf{R}}$. It is a simplicial scheme (cf. **[37]**) and there is a universal simplicial bundle $\mathscr{E}$ of rank N over $B.\mathrm{GL}_{N/\mathbf{R}}$. The Betti cohomology of $B.\mathrm{GL}_{N/\mathbf{R}}$ is well known:

$$H^{2n-1}(B.\mathrm{GL}_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}) = 0,$$
$$\bigoplus_{n \geq 0} H^{2n}(B.\mathrm{GL}_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n)) = \mathbf{Q}[c_1, \dots, c_N],$$

where

$$c_i = c_i(\mathscr{E}) \in H^{2i}(B.\mathrm{GL}_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(i))$$

are Chern classes of the universal bundle $\mathscr{E}$. As $\mathscr{E}$ is defined over $\mathbf{R}$, all c_i are fixed by ϕ_∞. According to **[19,** 9.1.1], all cohomology groups $H^{2n}(B.\mathrm{GL}_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n))$ have pure Hodge type $(0, 0)$. From (3.4.1), we get isomorphisms

$$H^{2n}_{\mathscr{MH}^+_\mathbf{R}}(B.\mathrm{GL}_{N/\mathbf{R}}, n) \xrightarrow{\sim} H^{2n}(B.\mathrm{GL}_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n)) \otimes \mathbf{R} \qquad (n \geq 0)$$

and, therefore, may view c_i as elements of the first group.

Let A be an $\mathbf{R}$-algebra of finite type. We have the evaluation map

$$\mathrm{ev} : \mathrm{Spec}(A) \times B.\mathrm{GL}_N(A) \to B.\mathrm{GL}_{N/\mathbf{R}},$$

which is a morphism of (simplicial) $\mathbf{R}$-schemes. Let us compute in general $H^p_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}} \times Y., q)$ for $X_{/\mathbf{R}}$ a separated scheme and $Y.$ a simplicial set. If we denote by $C^\cdot(Y., \mathbf{R})$ the cochain complex of $Y.$ (with real coefficients), then, in the notation of (3.4),

$$\underline{R\Gamma}(X_{/\mathbf{R}} \times Y., q) = s(\underline{R\Gamma}(X_{/\mathbf{R}}, q) \otimes_\mathbf{R} C^\cdot(Y., \mathbf{R}));$$

hence, the Künneth formula and the cap product

$$H^p_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}} \times Y., q) \xrightarrow{\sim} \bigoplus_j H^{p-j}_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}}, q) \otimes_\mathbf{R} H^j(Y., \mathbf{R}),$$

$$\cap : H^{2n}_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}} \times Y., n) \otimes H_i(Y., \mathbf{R}) \to H^{2n-i}_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}}, n).$$

For $i > 0$, the Hurewicz map

$$K_i(A) \to H_i(\mathrm{GL}(A), \mathbf{R}) = \varinjlim_N H_i(B.\mathrm{GL}_N(A), \mathbf{R})$$

and cap product with $\mathrm{ev}^*(c_n)$ induce a homomorphism

$$c_{i,n} : K_i(A) \to H^{2n-i}_{\mathscr{MH}^+_\mathbf{R}}(\mathrm{Spec}(A), n)$$

Suppose now that $X_{/\mathbf{R}}$ is a smooth quasi-projective variety. By Jouanolou's trick **[45]**, there is an affine variety $\mathrm{Spec}(A)$ and a morphism $\pi : \mathrm{Spec}(A) \to X_{/\mathbf{R}}$ which makes $\mathrm{Spec}(A)$ a vector bundle over $X_{/\mathbf{R}}$. By the homotopy property, π induces isomorphisms both on K_i and $H^i_{\mathscr{MH}^+_\mathbf{R}}$, and we define

$$c_{i,n} : K_i(X_{/\mathbf{R}}) \to H^{2n-i}_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}}, n)$$

by transport of structure.

Finally, one defines Chern *character* maps

$$\mathrm{ch}_i = \sum_{n \geq 0} \frac{(-1)^{n-1}}{(n-1)!} c_{i,n} : K_i(X_{/\mathbf{R}}) \otimes \mathbf{Q} \to \bigoplus_n H^{2n-i}_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}}, n)$$

for $i > 0$ and ch_0 as the usual Chern character

$$K_0(X_{/\mathbf{R}}) \otimes \mathbf{Q} \to \bigoplus_{n \geq 0} H^{2n}_{\mathscr{MH}^+_\mathbf{R}}(X_{/\mathbf{R}}, n).$$

The weight properties of Chern classes imply that $(K_i(X_{/\mathbf{R}})\otimes\mathbf{Q})^{(n)}$ is mapped by ch_i into $H^{2n-i}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$.

For a smooth quasi-projective variety $X_{/\mathbf{Q}}$, Beilinson's regulator $r_{\mathcal{H}}$ is defined as

$$\begin{aligned} r_{\mathcal{H}} &: H^i_{\mathcal{M}}(X, \mathbf{Q}(j)) \\ &= (K_{2j-i}(X)\otimes\mathbf{Q})^{(j)} \to (K_{2j-i}(X_{/\mathbf{R}})\otimes\mathbf{Q})^{(j)} \xrightarrow{\mathrm{ch}_j} H^i_{\mathcal{MH}^+_{\mathbf{R}}}(X, j). \end{aligned}$$

By multiplicativity of the Chern character, it satisfies

$$r_{\mathcal{H}}(x\cup y) = r_{\mathcal{H}}(x)\cup r_{\mathcal{H}}(y).$$

The basic example of Beilinson's regulator is the usual logarithm:

$$\begin{aligned} H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbf{R}), \mathbf{Q}(1)) = \mathbf{R}^*\otimes\mathbf{Q} &\to H^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathrm{Spec}(\mathbf{R}), 1) \\ &= \mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(1)) \xrightarrow{\sim} \mathbf{R} \end{aligned}$$

maps $x\otimes 1$ to $\log|x|$.

(5.3) Let X be, as before, a smooth projective variety over $\mathbf{Q}$. Choose a proper flat model $\mathcal{X}_{\mathbf{Z}}$ of X (it always exists), and put

$$H^i_{\mathcal{M}}(X, \mathbf{Q}(j))_{\mathbf{Z}} = \mathrm{Im}(K'_{2j-i}(\mathcal{X})\otimes\mathbf{Q}\to (K_{2j-i}(X)\otimes\mathbf{Q})^{(j)}).$$

Beilinson conjectures that this subgroup is independent of the choice of $\mathcal{X}$. This is true if we consider only proper and *regular* models of X; unfortunately, such models are rarely known to exist.

The localization sequence

$$\cdots\to K'_j(\mathcal{X})\to K_j(X)\to\bigoplus_p K'_{j-1}(\mathcal{X}\times_{\mathbf{Z}}\mathbf{F}_p)\to\cdots$$

and certain conjectures about K-theory of schemes over finite fields would imply that $H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))/H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}$ depends only on the bad fibres of $\mathcal{X}$ for $w = i-2n < -1$ and vanishes for $n > \max(i, \dim(X)) + 1$ (cf. [**1**, 2.4.2.2]).

(5.4) To sum up, Beilinson defines

$$H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}\hookrightarrow H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))\xrightarrow{r_{\mathcal{H}}} H^{i+1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$$

as a K-theoretic substitute for 4.5.1, which still remains only a wishful thinking.

6. Values of L-functions

We are now ready to formulate Beilinson's conjectures concerning the special values of L-functions. Let X be a smooth projective variety over $\mathbf{Q}$, and let $i\geq 0$ and $n\in\mathbf{Z}$ be integers satisfying $w = i-2n < 0$.

(6.1) CONJECTURE. *Assume* $w \le -3$. *Then* (1) $r_{\mathscr{H}} \otimes \mathbf{R} : H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \otimes \mathbf{R} \to H^{i+1}_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$ *is an isomorphism*; (2) $r_{\mathscr{H}}(\det H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}) = L(h^i(X), n)\mathscr{D}(M) = L^*(h^i(X), i+1-n)\mathscr{B}(M)$ *in* $\det H^{i+1}_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$, *with* $\mathscr{D}(M)$ *and* $\mathscr{B}(M)$ *defined in* (2.2) *for* $M = h^i(n)$.

According to (2.3.1), the order of vanishing of $L(h^i(X), s)$ at $s = i+1-n$ is equal to $\dim_{\mathbf{R}} H^{i+1}_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$. If $M = h^i(X)(n)$ is critical (see (2.2)), then (6.1) reduces to Deligne's conjecture **[21]**

$$L(h^i(X), n) = L(M, 0) \in c^+(M)\mathbf{Q}^*.$$

(6.2) For $w = -2$, $L(h^i(X), s)$ can have a pole at $s = n = 1 + i/2$. The order of the pole is predicted by Tate's conjecture:

$$-\operatorname{ord}_{s=n} L(h^{2n-2}(X), s) = \dim_{\mathbf{Q}} N^{n-1}(X),$$

where

$$N^{n-1}(X) = (CH^{n-1}(X)/CH^{n-1}(X)_0) \otimes \mathbf{Q} \stackrel{?}{=} \operatorname{Hom}_{\mathscr{M}\mathscr{M}_{\mathbf{Q}}}(\mathbf{Q}(0), h^{2n-2}(n-1)).$$

The cycle class in Betti cohomology $cl_{\mathrm{B}} : N^{n-1} \to \operatorname{Ker}(\alpha_{M(-1)})$ (for $M = h^{2n-2}(X)(n)$) and the map $\beta_M : \operatorname{Ker}(\alpha_{M(-1)}) \to \operatorname{Coker}(\alpha_M)$ of (2.1) define

$$r_B : N^{n-1}(X) \to \operatorname{Coker}(\alpha_M) = H^{2n-1}_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n).$$

(6.3) CONJECTURE. *Assume* $w = -2$. *Then*

(1) $(r_{\mathscr{H}} \oplus r_B) \otimes \mathbf{R} : H^{2n-1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \otimes \mathbf{R} \oplus N^{n-1}(X) \otimes \mathbf{R} \to H^{2n-1}_{\mathscr{M}\mathscr{H}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$ *is an isomorphism*;

(2)
$$\begin{aligned}(r_{\mathscr{H}} \oplus r_B)(\det (H^{2n-1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \oplus N^{n-1}(X)) &= L^*(h^{2n-2}(X), n-1)\mathscr{B}(M) \\ &= L^*(h^{2n-2}(X), n)\mathscr{D}(M).\end{aligned}$$

(6.4) In the remaining case of $w = -1$, the conjecture has to be modified. A new ingredient is the height pairing

$$h : CH^n(X)_0 \otimes \mathbf{Q} \otimes CH^{\dim X + 1 - n}(X)_0 \otimes \mathbf{Q} \to \mathbf{R}. \quad (6.4.1)$$

We shall not discuss possible definitions of h (unfortunately, all definitions proposed so far are conditional, except for $n = 1$) and refer to **[64]** for more details.

(6.5) CONJECTURE. *Assume* $w = -1$. *Then* (1) *the height pairing* h *in* (6.4.1) *is nondegenerate*; (2) $\operatorname{ord}_{s=n} L(h^{2n-1}(X), s) = \dim_{\mathbf{Q}} CH^n(X)_0 \otimes \mathbf{Q}$; (3) $L^*(h^{2n-1}(X), n) \in c^+(h^{2n-1}(X)(n)) \det(h)\mathbf{Q}^*$, *where* $c^+(M)$ *is Deligne's period defined in* (2.2).

(6.6) The only case when (6.1) is known is, essentially, if $X = \operatorname{Spec}(F)$ for a finite extension $F/\mathbf{Q}$. In many cases, however, one can verify the following

WEAK CONJECTURE. *Assume* $w = i - 2n < 0$. *Then* (6.1) (*or* (6.3)), *resp.* (6.5), *holds if* $H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}(\oplus N^{n-1})$, *resp.* $CH^{\cdot}(X)_0 \otimes \mathbf{Q}$, *is replaced by a suitable* $\mathbf{Q}$*-subspace.*

(6.7) For quite a few interesting L-series one needs a refined version of the above conjectures, when $L(h^i(X), s)$ is replaced by $L(M, s)$ for a suitable "submotive" $M \subset h^i(X)$. This is indeed possible for Chow motives, which include, among others, motives of Dirichlet characters. Fix a number field E—the field of coefficients. Let $\mathscr{V}_k$ be the category of smooth projective varieties over a field k. The category $\mathscr{MC}(k, E)$ of Chow motives over k with coefficients in E has as objects triples

$$M = (X, p, m),$$

where $X \in \operatorname{Ob}\mathscr{V}_k$, $p \in CH^{\dim X}(X \times X) \otimes E$ a projector ($p^2 = p$), and $m \in \mathbf{Z}$. For $N = (Y, q, n)$ one has

$$\operatorname{Hom}(M, N) = q\, CH^{\dim Y + m - n}(X \times Y) \otimes E\, p,$$

with composition of $f : M_1 \to M_2$ and $g : M_2 \to M_3$ defined by the intersection product: $g \circ f = p_{13*}(p_{12}^* f \cdot p_{23}^* g)$, where $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projections. There is a natural covariant functor $\mathscr{V}_k \to \mathscr{MC}(k, E)$, sending X to the triple $(X, \mathrm{id}, 0)$ and $f : X \to Y$ to the graph of f. This definition is due to Jannsen **[42]**; a more traditional construction works in two steps, first adjoining images of projectors and then adding Tate twists.

The functors $X \mapsto H^{\cdot}_{\mathscr{M}}(X, \mathbf{Q}(*))$ extend to $\mathscr{MC}(k, E)$: we set

$$H^i_{\mathscr{M}}((X, p, m), \mathbf{Q}(j)) = p^* H^{i+2m}_{\mathscr{M}}(X, \mathbf{Q}(j+m)) \otimes E$$

and, for $f = q \circ g \circ p \in \operatorname{Hom}((X, p, m), (Y, q, n))$,

$$f^* : H^i_{\mathscr{M}}((Y, q, n), \mathbf{Q}(j)) \to H^i_{\mathscr{M}}((X, p, m), \mathbf{Q}(j))$$

by $f^*(\alpha) = p^*(\pi_X)_*(f \cup \pi_Y^*(q^*\alpha))$, where

$$\begin{aligned}(\pi_X)_* : H^{i+2\dim Y}_{\mathscr{M}}&((X \times Y, \mathrm{id}, m - n), \mathbf{Q}(j + \dim Y)) \\ &\to H^i_{\mathscr{M}}((X, \mathrm{id}, m), \mathbf{Q}(j))\end{aligned}$$

is the Gysin map.

Similarly, for $k = \mathbf{Q}$, one can extend to $\mathscr{MC}(\mathbf{Q}, E)$ functors $X \mapsto H^{\cdot}_{\mathscr{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, *)$ and the regulator

$$r_{\mathscr{H}} : H^i_{\mathscr{M}}(M, \mathbf{Q}(j)) \to H^i_{\mathscr{MH}^+_{\mathbf{R}}}(M_{/\mathbf{R}}, j).$$

The L-series $L(M, s)$ has values in $E \otimes \mathbf{C}$, and the above regulator is expected to determine its special value modulo E^*. See **[1, 24, 42]** for more details.

Beilinson conjectures [**3**, 8.5.1] that $H^i_{\mathscr{M}}(-, \mathbf{Q}(j))$ and the regulator in fact extend to the category of motives modulo *homological* equivalence.

7. Some computations

(7.1) In this section we present some explicit formulas for Beilinson's regulator. First, we slightly modify the target $H^i_{\mathscr{MH}^+_{\mathbf{R}}}$ and use instead a weaker cohomology theory, known as Deligne cohomology (see [**31, 35**]).

Let us forget the weight filtration in the complex (2.4.1), which represents $R\operatorname{Hom}(\mathbf{R}(0), H)$ in $\mathscr{MH}_{\mathbf{R}}$. The functor that associates to $H \in \operatorname{Ob}\mathscr{MH}_{\mathbf{R}}$ the complex

$$H \oplus F^0 H_{\mathbf{C}} \xrightarrow{i_W - i_F} H_{\mathbf{C}} \tag{7.1.1}$$

extends to a functor $R\Gamma^w(\mathscr{MH}_{\mathbf{R}}, -) : D^b(\mathscr{MH}_{\mathbf{R}}) \to D^b(\mathbf{R}-\text{mod})$. For a separated scheme X of finite type over $\mathbf{C}$, its Deligne.cohomology is defined as

$$H^i_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(j)) = H^i(R\Gamma^w(\mathscr{MH}_{\mathbf{R}}, \underline{R\Gamma}(X, 0)(j))),$$

where $\underline{R\Gamma}(X, 0) \in \operatorname{Ob} D^b(\mathscr{MH}_{\mathbf{R}})$ is Beilinson's complex from (3.4).

Almost by definition, Deligne cohomology sits in an exact sequence

$$\begin{aligned} \cdots \to H^i(X(\mathbf{C}), \mathbf{R}(n)) \to H^i_{\mathrm{dR}}(X_{/\mathbf{C}})/F^n \to H^{i+1}_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \\ \to H^{i+1}(X(\mathbf{C}), \mathbf{R}(n)) \to H^{i+1}_{\mathrm{dR}}(X_{/\mathbf{C}})/F^n \to \cdots . \end{aligned} \tag{7.1.2}$$

In particular, for $n > i+1$, $H^{i+1}_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^i(X(\mathbf{C}), \mathbf{C}/\mathbf{R}(n))$.

In a similar fashion, for $H \in \operatorname{Ob}\mathscr{MH}^+_{\mathbf{R}}$, one replaces (2.5.1) by

$$H^+ \oplus F^0 H_{\mathrm{dR}} \longrightarrow H_{\mathrm{dR}}$$

to get $R\Gamma^w(\mathscr{MH}^+_{\mathbf{R}}, -) : D^b(\mathscr{MH}^+_{\mathbf{R}}) \to D^b(\mathbf{R}-\text{mod})$ and Deligne cohomology

$$H^i_{\mathscr{D}}(X_{/\mathbf{R}}, \mathbf{R}(j)) = H^i_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(j))^{\phi_\infty \otimes c = 1}$$

for $X_{/\mathbf{R}}$ separated of finite type.

There is a canonical map $H^i_{\mathscr{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, j) \to H^i_{\mathscr{D}}(X_{/\mathbf{R}}, \mathbf{R}(j))$. It is an isomorphism for $i \le j$, or even for $i \le 2j$ if X is proper. The composition with the regulator $r_{\mathscr{H}}$ defines a regulator with values in Deligne cohomology (for any quasi-projective variety X over $\mathbf{Q}$)

$$r_{\mathscr{D}} : H^i_{\mathscr{M}}(X, \mathbf{Q}(j)) \to H^i_{\mathscr{D}}(X_{/\mathbf{R}}, \mathbf{R}(j)).$$

(7.2) For smooth varieties, Deligne cohomology can be computed as hypercohomology of quite explicit complexes of sheaves.

Suppose first that $X_{/\mathbf{C}}$ is proper and smooth. The complex

$$\mathbf{R}(n)_{\mathscr{D}} = [\mathbf{R}(n) \to \mathscr{O}_X \to \cdots \to \Omega^{n-1}_X] \tag{7.2.1}$$

in degrees 0 to n is part of an exact triangle

$$\Omega_X^{<n}[-1] \to \mathbf{R}(n)_{\mathscr{D}} \to \mathbf{R}(n). \tag{7.2.2}$$

By GAGA and the degeneration of the Hodge spectral sequence we have

$$H^i(X(\mathbf{C}), \Omega_X^{<n}) \xrightarrow{\sim} H^i(X_{\mathrm{Zar}}, \Omega_{X/\mathbf{C}}^{<n}) = H^i_{\mathrm{dR}}(X_{/\mathbf{C}})/F^n.$$

Comparing (7.1.2) and the cohomology sequence of (7.2.2), we see that

$$H^i_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^i(X(\mathbf{C}), \mathbf{R}(n)_{\mathscr{D}}).$$

If $X_{/\mathbf{C}}$ is smooth and separated, then there exists an open immersion $j : X \hookrightarrow \overline{X}$ of X into a proper smooth variety $\overline{X}$ such that the complement $D = \overline{X} - X$ is a divisor with normal crossings. There are natural maps

$$Rj_*\mathbf{R}(n) \to Rj_*\Omega_X^{\cdot}, \qquad \Omega_{\overline{X}}^{\geq n}(\log D) \to Rj_*\Omega_X^{\cdot}$$

and using their difference we define

$$\mathbf{R}(n)_{\mathscr{D}} = \mathrm{Cone}(\Omega_{\overline{X}}^{\geq n}(\log D) \oplus Rj_*\mathbf{R}(n) \to Rj_*\Omega_X^{\cdot})[-1].$$

Again, the degeneration of the logarithmic Hodge spectral sequence

$$H^i(\overline{X}(\mathbf{C}), \Omega_{\overline{X}}^{\geq n}(\log D)) \xrightarrow{\sim} F^n H^i_{\mathrm{dR}}(X_{/\mathbf{C}})$$

and (7.1.2) imply that

$$H^i_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^i(\overline{X}(\mathbf{C}), \mathbf{R}(n)_{\mathscr{D}}).$$

(7.3) A version of the complex $\mathbf{R}(n)_{\mathscr{D}}$ more suitable for calculations is given by

$$\widetilde{\mathbf{R}(n)}_{\mathscr{D}} = \mathrm{Cone}(\Omega_{\overline{X}}^{\geq n}(\log D) \to j_*\mathscr{A}_X^{\cdot} \otimes \mathbf{R}(n-1))[-1],$$

where $\mathscr{A}_X^{\cdot}$ is the de Rham complex of smooth real-valued differential forms on $X(\mathbf{C})$. There is a canonical quasi-isomorphism between $\mathbf{R}(n)_{\mathscr{D}}$ and $\widetilde{\mathbf{R}(n)}_{\mathscr{D}}$ induced by quasi-isomorphisms

$$Rj_*\mathbf{C} \to Rj_*\Omega_X^{\cdot} \to j_*\mathscr{A}_X^{\cdot} \otimes \mathbf{C}$$

and the projection $\pi_{n-1} : \mathbf{C} \to \mathbf{R}(n-1)$ along $\mathbf{R}(n)$.

This gives an explicit description of

$$H^n_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^n(\overline{X}(\mathbf{C}), \widetilde{\mathbf{R}(n)}_{\mathscr{D}})$$

as the $\mathbf{R}$-vector space

$$\frac{\{\varphi \in H^0(X(\mathbf{C}), \mathscr{A}^{n-1} \otimes \mathbf{R}(n-1)) \mid d\varphi = \pi_{n-1}(\omega),\, \omega \in H^0(\overline{X}(\mathbf{C}), \Omega_{\overline{X}}^n(\log D))\}}{dH^0(X(\mathbf{C}), \mathscr{A}^{n-2} \otimes \mathbf{R}(n-1))} \tag{7.3.1}$$

The cup product of the classes of φ_m, φ_n in $H^m_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(m))$, resp. $H^n_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(n))$, is represented by

$$\varphi_m \cup \varphi_n = \varphi_m \wedge \pi_n\omega_n + (-1)^m \pi_m\omega_m \wedge \varphi_n. \tag{7.3.2}$$

See **[31]** for general formulas for the cup product in Deligne cohomology.

(7.4) Any successful attack on the Weak Conjecture (6.6) usually proceeds in three steps:

(1) Construction of elements in $H^i_{\mathcal{M}}(X, \mathbf{Q}(j))$,

(2) Calculation of their image under the regulator map,

(3) Comparison of the result with the value of the L-function.

Let us give a simple example. If X is a smooth quasi-projective variety over $\mathbf{Q}$, then the regulator

$$r_{\mathcal{D}} : \mathcal{O}(X)^* \otimes \mathbf{Q} = H^1_{\mathcal{M}}(X, \mathbf{Q}(1)) \to H^1_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(1)) \tag{7.4.1}$$

maps $f \in \mathcal{O}(X)^*$ into $\varphi = \log|f|$ (with corresponding $\omega = d\log(f)$). For $f_1, \dots, f_n \in H^1_{\mathcal{M}}(X, \mathbf{Q}(1))$, denote by $\{f_1, \dots, f_n\}$ their cup product in $H^n_{\mathcal{M}}(X, \mathbf{Q}(n))$. Both regulators $r_{\mathcal{H}}, r_{\mathcal{D}}$ are multiplicative; hence,

$$r_{\mathcal{D}}(\{f_1, \dots, f_n\}) = r_{\mathcal{D}}(f_1) \cup \dots \cup r_{\mathcal{D}}(f_n) \in H^n_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)).$$

In particular, if $f_1, \dots, f_n$ are any rational functions on X, write $U \subseteq X$ for the complement of their divisors. The symbol $\{f_1, \dots, f_n\}$ lies in $H^n_{\mathcal{M}}(U, \mathbf{Q}(n))$ and the problem is how to extend it to X. It is possible, sometimes, to exploit an inherent symmetry of the situation in question and construct a natural projector $\pi_{\mathcal{M}}$ from $H^n_{\mathcal{M}}(U, \mathbf{Q}(n))$ to $H^n_{\mathcal{M}}(X, \mathbf{Q}(n))$, thus completing Step 1.

As a next step, one copies the construction of $\pi_{\mathcal{M}}$ to get a similar projector $\pi_{\mathcal{D}}$ in Deligne cohomology. The regulator is then equal to

$$r_{\mathcal{D}}(\pi_{\mathcal{M}}(\{f_1, \dots, f_n\})) = \pi_{\mathcal{D}}(r_{\mathcal{D}}(f_1) \cup \dots \cup r_{\mathcal{D}}(f_n)).$$

Consider the simplest case of $H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$. Suppose that X is a smooth scheme over a field k. According to **[68,** Theorem 4], there is a spectral sequence

$$E_1^{pq} = \coprod_{x \in X^{(p)}} (K_{-p-q}(k(x)) \otimes \mathbf{Q})^{(j-p)} \Rightarrow (K_{-p-q}(X) \otimes \mathbf{Q})^{(j)},$$

where $X^{(p)}$ denotes the set of points of codimension p on X. For a field F and $i = 1, 2$, $(K_i(F) \otimes \mathbf{Q})^{(j)}$ vanishes unless $j = i$, so the spectral sequence reduces to

$$H^2_{\mathcal{M}}(X, \mathbf{Q}(2)) = \mathrm{Ker}(K_2(k(X)) \otimes \mathbf{Q} \xrightarrow{T} \coprod_{x \in X^{(1)}} k(x)^* \otimes \mathbf{Q}).$$

Here the value of $T(\{f, g\})$ at $x \in X^{(1)}$ is equal to the tame symbol

$$(-1)^{\mathrm{ord}_x(f) \cdot \mathrm{ord}_x(g)} \cdot \left(\frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}} \right)(x).$$

The following construction is due to Bloch. Suppose that X is a smooth projective curve over a number field k and $f, g \in k(X)^*$ two rational functions on X. Write D for the union of supports of the divisors of f and g and k_D for the splitting field of D. We assume that, for a fixed integer N, a difference of any two geometric points $P, Q \in D(k_D)$ in the Jacobian of X is torsion of order dividing N. Fix $O \in D(k_D)$. For any $P \in D(k_D) - \{O\}$ there is a rational function $f_P \in k_D(X)^*$ with divisor $N(P) - N(O)$. Write $c_P \in k_D^*$ for the value of the tame symbol $T(\{f, g\})$ at $P \in D(k_D)$. Then

$$\pi(\{f, g\}) = \{f, g\} + \sum_{O \neq P \in D(k_D)} \{f_P, c_P\} \otimes \frac{1}{N}$$

lies in $K_2(k_D(X)) \otimes \mathbf{Q}$, is $G(k_D/k)$-invariant, does not depend on the choice of f_P (as K_2 is torsion for number fields), and lies in the kernel of the tame symbol T (this is clear outside of O; vanishing of T at O then follows from Weil's reciprocity law) and, hence, represents an element of

$$H^2_{\mathscr{M}}(X \times_k k_D, \mathbf{Q}(2))^{G(k_D/k)} \xrightarrow{\sim} H^2_{\mathscr{M}}(X, \mathbf{Q}(2)).$$

Interestingly, there exist examples of families of elliptic curves $E_{/\mathbf{Q}}$ with elements in $H^2_{\mathscr{M}}(E, \mathbf{Q}(2))$ coming from functions with divisors supported at nontorsion points.

(7.5) Fix an embedding $k \hookrightarrow \mathbf{C}$. Let us compute the regulator $r_{\mathscr{D}}$ on $\pi(\{f, g\})$. Set $U = X - D$. According to (7.3.1), we have

$$H^2_{\mathscr{D}}(U_{/\mathbf{C}}, \mathbf{R}(2)) \xrightarrow{\sim} H^1(U(\mathbf{C}), \mathbf{R}(1)), \quad H^2_{\mathscr{D}}(X_{/\mathbf{C}}, \mathbf{R}(2)) \xrightarrow{\sim} H^1(X(\mathbf{C}), \mathbf{R}(1)).$$

By (7.4.1), $r_{\mathscr{D}}(f) \in H^1_{\mathscr{D}}(U_{/\mathbf{C}}, \mathbf{R}(1))$ is represented by $\varphi_f = \log|f|$ with $df = \pi_0(\omega_f)$ for $\omega_f = \partial f/f$.

According to (7.3.2), the cup product $r_{\mathscr{D}}(\{f, g\}) \in H^1(U(\mathbf{C}), \mathbf{R}(2))$ is represented by

$$\psi(f, g) = \varphi_f(\pi_1\omega_g) - (\pi_1\omega_f)\varphi_g = \log|f| d\arg(g) - \log|g| d\arg(f).$$

The pairing

$$\langle\ ,\ \rangle : \psi, \omega \mapsto \frac{1}{2\pi i}\int_{U(\mathbf{C})} \psi \wedge \omega$$

defines a map $H^1(U(\mathbf{C}), \mathbf{R}(1)) \to H^0(X(\mathbf{C}), \Omega^1_X)^*$, whose restriction to $H^1(X(\mathbf{C}), \mathbf{R}(1))$ is an isomorphism (of $\mathbf{R}$-vector spaces).

An elementary calculation shows that, modulo an exact form,

$$\psi(f, g) \wedge \omega \sim \log|f| d\log(\overline{g}) \wedge \omega,$$

which vanishes if g is a constant function. Consequently, for $\omega \in H^0(X(\mathbf{C}), \Omega^1_X)$,

$$\text{(7.5.1)} \qquad \langle r_{\mathscr{D}}(\pi(\{f, g\})), \omega\rangle = \frac{1}{2\pi i}\int_{X(\mathbf{C})} \log|f| d\log(\overline{g}) \wedge \omega,$$

completing Step 2 of the program formulated in (7.4).

For an elliptic curve, this integral can be computed explicitly in terms of certain Kronecker (-Eisenstein-Lerch) series. More precisely, suppose that X is an elliptic curve defined over $\mathbf{R}$ with complex points $X(\mathbf{C}) = \mathbf{C}/\Gamma$ for $\Gamma = \mathbf{Z}+\mathbf{Z}\tau$ $(\mathrm{Im}(\tau) > 0)$ and f, g two rational functions on X with divisors supported at torsion points. Writing dz for the canonical differential on $\mathbf{C}/\Gamma$ and

$$(z, \gamma) = \exp\left(\frac{2\pi i}{\tau - \overline{\tau}}(z\overline{\gamma} - \overline{z}\gamma)\right), \qquad z \in \mathbf{C}/\Gamma, \ \gamma \in \Gamma = \mathbf{Z} + \mathbf{Z}\tau,$$

for the duality between $\mathbf{C}/\Gamma$ and Γ, the regulator is given by the formula

$$\begin{aligned}&\langle r_{\mathscr{D}}(\pi(\{f, g\})), dz\rangle \\ (7.5.2) \qquad &= -\frac{1}{2}\sqrt{\frac{\tau - \overline{\tau}}{2\pi i}} \sum_{x, y \in X(\mathbf{C})} \mathrm{ord}_x(f)\, \mathrm{ord}_y(g) \sum_{0 \neq \gamma \in \Gamma} \frac{\overline{\gamma}}{|\gamma|^4}(y - x, \gamma).\end{aligned}$$

The reader may wish to consult **[1, 30, 57]** for the details of the computation. If the curve has complex multiplication, then its L-function at $s = 2$ is a series of the same type and it is relatively easy to compare the regulator in (7.5.2) with $L(X, 2)$. Historically, this computation was performed first by Bloch **[6, 7]**, using another definition of the regulator.

If $X_{/\mathbf{Q}}$ is an elliptic curve *without* complex multiplication but with a nontrivial torsion over $\mathbf{Q}$, one can repeat the construction of (7.4) and get an element of $H^2_{\mathscr{M}}(X, \mathbf{Q}(2))$ but not necessarily of $H^2(X, \mathbf{Q}(2))_{\mathbf{Z}}$. Amusingly, the obstruction to integrality is related to the third Bernoulli polynomial in **[11, 60]**. In fact, it was only after the calculations of **[11]** that the cohomology of the integral model was incorporated into the conjectures.

(7.6) Note that the above construction works in a family: if S is the (open) modular curve over $\mathbf{C}$ classifying elliptic curves with a full level N structure (for a fixed $N \geq 3$), consider the universal elliptic curve $p: X \to S$ and choose two sections u, v that generate the subgroup of N-torsion of X. There exist rational functions f, g on X with divisors equal to $N(u) - N(0)$, resp. $N(v) - N(0)$, and they can be normalized in such a way that $f_{|v} = 1$, $g_{|u} = 1$. The symbol $\{f, g\}$ then represents an element of $H^2_{\mathscr{M}}(X, \mathbf{Q}(2))$. Its restriction to each fibre X_s has the same field of definition as $s \in S(\mathbf{C})$.

In general, if $p: X \to S$ is a proper smooth map between two varieties over $\mathbf{C}$ and $a \in H^i_{\mathscr{M}}(X, \mathbf{Q}(n))$ a global element with $i < n$, then the regulator of its restriction to the fibre X_s

$$r_{\mathscr{D}}(a_s) \in H^i_{\mathscr{D}}(X_s, \mathbf{R}(n))$$

is "locally constant" as a function of s. The intuition is quite clear: for $i < n$, the groups $H^i_{\mathscr{D}}(X_s, \mathbf{R}(n)) \xrightarrow{\sim} H^{i-1}(X_s(\mathbf{C}), \mathbf{C}/\mathbf{R}(n))$ form a locally constant sheaf on $S(\mathbf{C})$. The formal argument goes as follows: $r_{\mathscr{D}}(a) \in H^i(\overline{X}(\mathbf{C}), \mathbf{R}(n)_{\mathscr{D}})$ defines a global section of the sheaf $R^i p_* j^* \mathbf{R}(n)_{\mathscr{D}}$, where

$\mathbf{R}(n)_{\mathscr{D}}$ is the complex defined in (7.3), living on a suitable compactification $j : X \hookrightarrow \overline{X}$. The complex $j^*\mathbf{R}(n)_{\mathscr{D}}$ is quasi-isomorphic to

$$\mathscr{F}_X = [\mathbf{R}(n) \to \mathscr{O}_X \to \cdots \Omega_X^{n-1}].$$

Denote by $\mathscr{F}_{X/S}$ the analogous complex in which we replace differentials on X by relative differentials.

We may assume that S is a curve. Then the exact triangle

$$p^*\Omega_S^1 \otimes \Omega^{\cdot}_{X/S}/F^{n-1}[-2] \to \mathscr{F}_X \to \mathscr{F}_{X/S}$$

induces an exact sequence

$$R^ip_*\mathscr{F}_X \to R^ip_*\mathscr{F}_{X/S} \xrightarrow{D} \mathscr{H}_{\mathrm{dR}}^{i-1}(X/S)/\mathscr{F}^{n-1} \otimes_{\mathscr{O}_S} \Omega_S^1.$$

Here the map D comes from an exact sequence

$$H_{\mathrm{B}}^{i-1}(X/S, \mathbf{R}(n)) \to \mathscr{H}_{\mathrm{dR}}^{i-1}(X/S)/\mathscr{F}^n \to R^ip_*\mathscr{F}_{X/S} \to 0$$

and the Gauss-Manin connection

$$\mathscr{H}_{\mathrm{dR}}^{i-1}(X/S)/\mathscr{F}^n \xrightarrow{\nabla} \mathscr{H}_{\mathrm{dR}}^{i-1}(X/S)/\mathscr{F}^{n-1},$$

with respect to which the Betti cohomology of the fibres is horizontal.

Finally, $r_{\mathscr{D}}(a_s) \in H^0(S, R^ip_*\mathscr{F}_{X/S})$ must be a section of $\operatorname{Ker}(D)$, which is (assuming $i < n$) nothing else than the locally constant sheaf of $H^{i-1}(X_s(\mathbf{C}), \mathbf{C}/\mathbf{R}(n))$.

This digression explains why, for example, there is no "uniform" construction of elements in $H^2_{\mathscr{M}}(E, \mathbf{Q}(n))$, for elliptic curves E and $n > 2$ (corresponding to the values $L(E, n) = L(h^1(E), n)$, conjecturally). Such a uniform construction works, however, for $H^{n+1}_{\mathscr{M}}(\mathrm{Sym}^n(h^1(E)), n+1)$ (Eisenstein symbols, constructed by Beilinson **[2]**; see also **[24, 29]**). For curves with complex multiplication, Deninger **[24]** defines a map

$$H^{2m}_{\mathscr{M}}(\mathrm{Sym}^{2m-1}(h^1(E)), 2m) \to H^2_{\mathscr{M}}(h^1(E), m+1)$$

into cohomology responsible for the value $L(E, m+1)$.

8. Evidence for the conjectures

In this section we make a survey of the progress toward the conjectures in various special cases; results on Deligne's conjecture in the critical case will not be mentioned.

(1) Zeta functions of number fields. For $\zeta_F(s)$, the strong conjecture has been proved by Borel **[14, 15]** at all integers (with $s = 0, 1$ being classical), using his own definition of a regulator

$$K_{2n-1}(\mathbf{C}) \to \mathbf{R}(n-1).$$

It is proved in **[1]** (cf. **[55]**) that Borel's and Beilinson's regulator coincide.

In recent years, there has been a resurgence of activity centered around classical polylogarithm functions, values of $\zeta_F(s)$, and their relation to Borel's regulator. See the articles **[5, 10, 36]** in these proceedings.

(2) Dirichlet L-series. For every Dirichlet character χ there is a Chow motive M_χ over $\mathbf{Q}$ such that $L(s, \chi) = L(M_\chi, s)$. Beilinson **[1]** proved the weak conjecture at all integers; the results of Borel (1) then imply the strong conjecture.

REMARK. These are the only cases when the strong conjecture has been proved, except for (9) below, dealing with the central point.

(3) Elliptic curves with complex multiplication. For $E_{/\mathbf{Q}}$ an elliptic curve with complex multiplication, the weak conjecture for $L(h^1(E), s)$ has been proved at $s = 2$ by Bloch **[6, 7]**, and Beilinson **[1]**; for elliptic curves of Shimura type, proved at all integers $s \geq 2$ by Deninger **[24]**.

(4) Motives of Hecke characters of an imaginary quadratic field. The weak conjecture is proved by Deninger **[25]**, where he also reproves (2).

(5) Modular forms. If f is a newform of weight $k+2$ on some congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, then there is a Grothendieck motive $M(f)$ associated to f **[62]** such that $L(f, s) = L(M(f), s)$. The motive $M(f)$ is constructed from a Chow motive M corresponding to all cusp forms of given type using a projector Π_f in Hecke algebra. For every integer $n \geq k+2$, Beilinson **[2]** for $k = 0$ and Scholl **[65]** in general construct a subspace $P_n \subseteq H^{k+2}_{\mathcal{M}}(M, \mathbf{Q}(n))$ such that $\det(r_{\mathcal{D}}(\Pi_f(P_n)))$ gives the value of $L(f, n)$. However, for $k > 0$, it is not known if P_n lies in $H^{k+2}_{\mathcal{M}}(M, \mathbf{Q}(n))_{\mathbf{Z}}$. See also **[29, 59]** for more details.

(6) Shimura curves over $\mathbf{Q}$. For any Shimura curve X coming from an automorphic form on an indefinite quaternion algebra B over $\mathbf{Q}$, Ramakrishnan **[52]** proves the weak conjecture for $L(h^1(X), s)$ at all integers $s \geq 2$. Ramakrishnan uses Jacquet-Langlands correspondence between automorphic forms on B and GL_2, together with Faltings isogeny theorem to deduce this result from the corresponding statement for modular forms on GL_2, proved by Beilinson (see (5) above).

(7) Product of two modular curves. For two modular curves C_1, C_2 defined over $\mathbf{Q}$, Beilinson **[1]** proves the weak conjecture for $L(h^2(C_1 \times C_2), s)$ at $s = 2$, but he makes an incorrect argument for the integrality of elements in $H^3_{\mathcal{M}}(C_1 \times C_2, \mathbf{Q}(2))$ that he constructs. A revised version of **[54]** is supposed to fill this gap.

(8) Hilbert-Blumenthal surfaces. Let X be a Hilbert-Blumenthal surface over a real quadratic field F for some congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_F)$. There is a smooth toroidal compactification $\overline{X}$ of X. Ramakrishnan **[54]** proves the weak conjecture for an incomplete L-series $L_S(h^2(\overline{X}), s)$ (with bad Euler factors removed) at $s = 2$. The integrality of relevant elements in the motivic cohomology is not known, however.

(9) Elliptic curves at the central point. If E is an elliptic curve over $\mathbf{Q}$ which is modular (i.e., admits a nontrivial map $X_0(N) \to E$) and the order of vanishing of $L(E, s)$ at $s = 1$ is equal to 0 or 1, then the conjecture of Birch and Swinnerton-Dyer is true for E, up to a controlled rational factor. This follows from the work of Kolyvagin **[46]**, combined with **[38]** and nonvanishing theorems about L-functions of modular forms (see **[51]**).

(10) Numerical evidence. In **[11]**, Bloch and Grayson give results of computations of the regulator on $H^2_{\mathscr{M}}(E, \mathbf{Q}(2))_{\mathbf{Z}}$ for certain elliptic curves without complex multiplication. The result compares favorably with the value $L(E, 2)$, as expected.

Mestre and Schappacher **[50]** report on similar computations for the symmetric square of an elliptic curve without complex multiplication.

9. Mixed motives revisited

(9.1) This section contains a reformulation of Beilinson's conjectures in terms of mixed motives, due to Scholl **[63]**. We have to assume that the formalism of motivic cohomology over $\mathbf{Q}$, described in §§3–4, makes sense and that the categories $\mathscr{MM}_{\mathbf{Z}}$, $\mathscr{MM}_{\mathbf{Q}}$ exist and behave as expected.

For the reader's convenience, we summarize a few relevant formulas. If X is a smooth projective variety over $\mathbf{Q}$ and $M = h^i(X)(n)$, then we should have

$$\mathrm{Ext}^0_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) = \begin{cases} 0 & \text{if } i \neq 2n, \\ CH^n(X)/CH^n(X)_0 \otimes \mathbf{Q} & \text{if } i = 2n; \end{cases}$$

$$\mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) = \begin{cases} H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} & \text{if } i+1 \neq 2n, \\ CH^n(X)_0 \otimes \mathbf{Q} & \text{if } i+1 = 2n. \end{cases}$$

The Ext group in $\mathscr{MM}_{\mathbf{Q}}$ should be given by the same formula with $H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}$ replaced by $H^{i+1}_{\mathscr{M}}(X, \mathbf{Q}(n))$.

A special case of the above formulas (for $M = h^2(\mathbf{P}^1)(2)$) is

$$\mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^* \otimes \mathbf{Q},$$
$$\mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Z}^* \otimes \mathbf{Q} = 0.$$

Finally, for $w = i - 2n < -1$, the regulator map should be none else than the Hodge realization

$$\mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), h^i(X)(n)) \to \mathrm{Ext}^1_{\mathscr{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), H^i(X(\mathbf{C}), \mathbf{R}(n))).$$

(9.2) Deligne's period map

$$\alpha_E : E^+_{\mathrm{B}} \otimes \mathbf{R} \to (E_{\mathrm{dR}}/F^0) \otimes \mathbf{R}$$

makes sense for any mixed motive $E \in \mathrm{Ob}\,\mathscr{MM}_{\mathbf{Q}}$. Scholl **[63]** calls E critical if α_E is an isomorphism.

For a critical mixed motive E, Deligne's period $c^+(E) \in \mathbf{R}^*/\mathbf{Q}^*$ is defined as the determinant of α_E with respect to the $\mathbf{Q}$-structures E_B^+ and E_{dR}/F^0.

One defines the L-function $L(E, s)$ of a mixed motive in the same way as $L(M, s)$ is defined in (1.4) (assuming the independence of ℓ of the local L-factors). If E is a mixed motive over $\mathbf{Z}$, then

$$L(E, s) = \prod_j L(Gr_j^W(E), s).$$

(9.3) Scholl **[63]** makes the following three conjectures:

(A) If $E \in \mathrm{Ob}\mathcal{MM}_{\mathbf{Q}}$ is critical, then $L(E, 0) \cdot c^+(E)^{-1} \in \mathbf{Q}$.

(B) For any mixed motive E over $\mathbf{Z}$,

$$\mathrm{ord}_{s=0} L(E, s) = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E^\vee(1)) - \dim \mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E^\vee(1)).$$

(C) If E is a mixed motive over $\mathbf{Z}$ satisfying

(9.3.1) $$\mathrm{Ext}^i_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E) = \mathrm{Ext}^i_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E^\vee(1)) = 0 \quad \text{for } i = 0, 1,$$

then E is critical.

(9.4) There is a natural construction which transforms any mixed motive M over $\mathbf{Z}$ into a new mixed motive E over $\mathbf{Z}$ satisfying (9.3.1), by taking successive universal extensions and killing inconvenient subgroups. It proceeds in four steps

$$0 \to \mathrm{Hom}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0) \to M \to M_1 \to 0,$$
$$0 \to M_1 \to M_2 \to \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M_1) \otimes \mathbf{Q}(0) \to 0,$$
$$0 \to M_3 \to M_2 \to \mathrm{Hom}(\mathbf{Q}(0), M_2^\vee(1))^\vee \otimes \mathbf{Q}(1) \to 0,$$
$$0 \to \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M_3^\vee(1))^\vee \otimes \mathbf{Q}(1) \to E \to M_3 \to 0,$$

which do not require special comment. In verifying that E indeed satisfies (9.3.1) one has to remember that $\mathrm{Ext}^1(\mathbf{Q}(0), \mathbf{Q}(1))$ is supposed to vanish in $\mathcal{MM}_{\mathbf{Z}}$.

The L-function of E is equal to

$$L(E, s) = L(M, s)\zeta(s)^a\zeta(s+1)^b,$$

where

$$a = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) - \dim \mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M),$$
$$b = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M^\vee(1)) - \dim \mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M^\vee(1)).$$

The conjectures (A)–(C) predict that this new motive E is critical, $L(E, 0) \neq 0$ and

(9.4.1) $$L^*(M, 0) \in L(E, 0) \cdot \mathbf{Q}^* = c^+(E) \cdot \mathbf{Q}^*.$$

(9.5) We shall now consider the case of a pure motive $M = h^i(X)(n)$ of weight $w = i - 2n$.

Suppose first that $w \leq -2$ and that $\mathrm{Hom}(\mathbf{Q}(0), M^{\vee}(1)) = 0$. For weight reasons and by semisimplicity of $\mathscr{M}_{\mathbf{Q}}$,

$$\mathrm{Ext}^0(\mathbf{Q}(0), M) = \mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M^{\vee}(1)) = 0.$$

This means that E is simply the universal extension

$$0 \to M \to E \to \mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0) \to 0;$$

as $\alpha_{\mathbf{Q}(0)} = 0$ and $\mathrm{Ker}(\alpha_M) = 0$, snake lemma implies that E is critical if and only if the connecting homomorphism

$$\mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \xrightarrow{\partial_M} \mathrm{Coker}(\alpha_M)$$

becomes an isomorphism after tensoring with $\mathbf{R}$. This map, however, is nothing else than the regulator and the canonical $\mathbf{Q}$-structure on its target is $\mathscr{D}(M)$, so (9.4.1) is equivalent to the first half of the conjecture (6.1.2)

$$\det(\partial_M) = L(h^i(X), n)\mathbf{Q}^* \in \mathbf{R}^*/\mathbf{Q}^*.$$

Continuing with the same notation, set $N = M^{\vee}(1)$. This is a pure motive of weight $-2-w \geq 0$ with $\mathrm{Hom}(\mathbf{Q}(0), N) = 0$ and its "universal extension" is

$$0 \to \mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M)^{\vee} \otimes \mathbf{Q}(1) \to E^{\vee}(1) \to N \to 0.$$

The period map for N is given by the connecting homomorphism

$$\partial_N : \mathrm{Ker}(\alpha_N) \to \mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M)^{\vee} \otimes \mathbf{R}.$$

Using (2.2.1) and the fact that ∂_M, ∂_N are adjoint maps, we see that (9.4.1) for N becomes

$$\det(\partial_N^{\vee}) = L^*(N, 0)\mathbf{Q}^* = L^*(h^i(X), i+1-n)\mathbf{Q}^* \in \mathbf{R}^*/\mathbf{Q}^*,$$

which is the second half of the conjecture (6.1.2).

If M is pure of weight -2 with no other restrictions, then E has to be constructed in two steps. Its period map accounts for Beilinson's "thickened regulator" $r_{\mathscr{H}} \oplus r_{\mathrm{B}}$ in (6.2), but is more canonical; the reader is invited to verify that (A)–(C) predict that the following sequence

$$0 \to \mathrm{Ext}^1_{\mathscr{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \otimes \mathbf{R} \to \mathrm{Coker}(\alpha_M) \to \mathrm{Hom}(\mathbf{Q}(0), M^{\vee}(1))^{\vee} \otimes \mathbf{R} \to 0$$

is exact. The natural $\mathbf{Q}$-structures, say, $\mathscr{E}(M)$, $\mathscr{D}(M)$, $\mathscr{H}(M)$ on determinants of all three terms, define the period

$$c^+(E) = \mathscr{E}(M) \cdot \mathscr{D}(M)^{-1} \cdot \mathscr{H}(M) \in \mathbf{R}^*/\mathbf{Q}^*,$$

equal to $L^*(M, 0)$ by (9.4.1). This is equivalent to the conjecture (6.3).

The case of $w = -1$, related to height pairings, is discussed in detail in **[64]**.

Scholl **[63]** shows that, modulo some other (reasonable) hypotheses about the structure of $\mathscr{MM}_{\mathbf{Q}}$, his conjectures (A)–(C) are in fact *equivalent* to Beilinson's conjectures (6.1), (6.3), (6.5).

Finally, let us mention that Fontaine and Perrin-Riou [**32, 33**] introduced a beautiful six-term exact sequence, which is effectively a translation of (A)–(C) but works for general mixed motives over $\mathbf{Q}$.

Note added in proof. M. Spivakovsky recently announced a general result on resolution of singularities, which implies that a regular proper flat model of X mentioned in §5.3 always exists.

REFERENCES

1. A. A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet Math. **30** (1985), 2036–2070.
2. ——, *Higher regulators of modular curves*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Part I, Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, pp. 1–34.
3. ——, *Notes on absolute Hodge cohomology*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Part I, Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, pp. 35–68.
4. ——, *Height pairing between algebraic cycles*, K-theory, Arithmetic, and Geometry; Seminar, Moscow 1984–1986 (Yu. I. Manin, ed.), Lecture Notes in Math., vol. 1289, Springer-Verlag, Berlin, Heidelberg and New York, 1987, pp. 1–26.
5. A. Beilinson and P. Deligne, *Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs*, these Proceedings, vol. 2, pp. 97–121.
6. S. Bloch, *Higher regulators, algebraic K-theory, and Zeta functions of elliptic curves*, preprint.
7. ——, *Lectures on algebraic cycles*, Duke Univ. Math. Ser., vol. IV, Duke University, Durham, 1980.
8. ——, *The dilogarithm and extensions of Lie algebras*, Algebraic K-theory, Evanston, 1980, Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin, Heidelberg and New York, 1981, pp. 1–23.
9. ——, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.
10. S. Bloch and I. Kříž, *Mixed Tate motives*, Ann. of Math. (to appear).
11. S. Bloch and D. Grayson, *K_2 and L-functions of elliptic curves: Computer calculations*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Part I, Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, pp. 79–88.
12. S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift I, Progr. Math., vol. 86, Birkhäuser, Boston, Basel, and Berlin, 1990, pp. 333–400, pp. 333–400.
13. S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201.
14. A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272.
15. ——, *Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), 613–636.
16. J. A. Carlson, *Extensions of mixed Hodge structures*, Journées de Géométrie Algébrique d'Angers 1979, Sijthoff & Nordhoff, Alphen an den Rijn, the Nederlands, 1980, pp. 107–127.
17. J. A. Carlson and R. M. Hain, *Extensions of variations of mixed Hodge structure*, Théorie de Hodge, Luminy 1987, Astérisque **179/180**, Société Mathématique France, Paris, 1989, pp. 39–65.
18. P. Deligne, *Théorie de Hodge.* II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57.
19. ——, *Théorie de Hodge.* III, Inst. Hautes Études Sci. Publ. Math. **44** (1975), 5–77.
20. ——, *La conjecture de Weil.* I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–308.

21. ——, *Périodes d'intégrales et valeurs de fonctions* L, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 313–346.
22. ——, Lettre à C. Soulé (1/20/1985).
23. ——, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over **Q**, Proceedings of a Workshop held March 23–27, 1987 (Y. Ihara, K. Ribet, and J.-P. Sierre, eds.), Math. Sci. Res. Inst. Publ., vol. 16, Springer-Verlag, Berlin, Heidelberg, and New York, 1987, pp. 79–297.
24. C. Deninger, *Higher regulators and Hecke* L*-series of imaginary quadratic fields.* I, Invent. Math. **96** (1989), 1–69.
25. ——, *Higher regulators and Hecke* L*-series of imaginary quadratic fields.* II, Ann. Math. (2) **132** (1990), 131–158.
26. ——, *On the* Γ*-factors attached to motives*, Invent. Math. **104** (1991), 245–261.
27. ——, *Local* L*-factors of motives and regularized determinants*, Invent. Math. **107** (1992), 135–150.
28. C. Deninger, *Motivic* L*-functions and regularized determinants*, these Proceedings, vol. 1, pp. 707–743.
29. C. Deninger and A. J. Scholl, *The Beilinson conjectures*, London Math. Soc. Lecture Notes Ser. (J. Coates and M. J. Taylor, eds.), Cambridge Univ. Press, Cambridge, New York, Port Chester, Melbourne and Sydney, vol. 153, 1991, pp. 173–209.
30. C. Deninger and K. Wingberg, *On the Beilinson conjectures for elliptic curves with complex multiplication*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 249–272.
31. H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 43–92.
32. J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato.* I, II, III, C. R. Acad. Sci., Paris, Sér. I Math. **313** (1991), 189–196, 349–356, 421–428.
33. J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: Cohomologie galoisienne et valeurs de fonctions* L, these Proceedings, vol. 1, pp. 599–706.
34. H. Gillet, *Riemann-Roch theorems for higher algebraic* K*-theory*, Adv. Math. **40** (1981), 203–289.
35. ——, *Deligne homology and Abel-Jacobi maps*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), 286–288.
36. A. B. Gončarov, *Polylogarithms and motivic Galois groups*, these Proceedings, vol. 2, pp. 43–96.
37. D. R. Grayson, *Weight filtrations in algebraic* K*-theory*, these Proceedings, vol. 1, pp. 207–237.
38. B. H. Gross and D. Zagier, *Heegner points and derivatives of* L*-series*, Invent. Math. **84** (1986), 225–320.
39. A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic Geometry, Tata Inst. Bombay 1968, Oxford Univ. Press, London, 1969, pp. 193–199.
40. U. Jannsen, *On the* ℓ*-adic cohomology of varieties over number fields and its Galois cohomology*, Galois Groups over **Q** (Y. Ihara, K. Ribet, and J.-P. Serre, eds.), Proceedings of a Workshop held March 23–27, 1987, Math. Sci. Res. Inst. Publ., vol. 16, Springer-Verlag, Berlin, Heidelberg and New York, 1987, pp. 315–360.
41. ——, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.
42. ——, *Deligne homology, Hodge-D-conjecture, and motives*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 305–372.
43. ——, *Mixed motives and algebraic* K*-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, Berlin, Heidelberg and New York, 1990.
44. ——, *Motives, numerical equivalence and semi-simplicity*, Invent. Math. **107** (1992), 447–452.
45. J. P. Jouanolou, *Une suite exacte de Mayer-Vietoris en* K*-théorie algébrique*, Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, Heidelberg and New York, 1973, pp. 293–316.
46. V. A. Kolyvagin, *Euler systems*, The Grothendieck Festschrift II, Progr. Math., vol. 87, Birkhäuser, Boston, Basel, and Berlin, 1990, pp. 435–483.

47. S. Lichtenbaum, *Values of zeta-functions, étale cohomology and algebraic K-theory*, Algebraic K-theory II: "Classical" Algebraic K-theory, and Connections with Arithmetic, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin, Heidelberg and New York, 1973, pp. 489–501.

48. ——, *Values of zeta-functions at non-negative integers*, Number Theory (Journées Arithmétiques, Noordwijkerhout, 1983) (H. Jager, ed.), Lecture Notes in Math., vol. 1068, Springer-Verlag, Berlin, Heidelberg and New York, 1984, pp. 127–138.

49. ——, *Motivic complexes*, these Proceedings, vol. 1, pp. 303–313.

50. J.-F. Mestre and N. Schappacher, *Séries de Kronecker et fonctions L des puissances symétriques de courbes elliptiques sur* $\mathbf{Q}$, Arithmetic Algebraic Geometry (Proceedings of a Conference on Texel Island, April, 1989) (G. van der Geer, F. Oort, and J. Steenbrink, eds.), Progr. Math., vol. 89, Birkhäuser, Boston, Basel, and Berlin, 1991, pp. 209–245.

51. B. Perrin-Riou, *Travaux de Kolyvagin et Rubin*, Séminaire Bourbaki, exposé 717, Astérisque **189/190**, Société Mathématique France, Paris, 1989/90, pp. 69–106.

52. D. Ramakrishnan, *Higher regulators on quaternionic Shimura curves and values of L-functions*, Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Part I, Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, pp. 377–387.

53. ——, *Regulators, algebraic cycles, values of L-functions*, Algebraic K-theory and Algebraic Number Theory, Proceedings of a Seminar held January 12–16, 1987, in Honolulu, Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 183–310.

54. ——, *Periods of integrals arising from* K_1 *of Hilbert-Blumentahl surfaces*, preprint.

55. M. Rapoport, *Comparison of the regulators of Beilinson and Borel*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 169–192.

56. M. Rapoport, N. Schappacher, and P. Schneider, eds., *Beilinson's Conjectures on Special Values of L-functions*, Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988.

57. D. Rohrlich, *Elliptic curves and values of L-functions*, Proc. of the CMS summer school on algebraic number theory, Montreal, 1985 (H. Kisilevsky and J. Labute, eds.), Amer. Math. Soc., Providence, RI, 1987, pp. 371–387.

58. M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), 849–995.

59. N. Schappacher and A. J. Scholl, *Beilinson's theorem on modular curves*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 273–304.

60. ——, *The boundary of the Eisenstein symbol*, Math. Ann. **290** (1991), 303–321.

61. P. Schneider, *Introduction to the Beilinson Conjectures*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 1–36.

62. A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.

63. ——, *Remarks on special values of L-functions*, London Math. Soc. Lecture Notes Ser. (J. Coates and M. J. Taylor, eds.), Cambridge Univ. Press, Cambridge, New York, Port Chester, Melbourne and Sydney, vol. 153, 1991, pp. 373–392.

64. ——, *Height pairings and special values of L-functions*, these Proceedings, vol. 1, pp. 163–187.

65. ——, *Higher regulators and special values of L-functions of modular forms*, in preparation.

66. A. Grothendieck, P. Berthelot, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois-Marie 1966/67, SGA 6, Lecture Notes in Math., vol. 225, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

67. C. Soulé, *Régulateurs*, Séminaire Bourbaki, exposé 644, Astérisque **133/134**, Société Mathématique France, Paris, 1986, pp. 237–253.

68. ——, *Opérations en K-théorie algébrique*, Canad. J. Math. **27** (1985), 488–550.

69. G. Tamme, *The theorem of Riemann-Roch*, Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 103–168.

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Height Pairings and Special Values of L-Functions

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Introduction

The object of this paper is to give a "motivic" interpretation of the height pairings on algebraic cycles introduced by Beilinson [2], Bloch [4], and Gillet and Soulé [14]. The existence of such an interpretation was sketched in [21, §V].

Let X be a smooth projective variety over $\mathbf{Q}$, and suppose that $a, b \geq 0$ are integers satisfying $a+b = \dim X + 1$. In the rest of this paper we shall use Beilinson's definition of the height pairing for cycles. Recall from [2] that (under suitable hypotheses) there are defined local pairings $\langle x, y\rangle_p \in \mathbf{Q}$, $\langle x, y\rangle_\infty \in \mathbf{R}$ for cycles x and y of codimensions a and b whose supports are disjoint. With these one constructs a global pairing $\langle x, y\rangle_{\mathbf{Q}} \in \mathbf{R}$, which depends only on the rational equivalence classes of x and y. By the moving lemma this defines a pairing on Chow groups

$$\langle\ ,\ \rangle_{\mathbf{Q}}: CH^a(X) \otimes CH^b(X) \to \mathbf{R}.$$

Our first result reinterprets the local height pairings. Fix disjoint closed subsets Y, Z of codimensions a, b, and let H, H' be the groups of cycles defined over $\overline{\mathbf{Q}}$ whose supports are contained in Y and Z, respectively. Then the local pairings can be described as follows. Consider the cohomology group $H^{2a-1}(\overline{X} - Y \operatorname{rel} Z, \mathbf{Q}_l(a))$. As a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-module it has a canonical filtration (by weight) with graded pieces $H' \otimes \mathbf{Q}_l(0)$, $H^{2a-1}(\overline{X}, \mathbf{Q}_l(a))$, and $H^\vee \otimes \mathbf{Q}_l(1)$. The essential observation is that (under suitable assumptions) the restriction of this representation to the inertia group at a finite prime $p \neq l$ is partially split; the constituent $H^{2a-1}(\overline{X}, \mathbf{Q}_l(a))$ is a direct summand, and one is left with an extension of $H' \otimes \mathbf{Q}_l(0)$ by $H^\vee \otimes \mathbf{Q}_l(1)$. By Kummer theory this is classified by a homomorphism $H \otimes H' \to \mathbf{Q}_l$, which turns out to be precisely the local pairing at p. For the infinite component one replaces

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the l-adic cohomology with cohomology with real coefficients, viewed as a mixed **R**-Hodge structure.

In other words, there is an interpretation of the (local) height pairing in terms of the "mixed motive" $h^{2a-1}(X - Y \operatorname{rel} Z)(a)$. However, in order to make this a truly motivic interpretation it is desirable to remove the dependence on the choice of supports Y, Z. The construction we adopt is suggested by the conjectures on periods and special values of L-functions, as reformulated in **[21]**; see also 7.8 below. These relate the behaviour at $s = 0$ of the L-function of the motive $M = h^{2a-1}(X)(a)$ to the cycle class groups $CH^a(X)^0$, $CH^b(X)^0$. In §§6 and 7 we attach to X a certain "mixed motive" $\widetilde{M}$ whose weight filtration has three nontrivial graded pieces, isomorphic to

$$(CH^b(X)^0)^\vee \otimes \mathbf{Q}(1),\, M \text{ and } CH^a(X)^0 \otimes \mathbf{Q}(0).$$

Associated to $\widetilde{M}$ is its period mapping $\widetilde{M}_B^+ \otimes \mathbf{R} \to (\widetilde{M}_{\mathrm{dR}}/F^0) \otimes \mathbf{R}$. We show that it can be described simply in terms of the period mapping for M and the (global) height pairing. This shows that the Birch–Swinnerton-Dyer–Beilinson–Bloch conjecture for the behaviour of the L-series of M at $s = 0$ is essentially equivalent to the critical value conjectures for mixed motives **[21**, Conjectures A–C]. At present we can only construct $\widetilde{M}$ under certain hypotheses; the most desirable (conjectural) situation is described in §§6.2–6.7.

To perform these constructions in an unconditional way, the first requirement is a theory of mixed motives. A candidate for such a theory has been constructed independently by Deligne and Jannsen; we recall their construction in §1.

In order to compare the "motivic" and "geometric" pairings it is useful to work over an arbitrary number field, which we do up until §6. However, for the construction of a unique "universal extension" we need the ground field to be $\mathbf{Q}$.

One recurrent problem in the comparison of heights is that of signs. I have tried very hard to ensure consistency of signs (see §0 for the necessary conventions), as the signature of the height pairing should be significant (see **[2]** for a precise conjecture).

Here are some related topics not covered in this account:

(i) A formulation of the motivic pairing for motives with arbitrary coefficients. However, this should present no essential difficulty.

(ii) A description of the relation between the heights considered here and bi-extensions (see **[18, 5]**). The canonical pairings of §3 are none other than splittings of local biextensions, as was pointed out to me by Beilinson.

(iii) A precise description of Brylinski's height pairings **[6]** for local systems on curves in the motivic setting.

(iv) p-adic pairings. For this, see the recent work of Nekovář **[19]**, in which a related—but rather more sophisticated—p-adic theory is developed.

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0. Notation and signs

If k is a number field and v is a finite place v of k we write $\mathscr{G}_v \subset \mathrm{Gal}(\overline{k}/k)$ for a decomposition group at v, $\mathscr{I}_v \subset \mathscr{G}_v$ for the inertia subgroup, and $\mathrm{Frob}_v \in \mathscr{G}_v$ for a geometric Frobenius element. The completion of k at v is k_v and the residue field $k(v)$; we write q_v for its cardinality. If there is no risk of confusion, we write $\mathscr{O}$ for the ring of integers of k. If X is a k-scheme, we write $\overline{X}$ for $X \otimes_k \overline{k}$.

Cohomology groups of schemes are (unless otherwise indicated) étale cohomology; for schemes not defined over an algebraically closed field we use continuous étale cohomology **[15]**. Likewise, all Galois cohomology is continuous group cohomology **[22]**.

For a (suitably good) scheme X, the group of codimension p cycles on X is denoted $\mathscr{Z}^p(X)$ and the Chow group $CH^p(X)$. If X is a scheme over a field k of characteristic zero, the subgroups of cycles and rational equivalence classes whose cohomology classes in $H^{2p}(\overline{X}, \mathbf{Q}_l(p))$ vanish are denoted $\mathscr{Z}^p(X)^0$ and $CH^p(X)^0$, respectively.

If A is an abelian group we often write $A_{\mathbf{Q}}$ in place of $A \otimes \mathbf{Q}$.

Signs. The definition of height pairings given in **[2]** involves exact sequences of cohomology and duality and, therefore, gives rise to problems of signs. We follow the "usual" conventions for signs; to avoid any confusion, this means that in the derived category we take for distinguished triangles those coming from semisplit short exact sequences of complexes (in agreement with SGA4$\frac{1}{2}$ "C.D." and **[3]**). Recall also that if $A^\bullet$ and $B^\bullet$ are complexes and $A^\bullet \otimes B^\bullet$ is their tensor product (with usual differential $d_{A\otimes B} = d_A \otimes id_B + (-1)^{\deg_A} id_A \otimes d_B$) then the canonical isomorphisms

$$A[1] \otimes B \xleftarrow{\sim} (A \otimes B)[1] \xrightarrow{\sim} A \otimes (B[1]) \tag{0.0.1}$$

are given by

$$a \otimes b \leftarrow\!\shortmid a \otimes b \mapsto (-1)^{\deg(a)} a \otimes b.$$

Useful references for signs in connection with tensor products are SGA4, Éxposé XVII, §1, and **[12]** (but note that the first reference takes the opposite convention for distinguished triangles in the derived category). We need the following compatibilities. Unless stated otherwise, ∂ denotes the connecting homomorphism in the long exact sequence for cohomology with supports.

0.1. Lemma. *Let X be a scheme, $K_1, K_2 \in D^b(X_{\text{ét}}, \mathbf{Z}/l^n)$, and $Y_1, Y_2 \subset X$ closed subsets. Then the diagram*

$$\begin{array}{ccc} H^p_{Y_1}(X, K_1) \otimes H^q(X - Y_2, K_2) & \xrightarrow{\cup} & H^{p+q}_{Y_1 - Y_2}(X - Y_2, K_1 \underline{\otimes}^{\mathbf{L}} K_2) \\ \downarrow \mathrm{id}\otimes\partial & & \downarrow \partial \\ H^p_{Y_1}(X, K_1) \otimes H^{q+1}_{Y_2}(X, K_2) & \xrightarrow{\cup} & H^{p+q+1}_{Y_1 \cap Y_2}(X, K_1 \underline{\otimes}^{\mathbf{L}} K_2) \end{array}$$

commutes with sign $(-1)^p$. (*The second vertical arrow is the boundary in the long exact sequence*

$$\cdots \to H^\bullet_{Y_1 \cap Y_2}(X) \to H^\bullet_{Y_1}(X) \to H^\bullet_{Y_1 - Y_2}(X) \to \cdots .)$$

PROOF. See [**12**, Corollary 2.3]. □

0.2. LEMMA. *Let* X *be a scheme and* Y *a closed subscheme. Let* $F, G \in D^b(X_{\text{ét}}, \mathbf{Z}/n)$. *Then the diagram*

$$\begin{array}{ccc} H^p(Y, i^*F) \otimes H^q(X - Y, j^*G) & \xrightarrow{\text{id}\otimes\partial} & H^p(Y, i^*F) \otimes H^{q+1}_Y(X, G) \\ \downarrow \partial\otimes\text{id} & & \downarrow \cup \\ H^{p+1}(X, j_!j^*F) \otimes H^q(X - Y, j^*G) & \xrightarrow{\cup} & H^{p+q+1}(X, F \underline{\otimes}^{\mathbf{L}} G) \end{array}$$

commutes up to sign $(-1)^{p+1}$.

PROOF. Compatibilities of this kind seem to be generally well known, but I could not find a reference for this, even in the special case required here (proof of 7.5), and so for completeness will give a proof. The general result we need from homological algebra is:

PROPOSITION. *Suppose that*

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{u} & B & \xrightarrow{v} & C & \to & 0, \\ 0 & \leftarrow & A' & \xleftarrow{u'} & B' & \xleftarrow{v'} & C' & \leftarrow & 0 \end{array}$$

are exact sequences of bounded-below complexes of R-modules, which have splittings $s\colon C^n \to B^n$, $s'\colon A'^n \to B'^n$ *in every degree. Let* $f\colon C \to A[1]$, $f'\colon A' \to C'[1]$ *be the usual connecting homomorphisms:* $f = ds - sd$, *etc. Let* $\beta\colon B \otimes B' \to D$ *be a pairing into another complex whose restriction to* $A \otimes C'$ *is chain homotopic to zero. Then there are defined canonical pairings*

$$\alpha\colon A \otimes A' \to D, \qquad \gamma\colon C \otimes C' \to D$$

such that in the diagram

$$\begin{array}{ccccc} C[-1] \otimes A' & \xrightarrow{f[-1]\otimes 1} & A \otimes A' & \xleftarrow{1\otimes u'} & A \otimes B' \\ & & \downarrow \alpha & & \downarrow u\otimes 1 \\ \wr(0.0.1) & & D & \xleftarrow{\beta} & B \otimes B' \\ & & \uparrow \gamma & & \uparrow 1\otimes v' \\ C \otimes A'[-1] & \xrightarrow{1\otimes f'[-1]} & C \otimes C' & \xleftarrow{v\otimes 1} & B \otimes C' \end{array}$$

the two right-hand squares commute up to homotopy and the left-hand pentagon is anticommutative up to homotopy.

PROOF. We give the necessary formulae: if $\beta \circ (u \otimes v') = dH + Hd$ for a homotopy H, then α and γ are defined by

$$\alpha(x\otimes y) = \beta(ux\otimes s'y) - (-1)^p H(x\otimes w'y), \quad \gamma(x\otimes y) = \beta(sx\otimes v'y) - H(wx\otimes y)$$

and the sum of the two maps $C[-1] \otimes A' \to D$ coming from the left-hand pentagon equals $dK + Kd$ with

$$K = \beta \circ (s \otimes s')\colon C^p \otimes A'^q \to D^{p+q}. \quad \square$$

In the present case we replace the triangle

$$j_!j_*F \to F \to i_*i^*F \to j_!j^*F[1]$$

by a short exact sequence of complexes of injective sheaves $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ and take $A = \Gamma(X, \mathscr{A})$, etc. For the second sequence we take

$$0 \leftarrow \Gamma(U, j^*\mathscr{I}) \leftarrow \Gamma(X, \mathscr{I}) \leftarrow \Gamma_Y(X, \mathscr{I}) \leftarrow 0$$

for an injective resolution $G \to \mathscr{I}$. Since the pairing

$$R\Gamma_Y(X, F)\overset{\mathbf{L}}{\underline{\otimes}}R\Gamma(X, j_!j^*G) \to R\Gamma(X, F\overset{\mathbf{L}}{\underline{\otimes}}G)$$

is zero in the derived category (it factors through $R\Gamma(X, i_*Ri^!F\overset{\mathbf{L}}{\underline{\otimes}}j_!j^*G) = 0$), we can choose a complex D representing $R\Gamma(X, F\overset{\mathbf{L}}{\underline{\otimes}}G)$ such that the hypotheses of the proposition are satisfied. □

0.3. Consider a first quadrant spectral sequence $E_2^{ij} \Rightarrow E_\infty^*$. Denoting the filtration on the abutment by $\mathrm{Fil}^\bullet$, recall that the differentials of the spectral sequence give rise to edge homomorphisms

$$e^0 \colon E_\infty^n = \mathrm{Fil}^0 E_\infty^n \to E_2^{0,n},$$
$$e^1 \colon \ker(e^0) = \mathrm{Fil}^1 E_\infty^n \to E_2^{1,n-1}.$$

0.4. Lemma. *Let $\mathscr{T}$ be an abelian rigid F-linear tensor category, where F is a field of characteristic zero. Let $K^\bullet, L^\bullet \in D^b(\mathscr{T})$. Write $H^i(\mathscr{T}, -) = \mathrm{Ext}^i_{\mathscr{T}}(\mathbf{1}_{\mathscr{T}}, -)$, and consider the edge homomorphisms e^i attached to the spectral sequences*

$$E_2^{ij} = H^i(\mathscr{T}, \mathscr{H}^j(K^\bullet)) \Rightarrow H^{i+j}(\mathscr{T}, K^\bullet),$$
$$E_2^{ij} = H^i(\mathscr{T}, \mathscr{H}^j(K^\bullet \otimes L^\bullet)) \Rightarrow H^{i+j}(\mathscr{T}, K^\bullet \otimes L^\bullet)$$

Then the following diagram is commutatve:

$$\begin{array}{ccc} \mathrm{Fil}^1 H^i(\mathscr{T}, K^\bullet) \otimes H^j(\mathscr{T}, L^\bullet) & \xrightarrow{\cup} & \mathrm{Fil}^1 H^{i+j}(\mathscr{T}, K^\bullet \otimes L^\bullet) \\ \downarrow e^1 \otimes e^0 & & \downarrow e^1 \\ H^1(\mathscr{T}, \mathscr{H}^{i-1}(K^\bullet)) \otimes H^0(\mathscr{T}, \mathscr{H}^j(L^\bullet)) & \xrightarrow{\cup} & H^1(\mathscr{T}, \mathscr{H}^{i+j-1}(K^\bullet \otimes L^\bullet)) \end{array}$$

(*We have written $\otimes$ for $\underline{\otimes}$ since by hypothesis tensor product is exact.*)

Proof. The edge homomorphisms can be described as follows: consider the distinguished triangle

$$\tau_{\leq n-1}K^\bullet \to \tau_{\leq n}K^\bullet \xrightarrow{\gamma_n} \mathscr{H}^n(K^\bullet)[-n].$$

Then e^0 and Fil^1 are given by the short exact sequence

$$\begin{array}{ccccc} 0 \to H^n(\mathscr{T}, \tau_{\leq n-1}K^\bullet) & \longrightarrow & H^n(\mathscr{T}, \tau_{\leq n}K^\bullet) & \xrightarrow{e^0 = H^n(\gamma_n)} & H^n(\mathscr{T}, \mathscr{H}^n(K^\bullet)[-n]) \\ \| & & \| & & \| \\ \mathrm{Fil}^1 H^n(\mathscr{T}, K^\bullet) & & H^n(\mathscr{T}, K^\bullet) & & H^0(\mathscr{T}, \mathscr{H}^n(K^\bullet)) \end{array}$$

and e^1 is the map

$$e^1 = H^n(\gamma_{n-1})\colon \operatorname{Fil}^1 H^n(\mathscr{T}, K^\bullet) = H^n(\mathscr{T}, \tau_{\leq n-1}K^\bullet) \to H^1(\mathscr{T}, \mathscr{H}^{n-1}(K^\bullet)).$$

We have a commutative diagram

$$\begin{array}{ccc} \tau_{\leq i-1}K^\bullet \otimes \tau_{\leq j}L^\bullet & \longrightarrow & \tau_{\leq i+j-1}(K^\bullet \otimes L^\bullet) \\ \downarrow & & \downarrow \\ \mathscr{H}^{i-1}(K^\bullet)[-i+1] \otimes \mathscr{H}^j(L^\bullet)[-j] & \xrightarrow{(*)} & \mathscr{H}^{i+j-1}(K^\bullet \otimes L^\bullet)[-i-j+1], \end{array}$$

where the map $(*)$ is given by the cup-product $\mathscr{H}^{i-1}(K^\bullet) \otimes \mathscr{H}^j(L^\bullet) \to \mathscr{H}^{i+j-1}(K^\bullet \otimes L^\bullet)$ in degree $i+j-1$; this differs from the "correct" map (0.0.1) by the factor $(-1)^j$. Taking cohomology and using the functoriality of cup-product, the claim of the lemma follows. □

1. A category of mixed motives

1.0. A manageable category of motives has been constructed by Deligne **[11]**, in which the morphisms are defined by absolute Hodge cycles. This construction has been extended independently by Deligne **[10**, §1] and Jannsen **[16**, part I] to give a category of mixed motives. We recall here some of the properties of these categories. For simplicity we consider only motives with coefficients in $\mathbf{Q}$.

1.1. Let k be a number field. The category $\mathscr{C}\mathscr{V}_k$ is defined to be the category whose objects are symbols $h(X)$ for smooth and projective varieties X over k and whose morphisms are homological correspondences defined by absolute Hodge cycles. The Tannakian category $\mathscr{M}_k$ of (unmixed) motives over k is constructed from $\mathscr{C}\mathscr{V}_k$ by adjoining the kernels of projectors and the Tate motive $\mathbf{Q}(1)$ and modifying the commutativity constraint (see **[11**, Chapter 2.6] for details). Associated to an object M of $\mathscr{M}_k$ are its various realisations M_l, M_{dR}, and M_σ (for each $\sigma\colon \bar{k} \to \mathbf{C}$) together with the comparison isomorphisms

$$I_{\sigma,l}\colon M_\sigma \otimes_{\mathbf{Q}} \mathbf{Q}_l \xrightarrow{\sim} M_l, \qquad I_{\sigma,\infty}\colon M_\sigma \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_\sigma \mathbf{C}.$$

A motive M over k is *pure of weight* w if the eigenvalues of an unramified Frobenius element $\mathrm{Frob}_v \in \mathrm{Gal}(\bar{k}/k)$ acting on M_l have absolute value $q_v^{w/2}$, and if for each σ the Hodge filtration induces a Hodge structure on M_σ which is pure of weight w. If X is smooth and projective over k, then $M = h^i(X)(m)$ is pure of weight $w = i - 2m$; and every unmixed motive is a direct sum of pure motives.

1.2. To define a category of mixed motives, Deligne and Jannsen first define a category of "mixed realisations" $\mathscr{M}\mathscr{R}_k$. An object R of $\mathscr{M}\mathscr{R}_k$ is given by the following data:

- For each $\sigma\colon \bar{k} \to \mathbf{C}$, a finite-dimensional $\mathbf{Q}$-vector space R_σ, depending only on the restriction of σ to k;

- A finite-dimensional k-vector space R_{dR}, with a decreasing filtration $F^\bullet$ (Hodge) and an increasing filtration $W_\bullet$ (weight);
- For every prime l a finite-dimensional continuous representation R_l of $\mathrm{Gal}(\overline{k}/k)$;
- A family of isomorphisms ("comparison")

$$I_{\sigma,\infty}\colon R_\sigma \otimes \mathbf{C} \xrightarrow{\sim} R_{\mathrm{dR}} \otimes_\sigma \mathbf{C},$$
$$I_{\sigma,l}\colon R_\sigma \otimes \mathbf{Q}_l \xrightarrow{\sim} R_l.$$

These are subject to certain conditions:

(i) For each σ the filtrations $F^\bullet$, $W_\bullet$ define (through $I_{\sigma,\infty}$) a mixed $\mathbf{Q}$-Hodge structure on R_σ;

(ii) If $\rho \in \mathrm{Gal}(\overline{k}/k)$ then $I_{\sigma\rho,l} = \rho \circ I_{\sigma,l}$;

(iii) The filtration $I_{\sigma,l_*} I^*_{\sigma,\infty} W_\bullet$ on R_l is $\mathrm{Gal}(\overline{k}/k)$-equivariant and independent of σ. (Note that the filtration $I^*_{\sigma,\infty} W_\bullet$ on $R_\sigma \otimes \mathbf{C}$ is defined over $\mathbf{Q}$, by (i).)

1.3. For any scheme X of finite type over k, the singular, de Rham and l-adic cohomology groups associate to X certain mixed realisations $h^i(X)$. (The same is true of the cohomology groups with support.) Jannsen then proves [**16**, Theorem 4.4] that $\mathcal{M}_k$ is equivalent (by the obvious functor) to the smallest Tannakian subcategory of $\mathcal{MR}_k$ containing the realisations $h^i(X)$ for X smooth and projective over k. One may then define $\mathcal{MM}_k$, the category of mixed motives over k, as the smallest Tannakian subcategory of $\mathcal{MR}_k$ contaning $h^i(X)$ for every X of finite type over k. The category $\mathcal{MM}_k$ contains $\mathcal{M}_k$ as a full, semisimple subcategory. Each mixed motive carries an increasing filtration (the weight filtration $W_\bullet$) whose graded pieces are pure motives.

It is also convenient to work with mixed realisations coming from cohomology with support or relative cohomology, as the comparison isomorphisms are defined in these cases also. See for example the discussion in [**8**] (for relative cohomology) as well as the treatment in [**16**, 6.11] where it is shown that all the long exact cohomology sequences are compatible with the comparison isomorphisms. In some cases this does not enlarge the category of realisations considered, as the following example shows.

1.4. Proposition. *Let U be quasi-projective over k, and let $j\colon Z \hookrightarrow U$ be a closed subset. Then there is for each i a mixed motive $h^i(U \operatorname{rel} Z)$ over k whose realisations are isomorphic to the relative cohomology groups of (U, Z).*

Proof. Let $Z \hookrightarrow W$ be a closed immersion of Z into some quasi-projective variety which factors through the inclusion $Z \hookrightarrow U$. (For example, take $W = U$.) Let $X \subset W \times \mathbf{A}^1$ denote the union of $U \times \{0\}$,

$\widetilde{Z} = Z \times \mathbf{A}^1$ and $W \times \{1\}$. We have a commutative diagram (in, say, singular cohomology):

$$\begin{array}{ccccccc} & & H^{i-1}(\widetilde{Z} \cup W) & \xrightarrow{\theta} & H^i(X \operatorname{rel} \widetilde{Z} \cup W) & \longrightarrow & H^i(X) \\ & & \downarrow \psi & & \downarrow \wr & & \downarrow \\ H^{i-1}(U) & \longrightarrow & H^{i-1}(Z) & \longrightarrow & H^i(U \operatorname{rel} Z) & \longrightarrow & H^i(U) \end{array}$$

Now the inclusion of W in $\widetilde{Z} \cup W$ induces an isomorphism on cohomology, and the restriction $H^{i-1}(W) \to H^{i-1}(Z)$ factors through $H^{i-1}(U)$. Therefore, the map ψ factors through $H^{i-1}(U)$ as indicated by the dotted arrow and so θ is zero. Hence, $H^i(U \operatorname{rel} Z) \xrightarrow{\sim} \ker(H^i(X) \to H^i(\widetilde{Z} \cup W))$, defining the motive $h^i(U \operatorname{rel} Z)$. □

1.5. REMARKS. (i) In particular, the compatibility with the comparison isomorphisms shows that there is a long exact sequence of mixed motives

$$\to h^{i-1}(Z) \to h^j(U \operatorname{rel} Z) \to h^j(U) \to h^j(Z) \to \cdots .$$

(ii) Jannsen's definition in **[16**, §4] of $\mathcal{MM}_k$ uses only smooth quasi-projective varieties. This is probably inadequate for our purposes. He has suggested the possibility of enlarging $\mathcal{MM}_k$ to include $h^i(X)$ for simplicial varieties X; then $\mathcal{MM}_k$ would automatically contain motives attached to mapping cones of arbitrary proper morphisms. See **[16**, Appendix C2–3] for further discussion.

1.6. We may define the L-function of a mixed motive E over k to be the Euler product over finite places v of k

$$L(E, s) = \prod_v L_{(v)}(E, s),$$

where

$$L_{(v)}(E, s) = \det(1 - q_v^{-s} \operatorname{Frob}_v | E_l^{\mathcal{I}_v})^{-1}, \qquad v \nmid l.$$

Here it is tacitly assumed that the Euler factors are independent of l (which in this generality is not even known for the good factors). Since the graded pieces $\operatorname{Gr}_j^W E$ of E are pure motives, $L(E, s)$ will in general differ from the product $\prod L(\operatorname{Gr}_j^W E, s)$ by a finite number of Euler factors (as the passage to invariants under inertia is not an exact functor). There is one obvious case in which we have equality.

1.7. DEFINITION. E is a *mixed motive over* $\mathcal{O}$ if the weight filtration on E_l splits over $\mathcal{I}_v$, for every l, v with $v \nmid l$.

1.8. The mixed motives over $\mathcal{O}$ form a full Tannakian subcategory $\mathcal{MM}_{\mathcal{O}}$ of $\mathcal{MM}_k$, containing $\mathcal{M}_k$. We denote the Yoneda extension groups in $\mathcal{MM}_k$, $\mathcal{MM}_{\mathcal{O}}$ by $\operatorname{Ext}_k^i(\cdot, \cdot)$, $\operatorname{Ext}_{\mathcal{O}}^i(\cdot, \cdot)$. If E, E' are mixed motives over $\mathcal{O}$, then it is clear that $\operatorname{Ext}_{\mathcal{O}}^0 = \operatorname{Ext}_k^0 = \operatorname{Hom}$, and $\operatorname{Ext}_{\mathcal{O}}^1(E, E')$ is the subgroup of $\operatorname{Ext}_k^1(E, E')$ comprising the classes of extensions $E' \to E'' \to E$ whose l-adic realisation splits over $\mathcal{I}_v$, for all l and all $v \nmid l$.

1.9. Remark. Definition 1.7 is not the only possibility. One could insist that the weight filtration splits as a representation of the whole decomposition group. However, if the pure parts $\mathrm{Gr}_j^W E_l$ satisfy Deligne's conjecture on the purity of the monodromy filtration (see 3.6) one can show that these definitions are equivalent. We prefer to use the inertia group, although at times will need to know that the splitting is in fact invariant under $\mathscr{G}_v$. (I am grateful to Jan Nekovář for drawing this last point to my attention.)

2. Kummer theory

2.0. In this section we recall certain properties of extensions of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$ and local analogues.

2.1. We first review some facts about extensions of Hodge structures. A basic reference for most of this is [7]. For $A = \mathbf{Q}$ or $\mathbf{R}$, let $\mathscr{H}_A$ denote the category of mixed A-Hodge structures and $\mathscr{H}_A^+$ the category of mixed A-Hodge structures over $\mathbf{R}$ (that is, with a Frobenius at infinity Φ_∞). For $H \in \mathscr{H}_{\mathbf{R}}^+$ one has the "de Rham" $\mathbf{R}$-structure $H_{\mathrm{dR}} \subset H_{\mathbf{C}}$, namely, the invariants of the semilinear extension of Φ_∞ to $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes \mathbf{C}$. As a matter of notation, for any mixed motive M over $\mathbf{Q}$ we write M_A for the associated mixed A-Hodge structure (with Frobenius at infinity where appropriate).

2.2. Proposition. (i) *Let $H \in \mathscr{H}_{\mathbf{R}}$ be a pure Hodge structure. Then*

$$\mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}}(\mathbf{R}(0), H) = H_{\mathbf{R}} \backslash H_{\mathbf{C}} / F^0(H_{\mathbf{C}}).$$

(ii) *Let $H \in \mathscr{H}_{\mathbf{R}}^+$ be a pure Hodge structure over $\mathbf{R}$. Then*

$$\mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}^+}(\mathbf{R}(0), H) = H_{\mathbf{R}}^+ \backslash H_{\mathrm{dR}} / F^0(H_{\mathrm{dR}}). \quad \square$$

2.3. Corollary. (i) *If $w(H) = -1$ then*

$$\mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}}(\mathbf{R}(0), H) = \mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}^+}(\mathbf{R}(0), H) = 0.$$

(ii) $\mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(1)) = \mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}^+}(\mathbf{R}(0), \mathbf{R}(1)) = \mathbf{R}$. $\square$

2.4. In the second part of the corollary, we normalise the isomorphism so that $t \in \mathbf{R}$ corresponds to the following extension H_t of $\mathbf{R}(0)$ by $\mathbf{R}(1)$:

- $H_t = \mathbf{C}$, as real vector space, with complex conjugation for Φ_∞; there is an obvious exact sequence $\mathbf{R}(1) \to H_t \to \mathbf{R}$;
- $F^0(H_t \otimes \mathbf{C})$ is generated by

$$1 \otimes 1 - 2\pi i \otimes \frac{t}{2\pi i} \in H_t \otimes \mathbf{C} = \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}.$$

For the proof, see [7] or [16].

2.5. Now let K be a non-Archimedean local field, with $\mathrm{Gal}(\overline{K}/K) = \mathscr{G}$ and inertia group $\mathscr{I}$. Let V be a continuous finite-dimensional l-adic representation of $\mathcal{G}$, where l is a prime number different from the residue characteristic of K. Then we have a canonical isomorphism

$$H^1(\mathcal{I}, V) \xrightarrow{\sim} V(-1)_{\mathcal{I}}$$

compatible with the action of $\mathcal{G}/\mathcal{I}$. In particular, if 1 is not an eigenvalue of Frobenius on $V(-1)_{\mathcal{I}}$, any extension $V \to E \to \mathbf{Q}_l$ of $\mathcal{G}$-modules splits over $\mathcal{I}$. Dually, if 1 is not an eigenvalue of Frobenius on $V^{\mathcal{I}}$, then any extension $\mathbf{Q}_l(1) \to E \to V$ of $\mathcal{G}$-modules splits over $\mathcal{I}$.

2.6. When $V = \mathbf{Q}_l(1)$ we recover the isomorphism

$$\mathrm{Ext}^1_{\mathcal{I}}(\mathbf{Q}_l, \mathbf{Q}_l(1)) \xrightarrow{\sim} H^1(\mathcal{I}, \mathbf{Q}_l(1)) \xrightarrow{\sim} \mathbf{Q}_l, \tag{2.6.1}$$

the second isomorphism being given by Kummer theory. We fix the normalisation of this isomorphism so that $1 \in \mathbf{Q}_l$ corresponds to the extension

$$0 \to \mathbf{Q}_l(1) \to T_l(\overline{K}^*/\pi^{\mathbf{Z}}) \otimes \mathbf{Q}_l \to \mathbf{Q}_l \to 0$$

with π a uniformiser in K.

2.7. The 1-motives defined in [8] can be regarded as mixed motives. In particular, for every $x \in k^*$ there is a 1-motive $K\langle x\rangle = [\mathbf{Z} \xrightarrow{x} \mathbf{G}_m]$ which is an extension of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$, trival if and only if x is a root of unity. Recall from [8] how $K\langle x\rangle$ may be constructed geometrically:

Let C be the singular curve obtained from $\mathbf{G}_m/k$ by identifying the points 1 and x. Then $h^1(C)(1) = h^1(\mathbf{G}_m \operatorname{rel}\{1, x\})$ sits in an exact sequence

$$0 \to A \to h^1(C)(1) \to B \to 0,$$

where

$$A = \operatorname{coker}(h^0(\mathbf{P}^1)(1) \to h^0(\{1, x\})(1)),$$
$$B = \ker(h^2_{\{0,\infty\}}(\mathbf{P}^1)(1) \to h^2(\mathbf{P}^1)(1)).$$

We fix an isomorphism $A \xrightarrow{\sim} \mathbf{Q}(1)$ by evaluation at x and an isomorphism $\mathbf{Q}(0) \xrightarrow{\sim} B$ by the difference $[\infty] - [0]$ of the cohomology classes of $0, \infty$. In terms of these isomorphisms $h^1(C)(1)$ is isomorphic as an extension to $K\langle x\rangle$. We now recall from §10.3 of loc. cit. some of its realisations:

- (l-adic) The class of extension $K\langle x\rangle_l$ is

  $$x \otimes 1 \in \hat{k}^* \otimes \mathbf{Q}_l = H^1(\overline{k}/k, \mathbf{Q}_l(1)) = \mathrm{Ext}^1_{\mathrm{Gal}(\overline{k}/k)}(\mathbf{Q}_l(0), \mathbf{Q}_l(1)).$$

- (Hodge) The $\mathbf{R}$-Hodge structure $K\langle x\rangle_{\sigma/\mathbf{R}}$ is isomorphic to H_t with $t = \log|\sigma(x)|$.

2.8. It is hoped (see for example [10, §2.4]) that these 1-motives generate $\mathrm{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1))$, i.e., that $\mathrm{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1)) = k^* \otimes \mathbf{Q}$. One consequence would be that *any* extension E of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$ is uniquely determined by the following local data:

- for finite v and each l prime to q_v, the action of $\mathscr{I}_v$ on E_l, given by the invariant

$$t_{v,l}\colon \operatorname{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1)) \to \operatorname{Ext}^1_{\operatorname{Gal}(\bar{k}/k)}(\mathbf{Q}_l(0), \mathbf{Q}_l(1)) \xrightarrow{\sim} k^* \hat{\otimes} \mathbf{Q}_l \xrightarrow{v} \mathbf{Q}_l;$$

- the classes of the $\mathbf{R}$-Hodge structures $E_{\sigma/\mathbf{R}}$, given by the invariants

$$t_\sigma\colon \operatorname{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1)) \to \operatorname{Ext}^1_{\mathscr{H}_\mathbf{R}}(\mathbf{R}(0), \mathbf{R}(1)) \to \mathbf{R}.$$

Moreover, $t_v = t_{v,l}$ would be $\mathbf{Q}$-valued and independent of l, and there would be a "product" formula:

$$\sum_{v \text{ finite}} -\log q_v \cdot t_v + \sum_{\sigma\,:\,k \hookrightarrow \mathbf{C}} t_\sigma = 0. \tag{2.8.1}$$

For motives over $\mathscr{O}$ we would have $\operatorname{Ext}^1_{\mathscr{O}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathscr{O}^* \otimes \mathbf{Q}$ and in particular $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$.

2.9. Remark. It is in keeping with the formalism of Beilinson's conjectures to hope that something much more general is true: namely, that if M is any pure motive of weight > -1, then an extension E of M by $\mathbf{Q}(1)$ is determined by the extensions of $\mathbf{R}$-Hodge structures $E_{\sigma/\mathbf{R}}$ together with the extensions of $\mathscr{I}_v$-modules E_l (for every l and every finite $v \nmid l$). For motives E over $\mathscr{O}$ this would essentially amount to the injectivity of Beilinson's regulator.

3. Pairings attached to certain mixed motives

3.0. Let G be a finite-dimensional representation of $\operatorname{Gal}(\bar{k}/k)$ over $\mathbf{Q}$. There is associated to G in the usual way an Artin motive, which we denote $G(0)$. Thus the σ-realisation of $G(0)$ is simply G itself, for any σ. Write $G(n)$ for the Tate twist $G(0) \otimes \mathbf{Q}(n)$.

3.1. Now let E be a mixed motive over k with

$$\operatorname{Gr}^W_{-1}(E) = M, \qquad \operatorname{Gr}^W_0(E) = G_1(0), \qquad \operatorname{Gr}^W_{-2}(E) = G_2(1)$$

and $\operatorname{Gr}^W_i(E) = 0$ for $i < -2$ and $i > 0$, for Galois representations G_1, G_2 as above. We will construct in this section local pairings

$$b_v = b_{v,E}\colon G_1 \times G_2^\vee \to \begin{cases} \mathbf{R} & \text{for } v \text{ an infinite place of } k, \\ \mathbf{Q}_l & \text{for } v \text{ a finite place, } v \nmid l, \end{cases}$$

under certain hypotheses.

3.2. The pairings will transform as follows under finite field extensions k'/k. Let v' be any place of k' over k, of ramification degree $e(v'/v)$. Write E' for the basechange of E to k'. Then

$$b_{v',E'} = e(v'/v) \cdot b_{v,E}.$$

3.3. The pairing at an Archimedean place can be constructed unconditionally. By 2.3 there is a canonical splitting in $\mathscr{H}_{\mathbf{R}}$:

$$E_{\sigma/\mathbf{R}} = M_{\sigma/\mathbf{R}} \oplus V_{\sigma/\mathbf{R}},$$

where $V_{\sigma/\mathbf{R}}$ is an extension

$$0 \to G_2(1)_{\mathbf{R}} \to V_{\sigma/\mathbf{R}} \to G_1(0)_{\mathbf{R}} \to 0.$$

This extension is classified by an element of

$$\begin{aligned}\mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}}(G_1(0)_{\mathbf{R}}, G_2(1)_{\mathbf{R}}) &= \mathrm{Hom}(G_1, G_2) \otimes \mathrm{Ext}^1_{\mathscr{H}_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(1)) \\ &= \mathrm{Hom}(G_1, G_2) \otimes \mathbf{R}\end{aligned}$$

(where the isomorphism is normalised as in 2.4) and thus determines a pairing $G_1 \times G_2^{\vee} \to \mathbf{R}$ which we define to be b_v if v is a real place corresponding to the embedding σ and $\frac{1}{2}b_v$ if v is complex. It is obvious that this satsfies the compatibility of 3.2.

3.4. To define the pairings $b_{v,E}$ at finite places, we need a hypothesis. Denote by M_1, M_2 the intermediate layers of the extension E:

$$M_1 = E/W_{-2}(E), \qquad M_2 = W_{-1}(E).$$

3.5. Hypothesis. *The motives M_1, M_2 are motives over $\mathscr{O}$.*

By 2.5, this would follow automatically from the following hypothesis involving only M:

3.6. Hypothesis. *For every l, v with $v \nmid l$, no eigenvalue of* Frob_v *on $M_l^{\mathscr{I}_v}$ or $M_l(-1)_{\mathscr{I}_v}$ is a root of unity.*

(Recall that for $M = h^{2n-1}(X)(n)$, X smooth and proper over k, this would in turn be a consequence of Deligne's conjecture [**20**, 3.8] on the purity of the monodromy filtration.)

3.7. First assume that $\mathrm{Gal}(\bar{k}/k)$ acts trivially on G_i. Then there is a splitting of the extension of $\mathscr{I}_v$-modules, unique up to isomorphism

$$E_l = M_l \oplus V_{v,l}$$

for every l with $v \nmid l$, where $V_{v,l}$ is an extension

$$0 \to G_2 \otimes \mathbf{Q}_l(1) \to V_{v,l} \to G_1 \otimes \mathbf{Q}_l \to 0,$$

unique up to isomorphism (as an extension). In fact, by hypothesis both the extensions of $\mathscr{I}_v$-modules $W_{-1}(E_l)$, $E_l/W_{-2}(E_l)$ are split. So there is a short exact sequence

$$\begin{aligned}0 \to \mathrm{Ext}^1_{\mathscr{I}_v}(G_1 \otimes \mathbf{Q}_l, G_2 \otimes \mathbf{Q}_l(1)) &\to \mathrm{Ext}^1_{\mathscr{I}_v}(G_1 \otimes \mathbf{Q}_l, M_{2,l}) \\ &\to \mathrm{Ext}^1_{\mathscr{I}_v}(G_1 \otimes \mathbf{Q}_l, M_l) \to 0\end{aligned}$$

in which the class $[E_l] \in \mathrm{Ext}^1_{\mathcal{J}_v}(G_1 \otimes \mathbf{Q}_l, M_{2,l})$ maps to zero in $\mathrm{Ext}^1_{\mathcal{J}_v}(G_1 \otimes \mathbf{Q}_l, M_l)$. The class of the extension $V_{v,l}$ is then the inverse image of $[E_l]$ in $\mathrm{Ext}^1_{\mathcal{J}_v}(G_1 \otimes \mathbf{Q}_l, G_2 \otimes \mathbf{Q}_l(1))$. The isomorphism (2.6.1) gives

$$\mathrm{Ext}^1_{\mathcal{J}_v}(G_1 \otimes \mathbf{Q}_l, G_2 \otimes \mathbf{Q}_l(1)) = \mathrm{Hom}(G_1, G_2) \otimes \mathbf{Q}_l,$$

and the class $[V_{v,l}]$ therefore defines a pairing b_v.

3.8. (The general case). It is simple to check (using 2.6) that the pairings just defined satisfy the basechange property 3.2. In olrder to define them in general, choose an extension k'/k such that $\mathrm{Gal}(\bar{k}/k')$ acts trivially on G_1 and G_2 and set

$$b_{v,E} = \frac{1}{e(v'/v)} b_{v',E'}.$$

3.9. CONJECTURE. *The pairings $b_{v,E}$ for finite v are $\mathbf{Q}$-valued and independent of l.*

3.10. We need some formal properties of these pairings. The proofs are straightforward consequences of the definitions. In each case the hypothesis 3.5 is assumed to hold.

3.11. PROPOSITION. *Let $\phi_1: G_1 \to H_1$, $\phi_2: H_2 \to G_2$ be $\mathrm{Gal}(\bar{k}/k)$-homomorphisms, and let E' be a mixed motive over k with graded pieces $H_2(1)$, M, and $H_1(0)$. Write E for the motive with graded pieces $G_2(1)$, M, and $G_1(0)$ which is obtained from E' by pullback and pushout via ϕ_1 and ϕ_2. Then the pairings attached to E, E' satisfy*

$$b_{v,E}(x_1, x_2) = b_{v,E'}(\phi_1(x_1), x_2 \circ \phi_2). \quad \square$$

3.12. PROPOSITION. *Let E be as above, and suppose there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M_1' & \to & G_1(0) & \to & 0 \\ & & \downarrow & & \downarrow \omega & & \| & & \\ 0 & \to & M & \to & E/W_{-2}(E) & \to & G_1(0) & \to & 0 \end{array}$$

in which M' is pure of weight -1. Let E' be the mixed motive with graded pieces $G_2(1)$, M', $G_1(0)$ obtained from E by pullback with ω. Then $b_{v,E'} = b_{v,E}$. $\square$

3.13. PROPOSITION. *Let E, E' both be extensions of $G_1(0)$ by M_2, and let E'' be their Baer sum. Then $b_{v,E''} = b_{v,E} + b_{v,E'}$.* $\square$

3.14. PROPOSITION. *Let $M = 0$, and let E be the extension $K\langle x \rangle$ of the previous section, with $G_1 = G_2 = \mathbf{Q}$ (with trivial Galois action), and $x \in k^*$. Then*

$$b_v(1, 1) = \begin{cases} \log|x|_v & \text{for } v \text{ infinite}, \\ \mathrm{ord}_v(x) & \text{for } v \text{ finite}. \end{cases} \quad \square$$

3.15. Proposition. *Assume that G_i are trivial Galois modules and that $k = \mathbf{Q}$. Then the pairing $b_{\infty,E}$ is a perfect pairing if and only if E is a* **critical** *mixed motive, and if this is the case then*

$$c^+(E) = c^+(M)\cdot \det b_{\infty,E}. \quad \square$$

Recall **[21]** that E is *critical* if the period mapping $I^+_\infty(E)$, defined by the commutative diagram

$$\begin{array}{ccccc} E_B\otimes\mathbf{C} & \supset & E_B^+\otimes\mathbf{R} & \xrightarrow{\quad I^+_\infty(E)\quad} & \\ \downarrow\wr\, I_\infty(E) & & \downarrow & & \\ E_{\mathrm{dR}}\otimes\mathbf{C} & \supset & E_{\mathrm{dR}}\otimes\mathbf{R} & \rightarrow & (E_{\mathrm{dR}}/F^0)\otimes\mathbf{R} \end{array}$$

is an isomorphism and that if this is the case the period $c^+(E)\in\mathbf{R}^*/\mathbf{Q}^*$ is the determinant of $I^+_\infty(E)$, calculated with respect to the $\mathbf{Q}$-structures E_B^+, E_{dR}/F^0.

3.16. Remark. We tacitly assume, in the construction of the height pairing for extensions of motives, that the local pairings $b_{v,E}$ vanish for all but a finite number of v. Working in the category of mixed motives proposed by Jannsen this is automatic; for the l-adic realisation E_l of any mixed motive is obtained by tensor operations from the l-adic cohomology of some varieties over k. By the theorems on constructibility and generic basechange in l-adic cohomology, E_l therefore extends to a smooth $\mathbf{Q}_l$-sheaf of an open subset $U\subset \operatorname{Spec}\mathscr{O}$, and for a finite prime $v\in U$ we then will have $b_{v,E}=0$.

4. The local geometric pairings

4.0. For this section let X be a smooth and projective scheme over a number field k, equidimensional of dimension N. We will relate the pairings of the preceding section to the local height pairings (or link indices) as described in **[2**, §2**]**, whose definition we now recall.

4.1. Assume that X extends to a regular scheme $\mathscr{X}$ which is flat and proper over $\mathscr{O}$. Then there is an intersection pairing (see, e.g., **[13**, §6**]**)

$$\langle\ ,\ \rangle_{\mathscr{X}}\colon CH^a(\mathscr{X})^0\otimes CH^b(\mathscr{X})^0\to\mathbf{R},$$

where a and b satisfy $a+b=N+1$, $CH^n(\mathscr{X})$ is the Chow group of codimension n cycles on $\mathscr{X}$ modulo rational equivalence, and $CH^n(\mathscr{X})^0 = \ker\{CH^n(\mathscr{X})\to H^{2n}(X\otimes\bar{k},\mathbf{Q}_l(n))\}$.

4.2. Under some restrictions the pairing can be defined at the level of cycles on X rather than $\mathscr{X}$. For this, write $CH^n(X)^{00}_{\mathbf{Q}}$ for the image in $CH^n(X)_{\mathbf{Q}}$ of

$$\bigcap_{\substack{v,l\\ v\nmid l}}\ker\left\{\mathscr{Z}^n(\mathscr{X})_{\mathbf{Q}}\to H^{2n}(\mathscr{X}\otimes\overline{k(v)},\mathbf{Q}_l(n))\right\}.$$

If ξ, η are elements of $CH^*(X)^{00}_{\mathbf{Q}}$, they can be lifted to cycles ξ', η' on $\mathscr{X}$ (with $\mathbf{Q}$-coefficients) whose classes in $H^{2*}(\mathscr{X} \otimes \overline{k(v)}, \mathbf{Q}_l(*))$ are zero for every v, l with $v \nmid l$, and one can then define

$$\langle \xi, \eta \rangle_X = \langle \xi', \eta' \rangle_{\mathscr{X}}$$

which depends only on ξ, η.

4.3. Conjecture 2.2.5 of [2] asserts that

$$CH^n(X)^{00}_{\mathbf{Q}} = CH^n(X)^0_{\mathbf{Q}} \stackrel{\text{def}}{=} \ker\{CH^n(X)_{\mathbf{Q}} \to H^{2n}(X \otimes \bar{k}, \mathbf{Q}_l(n))\}.$$

Note that since $H^{2n}(\mathscr{X} \otimes \overline{k(v)}, \mathbf{Q}_l(n)) = H^{2n}(\mathscr{X} \otimes \mathscr{O}_v^{\text{nr}}, \mathbf{Q}_l(n))$ by the proper basechange theorem, the "absolute" cycle map

$$CH^n(X)_{\mathbf{Q}} \to H^{2n}(X \otimes k_v^{\text{nr}}, \mathbf{Q}_l(n)) \tag{4.3.1}$$

is zero on $CH^n(X)^{00}_{\mathbf{Q}}$ for every l and every $v \nmid l$.

4.4. The pairing $\langle\ ,\ \rangle_{\mathscr{X}}$ is defined as a sum of local terms. In [2] Beilinson expresses the pairing $\langle\ ,\ \rangle_X$ as a sum of local terms, each defined cohomologically. The terms for the finite and infinite primes are completely analogous. To describe them in a unified way, introduce a rigid abelian tensor category $\mathscr{T}$, with coefficient ring $A = \operatorname{End}_{\mathscr{T}}(\mathbf{1})$ and objects $\underline{R\Gamma}_c(X)$, $\underline{R\Gamma}_Y(X)$ in the derived category $D^b(\mathscr{T})$ for schemes of finite type X/F and closed subsets $Y \subset X$. Write $\underline{R\Gamma}(X) = \underline{R\Gamma}_X(X)$. The cases we need to consider are:

(i) F is either a number field or a finite extension of $\mathbf{Q}_p^{\text{nr}}$, $\mathscr{T}$ is the category of continuous finite-dimensional representations of $\operatorname{Gal}(\overline{F}/F)$ over $A = \mathbf{Q}_l$, and $\underline{R\Gamma}_\bullet(X) = R\Gamma_\bullet(\overline{X}_{\text{ét}}, \mathbf{Q}_l)$;

(ii) $F = \mathbf{R}$ or $\mathbf{C}$, $\mathscr{T}$ is the category of mixed $\mathbf{R}$-Hodge structures over F, and $\underline{R\Gamma}_\bullet(X)$ is the Hodge complex constructed in [1].

4.5. In both cases there is a "Tate object" $A(1)$ of $\mathscr{T}$ ($A = \mathbf{Q}_l$ or $\mathbf{R}$), and we write $\underline{R\Gamma}_\bullet(X, n) = \underline{R\Gamma}_\bullet(X) \otimes A(n)$. The corresponding cohomology objects in $\mathscr{T}$ will be denoted $\underline{H}^i_\bullet(X, n)$. We then obtain "absolute" cohomology complexes and groups:

$$R\Gamma_{\mathscr{T},\bullet}(X, n) = R\operatorname{Hom}(\mathbf{1}_{\mathscr{T}}, \underline{R\Gamma}_\bullet(X, n)) \in D(A),$$
$$H^i_{\mathscr{T},\bullet}(X, n) = H^i(R\Gamma_{\mathscr{T}\bullet}(X, n))$$

and the "Hochshild-Serre" spectral sequence

$$E_2^{ij} = H^i(\mathscr{T}, \underline{H}^j_\bullet(X, n)) = \operatorname{Ext}^i(\mathbf{1}_{\mathscr{T}}, \underline{H}^j_\bullet(X, n)) \Rightarrow H^{i+j}_{\mathscr{T},\bullet}(X, n)).$$

In the case (i) $H_{\mathscr{T}}(-, n)$ is the continuous étale cohomology [15] with coefficients $\mathbf{Q}_l(n)$; in (ii) it is the absolute Hodge (or Deligne-Beilinson) cohomology [1] $H_{\mathscr{H}}(-, A(n))$.

4.6. The functors $\underline{R\Gamma}_\bullet$ enjoy the usual properties of cohomology with supports. For example, there are triangles:

$$\underline{R\Gamma}_Y(X) \to \underline{R\Gamma}(X) \to \underline{R\Gamma}(X-Y) \to \underline{R\Gamma}_Y(X)[1],$$
$$\underline{R\Gamma}_c(X-Y) \to \underline{R\Gamma}_c(X) \to \underline{R\Gamma}_c(Y) \to \underline{R\Gamma}_c(X-Y)[1],$$

duality pairings

$$\underline{R\Gamma}_Y(X) \otimes \underline{R\Gamma}(Y) \to \underline{R\Gamma}_Y(X), \qquad \underline{R\Gamma}_c(X) \otimes \underline{R\Gamma}(X) \to \underline{R\Gamma}_c(X),$$

and a trace map

$$\mathrm{Tr}\colon \underline{R\Gamma}_c(X) \to A(-N)[-2N]$$

if X is smooth of dimension N.

4.7. For X smooth and $Y \subset X$ of codimension d one has the purity

$$\underline{H}^i_Y(X) = 0 \quad \text{for } i < 2d$$

and the cycle class map

$$cl_Y\colon A(-d) \to \underline{H}^{2d}_Y(X)$$

which is an isomorphism if Y is absolutely irreducible. This induces an absolute cycle map

$$cl_{\mathcal{T},Y}\colon \mathcal{Z}^d_Y(X) \to H^{2d}_{\mathcal{T},Y}(X, d)$$

which becomes an isomorphism when tensored with A.

4.8. For l-adic cohomology these are all standard facts, simply because $\mathrm{Gal}(\overline{F}/F)$ acts by transport of structure. In the case of Hodge cohomology the fact that the various arrows are compatible with the Hodge structures is not always obvious, but follows from the results of **[1, 8]**.

4.9. We will refer to case (i) with F a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$ and case (ii) as the *local cases*. In the local cases there is a canonical isomorphism

$$H^1(\mathcal{T}, A(1)) = \mathrm{Ext}^1_{\mathcal{T}}(A(0), A(1)) \to A \tag{4.9.1}$$

given by 2.6 in case (i) and by 2.4 in case (ii).

4.10. In all the cases we are considering, the functor $\underline{H}^\bullet$ factors through $\mathcal{MM}_k$ when $k \subset F$, so we can speak of the $\mathcal{T}$-realisation of a mixed motive over k.

Ideally one would like to be able to take $\mathcal{T} = \mathcal{MM}_k$ itself, but at present this is unknown. In this case the groups $H_{\mathcal{T}}$ would hopefully be the motivic cohomology groups $H_{\mathcal{M}}$. This would be part of the formalism of a "derived category of motivic sheaves", as explained in **[2**, §5.10**]**—see also **[9]** (both these references are discussed in **[17]**).

4.11. Although Beilinson's local pairings can be defined in a purely local setting, we will assume that we are in the setting of 4.1 and that one of the local cases for $\mathscr{T}$, with $F \supset k$, has been fixed. To simplify the notation, in the cohomology groups we will write X in place of $X \otimes F$, etc. Let ξ, η be cycles on X of codimensions a, b respectively, with disjoint supports Y, Z. Assume that their global absolute cohomology classes in $H^{2*}_{\mathscr{T}}(X, *)$ vanish; this is true if their rational equivalence classes lie in $CH^*(X)^{00}_{\mathbf{Q}}$; cf. (4.3.1). Write $V = X - Z$, and let $\widetilde{cl}_{\mathscr{T}}(\eta) \in H^{2b-1}_{\mathscr{T}}(V, b)$ be any cohomology class whose image in $H^{2b}_{\mathscr{T},Z}(X_k, b)$ is $cl_{\mathscr{T},Z}(\eta)$. The local pairing $\langle \xi, \eta \rangle_{X,\mathscr{T}}$ is by definition the image of $-cl_{\mathscr{T},Y}(\xi) \otimes \widetilde{cl}_{\mathscr{T}}(\eta)$ under the composite map

$$\begin{array}{ccccc} H^{2a}_{\mathscr{T},Y}(V, a) \otimes H^{2b-1}_{\mathscr{T}}(V, b) & \xrightarrow{\cup} H^{2N+1}_{\mathscr{T},Y}(V, N+1) & \xrightarrow{\mathrm{Tr}} & H^{1}_{\mathscr{T}}(\mathscr{T}, 1) \\ \downarrow \wr & & & \downarrow \wr (4.9.1) \\ H^{2a}_{\mathscr{T},Y}(X, a) \otimes H^{2b-1}_{\mathscr{T}}(V, b) & \dashrightarrow & & A \end{array}$$

4.12. The cases of interest here are the non-Archimedean case 4.4(i) with $F = k_v^{\mathrm{nr}}$ and the Archimedean case 4.4(ii) with $F = k_v$ (v infinite). We then write $\langle\ ,\ \rangle_{X,v}$ instead of $\langle\ ,\ \rangle_{X,\mathscr{T}}$.

If ξ, η are cycles with disjoint support whose rational equivalence classes belong to $CH^*(X)^{00}_{\mathbf{Q}}$, then for v finite $\langle \xi, \eta \rangle_{X,v}$ is in $\mathbf{Q}$ and (as the notation suggests) is independent of l. The global pairing is given by

$$\langle \xi, \eta \rangle_X = \sum_{v|\infty} \langle \xi, \eta \rangle_{X,v} - \sum_{v} \log q_v \langle \xi, \eta \rangle_{X,v}.$$

4.13. Since we want to be precise about the signs, we give some details of the proof of the above compatibility in the case of a finite place. Write R for the ring of integers of $F = k_v^{\mathrm{nr}}$, and use $\bullet$ to denote supports in the closed fibre $\overline{\mathscr{X}}_v$. Let $\pi: \mathscr{X} \to \operatorname{Spec}\mathscr{O}$ be the structural morphism, and let Y, Z be the supports of ξ, η in X. We then have Diagram 1 (see p. 588).

Let $\hat{\xi} \in H^{2a-2N}_{Y\otimes F \cup \bullet}(\mathscr{X}_R, \pi^! \mathbf{Q}_l(a-N))$ be the cohomology class of the extension of ξ to $\mathscr{X}$, whose existence is assured by the definition of $CH(X)^{00}_{\mathbf{Q}}$. The image of $\hat{\xi} \otimes \widetilde{cl}_v(\eta)$ in the first column is the intersection pairing on $\mathscr{X}_R$; the image of $-\hat{\xi} \otimes \widetilde{cl}_v(\eta)$ around the extreme right-hand edge of the diagram is the cohomological definition from 4.11. The parts of the diagram labelled ②, ③ clearly commute; ④ commutes because the trace map $\pi_! \pi^! \to \mathrm{id}$ is compatible with the boundary in the long exact cohomology sequence; and ⑤ is anticommutative (see SGA4$\frac{1}{2}$, "Cycle" 2.1.3). The desired compatibility follows from the commutativity of ①, which is a consequence of 0.1.

4.14. Using our sign conventions, the sign in Beilinson's definition [2, 2.1.1(i)] should be reversed; (iii) is correct as it stands; and the sign in (ii) depends on the normalisation of the signs in the Mayer-Vietoris sequence.

$$
\begin{array}{ccccccc}
H^{2a-2N}_{Y\cup\bullet}(\mathscr{X}^{\cdot}_R, \pi^!\mathbf{Q}_l(a-N)) \otimes H^{2b-1}(V_F, \mathbf{Q}_l(b)) & \rightarrow & H^{2a-2N}_{Y}(\mathscr{X}^{\cdot}_F, \pi^!\mathbf{Q}_l(a-N)) \otimes H^{2b-1}(V_F, \mathbf{Q}_l(b)) & = & \mathrm{H}^{2a}_Y(X_F, \mathbf{Q}_l(a)) \otimes H^{2b-1}(V_F, \mathbf{Q}_l(b)) \\
\downarrow \mathrm{id}\otimes\partial & & \downarrow \cup & ② & \downarrow \cup \\
H^{2a-2N}_{Y\cup\bullet}(\mathscr{X}^{\cdot}_R, \pi^!\mathbf{Q}_l(a-N)) \otimes H^{2b}_{Z\cup\bullet}(\mathscr{X}^{\cdot}_R, \mathbf{Q}_l(b)) & ① & H^1_Y(V_F, \pi^!\mathbf{Q}_l(1)) & = & H^{2N+1}_Y(V_F, \mathbf{Q}_l(N+1)) \\
\downarrow \cup & & \downarrow & ③ & \\
H^2_\bullet(\mathscr{X}^{\cdot}_R, \pi^!\mathbf{Q}_l(1)) & \overset{\partial}{\leftarrow} & H^1(X_F, \pi^!\mathbf{Q}_l(1)) & & \swarrow \mathrm{Tr} \\
\downarrow \mathrm{Tr} & ④ & \downarrow \mathrm{Tr} & & \\
H^2_\bullet(\mathrm{Spec}\, R, \mathbf{Q}_l(1)) & \overset{\partial}{\leftarrow} & H^1(F, \mathbf{Q}_l(1)) & & \\
\uparrow cl(\bullet) & ⑤ & \downarrow (2.6.1) & & \\
\mathbf{Q}_l & = & \mathbf{Q}_l & &
\end{array}
$$

Diagram 1

5. Comparison of the local pairings

5.0. Let X be as in 4.0, and let $a, b \geq 1$ be integers with $a+b=N+1$. To make the comparison between the motivic and geometric pairings, we assume that X admits a regular model over $\mathscr{O}$ as 4.1 and that the hypothesis 3.6 holds for the motive $M = h^{2a-1}(a)$.

5.1. Let Y, Z, be disjoint closed subschemes of X of codimensions a, b respectively. Let $U = X - Y$, $V = X - Z$. We introduce some further notation. Write $\mathscr{Z}_Y^a(X)$ for the group of cycles of codimension a with coefficients in $\mathbf{Q}$ which are supported on Y, and likewise for Z. Write $\mathscr{Z}_Y^a(X)^0$ for the subgroup of cycles homologically equivalent to zero. Analogously, define

$$H_Y^{2a}(\overline{X}, \mathbf{Q}_l(a))^0 = \ker(H_{\overline{Y}}^{2a}(\overline{X}, \mathbf{Q}_l(a)) \to H^{2a}(\overline{X}, \mathbf{Q}_l(a))).$$

Write $H = \mathscr{Z}_{\overline{Z}}^b(\overline{X})^0$ and $H' = \mathscr{Z}_{\overline{Y}}^a(\overline{X})^0$. These spaces are equipped with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and may be viewed as Artin motives. The cycle class map gives isomorphisms

$$H \otimes \mathbf{Q}_l \simeq H_Z^{2b}(\overline{X}, \mathbf{Q}_l(b))^0, \qquad H' \otimes \mathbf{Q}_l \simeq H_Y^{2a}(\overline{X}, \mathbf{Q}_l(a))^0,$$

which we use without further comment.

5.2. Consider the motive $E = h^{2a-1}(U \operatorname{rel} Z)(a)$ (as in 1.4). There are exact sequences

$$h^{2a-2}(U)(a) \to h^{2a-2}(Z)(a) \to E \to h^{2a-1}(U)(a) \to 0$$

and

$$0 \to M \to h^{2a-1}(U)(a) \to H'(0) \to 0$$

(since $\dim Z = a-1$ and $\operatorname{codim} Y = a$). The trace map defines an isomorphism

$$\frac{h^{2a-2}(Z)(a)}{\mathrm{Im}(h^{2a-2}(U)(a))} \xrightarrow{\sim} H^{\vee}(1).$$

Therefore, E is a motive with $\mathrm{Gr}_\bullet^W E = H^\vee(1) \oplus M \oplus H'(0)$. By 2.5 and the hypotheses of 5.0 the motives $W_{-1}E$ and $E/W_{-2}E$ are motives over $\mathscr{O}$, whence there are the local pairings of §3:

$$b_{v,E}\colon H \times H' \to \begin{cases} \mathbf{Q}_l & \text{for finite } v, \\ \mathbf{R} & \text{for infinite } v. \end{cases}$$

5.3. THEOREM. *If $\xi \in H'$, $\zeta \in H$ and their rational equivalence classes belong to $CH^*(X)_{\mathbf{Q}}^{00}$ then $b_{v,E}(\xi, \zeta) = \langle \xi, \zeta \rangle_{X,v}$ for every place v of k.*

PROOF. We first recall that the local geometric pairing enjoys a basechange property analogous to 3.2—this is almost automatic from the definition.

Therefore, we may assume (enlarging k if necessary) that the action of $\mathrm{Gal}(\bar{k}/k)$ on H and H' is trival. Fix v and take the corresponding $\mathscr{T}$-cohomology as in 4.4. By abuse of notation write $H(n)$ for the objects $H \otimes A(n)$ of $\mathscr{T}$. Define a map θ by the commutativity of the diagram:

$$\begin{array}{ccccccc} & H' & \xrightarrow{cl_{\mathscr{T},Y}} & H^{2a}_{\mathscr{T},Y}(X, a) & \longrightarrow & H^0(\mathscr{T}, \underline{H}^{2a}(X, a)) \\ & \downarrow & & \downarrow & & \downarrow \wr \\ \theta & \mathrm{Fil}^1 H^{2a}_{\mathscr{T},c}(V, a) & \longrightarrow & H^{2a}_{\mathscr{T},c}(V, a) & \xrightarrow{e^0} & H^0(\mathscr{T}, \underline{H}^{2a}_c(V, a)) \\ & \downarrow e^1 & & & & \\ & H^1(\mathscr{T}, \underline{H}^{2a-1}_c(V, a)) & & & & \end{array}$$

(Here $\mathrm{Fil}^\bullet$ and e^i are the filtration and edge homomorphisms coming from the "Hochschild-Serre" spectral sequence

$$E_2^{ij} = H^i(\mathscr{T}, \underline{H}^j_c(V, a)) \Rightarrow H^*_{\mathscr{T},c}(V, a);$$

cf. 0.3 above.)

5.4. PROPOSITION. *θ is the classifying map for the extension:*

$$\begin{array}{l} 0 \to \underline{H}^{2a-1}(X \,\mathrm{rel}\, Z, a) \to \underline{H}^{2a-1}(U \,\mathrm{rel}\, Z, a) \to H'(0) \to 0 \\ \qquad \| \\ \quad \underline{H}^{2a-1}_c(V, a) \end{array}$$

This is a very mild generalisation of Lemmas 9.4 (l-adic) and 9.2 and Remark 9.7(c) (Hodge) in **[16]**. □

We now have split short exact sequences

$$0 \to H^\vee(1) \xrightarrow{-\partial_c} \underline{H}^{2a-1}_c(V, a) \to \underline{H}^{2a-1}(X, a) \to 0,$$
$$0 \to \underline{H}^{2b-1}(X, b) \to \underline{H}^{2b-1}(V, b) \xrightarrow{\partial} H(0) \to 0,$$

which are dual, by Lemma 0.2. Choose splittings in $\mathscr{T}$

$$\sigma: H(0) \to \underline{H}^{2b-1}(V, b),$$
$$\tau: \underline{H}^{2a-1}_c(V, a) \to H^\vee(1)$$

which are adjoint with respect to this duality; in other words, so that the diagram

$$\begin{array}{ccc} \underline{H}^{2a-1}_c(V, a-1) \otimes H & \xrightarrow{1\otimes\sigma} & \underline{H}^{2a-1}_c(V, a-1) \otimes \underline{H}^{2b-1}(V, b) \\ \downarrow \tau\otimes 1 & & \downarrow \text{P.D.} \\ H^\vee \otimes H(0) & \xrightarrow{\text{contract}} & A(0) \end{array} \tag{5.4.1}$$

is commutative. Without loss of generality we may also assume (by choice of $\widetilde{cl}_{\mathscr{T}}$ in 4.8) that the diagram

$$\begin{array}{ccc} H \otimes A & = & H^0(\mathscr{T}, H(0)) \\ \downarrow \widetilde{cl}_{\mathscr{T}} & & \downarrow H^0(\sigma) \\ H^{2b-1}_{\mathscr{T}}(V, b) & \xrightarrow{e^0} & H^0(\mathscr{T}, \underline{H}^{2b-1}(V, b)) \end{array}$$

commutes. We now have a large diagram (see Diagram 2 on p. 592) in which the only group that remains to be defined is

$$H^{2a}_{\mathcal{T},Y}(X, a)^0 \stackrel{\text{def}}{=} \ker(H^{2a}_{\mathcal{T},Y}(X, a) \to H^0(\mathcal{T}, \underline{H}^{2a}(X, a)).$$

The map $H' \otimes H \to H^1(\mathcal{T}, 1)$ obtained by following the arrows around the top of the diagram is the negative of the local geometric pairing, whereas that obtained by going around the bottom is the negative of the local motivic pairing b_v. The diagram is clearly commutative, except possibly for the parts labelled **A** and **B**; and these are consequences of Lemma 0.4 and (5.4.1), respectively. □

6. A motivic height pairing

6.0. We assume the notation of §3, but assume now that $k = \mathbf{Q}$ and that G_1, G_2 have trivial $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ action. We aim to assemble the local pairings b_v to make a global "motivic height pairing".

6.1. PROPOSITION. *Let E' be a mixed motive over $\mathbf{Q}$, such that*

$$\mathrm{Gr}^W_{-1}(E') = M, \qquad \mathrm{Gr}^W_0(E') = G_1(0), \qquad \mathrm{Gr}^W_{-2}(E') = G_2(1),$$

and $\mathrm{Gr}^W_i(E') = 0$ for $i > 0$, $i < -2$. Let

$$M_1 = E'/W_{-2}(E'), \qquad M_2 = W_{-1}(E'),$$

and assume that M_1, M_2 are motives over $\mathbf{Z}$. Assume the truth of Conjecture 3.9 for the pairings $b_{p,E'}$. Then there is a motive E over $\mathbf{Z}$ with $W_{-1}E = M_2$, $E/W_{-2}(E') = M_1$ and

$$b_{\infty,E} = b_{\infty,E'} - \sum_p \log p \cdot b_{p,E'}.$$

PROOF. We have

$$\begin{aligned}\mathrm{Ext}^1_{\mathbf{Q}}(G_1(0), G_2(1)) &= \mathrm{Hom}(G_1, G_2) \otimes \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(1)) \\ &= \mathrm{Hom}(G_1 \otimes G_2^\vee, \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(1))).\end{aligned}$$

Write $K\langle p\rangle$ for the 1-motive $[\mathbf{Z} \xrightarrow{\phi} \mathbf{G}_m]$, where $\phi(1) = p$ (as in §1), and consider

$$\sum_p [K\langle p\rangle] b_{p,E'} \in \mathrm{Hom}(G_1 \otimes G_2^\vee, \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(1))).$$

Let $[F]$ denote its image under the natural map

$$\mathrm{Ext}^1_{\mathbf{Q}}(G_1(0), G_2(1)) \to \mathrm{Ext}^1_{\mathbf{Q}}(G_1(0), M_2).$$

Then, by 3.12, 3.14, $b_{p,F} = b_{p,E'}$ and $b_{\infty,F} = \sum_p b_{p,E'} \otimes b_{\infty,K\langle p\rangle} = \sum_p \log p \cdot b_{p,E'}$. By 3.13 we can therefore take E to be an extension in the class of $[E'] - [F]$. □

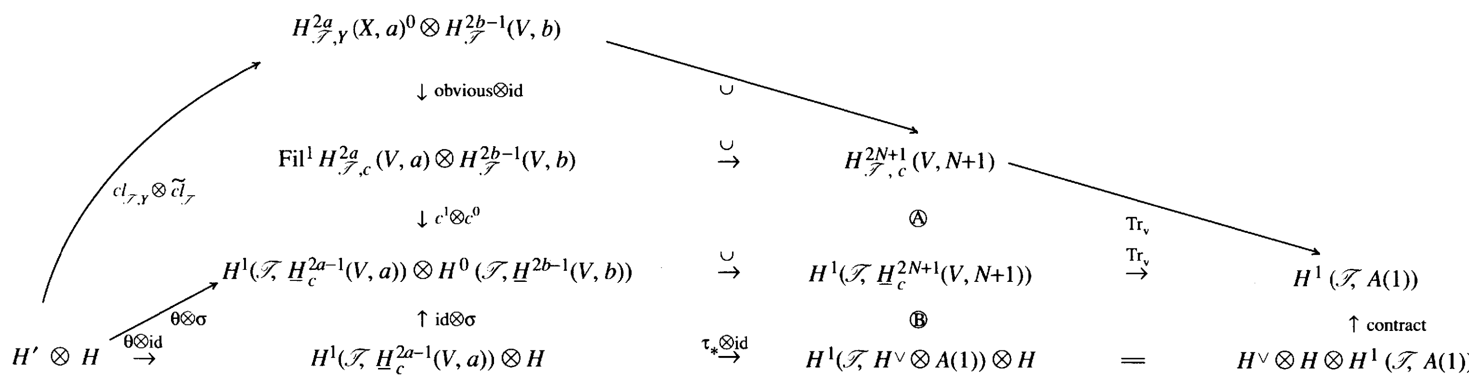
$H^{2a}_{\mathcal{T},Y}(X,a)^0 \otimes H^{2b-1}_{\mathcal{T}}(V,b)$
↓ obvious⊗id
$\mathrm{Fil}^1 H^{2a}_{\mathcal{T},c}(V,a) \otimes H^{2b-1}_{\mathcal{T}}(V,b)$
↓ $c^1 \otimes c^0$
$H^1(\mathcal{T}, \underline{H}^{2a-1}_c(V,a)) \otimes H^0(\mathcal{T}, \underline{H}^{2b-1}(V,b))$
↑ id⊗σ
$H^1(\mathcal{T}, \underline{H}^{2a-1}_c(V,a)) \otimes H$
$H' \otimes H$
$cl_{\mathcal{T},Y} \otimes \widetilde{cl}_{\mathcal{T}}$
θ⊗σ
θ⊗id
∪
$H^{2N+1}_{\mathcal{T},c}(V,N+1)$
Ⓐ
$H^1(\mathcal{T}, \underline{H}^{2N+1}_c(V,N+1))$
Ⓑ
$\tau_* \otimes \mathrm{id}$
$H^1(\mathcal{T}, H^\vee \otimes A(1)) \otimes H$
Tr_v
$H^1(\mathcal{T}, A(1))$
↑ contract
$H^\vee \otimes H \otimes H^1(\mathcal{T}, A(1))$

Diagram 2

6.2. [1] The motivation for the preceding proposition comes from the following "thought-experiment". Let us imagine that the following hypotheses were known to be true (cf. 2.8):

6.3. HYPOTHESIS. $\mathrm{Ext}^2_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$, *and* $\mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(1))$ *is generated by the classes of the* 1*-motives* $K\langle p\rangle$.

Notice that the second part of the hypothesis implies that $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$. As for the first part, one should expect $\mathrm{Ext}^j_{\mathbf{Z}}(M, M') = \mathrm{Ext}^j_{\mathbf{Q}}(M, M') = 0$ for all pure motives M, M' and every $j > 1$. This is closely related to the conjectural injectivity of the Abel-Jacobi map (for varieties over number fields).

6.4. Now let M be pure of weight -1, and let G, G' be any finite-dimensional subspaces

$$G \subset \mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1)),$$
$$G' \subset \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M).$$

Then there are corresponding motives M_i over $\mathbf{Z}$:

$$0 \to M \to M_1 \to G'(0) \to 0,$$
$$0 \to G^\vee(1) \to M_2 \to M \to 0.$$

In our hypothetical setting, we can apply $\mathrm{Ext}^1_{\mathbf{Z}}(G'(0), -)$ to the second sequence to see that there is (up to isomorphism) a unique object E of $\mathcal{MM}_{\mathbf{Z}}$ together with isomorphisms

$$\alpha_1 \colon W_{-1}E \xrightarrow{\sim} M_1, \qquad \alpha_2 \colon E/W_{-2}E \xrightarrow{\sim} M_2$$

such that the induced isomorphisms

$$\mathrm{Gr}^W_{-1}(\alpha_i) \colon \mathrm{Gr}^W_{-1}(E) \xrightarrow{\sim} \mathrm{Gr}^W_{-1}(M_i) = M$$

for $i = 1, 2$ are equal.

6.5. This defines a canonical pairing

$$b_{\infty, E} \colon G' \times G \to \mathbf{R},$$

and it follows from 3.11 that if $H \subset G$ and $H' \subset H$ then the pairing between H and H' is simply the restriction of that between G and G'. By passing to the inductive limit (not required if the $\mathrm{Ext}^1_{\mathbf{Z}}$-groups are finite dimensional), this defines a canonical pairing, the *motivic height pairing*

$$\langle \cdot, \cdot \rangle_{\mathcal{M}} \colon \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M) \otimes \mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1)) \to \mathbf{R}.$$

[1] The hypothetical discussion that follows is independent of the rest of the paper.

6.6. Let us attempt to clarify the relationship between Hypothesis 6.3 and the hypotheses of 6.1. Assume the truth of 6.3, and suppose M_1, M_2 are motives over $\mathbf{Z}$ as in 6.1. Then by contemplating the exact sequences

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathbf{Z}}(G_1(0), G_2(1)) & \to & \mathrm{Ext}^1_{\mathbf{Z}}(G_1(0), M_2) & \to & \mathrm{Ext}^1_{\mathbf{Z}}(G_1(0), M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathbf{Q}}(G_1(0), G_2(1)) & \to & \mathrm{Ext}^1_{\mathbf{Q}}(G_1(0), M_2) & \to & \mathrm{Ext}^1_{\mathbf{Q}}(G_1(0), M) \end{array}$$

one sees that the set of isomorphism classes of motives E' over $\mathbf{Q}$ with $W_{-1}E' = M_2$, $E'/W_{-2}E' = M_1$ is a homogeneous space under $\mathrm{Hom}(G_1 \otimes G_2^\vee, \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(1)))$. From the above discussion we see that this set contains a unique isomorphism class $[E]$, where E is a motive over $\mathbf{Z}$.

In other words, 6.3 not only gives the conclusions of 6.1 but also shows that E is unique and depends only on M_1 and M_2; and the auxiliary motive E' is not needed for the construction of E.

6.7. Remark. Without assuming that $k = \mathbf{Q}$ it is still possible to write down a height pairing under the same kind of hypotheses as 6.3. We can go through the arguments of 6.4 with $\mathbf{Z}$ replaced by $\mathscr{O}$. The difference is that as in general $\mathrm{Ext}^1_{\mathscr{O}}(\mathbf{Q}(0), \mathbf{Q}(1)) \neq 0$, the iterated extension E will no longer be unique. Picking any such extension we may nevertheless consider the sum

$$\sum_{v|\infty} b_{v,E} \colon G' \times G \to \mathbf{R}$$

which by the product formula (2.8.1) should be independent of the choice of E.

7. Comparison of global pairings; special values of L-functions

7.0. Let X be a projective and smooth variety, as in §§4 and 5, and additionally assume that $k = \mathbf{Q}$.

7.1. Let $M = h^{2a-1}(X)(a)$. There are canonical homomorphisms

$$\alpha \colon CH^a(X)^0 \to \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), M), \qquad \beta \colon CH^b(X)^0 \to \mathrm{Ext}^1_{\mathbf{Q}}(M, \mathbf{Q}(1))$$

defined as follows. Let $Y \subset X$ be a closed subscheme of codimension a, and let $\xi \in \mathscr{Z}^a_Y(X)^0$ be a cycle supported in Y, homologically equivalent to zero. If $H(-,\cdot)$ denotes the singular, l-adic or de Rham cohomology, then the exact cohomology sequence

$$0 \to H^{2a-1}(X, a) \to H^{2a-1}(X - Y, a) \to H^{2a}_Y(X, a) \to H^{2a}(X, a)$$

determines a unique element of $H^{2a-1}(X-Y, a)/H^{2a-1}(X, a)$ whose image in $H^{2a}_Y(X, a)$ is the cohomology class of ξ. This gives a morphism $\mathbf{Q}(0) \xrightarrow{\xi} h^{2a-1}(X - Y)(a)/h^{2a-1}(X)(a)$ in $\mathscr{M}\mathscr{M}_{\mathbf{Q}}$, whence by pullback an extension

$$0 \to M \to E_\xi \to \mathbf{Q}(0) \to 0.$$

7.2. Theorem **[16**, §9]. (i) *The mapping* $\xi \mapsto [E_\xi]$ *is a homomorphism, and* $[E_\xi]$ *depends only on the rational equivalence class of* ξ.

(ii) *The class of the* $\mathcal{T}$*-realisation of* E_ξ *in* $\mathrm{Ext}^1_{\mathcal{T}}(\mathbf{1}_{\mathcal{T}}, M_{\mathcal{T}}) = H^1(\mathcal{T}, \underline{H}^{2a-1}(X, a))$ *is the same as the image of the cycle class* $cl_l(\xi)$ *under the edge homomorphism*

$$\ker(H^{2a}_{\mathcal{T}}(X, a) \overset{\text{res}}{\to} H^0(\mathcal{T}, \underline{H}^{2a}(X, a))) \to H^1(\mathcal{T}, \underline{H}^{2a-1}(X, a))$$

of the Hochschild-Serre spectral sequence.

7.3. By passage to the limit over Y we obtain the desired homomorphism α. Now by Poincaré duality there is an isomorphism $h^{2b-1}(X)(b) \simeq M^\vee(1)$, which in accordance with the conventions for signs (cf. Lemma 0.2) is normalised by taking the cup-product in the order

$$h^{2a-1}(X)(a) \times h^{2b-1}(X)(b) \to h^{2N}(X)(N+1) \overset{\text{Tr}}{\to} \mathbf{Q}(1).$$

We thus obtain a homomorphism

$$\begin{aligned} CH^b(X)^0 \to \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), h^{2b-1}(X)(b)) &= \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), M^\vee(1)) \\ &= \mathrm{Ext}^1_{\mathbf{Q}}(M, \mathbf{Q}(1)), \end{aligned}$$

which we take to be $-\beta$.

7.4. In a moment we will need an alternative description of β. Consider the exact sequence
(7.4.1)

$$h^{2a-2}(X)(a) \to h^{2a-2}(Z)(a) \to h^{2a-1}(X \operatorname{rel} Z)(a) \to h^{2a-1}(X)(a) \to 0.$$

A cycle $\zeta \in \mathcal{Z}^b_Z(X)$ gives a trace map $h^{2a-2}(Z) \to \mathbf{Q}(-a+1)$, which vanishes on the image of $h^{2a-2}(X)$ if ζ is homologically equivalent to zero. By pushout we obtain directly an extension of M by $\mathbf{Q}(1)$, whose class we denote $\beta'(\zeta)$.

7.5. Proposition. *The extension classes* $\beta(\zeta)$ *and* $\beta'(\zeta)$ *are equal.*

Proof. By the compatibility 0.2, the l-adic realisation of the exact sequence (7.4.1) is dual to the local cohomology sequence

$$0 \to H^{2b-1}(X, b-1) \to H^{2b-1}(U, b-1) \overset{-\partial}{\to} H^{2b}_Z(X, b-1) \to H^{2b}(X, b-1).$$

By the comparison theorems the same is true in the other realisations (in a way compatible with the comparison isomorphisms) and the trace map is dual to the cycle class map $\mathcal{Z}^b_Z(X) \to H^{2b}_Z(X, b)$. The extension classes therefore agree. □

7.6. One hopes that $\alpha\otimes\mathbf{Q}$, $\beta\otimes\mathbf{Q}$ are isomorphisms, and one might dream that $\langle\alpha(x), \beta(y)\rangle_{\mathcal{M}} = \langle x, y\rangle_X$. In the absence of 6.3 we will be content with the following motivic interpretation of $\langle -, -\rangle_X$.

7.7. Theorem. *Let $G \subset CH^b(X)^{00}_{\mathbf{Q}}$, $G' \subset CH^a(X)^{00}_{\mathbf{Q}}$ be any finite-dimensional subspaces. Assume the hypotheses of* 5.0 *hold. Then there is a motive $\widetilde{M}$ over $\mathbf{Z}$ with the following properties*:

(i) $\mathrm{Gr}^W_0 \widetilde{M} = G'(0)$, $\mathrm{Gr}^W_{-1} \widetilde{M} = M$, *and* $\mathrm{Gr}^W_{-2} \widetilde{M} = G^\vee(1)$; $\mathrm{Gr}^W_i \widetilde{M} = 0$ *for all other* i.

(ii) *The classes of the intermediate extensions*

$$0 \to M \to \widetilde{M}/W_{-2}\widetilde{M} \to G'(0) \to 0,$$
$$0 \to G^\vee(1) \to W_{-1}\widetilde{M} \to M \to 0$$

are given by α, β *respectively.*

(iii) *If* $x \in G'$, $y \in G$ *then* $b_{\infty, \widetilde{M}}(x, y) = \langle x, y\rangle_X$.

Proof. We first construct a motive E' over $\mathbf{Q}$ satisfying (i) and (ii). By the moving lemma, there are disjoint closed subschemes Y, Z of X, of codimensions a, b respectively, such that any element of G' (resp. G) is rationally equivalent to a cycle supported in Y (resp. Z). Using the same notations as in 5.1, we define $E'' = h^{2a-1}(U \operatorname{rel} Z)(a)$.

Choose splittings over G and G'

$$spl_Y\colon G' \to \mathscr{Z}^a_Y(X)^0 \subset H', \qquad spl_Z\colon G \to \mathscr{Z}^b_Z(X)^0 \subset H$$

of the cycle class maps. Applying to E'' pullback by spl_Y and pushout by the transpose of spl_Z, we obtain a motive E' with $\mathrm{Gr}^W_\bullet E' = G^\vee(1) \oplus M \oplus G'(0)$. From the construction and 7.2 it is clear that the extensions $E'/W_{-2}E'$ and $W_{-1}E'$ are classified by the homomorphisms α, β and in particular do not depend on the choice of splittings. (Of course, E' itself does in general depend on this choice.)

By 5.3 and 3.11 it follows that, for any $y \in G'$ and $z \in G$,

$$b_{v, E'}(y, z) = \langle spl_Y(y), spl_Z(z)\rangle_{\mathscr{G}, v} \quad \text{for } v = \infty \text{ or } p.$$

This and the hypotheses imply that the pairings $b_{p, E'}$ are $\mathbf{Q}$-valued, independent of l. By 6.1 we may therefore construct $\widetilde{M} = E$, a motive over $\mathbf{Z}$, satisfying all the requirements of the theorem. □

7.8. To explain the connection with special values of L-functions, we first briefly review the reformulation of the conjectures of Beilinson and others in terms of periods of mixed motives [**21**]. If M is a pure motive of weight $w \in \mathbf{Z}$, then under some general hypotheses it is possible to construct a certain associated mixed motive, which in the present context we have denoted $\widetilde{M}$, in four steps:

(i) First remove any submotive isomorphic to $\mathbf{Q}(0)$. This means replace M by the quotient motive M_1 in the exact sequence

$$0 \to \operatorname{Hom}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0) \to M \to M_1 \to 0.$$

(ii) Next remove any quotient motive isomorphic to $\mathbf{Q}(1)$. This replaces M_1 by M_2, which is the kernel:

$$0 \to M_2 \to M_1 \to \operatorname{Hom}(M_1, \mathbf{Q}(1))^\vee \otimes \mathbf{Q}(1) \to 0.$$

(iii) Now take the universal extension of M_2 by $\mathbf{Q}(1)$ on the left and by $\mathbf{Q}(0)$ on the right.

This comes from the two exact sequences

$$0 \to \operatorname{Ext}^1_{\mathbf{Z}}(M_2, \mathbf{Q}(1))^\vee \otimes \mathbf{Q}(1) \to M_3 \to M_2 \to 0,$$
$$0 \to M_3 \to \widetilde{M} \to \operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M_3) \otimes \mathbf{Q}(0) \to 0.$$

Using the hypothesis $\operatorname{Ext}^i_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$ it is easily seen that in step (iii) the order in which the extensions are made is immaterial, and that $\widetilde{M}$ has a three-step filtration, with associated graded pieces $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0)$, M_2, and $\operatorname{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1))^\vee \otimes \mathbf{Q}(1)$.

7.9. In **[21]** it was explained that the conjectures of Beilinson and Bloch are equivalent to the conjecture:

The mixed motive $\widetilde{M}$ *is critical* (see 3.15) *and* $L(\widetilde{M}, 0)/c^+(\widetilde{M}) \in \mathbf{Q}^*$

7.10. This is a special case of the conjectures A–C of **[21]**. In fact it was shown in §VI of that paper that this is in some sense the essential case of those conjectures. We now explain this in greater detail in the present case when $M = h^{2a-1}(a)$ has weight -1. This is the only case in which both of the groups $\operatorname{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1))$ and $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M)$ can be nonzero. We write ρ, ρ' for their dimensions (assumed to be finite in the entire discussion). To make the link between the motivic and the geometric setting we need to assume that the maps α, β are isomorphisms.

The L-function of $\widetilde{M}$ is

$$L(\widetilde{M}, s) = L(M, s) \cdot \zeta(s+1)^\rho \cdot \zeta(s)^{\rho'}.$$

(Note that we have exact equality here as $\widetilde{M}$ is a motive over $\mathbf{Z}$; were this not the case we would have to remove one or more Euler factors of the form $(1 - p^{-s})^{-1}$, which would change the order of $L(\widetilde{M}, s)$ at $s = 0$.) On the other hand, combining 7.7(iii) with 3.15, we see that $\widetilde{M}$ is critical if and only if the height pairing $\langle \cdot, \cdot \rangle_{\mathscr{G}}$ is nonsingular; and if this is the case, then $c^+(\widetilde{M}) = c^+(M) \cdot \det\langle \cdot, \cdot \rangle_{\mathscr{G}}$. Thus the "motivic" conjecture 7.9 is in this case equivalent to the conjunction of the statements:

- The geometric height pairing $\langle \cdot, \cdot \rangle_{\mathscr{G}}$ is nonsingular.
- $\operatorname{ord}_{s=0} L(M, s) = \rho$, and the leading coefficient in the Taylor series of $L(M, s)$ about $s = 0$ is a rational multiple of $c^+(M) \cdot \det\langle \cdot, \cdot \rangle_{\mathscr{G}}$.

This is precisely the generalisation by Beilinson and Bloch of the Birch–Swinnerton-Dyer conjectures to arbitrary Chow groups.

REFERENCES

1. A. A. Beilinson, *Notes on absolute Hodge cohomology*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Contemp. Math. **55** (1986), 35–68.
2. ——, *Height pairing between algebraic cycles*, Current Trends in Arithmetical Algebraic Geometry (K. Ribet, ed.), Contemp. Math. **67** (1987), 1–24.
3. A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analyse et Topologie sur les Espaces Singuliers I, Astérisque **100** (1982).
4. S. Bloch, *Height pairing for algebraic cycles*, J. Pure Appl. Algebra **34** (1984), 119–145.
5. ——, *Cycles and biextensions*, Algebraic K-Theory and Algebraic Number Theory (M. R. Stein and R. K. Dennis, eds.), Contemp. Math. **83** (1989), 19–30.
6. J.-L. Brylinski, *Heights for local systems on curves*, Duke Math. J. **59** (1989), 1–26.
7. J. Carlson, *Extensions of mixed Hodge structures*, J. Géom. Algébrique Angers (1979), 107–127; Sijthoff & Noordhoff, Alphen aan den Rijn, Germantown, MD, 1980.
8. P. Deligne, *Théorie de Hodge*. III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–78.
9. ——, Letter to C. Soulé, January 20, 1985.
10. ——, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over **Q** (Y. Ihara, K. Ribet, and J.-P. Serre, eds.), Math. Sci. Res. Inst. Publ. **16**, Springer-Verlag, New York and Berlin, 1989.
11. P. Deligne, J. Milne, A. Ogus, and K.-y. Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math., vol. 900, Springer-Verlag, New York and Berlin, 1982.
12. J. Gamst and K. Hoechsmann, *Products in sheaf cohomology*, Tôhoku Math. J. **22** (1970), 143–162.
13. H. Gillet, *An introduction to higher dimensional Arakelov theory*, Current Trends in Arithmetical Algebraic Geometry (K. Ribet, ed.), Contemp. Math. **67** (1987), 209–228.
14. H. Gillet and C. Soulé, *Intersection sur les variétés d'Arakelov*, C. R. Acad. Sci. Paris Sér. I Math. **12** (1984), 563–566.
15. U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.
16. ——, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, New York, 1990.
17. ——, *Deligne homology, Hodge D-conjecture, and motives* (M. Rapoport, N. Schappacher, and P. Schneider, eds.), Beilinson's Conjectures on Special Values of L-Functions, Academic Press, New York, 1988, pp. 305–372.
18. B. Mazur and J. Tate, *Biextensions and height pairings*, Arithmetic and Geometry, vol. I (M. Artin and J. Tate, ed.), **28**, Birkhäuser, Basel, 1983, pp. 195–237.
19. J. Nekovář, *On p-adic height pairings*, in Séminaire de théories des nombres de Paris, 1990/91, Progress in Math. **108** (S. David, ed.), Birkhaüser, Boston, 1993, pp. 127–202.
20. M. Rapoport, *On the bad reduction of Shimura varieties*, Automorphic Forms, Shimura Varieties, and L-Functions (L. Clozel and J. S. Milne, eds.), Academic Press, New York, 1990, pp. 253–322.
21. A. J. Scholl, *Remarks on special values of L-functions*, L-Functions and Arithmetic (J. H. Coates and M. J. Taylor, eds.), Cambridge Univ. Press, Cambridge and New York, 1991, pp. 373–392.
22. J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. **36** (1976), 257–274.

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Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L

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Introduction

Ce texte comprend trois chapitres: le premier (resp. le second) concerne la cohomologie galoisienne des représentations ℓ-adiques du groupe de Galois d'une extension finie de $\mathbb{Q}_p$ (resp. $\mathbb{Q}$) et le troisième, de nature presqu'entièrement spéculative, les valeurs des fonctions L des motifs sur un corps de nombres.

Commençons par expliquer l'objectif du troisième. Soit F une extension finie de $\mathbb{Q}$. Soient X une variété projective lisse sur F, $i \in \mathbb{N}$ et $m \in \mathbb{Z}$. Des travaux de Deligne, Beilinson, et Bloch (**[De79, Be85, Be87, Bl84]**, ...) ont permis de donner l'ordre conjectural r en $s = m$ du zéro (ou pôle) éventuel de la fonctions $L(h^i(X), s)$ ainsi que la valeur, à multiplication par un nombre rationnel non nul près, de

$$L^*(h^i(X), m) = \lim_{s \mapsto m} (s-m)^r . L(h^i(X), s).$$

Ces travaux étaient inspirés par les conjectures de Birch et Swinnerton-Dyer qui donnent la valeur *exacte* de $L^*(h^1(X), 1)$ lorsque X est une variété abélienne. Il était tentant de généraliser ces conjectures pour obtenir une valeur exacte de $L^*(h^i(X), m)$ pour X, i et m arbitraires. C'est ce qu'ont réussi à faire Bloch et Kato **[BK90]**.

Dans les conjectures de Birch et Swinnerton-Dyer, un point essentiel est que, si ℓ est un nombre premier, $T_\ell(X)$ le module de Tate de X et $V_\ell(X) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(X)$, la variété abélienne X permet de fabriquer un sous-$\mathbb{Q}_\ell$-espace vectoriel $H^1_f(F, V_\ell(X))$ de $H^1(F, V_\ell(X))$: c'est celui qui est engendré par l'image de $X(F)$.

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Dans le cas général, si $V_\ell(X)$ est remplacé par $h^i(X)_\ell(m) = H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell(m))$ (où $\overline{F}$ est une clôture algébrique de F), Bloch et Kato définissent le sous-groupe $H^1_f(F, h^i(X)_\ell(m))$ de $H^1(F, h^i(X)_\ell(m))$ par des conditions locales (cf. I, n°3.3 et II, n°1.3 ; par exemple si $\mathfrak{p}$ est une place de bonne réduction, i.e. telle que $h^i(X)_\ell(m)$ est non ramifiée (resp. cristalline) en $\mathfrak{p}$ si $\mathfrak{p}$ ne divise pas (resp. divise) ℓ, on demande que l'extension de $\mathbb{Q}_\ell$ par $h^i(X)_\ell(m)$ correspondante ait encore bonne réduction).

Le but essentiel du chapitre III est d'énoncer des variantes des conjectures de Bloch et Kato (et donc aussi de celles de Deligne et Beilinson). Par rapport au point de vue initial de Bloch et Kato, il y a deux changements importants (largement inspirés des travaux de Bloch sur les conjectures de Birch et Swinnerton-Dyer **[Bl80]** et de certaines idées de Deligne **[De85]**, Scholl **[Sc91]** et Lichtenbaum **[Li72]**). La philosophie de ces changements est la suivante.

a) Les groupes de K-théorie algébrique qui interviennent dans le point de vue "classique" doivent pouvoir s'interpréter comme certains Ext^i de la catégorie des "motifs mixtes sur F"; on y gagne de ne plus avoir besoin de K-théorie et de pouvoir travailler avec des motifs mixtes arbitraires; on y perd d'avoir à travailler avec une catégorie dont on n'est même pas sûr de connaître la bonne définition!

b) Les formules compliquées faisant intervenir des nombres de Tamagawa, des ordres de groupes de Shafarevich, ne sont que le résultat du calcul explicite d'une formule "intrinsèque" faisant intervenir certaines caractéristiques d'Euler-Poincaré; on y gagne une grande simplification de la formulation des conjectures, une plus grande sécurité aussi (on craint moins d'oublier une puissance de $2, \ldots$), une plus grande généralité également, la formule gardant un sens dans tous les cas, la possibilité de vérifier relativement facilement différentes compatibilités; cela ne dispense pas, dans les cas particuliers auxquels on s'intéresse, de faire des calculs si l'on veut une formule "explicite".

Dans un souci de simplification, nous n'avons parlé ici que de motifs sur F à coefficients dans $\mathbb{Q}$, mais ce genre de conjectures s'étend aux motifs à coefficients dans un corps de nombres arbitraire (cf. **[Fo92]**).

Nous n'avons pas non plus:

i) explicité comment, modulo des interprétations convenables des groupes de K-théorie, des régulateurs et des hauteurs qui interviennent, ces conjectures redonnent les conjectures de Deligne-Beilinson et Bloch-Kato;

ii) donné d'exemples;

iii) expliqué quels sont les résultats connus.

Pour ces trois points, voir **[FP91, Fo92]** et les textes de Nekovar et Scholl dans ce volume; nous conseillons vivement au lecteur de regarder lui-même ce que donne la théorie développée ici appliquée à ses motifs "préférés".

Disons quelques mots des deux premiers chapitres: on y introduit toute une hiérarchie des représentations ℓ-adiques du groupe de Galois d'une extension finie de $\mathbb{Q}_p$ (chap. I) et de $\mathbb{Q}$ (chap. II) et on étudie quelques propriétés de la cohomologie de ces représentations. Les développements qu'on y trouvera sont plus importants que ce qui est strictement nécessaire pour le chapitre III. Nous avons trouvé que certains d'entre eux donnent des indications sur les propriétés que semble avoir la catégorie des motifs mixtes sur un corps de nombres et nous espérons qu'une partie d'entre eux devrait se révéler utile dans certains des travaux qui ne manqueront pas de se développer autour des conjectures de Bloch et Kato. Nous avons adopté un point de vue résolument élémentaire, i.e., nous avons travaillé avec la cohomologie galoisienne, bien que l'utilisation de la topologie étale simplifie certaines descriptions (cf. **[Fo92]**).

On aura compris que cet article reprend beaucoup des idées et résultats de l'article original de Bloch et Kato et nous avons renoncé à les citer à chaque fois que nous reprenons ce qu'ils ont fait.[1] De même nous avons renoncé à citer Deligne, Jannsen, et Scholl quand nous utilisons certaines de leurs idées dans le chapitre III. Signalons aussi qu'il y a une intersection importante des chapitres I et II avec des travaux de Niziol **[Ni]** et de Kato **[Ka]**. Enfin, les notes **[FP91]** peuvent être considérées comme une introduction ou comme un résumé du présent texte.

Plan

[1] Nous nous sommes efforcés d'utiliser des notations compatibles avec celles de Bloch et Kato, voir toutefois les remarques des n°I, 3.3.3 et II, 1.3.1.

Outils, conventions et notations

Le lecteur est invité à ne lire ce paragraphe qu'en cas de besoin.

0.1. Catégories tannakiennes et représentations ℓ-adiques. On renvoit à **[DM82]** pour les généralités sur les catégories tannakiennes (dont le rôle ici est essentiellement figuratif). Dans ce texte, sauf mention explicite du contraire, les *catégories tannakiennes* que l'on considère sont des catégories tannakiennes *neutres* sur un corps E. Si $\mathscr{C}$ est une telle catégorie, le choix d'un foncteur fibre permet d'identifier $\mathscr{C}$ à la catégorie des représentations E-linéaires de dimension finie d'un groupe pro-algébrique G défini sur E. *L'objet-unité* $\mathbf{1}_{\mathscr{C}}$ de $\mathscr{C}$ est E muni de l'action triviale de G.

Pour tout objet V de $\mathscr{C}$, on pose $H^0(\mathscr{C}, V) = \mathrm{Hom}_{\mathscr{C}}(\mathbf{1}_{\mathscr{C}}, V)$ et, pour tout $i \in \mathbb{Z}$, on note $H^i(\mathscr{C}, -)$ le i-ième foncteur dérivé de $H^0(\mathscr{C}, -)$ (on a $H^i(\mathscr{C}, V) = 0$ si $i < 0$).

Une *sous-catégorie tannakienne* de $\mathscr{C}$ est une sous-catégorie pleine, contenant $\mathbf{1}_{\mathscr{C}}$, stable par sous-objet, quotient, somme directe, produit tensoriel et dual (et s'identifie à la catégorie des représentations E-linéaires de dimension finie d'un quotient de G).

Si $\mathscr{D}$ est une sous-catégorie tannakienne de $\mathscr{C}$, l'inclusion $j^*: \mathscr{D} \to \mathscr{C}$ admet un adjoint à gauche $j_*: \mathscr{C} \to \mathscr{D}$ ($j_*(V)$ est le plus grand sous-objet de V qui est dans $\mathscr{D}$) et, pour tout i, on dispose d'une application naturelle

$$H^i(\mathscr{D}, j_*V) \to H^i(\mathscr{C}, V)$$

(qui est un isomorphisme pour $i = 0$ et est injective pour $i = 1$).

Soient G un groupe profini et ℓ un nombre premier. Une *représentation ℓ-adique de G* est un $\mathbb{Q}_\ell$-espace vectoriel de dimension finie, muni d'une action linéaire et continue de G. Ces représentations forment de manière naturelle une catégorie tannakienne neutre sur $\mathbb{Q}_\ell$, que l'on note $\mathbf{Rep}_{\mathbb{Q}_\ell}(G)$ (le groupe pro-algébrique correspondant est la limite projective de la clôture zariskienne de l'image de G dans chacune de ces représentations). Pour tout $i \in \mathbb{Z}$ et toute représentation ℓ-adique V de G, on pose $H^i(G, V) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell}(G), V)$.

0.2. Représentations $\mathbb{Z}_\ell$-adiques et cohomologie continue ([Ta76]). Soient G un groupe profini et ℓ un nombre premier. On appelle *représentation $\mathbb{Z}_\ell$-adique de G* la donnée d'un $\mathbb{Z}_\ell$-module de type fini muni d'une action linéaire et continue de G.

Plus généralement, si T est un $\mathbb{Z}_\ell$-module topologique muni d'une action linéaire et continue de G, pour tout $i \in \mathbb{N}$, on note $C(G, T)$ *le complexe des cochaînes continues de G à valeurs dans T*. On a donc $C^i(G, T) = 0$

pour i entier < 0, tandis que si $i \in \mathbb{N}$, $C^i(G, T)$ est le $\mathbb{Z}_\ell$-module des applications continues de G^i dans T, le cobord

$$\delta\colon C^{i-1}(G, T) \to C^i(G, T)$$

étant défini par la formule usuelle

$$\begin{aligned}\delta f(g_1, g_2, \dots, g_i) = g_1(f(g_2, g_3, \dots, g_i)) \\ + \sum_{1 \le r < i} f(g_1, \dots, g_{r-1}, g_r g_{r+1}, g_{r+2}, \dots, g_i) \\ + (-1)^i \cdot f(g_1, g_2, \dots, g_{i-1}).\end{aligned}$$

On note $Z^i(G, T)$ (resp. $B^i(G, T)$) le groupe des i-cocycles (resp. i-cobords) et $H^i_{\mathrm{cont}}(G, T) = Z^i(G, T)/B^i(G, T)$ de i-ième groupe de cohomologie continue.

Toute suite exacte courte

$$0 \to T' \to T \to T'' \to 0$$

de $\mathbb{Z}_\ell$-modules topologiques munis d'une action linéaire et continue de G *telle que la projection de* T *sur* T'' *admet une section continue* (c'est le cas si T est un $\mathbb{Z}_\ell$-module de type fini ou un $\mathbb{Q}_\ell$-espace vectoriel de dimension finie) induit une suite exacte longue de cohomologie

$$\cdots \to H^i_{\mathrm{cont}}(G, T') \to H^i_{\mathrm{cont}}(G, T) \to H^i_{\mathrm{cont}}(G, T'') \to \cdots.$$

0.3. Torsion à la Tate. Soient E un corps quelconque, $\overline{E}$ une clôture séparable de E, $G_E = \mathrm{Gal}(\overline{E}/E)$ et ℓ un nombre premier différent de la caractéristique de E. Comme d'habitude, $\mathbb{Z}_\ell(1) = \text{lim.proj.}\, \mu_{\ell^n}(\overline{E})$ désigne le module de Tate du groupe multiplicatif; pour tout $i \in \mathbb{N}$, $\mathbb{Z}_\ell(i)$ est sa puissance tensorielle i-ième et $\mathbb{Z}_\ell(-i)$ son dual; pour tout $\mathbb{Z}_\ell$-module M sur lequel G_E opère et tout $i \in \mathbb{Z}$, on pose $M(i) = M \otimes \mathbb{Z}_\ell(i)$.

0.4. Déterminants: le cas des espaces vectoriels. Soit E un corps. Pour tout E-espace vectoriel V de dimension finie d, on pose $\det_E V = \bigwedge^d_E(V)$ (en particulier, $\det(\{0\}) = E$). Si L est une droite et si $i \in \mathbb{N}$, on note L^i sa puissance symétrique i-ième et L^{-i} la droite duale.

Soit

$$0 \to V' \to V \to V'' \to 0$$

une suite exacte courte d'espaces vectoriels de dimension finie. On dispose d'un isomorphisme canonique

$$(\det_E V') \otimes (\det_E V'') \to \det_E V;$$

c'est celui qui, à $(v_1 \wedge v_2 \wedge \cdots \wedge v_r) \otimes (w_1 \wedge w_2 \wedge \cdots \wedge w_s)$ associe $v_1 \wedge v_2 \wedge \cdots \wedge v_r \wedge \hat{w}_1 \wedge \hat{w}_2 \wedge \cdots \wedge \hat{w}_s$ (où l'on a noté encore v_i l'image de $v_i \in V'$ dans V et $\hat{w}_j$ un relèvement arbitraire de $w_j \in V''$ dans V).

Soit

$$(\mathrm{C}) \qquad \cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

un complexe de E-espaces vectoriels tel que les $H^i(C)$ sont de dimension finie, presque tous nuls. On pose $\det_E C = \bigotimes_{i\in\mathbf{Z}}(\det_E H^i(C))^{(-1)^i}$. Si les C^i sont eux mêmes de dimension finie et presque tous nuls, on voit que $\det_E C$ s'identifie à $\bigotimes_{i\in\mathbf{Z}}(\det_E C^i)^{(-1)^i}$. En particulier, toute suite exacte

$$\cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

de E-espaces vectoriels de dimension finie et presque tous nuls définit un isomorphisme canonique $\bigotimes_{i\in\mathbf{Z}}(\det_E C^i)^{(-1)^i} \to E$.

Si l'on s'est donné une décomposition

$$V = \bigoplus_{i\in I} V_i$$

d'un espace vectoriel de dimension finie en somme directe de sous-espaces vectoriels, l'isomorphisme naturel $\det_E V \to \bigotimes_{i\in\mathbf{Z}} \det_E V_i$ dépend par un signe (du moins si tous les V_i ne sont pas de dimension paire) de l'ordre dans lequel on a rangé les V_i. Comme ces problèmes de signe n'ont pas d'importance pour ce que nous allons faire, nous appelons *s-isomorphisme* entre deux droites un isomorphisme qui n'est défini qu'à la multiplication par -1 près.

0.5. Déterminants: le cas des modules sur les anneaux principaux. Les considérations qui précèdent s'étendent aux modules de type fini sur un anneau principal A (et même aux modules de tor-dimension finie sur un anneau commutatif arbitraire, mais nous n'en aurons pas besoin).

Si M est un A-module libre de rang fini d, on pose $\det_A M = \bigwedge_A^d(M)$. Si $d = 1$, on pose $M^1 = M$ et on note M^{-1} son dual. Tout complexe borné à gauche

$$(\mathrm{C}) \qquad \cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

de A-modules tel que les $H^i(C)$ sont de type fini et presque tous nuls est quasi-isomorphe à un complexe parfait, i.e. un complexe

$$(\mathrm{D}) \qquad \cdots \to D^{i-1} \to D^i \to D^{i+1} \to \cdots$$

où les D^i sont des A-modules libres de type fini presque tous nuls. Si, pour un tel complexe, on pose $\det_A D = \bigotimes_{i\in\mathbf{Z}}(\det_A D^i)^{(-1)^i}$, tout quasi-isomorphisme entre D et un autre complexe parfait D' induit un isomorphisme canonique entre $\det_A D$ et $\det_A D'$. On peut donc définir $\det_A C$ comme étant le A-module (libre de rang 1) $\det_A D$, pour n'importe quel D parfait quasi-isomorphe à C.

En particulier, on peut parler de $\det_A M$ pour tout A-module de type fini (voir M comme un complexe concentré en degré 0). Si

$$0 \to L_{-1} \to L_0 \to M \to 0$$

est une résolution de M par des A-modules libres de rang fini, $\det_A M = \det_A L_0 \otimes (\det_A L_{-1})^{-1}$.

Si dans le complexe C, les C^i sont des A-modules de type fini et presque tous nuls, on a $\det_A C = \bigotimes_{i\in\mathbb{Z}} (\det_A C^i)^{(-1)^i}$. Ici encore, toute suite exacte

$$\cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

de A-modules de type fini et presque tous nuls définit un isomorphisme canonique $\bigotimes_{i\in\mathbb{Z}} (\det_A C^i)^{(-1)^i} \to A$.

Enfin, si $E = \operatorname{Frac} A$, si C est comme au début de ce paragraphe et si C_E est le complexe déduit de C par extension des scalaires, $\det_A C$ s'identifie à un réseau de $\det_E C_E$.

0.6. Diagrammes rectangulaires tordus. Le groupe $\mathbb{Z} \times \mathbb{Z}$ est le produit direct du groupe $L = \{(m, 0) | m \in \mathbb{Z}\}$ par le groupe $C = \{(0, n) | n \in \mathbb{Z}\}$. Se donner une action de $\mathbb{Z} \times \mathbb{Z}$ sur un ensemble Λ revient à se donner deux permutations ℓ, c de Λ vérifiant $c \circ \ell = \ell \circ c$ (il suffit de poser $\ell(\lambda) = (1, 0) + \lambda$ et $c(\lambda) = (0, 1) + \lambda$).

Un *diagramme rectangulaire tordu* dans une catégorie abélienne $\mathscr{C}$ est la donnée

i) d'un ensemble Λ sur lequel $\mathbb{Z} \times \mathbb{Z}$ opère transitivement, le noyau de l'action étant engendré par $(r, -s)$ où r et s sont deux entiers ≥ 1 ;

ii) pour chaque $\lambda \in \Lambda$ d'un objet M^λ et de deux flèches

$$f_\lambda : M^\lambda \to M^{c(\lambda)} \quad \text{et} \quad g_\lambda : M^\lambda \to M^{\ell(\lambda)}$$

de $\mathscr{C}$.

On dit qu'un tel diagramme est *commutatif* si $f_{\ell(\lambda)} \circ g_\lambda = g_{c(\lambda)} \circ f_\lambda$, pour tout $\lambda \in \Lambda$. On dit qu'il est *exact* si, pour tout $\lambda \in \Lambda$, les suites

$$M^{c^{-1}(\lambda)} \to M^\lambda \to M^{c(\lambda)} \quad \text{et} \quad M^{\ell^{-1}(\lambda)} \to M^\lambda \to M^{\ell(\lambda)}$$

sont exactes.

A chaque orbite de Λ sous l'action de C correspond une *ligne* du diagramme

$$\cdots \to M^{c^{i-1}(\lambda)} \to M^{c^i(\lambda)} \to M^{c^{i+1}(\lambda)} \to \cdots ,$$

tandis qu'à chaque orbite sous l'action de L correspond une *colonne*

$$\cdots \to M^{\ell^{i-1}(\lambda)} \to M^{\ell^i(\lambda)} \to M^{\ell^{i+1}(\lambda)} \to \cdots .$$

Un tel diagramme a r lignes et s colonnes. Si l'on choisit une origine $\lambda_0 \in \Lambda$ et si l'on pose $M^{(i,j)} = M^{(i+j)+\lambda_0}$, un tel diagramme peut se représenter schématiquement sous la forme

$$
\begin{array}{ccccccccccccc}
 & \vdots & & \vdots & & \vdots & & (s-1) & & (0) & & (1) & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & M^{(0,-1)} & \to & M^{(0,0)} & \to & M^{(0,1)} & \to \cdots \to & M^{(0,s-1)} & \to & M^{(0,s)} & \to & M^{(0,s+1)} & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & M^{(i,-1)} & \to & M^{(i,0)} & \to & M^{(i,1)} & \to \cdots \to & M^{(i,s-1)} & \to & M^{(i,s)} & \to & M^{(i,s+1)} & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & M^{(r-1,-1)} & \to & M^{(r-1,0)} & \to & M^{(r-1,1)} & \to \cdots \to & M^{(r-1,s-1)} & \to & M^{(r-1,s)} & \to & M^{(r-1,s+1)} & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & (s-1) & & (0) & & (1) & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Dire que le diagramme est exact revient à dire que ses lignes et ses colonnes sont exactes.

On vérifie facilement que, si dans un diagramme rectangulaire tordu commutatif toutes les lignes et toutes les colonnes, sauf peut-être une, sont exactes, et si cette dernière est un complexe borné à gauche, alors elle est exacte.

CHAPITRE I

REPRÉSENTATIONS ℓ-ADIQUES DES CORPS p-ADIQUES

Dans tout ce chapitre, K est un corps de caractéristique 0, complet pour une valuation discrète, dont le corps résiduel k est supposé fini de caractéristique $p > 0$. On choisit une clôture algébrique $\overline{K}$ de K et on pose $G_K = \mathrm{Gal}(\overline{K}/K)$. On désigne par $\overline{k}$ le corps résiduel de $\overline{K}$, on pose $G_k = \mathrm{Gal}(\overline{k}/k)$ et on note I_K le groupe d'inertie. On a donc une suite exacte

$$1 \to I_K \to G_K \to G_k \to 1 .$$

On note $\mathbb{Q}_p^{nr}$ l'extension maximale non ramifiée de $\mathbb{Q}_p$ contenue dans $\overline{K}$, K_0 (resp. $\mathbb{Q}'_p$) le corps des fractions de l'anneau $W(k)$ (resp. $W(\overline{k})$) des vecteurs de Witt à coefficients dans k (resp. $\overline{k}$). On a donc $K_0 = K \cap \mathbb{Q}_p^{nr}$ et $\mathbb{Q}'_p$ est le complété de $\mathbb{Q}_p^{nr}$. On pose $e = [K : K_0]$ et on note σ le Frobenius absolu agissant sur k (par $x \mapsto x^p$), K_0, $\mathbb{Q}_p^{nr}$, et $\mathbb{Q}'_p$. On note $q = p^h$ le cardinal de k et f_k le Frobenius géométrique relatif à k (i.e., le générateur topologique de G_k vérifiant $f_k(x) = x^{q^{-1}}$, pour tout $x \in \overline{k}$).

Enfin ℓ est un nombre premier (égal ou non à p).

1. Autour du groupe de Weil-Deligne

Dans tout ce paragraphe, E est un corps de caractéristique 0.

1.1. Représentations du groupe de Weil-Deligne (cf. [De73, Ta79]).

1.1.1. Soit W_K le groupe de Weil de K, i.e. le sous-groupe de G_K formé des w dont l'image dans G_k est de la forme $f_k^{\nu(w)}$, avec $\nu(w) \in \mathbb{Z}$. On note $\mathbf{Rep}_E(W_K)$ la catégorie des E-espaces vectoriels de dimension finie munis d'une action de W_K dont la restriction à I_K est continue (le E-espace vectoriel étant muni de la topologie discrète). Un objet de $\mathbf{Rep}_E(W_K)$ est donc un couple formé d'un E-espace vectoriel Δ et d'un homomorphisme $\rho = \rho_\Delta \colon W_K \to \mathrm{Aut}_E(\Delta)$ tel que $I_K \cap \mathrm{Ker}\,\rho$ soit ouvert dans I_K; un morphisme de $\mathbf{Rep}_E(W_K)$ est une application E-linéaire W_K-équivariante.

La catégorie $\mathbf{Rep}_E(W_K)$ est, de façon évidente, une catégorie tannakienne neutre sur E (n°01). L'objet-unité $\mathbf{1}$ est E muni de l'action triviale de W_K.

1.1.2. On dit qu'on objet Δ de $\mathbf{Rep}_E(W_K)$ est *non ramifié* si I_K opère trivialement, i.e. si ρ se factorise à travers un homomorphisme $\overline{\rho}$ du groupe cyclique engendré par f_k dans $\mathrm{Aut}_E(\Delta)$. Se donner un tel objet revient donc à se donner $\overline{\rho}(f_k)$ qui peut être n'importe quel élément de $\mathrm{Aut}_E(\Delta)$.

Appelons *objet de Tate* tout objet Ta de $\mathbf{Rep}_E(W_K)$ non ramifié et de dimension 1 tel que $\overline{\rho}(f_k) = q^{-1}$. Un tel objet est unique à isomorphisme non unique près.

1.1.3. On *choisit* un objet de Tate Ta et on note Ta^{-1} son dual. Pour tout objet Δ de $\mathbf{Rep}_E(W_K)$, on pose $\Delta\{-1\} = \Delta \otimes Ta^{-1}$. *Une représentation de* $'W_K$ *ou représentation du groupe de Weil-Deligne de* K (à coefficients dans E et relativement au choix de Ta) est un triplet (Δ, ρ, N), où (Δ, ρ) est un objet de $\mathbf{Rep}_E(W_K)$ et où $N = N_\Delta \colon \Delta \to \Delta\{-1\}$ est un morphisme de $\mathbf{Rep}_E(W_K)$.

Les représentations de $'W_K$ forment, de manière évidente, une catégorie abélienne $\mathbf{Rep}_{E,Ta}('W_K)$ (un morphisme est un morphisme des objets de $\mathbf{Rep}_E(W_K)$ sous-jacents qui commute à N) et même une catégorie tannakienne neutre sur E (attention que, avec les conventions que l'on imagine, $N_{\Delta_1 \otimes \Delta_2} = \mathrm{id}_{\Delta_1} \otimes N_{\Delta_2} + N_{\Delta_1} \otimes \mathrm{id}_{\Delta_2}$). La catégorie $\mathbf{Rep}_E(W_K)$ s'identifie à la sous-catégorie pleine de $\mathbf{Rep}_{E,Ta}('W_K)$ dont les objets sont ceux pour lesquels $N = 0$.

1.1.4. Remarques. a) Si Ta_0 désigne l'objet de Tate dont le E-espace vectoriel sous-jacent est E lui-même, on voit qu'un objet de $\mathbf{Rep}_{E,Ta_0}('W_K)$ est un triplet (Δ, ρ, N_0) où (Δ, ρ) est un objet de $\mathbf{Rep}_E(W_K)$ et $N_0 \colon \Delta \to \Delta$ une application E-linéaire telle que

$$N_0 \circ \rho(w) = q^{\nu(w)} \cdot \rho(w) \circ N_0 \quad \text{pour tout } w \in W_K.$$

Le choix d'une base t de Ta définit une $\otimes$-équivalence ι_t entre $\mathbf{Rep}_{E,Ta}({}'W_K)$ et $\mathbf{Rep}_{E,Ta_0}({}'W_K)$: c'est celle qui, à (Δ, ρ, N) associe (Δ, ρ, N_0) où N_0 est défini par $Nd = N_0 d \otimes t^{-1}$, pour tout $d \in \Delta$; si t et t' sont deux bases de Ta, les deux $\otimes$-foncteurs ι_t et $\iota_{t'}$ sont isomorphes. Si Ta et Ta' sont deux objets de Tate, le choix d'une base de $Ta' \otimes Ta^{-1}$ permet de définir une $\otimes$-équivalence entre $\mathbf{Rep}_{E,Ta}({}'W_K)$ et $\mathbf{Rep}_{E,Ta'}({}'W_K)$.

b) L'action de N sur un objet de $\mathbf{Rep}_{E,Ta}({}'W_K)$ est nilpotente (i.e. le E-endomorphisme N_0 de tout objet de $\mathbf{Rep}_{E,Ta_0}({}'W_K)$ est nilpotent).

1.1.5. Dans ce qui suit, on suppose choisi un objet de Tate Ta et on pose $\mathbf{Rep}_E({}'W_K) = \mathbf{Rep}_{E,Ta}({}'W_K)$. On se donne un objet $\Delta = (\Delta, \rho, N)$ de $\mathbf{Rep}_E({}'W_K)$.

On dit que Δ est *F-semi-simple* si la représentation ρ de W_K sous-jacente est semi-simple. Si l'on choisit un élément $w \in W_K$ qui n'est pas dans I_K, il revient au même de demander que $\rho(w)$ soit semi-simple.

On dit que Δ est *semi-stable* si le noyau de ρ contient I_K et on note $\mathbf{Rep}_{E,g}({}'W_K)$ la sous-catégorie pleine de $\mathbf{Rep}_E({}'W_K)$ dont les objets sont ceux qui sont semi-stables. On note $\mathbf{Rep}_{E,h}({}'W_K)$ (resp. $\mathbf{Rep}_{E,f}({}'W_K)$, resp. $\mathbf{Rep}_{E,e}({}'W_K)$) la sous-catégorie pleine de $\mathbf{Rep}_{E,g}({}'W_K)$ formée des objets qui sont F-semi-simples (resp. pour lesquels $N = 0$, resp. qui sont F-semi-simples et pour lesquels $N = 0$).

1.1.6. Si L est une extension finie de K contenue dans $\overline{K}$, le groupe de Weil W_L s'identifie à un sous-groupe de W_K et on peut restreindre ρ à W_L, ce qui permet de considérer également Δ comme un objet de $\mathbf{Rep}_E({}'W_L)$. Pour $\nu \in \{e, f, h, g\}$, on note $\mathbf{Rep}_{E,p\nu}({}'W_K)$ la sous-catégorie pleine de $\mathbf{Rep}_E({}'W_K)$ formée des Δ pour lesquels on peut trouver une extension finie L de K contenue dans $\overline{K}$ telle que Δ vu comme objet de $\mathbf{Rep}_E({}'W_L)$ soit dans $\mathbf{Rep}_{E,\nu}({}'W_L)$.

Pour $\nu \in \{e, f, h, g, pe, pf, ph, pg\}$, $\mathbf{Rep}_{E,\nu}({}'W_K)$ est une sous-catégorie tannakienne de $\mathbf{Rep}_E({}'W_K)$. On a $\mathbf{Rep}_{E,pg}({}'W_K) = \mathbf{Rep}_E({}'W_K)$, $\mathbf{Rep}_{E,pf}({}'W_K) = \mathbf{Rep}_E(W_K)$ et $\mathbf{Rep}_{E,ph}({}'W_K)$ est la sous-catégorie pleine de $\mathbf{Rep}_E({}'W_K)$ dont les objets sont ceux qui sont F-semi-simples; on laisse au lecteur le soin de décrire les différentes inclusions qui existent entre ces différentes catégories.

1.2. Facteurs locaux et ε, questions de rationalité et de compatibilité (cf. **[De73, Ta79]**).

1.2.1. On conserve les hypothèses et notations du n°1.1.5. Pour tout $w \in W_K$, on peut écrire, de manière unique, $\rho(w) = \rho^{ss}(w) \circ \rho^{u}(w)$ où $\rho^{ss}(w)$ et $\rho^{u}(w)$ sont des automorphismes du E-espace vectoriel Δ qui commutent entre eux, le premier étant semi-simple et le second unipotent. Le triplet

(Δ, ρ^{ss}, N) est alors un objet de $\mathbf{Rep}_{E,ph}('W_K)$ que nous notons Δ^{ss} et appelons le *F-semi-simplifié de* Δ. On note Δ_g (resp. Δ_f) le plus grand sous-objet de Δ qui est dans $\mathbf{Rep}_{E,g}('W_K)$ (resp. $\mathbf{Rep}_{E,f}('W_K)$) ; on a donc $\Delta_g = \Delta^{I_K}$ tandis que Δ_f est le noyau de la restriction de N à Δ_g. Le Frobenius f_k opère sur Δ_g et Δ_f et on pose

$$P_K(\Delta, u) = \det(1 - f_k u|\Delta_f) \in E[u].$$

Remarquons que $P_K(\Delta, u) = P_K(\Delta^{ss}, u)$.

1.2.2. Choisissons un corps algébriquement clos E' contenant E et un homomorphisme non trivial $\psi: K \to E'^*$ dont le noyau contient une puissance de l'idéal maximal de l'anneau des entiers de K et une mesure de Haar μ sur le groupe additif K à valeurs dans E'.

Si l'on oublie l'action de N,

$$\rho: W_K \to \mathrm{Aut}_E(\Delta)$$

définit une représentation E-linéaire de dimension finie de W_K et on sait lui associer un entier $a(\rho)$, *son conducteur d'Artin*, et un élément non nul $\varepsilon(\rho, \psi, \mu) \in E'^*$, *son facteur* ε (cf. **[De73]**, §3). On sait que $a(\rho)$ et $\varepsilon(\rho, \psi, \mu)$ ne dépendent en fait que du semi-simplifié ρ^{ss} de ρ.

Rappelons que le *conducteur* $a(\Delta)$ est alors l'entier

$$a(\Delta) = a(\rho) + \dim_E(\Delta_g/\Delta_f),$$

tandis que son *facteur* ε est

$$\varepsilon(\Delta, \psi, \mu) = \varepsilon(\rho, \psi, \mu) \cdot \det(-f_k|\Delta_g/\Delta_f).$$

On voit que $a(\Delta) = a(\Delta^{ss})$ et $\varepsilon(\Delta, \psi, \mu) = \varepsilon(\Delta^{ss}, \psi, \mu)$.

1.2.3. Soient

$$(\beta) \qquad 0 \to \Delta' \to \Delta \to \Delta'' \to 0$$

une suite exacte courte de $\mathbf{Rep}_E('W_K)$ et

$$(\beta^*(Ta)) \qquad 0 \to \Delta''^*(Ta) \to \Delta^*(Ta) \to \Delta'^*(Ta) \to 0$$

la suite exacte que l'on obtient en tordant par Ta la suite exacte duale. Posons

$$P_K(\beta, u) = P_K(\Delta, u)/P_K(\Delta', u) \cdot P_K(\Delta'', u),$$
$$P_K(\beta^*(Ta), u) = P_K(\Delta^*(Ta), u)/P_K(\Delta''^*(Ta), u) \cdot P_K(\Delta'^*(Ta), u),$$
$$a(\beta) = a(\Delta) - a(\Delta') - a(\Delta'') \quad \text{et} \quad \varepsilon(\beta) = \varepsilon(\Delta, \psi, \mu)/\varepsilon(\Delta', \psi, \mu) \cdot \varepsilon(\Delta'', \psi, \mu)$$

(cette dernière quantité est indépendante du choix de ψ et μ).

PROPOSITION. *Avec les notations ci-dessus, on a*

$$P_K(\beta^*(Ta), u^{-1}) = \varepsilon(\beta) \cdot u^{a(\beta)} \cdot P_K(\beta, u).$$

Preuve. Avec des notations évidentes (cf. n°1.2.1), on a le diagramme commutatif dont les lignes sont exactes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta'_g & \longrightarrow & \Delta_g & \longrightarrow & \Delta''_g & \longrightarrow & 0 \\ & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & \\ 0 & \longrightarrow & \Delta'_g\{-1\} & \longrightarrow & \Delta_g\{-1\} & \longrightarrow & \Delta''_g\{-1\} & \longrightarrow & 0. \end{array}$$

On en déduit par le lemme du serpent la suite exacte

$$0 \to \Delta'_f \to \Delta_f \to \Delta''_f \to \Delta'_g\{-1\}/N\Delta'_g \to \Delta_g\{-1\}/N\Delta_g \to \Delta''_g\{-1\}/N\Delta''_g \to 0.$$

Remarquons que le dual de $\Delta_g\{-1\}/N\Delta_g$ (en tant que E-espace vectoriel) est $\Delta^*(Ta)_f$. La suite exacte précédente devient

$$0 \to \Delta'_f \to \Delta_f \to \Delta''_f \to (\Delta'^*(Ta)_f)^* \to (\Delta^*(Ta)_f)^* \to (\Delta''^*(Ta)_f)^* \to 0.$$

Comme

$$P_K(\beta^*(Ta), u^{-1}) = \det(1 - f_k \cdot u^{-1}|\Delta^*(Ta)_f) \cdot \det(1 - f_k \cdot u^{-1}|\Delta'^*(Ta)_f)^{-1} \cdot \det(1 - f_k \cdot u^{-1}|\Delta''^*(Ta)_f)^{-1},$$

la suite exacte précédente implique

$$\begin{aligned} P(\beta^*(Ta), u^{-1}) &= \det(1 - f_k \cdot u^{-1}|(\Delta_f)^*) \cdot \det(1 - f_k \cdot u^{-1}|(\Delta'_f)^*)^{-1} \\ &\quad \cdot \det(1 - f_k \cdot u^{-1}|(\Delta''_f)^*)^{-1} \\ &= \det(1 - f_k^{-1} \cdot u^{-1}|\Delta_f) \cdot \det(1 - f_k^{-1} \cdot u^{-1}|\Delta'_f)^{-1} \\ &\quad \cdot \det(1 - f_k^{-1} \cdot u^{-1}|\Delta''_f)^{-1} \\ &= u^{a(\beta)} \cdot \det(-f_k|\Delta'_f) \cdot \det(-f_k|\Delta''_f) \\ &\quad \cdot \det(-f_k|\Delta_f)^{-1} \cdot P_K(\beta, u) \\ &= u^{a(\beta)} \cdot \varepsilon(\beta) \cdot P_K(\beta, u) \end{aligned}$$

(on a en effet $a(\beta) = \dim \Delta'_f + \dim \Delta''_f - \dim \Delta_f$).

1.2.4. Soient maintenant E_0 un sous-corps de E et E' un corps algébriquement clos contenant E_0 tel qu'il existe un E_0-plongement $\tau: E \to E'$. Pour un tel τ, notons Δ_τ l'objet de $\mathbf{Rep}_{E'}('W_K)$ déduit de Δ par l'extension des scalaires τ. On dit que Δ est *rationnelle sur* E_0 si, quelque soient τ et τ' E_0-plongements de E dans E', il existe un isomorphisme de Δ_τ sur $\Delta_{\tau'}$. Cette définition est indépendante du choix de E'.

Si Δ est rationnelle sur E_0, on vérifie que $P_K(\Delta, u) \in E_0[u]$ et que $\varepsilon(\Delta, \tau \circ \psi, \tau \circ \mu) = \varepsilon(\Delta, \psi, \mu)$ pour tout automorphisme τ de E laissant stable E_0.

1.2.5. Soient E_0 un corps, E_1 et E_2 deux extensions de E_0 et, pour $i \in \{1, 2\}$, Δ_i un objet de $\mathbf{Rep}_{E_i}('W_K)$. On dit que Δ_1 et Δ_2 sont E_0-*compatibles* si elles sont rationnelles sur E_0 et s'il existe une extension E' de E_0, des E_0-plongements $\tau_1 : E_1 \to E'$ et $\tau_2 : E_2 \to E'$ tels que les objets de $\mathbf{Rep}_{E'}('W_K)$ déduits de Δ_1 par τ_1 et de Δ_2 par τ_2 soient isomorphes.

Si Δ_1 et Δ_2 sont E_0-compatibles, on a $P_K(\Delta_1, u) = P_K(\Delta_2, u)$ et $\varepsilon(\Delta_1, \psi, \mu) = \varepsilon(\Delta_2, \psi, \mu)$.

Si $(E_i)_{i\in I}$ est une famille de corps contenant E_0 et, si pour tout $i \in I$, on s'est donné un objet Δ_i de $\mathbf{Rep}_{E_i}('W_K)$, on dit que la famille des Δ_i est E_0-*compatible* si les Δ_i sont deux à deux E_0-compatibles.

1.3. Modules de Dieudonné-Weil-Deligne.

1.3.1. Appelons $\mathbb{Q}_p$-*DWD-module sur* K (DWD = Dieudonné-Weil-Deligne) la donnée d'un $\mathbb{Q}_p^{nr}$-space vectoriel Δ de dimension finie muni

- d'une action semi-linéaire de W_K, i.e. d'un homomorphisme

$$\rho_{sl} : W_K \to \mathrm{Aut}_{\mathbb{Q}_p}(\Delta)$$

vérifiant $\rho_{sl}(w)(\lambda x) = f_k^{\nu(w)}(\lambda) \cdot \rho_{sl}(w)(x)$ pour $w \in W_K$, $\lambda \in \mathbb{Q}_p^{nr}$, $x \in \Delta$, (remarquer que l'action de I_K est donc linéaire) telle que le noyau de la restriction à I_K est un sous-groupe ouvert,

- d'une application $\varphi : \Delta \to \Delta$, σ-linéaire, i.e. additive et vérifiant $\varphi(\lambda x) = \sigma(\lambda) \cdot \varphi(x)$, commutant à l'action de W_K,
- d'un endomorphisme N commutant à l'action de W_K et vérifiant

$$N\varphi = p\varphi N.$$

Les $\mathbb{Q}_p$-DWD-modules sur K forment, de manière évidente, une catégorie abélienne $\mathbb{Q}_p$-linéaire (les morphismes sont les applications $\mathbb{Q}_p^{nr}$-linéaires qui commutent à l'action de W_K et à φ et N) que nous notons $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$.

On laisse au lecteur le soin de définir le produit tensoriel de deux objets de cette catégorie et de vérifier que l'on obtient ainsi une catégorie tannakienne sur $\mathbb{Q}_p$, *qui n'est pas neutre.*

1.3.2. Soit Δ un objet de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$. On peut munir Δ d'une action *linéaire* de W_K en posant (rappelons que $\sigma = f_k^{-h}$)

$$\rho(w) = \rho_{sl}(w) \circ \varphi^{h\nu(w)}.$$

On peut alors considérer $N : \Delta \to \Delta$ comme un morphisme

$$\Delta \to \Delta(Ta_0^{-1})$$

dans la catégorie des représentations $\mathbb{Q}_p^{nr}$-linéaires de $'W_K$, autrement dit on a muni Δ d'une structure d'objet de $\mathbf{Rep}_{\mathbb{Q}_p^{nr}}('W_K) := \mathbf{Rep}_{\mathbb{Q}_p^{nr}, Ta_0}('W_K)$.

On peut considérer cette construction comme un foncteur $\mathbb{Q}_p$-linéaire de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$ dans $\mathbf{Rep}_{\mathbb{Q}_p^{nr}}('W_K)$. Il n'est pas difficile de vérifier (cf. **[Bu,** Exp. VIII]) que c'est un $\otimes$-foncteur exact et fidèle et que, si Δ_1 et Δ_2 sont deux objets de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$, l'application naturelle

$$\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} \mathrm{Hom}_{\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)}(\Delta_1, \Delta_2) \to \mathrm{Hom}_{\mathbf{Rep}_{\mathbb{Q}_p^{nr}}('W_K)}(\Delta_1, \Delta_2)$$

est un isomorphisme.

1.3.3. Remarques. i) Soit Δ un $\mathbb{Q}_p^{nr}$-espace vectoriel de dimension finie. Pour se donner une structure d'objet de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$, il faut se donner ρ_{sl}, N et φ. On voit qu'il revient au même de se donner ρ, N et φ, ce qui fait que l'on peut voir un objet de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$ comme un objet Δ de $\mathbf{Rep}_{\mathbb{Q}_p^{nr}}('W_K)$ muni d'une application $\varphi: \Delta \to \Delta$, σ-semi-linéaire et commutant à l'action de ρ et de N.

ii) Il n'est pas difficile de voir que, si Δ est un objet de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$, alors l'objet de $\mathbf{Rep}_{\mathbb{Q}_p^{nr}}('W_K)$ sous-jacent est $\mathbb{Q}_p$-rationnel (cf. 1.2.4): on peut observer qu'il existe une extension finie galoisienne L de $\mathbb{Q}_p$ et un sous L-espace vectoriel Δ_L de Δ stable par N, φ et par $\rho(w)$ pour tout $w \in W_K$, tels que l'application naturelle $\mathbb{Q}_p^{nr} \otimes_L \Delta_L \to \Delta$ soit un isomorphisme; il suffit alors de vérifier que Δ_L est $\mathbb{Q}_p$-rationnel, ce qui résulte de ce que, pour tout $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, il existe un entier r tel que φ^r définit un isomorphisme de $L \otimes_{\tau} {}_L\Delta_L$ sur Δ_L dans $\mathbf{Rep}_L('W_K))$. En particulier (cf. 1.2.4), $P_K(\Delta, u) \in \mathbb{Q}_p[u]$.

1.3.4. Soit maintenant ℓ un nombre premier $\neq p$. Le groupe $W_K \subset G_K$ opère sur $\mathbb{Q}_\ell(1)$ qui devient ainsi un objet de Tate de $\mathbf{Rep}_{\mathbb{Q}_\ell}(W_K)$. Il est commode de poser $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K) = \mathbf{Rep}_{\mathbb{Q}_\ell, \mathbb{Q}_\ell(1)}('W_K)$, d'appeler $\mathbb{Q}_\ell$-*DWD-modules sur* K les objets de cette catégorie et de convenir que $\rho_{sl} = \rho$.

1.3.5. Soient ℓ un nombre premier et $\nu \in \{e, f, h, g, pe, pf, ph, pg\}$. Si $\ell \neq p$, on pose $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K) = \mathbf{Rep}_{\mathbb{Q}_\ell, \nu}('W_K)$. Si $\ell = p$, on note $\mathbf{Rep}_{\mathbb{Q}_p, \nu}(D'W_K)$ la sous-catégorie pleine de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$ dont les objets sont ceux dont le module de Weil-Deligne sous-jacent est un objet de $\mathbf{Rep}_{\mathbb{Q}_p^{nr}, \nu}('W_K)$. En particulier, $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(D'W_K) = \mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$.

1.3.6. Nous notons $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'G_K)$ la sous-catégorie pleine de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$ formée des objets Δ tels que l'action ρ_{sl} de W_K se prolonge en une action *continue* de G_K (ce prolongement est nécessairement unique). On voit que c'est une sous-catégorie tannakienne stable par extensions. Pour tout ν, on note aussi $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'G_K)$ "l'intersection" de $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$ et $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'G_K)$.

1.4. Cohomologie des DWD-modules. Dans ce n°, ℓ est un nombre premier arbitraire et Δ est un objet de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$.

1.4.1. Soit $\nu \in \{e, f, h, g, pe, pf, ph, pg\}$. Si $j_{\nu *}: \mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K) \to \mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$ désigne l'adjoint à gauche de l'inclusion de $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$ dans $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$, pour tout $i \in \mathbb{Z}$, on pose (n°01)

$$H^i_\nu(K, \Delta) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K), j_{\nu *}(\Delta)).$$

En particulier, $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K), \Delta) = H^i_{pg}(K, \Delta)$.

1.4.2. PROPOSITION. *Pour* $\nu \in \{e, f, h, g\}$ *et pour tout* i*, la flèche naturelle*

$$H^i_\nu(K, \Delta) \to H^i_{p\nu}(K, \Delta)$$

est un isomorphisme. En particulier, $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K), \Delta) = H^i_g(K, \Delta)$.

PREUVE. Soit L une extension finie galoisienne de K contenue dans $\overline{K}$ et soit J le groupe d'inertie de l'extension L/K. Si l'on note $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu, L}(D'W_K)$ la sous-catégorie pleine de $\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(D'W_K)$ formée des objets qui "deviennent ν" sur L, on voit qu'il suffit de vérifier que, si Δ est dans $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu, L}(D'W_K)$ et si Δ_0 désigne le plus grand sous-objet de Δ qui est dans $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$, alors la flèche naturelle

$$H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K), \Delta_0) \to H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(D'W_K)), \Delta)$$

est un isomorphisme. Mais $\Delta_0 = \Delta^J$ et cela résulte immédiatement de ce que, comme $\mathbb{Q}_\ell$ est de caractéristique 0 et J est un groupe fini, le foncteur $\Delta \mapsto \Delta_0$ est exact.

1.4.3. Dans la suite, Ta désigne *l'objet de Tate* choisi de la catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$:

- si $\ell \neq p$, on a donc $Ta = \mathbb{Q}_\ell(1)$, W_K agissant via l'inclusion de W_K dans G_K et $N = 0$;
- si $\ell = p$, on a $Ta = \mathbb{Q}_p^{nr}$, W_K agissant également via l'inclusion de W_K dans G_K, φ étant $p\sigma$ et $N = 0$.

Si $\ell \neq p$, notons Δ_{st} (resp. $\Delta_{st}\{-1\}$) le sous-$\mathbb{Q}_\ell$-espace vectoriel de Δ (resp. $\Delta\{1\}$) fixe par I_K (agissant via ρ et l'inclusion de I_K dans W_K). On voit que $\Delta_{st}\{-1\}$ s'identifie à $\Delta_{st}(-1)$. On note φ l'endomorphisme de Δ qui est l'action de $f_k \in W_K/I_K$ sur Δ_{st} et sur $\Delta_{st}\{-1\}$. On dispose d'une application $\mathbb{Q}_\ell$-linéaire $N: \Delta_{st} \to \Delta_{st}\{-1\}$ qui commute à l'action de φ.

Si $\ell = p$, notons Δ_{st} (resp. $\Delta_{st}\{-1\}$) le sous-K_0-espace vectoriel de Δ (resp. $\Delta\{-1\}$) fixe par W_K (agissant via ρ_{sl}). Ce sont des K_0-espaces vectoriels de dimension finie, munis d'une action σ-semi-linéaire de φ; le K_0-espace vectoriel sous-jacent à $\Delta_{st}\{-1\}$ s'identifie à Δ_{st}, l'action de

φ étant tordue par p^{-1}. On dispose encore d'une application K_0-linéaire $N\colon \Delta_{st} \to \Delta_{st}\{-1\}$ qui commute à l'action de φ.

Considérons les complexes suivants de $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie (où le premier terme écrit est en degré 0)

$$C_e(K, \Delta) : \Delta_{st, \varphi=1, N=0} \to 0 \to 0 \to 0 \to \cdots ;$$
$$C_f(K, \Delta) : \Delta_{st, N=0} \to \Delta_{st, N=0} \to 0 \to 0 \to \cdots \quad \text{(défini par } d \mapsto (\varphi - 1)(d));$$
$$C_h(K, \Delta) : \Delta_{st, \varphi=1} \to \Delta_{st}\{-1\}_{\varphi=1} \to 0 \to 0 \to \cdots \quad \text{(défini par } d \mapsto Nd);$$
$$C_g(K, \Delta) : \Delta_{st} \to \Delta_{st} \oplus \Delta_{st}\{-1\} \to \Delta_{st}\{-1\} \to 0 \to \cdots$$
$$\text{(défini par } d \mapsto ((\varphi - 1)(d), Nd) \text{ et } (d_1, d_2) \mapsto Nd_1 - (\varphi - 1)(d_2)).$$

1.4.4. Pour tout objet Δ de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$, notons Δ^* son dual et posons $\Delta^*(Ta) = \Delta^* \otimes Ta$.

Proposition. *Le complexe* $C_g(K, \Delta^*(Ta))$ *s'identifie au dual du complexe* $C_g(K, \Delta)$.

Preuve. C'est immédiat une fois observé que $(\Delta^*(Ta))_{st}$ s'identifie au dual de $\Delta_{st}\{-1\}$ et $(\Delta^*(Ta))_{st}(-1)$ à celui de Δ_{st} (dans le cas $\ell = p$, utiliser que, pour tout K_0-espace vectoriel D, l'application naturelle du K_0-dual de D dans le $\mathbb{Q}_p$-dual induite par la trace de K_0 à $\mathbb{Q}_p$ est un isomorphisme de $\mathbb{Q}_p$-espaces vectoriels).

1.4.5. Proposition. *Pour* $\nu \in \{e, f, h, g\}$, $H^i_\nu(K, \Delta)$ *s'identifie canoniquement et fonctoriellement à la cohomologie du complexe* $C_\nu(K, \Delta)$.

Preuve. On commence par remarquer que, si Δ_ν désigne le plus grand sous-objet de Δ qui est dans $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$, alors $C_\nu(K, \Delta) = C_\nu(K, \Delta_\nu)$. On observe alors que toute suite exacte courte

$$0 \to \Delta_1 \to \Delta_2 \to \Delta_3 \to 0$$

de $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$ induit une suite exacte de complexes

$$0 \to C_\nu(K, \Delta_1) \to C_\nu(K, \Delta_2) \to C_\nu(K, \Delta_3) \to 0$$

ce qui fait de $H^*(C_\nu(K, -))$ un foncteur cohomologique. Il est clair que, si Δ est dans $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$, alors $H^0(C_\nu(K, \Delta))$ s'identifie à $H^0_\nu(K, \Delta)$; il est facile de vérifier directement que $H^1(C_\nu(K, \Delta))$ s'identifie à $H^1_\nu(K, \Delta)$; par exemple, si $\nu = g$ et si $\ell = p$, on a $\Delta_{st} = \Delta$ et $H^1_\nu(K, \Delta)$ classifie les extensions de l'objet-unité $\mathbb{Q}_\ell$ par Δ; se donner une telle extension revient à se donner une suite exacte courte

$$0 \to \Delta \to D \to \mathbb{Q}_\ell \to 0$$

de $\mathbb{Q}_\ell$-espaces vectoriels et à se donner l'action de φ et de N sur un relèvement $\hat{1}$ de 1; on doit avoir $\varphi\hat{1} = \hat{1} + d_1$ avec $d_1 \in \Delta$ et $N\hat{1} = d_2$ avec $d_2 \in \Delta\{-1\}$; en écrivant que $(\varphi N)(\hat{1}) = (N\varphi)(\hat{1})$, on trouve que (d_1, d_2)

doit être un 1-cocycle de $C_g(K, \Delta)$; enfin $\hat{1}$ est défini à l'addition d'un élément de Δ près et cette addition change (d_1, d_2) par un cobord.

Ceci termine la démonstration pour $\nu = e, f$, et h. Cela montre aussi que $H_g^i(K, \Delta) = H^i(C_g(K, \Delta))$ pour $i \neq 2$, que l'application naturelle $H_g^2(K, \Delta) \to H^2(C_g(K, \Delta))$ est injective et que le foncteur $\Delta \mapsto H^2(C_g(K, \Delta))/H_g^2(K, \Delta)$ est exact. Comme le foncteur contravariant $\Delta \mapsto \Delta^*(Ta)$ est aussi exact, on voit, en utilisant la proposition précédente que, si $H_g^2(K, -) \neq H^2(C_g(K, -))$, il existerait un sous-foncteur exact, non réduit à 0, du foncteur $H_g^0(K, -)$. Pour voir que ce n'est pas le cas, il suffit de vérifier que, pour tout objet Δ de $\mathbf{Rep}_{\mathbb{Q}_\ell, g}(D'W_K)$ et tout $x \in H_g^0(K, \Delta)$, il existe un épimorphisme $D \to \Delta$ de $\mathbf{Rep}_{\mathbb{Q}_\ell, g}(D'W_K)$ tel que x n'est pas dans l'image de $H_g^0(K, D)$, exercice que nous laissons au lecteur.

1.4.6. Corollaire. *Pour Δ et ν comme ci-dessus, les $H_\nu^i(K, \Delta)$ sont des $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie, nuls si $i \neq 0, 1, 2$. On a aussi $H_\nu^2(K, \Delta) = 0$ si $\nu \neq g$ et $H_e^1(K, \Delta) = 0$. En outre*

$$\sum_{i \in \mathbb{Z}} (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H_g^i(K, \Delta) = 0, \qquad \dim_{\mathbb{Q}_\ell} H_f^0(K, \Delta) = \dim_{\mathbb{Q}_\ell} H_f^1(K, \Delta),$$

$$\dim_{\mathbb{Q}_\ell} H_h^0(K, \Delta) - \dim_{\mathbb{Q}_\ell} H_h^1(K, \Delta) = \dim_{\mathbb{Q}_\ell} \Delta_{st, \varphi=1} - \dim_{\mathbb{Q}_\ell} \Delta_{st}\{-1\}_{\varphi=1}.$$

Cela résulte immédiatement de la structure des complexes $C_\nu(K, \Delta)$ pour $\nu \in \{e, f, h, g\}$.

1.4.7. Si $\ell \neq p$, $H_g^*(K, Ta)$ est la cohomologie du complexe

$$\mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell \to \mathbb{Q}_\ell \to 0 \to 0 \to \cdots$$

où la flèche $\mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell \to \mathbb{Q}_\ell$ est la flèche nulle et $H_g^2(K, Ta)$ s'identifie à $\mathbb{Q}_\ell$.

Si $\ell = p$, $H_g^*(K, Ta)$ est la cohomologie du complexe

$$K_0 \to K_0 \oplus K_0 \to K_0 \to 0 \to 0 \to \cdots$$

où la flèche $K_0 \oplus K_0 \to K_0$ est la flèche $(a, b) \mapsto \sigma b - b$ et la trace de K_0 à $\mathbb{Q}_p$ identifie $H_g^2(K, Ta)$ à $\mathbb{Q}_\ell$.

Proposition. i) *Pour tout $i \in \mathbb{Z}$, le cup-produit*

$$H_g^i(K, \Delta) \times H_g^{2-i}(K, \Delta^*(Ta)) \to H_g^2(K, Ta) = \mathbb{Q}_\ell$$

n'est autre que l'accouplement fourni par la Proposition 1.4.4 (*en particulier, c'est un accouplement parfait*);

ii) *dans la dualité*

$$H_g^1(K, \Delta) \times H_g^1(K, \Delta^*(Ta)) \to \mathbb{Q}_\ell,$$

l'orthogonal de $H^1_f(K, \Delta)$ *s'identifie à* $H^1_f(K, \Delta^*(Ta))$ *et celui de* $H^1_h(K, \Delta)$ *à* $H^1_h(K, \Delta^*(Ta))$;

iii) *si* Δ *est* F*-semi-simple, i.e. si c'est un objet de* $\mathbf{Rep}_{\mathbb{Q}_\ell, ph}(D'W_K)$, *l'application naturelle* $H^1_f(K, \Delta) \oplus H^1_h(K, \Delta) \to H^1_g(K, \Delta)$ *est un isomorphisme.*

Preuve. Montrons (i). Les propriétés fonctorielles des deux accouplements impliquent que, pour chacun d'entre eux et pour toute extension $0 \to \Delta \to \Delta_1 \to \mathbb{1} \to 0$ (où $\mathbb{1}$ est l'objet-unité), on a le diagramme commutatif

$$\begin{array}{ccccc} H^0_g(K, \mathbb{1}) & \times & H^2_g(K, Ta)) & \to & H^2_g(K, Ta) = \mathbb{Q}_\ell \\ \downarrow & & \uparrow & & \| \\ H^1_g(K, \Delta) & \times & H^1_g(K, \Delta^*(Ta)) & \to & H^2_g(K, Ta) = \mathbb{Q}_\ell . \end{array}$$

En remarquant que pour tout élément x de $H^1_g(K, \Delta)$, il existe une telle extension pour laquelle x appartient à l'image de $H^0_g(K, \mathbb{1})$, on en déduit qu'il suffit de montrer que les deux accouplements $H^0_g(K, \mathbb{1}) \times H^2_g(K, Ta)) \to H^2_g(K, Ta)$ coïncident, ce qui est clair car dans les deux cas, il s'agit de $(a, b) \mapsto a \cdot b$ (multiplication par le scalaire a).

Montrons iii). Si l'on pose $\Delta^*(Ta) = \Delta'$, on a

$$\begin{aligned} \dim H^1_f(K, \Delta) + \dim H^1_f(K, \Delta') &= \dim H^0_g(K, \Delta) + \dim H^0_g(K, \Delta') \\ &= \dim H^0_g(K, \Delta) + \dim H^2_g(K, \Delta) \\ &= \dim H^1_g(K, \Delta). \end{aligned}$$

En outre, le dual de $\Delta_{st}\{-1\}_{\varphi=1}$ (resp. $\Delta_{st, \varphi=1}$) s'identifie à $\Delta'_{st}/(\varphi-1)(\Delta'_{st})$ (resp. $\Delta'_{st}\{-1)/(\varphi-1)\Delta'_{st}\{-1))$ et a la même dimension que $\Delta'_{st, \varphi=1}$ (resp. $\Delta'_{st}\{-1)_{\varphi=1}$) d'où l'on déduit que

$$\begin{aligned} \dim H^1_h(K, \Delta) - \dim H^0_h(K, \Delta) &= \dim_{\mathbb{Q}_\ell} \Delta_{st, \varphi=1} - \dim_{\mathbb{Q}_\ell} \Delta_{st}\{-1\}_{\varphi=1} \\ &= \dim_{\mathbb{Q}_\ell} \Delta'_{st}\{-1)_{\varphi=1} - \dim_{\mathbb{Q}_\ell} \Delta'_{st, \varphi=1} \\ &= -\dim H^1_h(K, \Delta') + \dim H^0_h(K, \Delta') \end{aligned}$$

et donc

$$\begin{aligned} \dim H^1_h(K, \Delta) + \dim H^1_h(K, \Delta') &= \dim H^0_h(K, \Delta) + \dim H^0_h(K, \Delta') \\ &= \dim H^1_g(K, \Delta). \end{aligned}$$

Si maintenant $x \in H^1_g(K, \Delta)$ et $y \in H^1_g(K, \Delta')$, on voit que $\langle x, y\rangle = 0$ si $x \in H^1_f(K, \Delta)$ et $y \in H^1_f(K, \Delta')$ ou si $x \in H^1_h(K, \Delta)$ et $y \in H^1_h(K, \Delta')$. Compte-tenu des dimensions des différents espaces vectoriels, (ii) est clair.

Pour montrer iii), commençons par vérifier que $\dim H^1_h(K, \Delta) + \dim H^1_f(K, \Delta) = \dim H^1_g(K, \Delta)$ ou encore compte-tenu de ce qui précède que $\dim H^1_h(K, \Delta) = \dim H^1_f(K, \Delta')$. Le dual de $\Delta_{st}\{-1\}_{\varphi=1}/N\Delta_{st}\{-1\}_{\varphi=1}$ s'identifie à $\Delta'_{st, N=0}/(\varphi-1)(\Delta'_{st, N=0})$. On en déduit que

$$\begin{aligned}\dim H^1_h(K, \Delta) &= \dim \Delta_{st}\{-1\}_{\varphi=1}/N \\ &= \dim \Delta'_{st, N=0}/(\varphi-1)(\Delta'_{st, N=0}) = \dim H^1_f(K, \Delta').\end{aligned}$$

Il reste à montrer que lorsque Δ est F-semi-simple, l'intersection de $H^1_h(K, \Delta)$ et de $H^1_f(K, \Delta)$ dans $H^1_g(K, \Delta)$ est nulle. Soit x un élément de l'intersection: comme $x \in H^1_f(K, \Delta)$, il est représenté dans $\Delta_{st} \oplus \Delta_{st}\{-1\}$ par $(a, 0)$ avec $a \in \Delta_{st, N=0}$; comme $x \in H^1_h(K, \Delta)$, il existe $b \in \Delta_{st}\{-1\}_{\varphi=1}$ et $d \in \Delta_{st}$ tels que $a = (\varphi-1)d$, $b = Nd$. On a donc $N(1-\varphi)d = 0$. La F-semi-simplicité de Δ implique que $d \in \Delta_{st, \varphi=1} + \Delta_{st, N=0}$. On a donc $a \in (\varphi-1)\Delta_{st, N=0}$, ce qui signifie que $x = 0$.

1.4.8. Remarques. a) Les résultats de ce paragraphe et leurs démonstrations restent les mêmes si l'on remplace les catégories $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'W_K)$ par $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'G_K)$. En particulier, si Δ est un objet de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'G_K)$, $j_{\nu*}(\Delta)$ est un objet de $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'G_K)$ et

$$H^1(\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'G_K), j_{\nu*}(\Delta)) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(D'G_K), j_{\nu*}(\Delta)).$$

b) Pour $\ell \neq p$, $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(D'W_K) = \mathbf{Rep}_{\mathbb{Q}_\ell, \mathbb{Q}_\ell(1)}('W_K)$. Les résultats de ce paragraphe et leurs démonstrations restent les mêmes si l'on remplace les catégories $\mathbf{Rep}_{\mathbb{Q}_\ell, \mathbb{Q}_\ell(1)}('W_K)$ par $\mathbf{Rep}_{E, Ta}('W_K)$ où E est un corps de caractéristique 0 et Ta un objet de Tate.

2. La hiérarchie des représentations ℓ-adiques

2.1. Les anneaux $B_{st,\ell}$.

2.1.1. Nous ne refaisons pas ici la théorie des représentations p-adiques de Hodge–Tate, de de Rham ou cristallines de G_K : voir **[Fl]** pour un "précis" ou **[Bu,** exp. II et III] pour une étude détaillée de cette théorie qui a son origine (Barsotti, Grothendieck, **[Se67, Ta67]**) dans la notion de représentation de Hodge–Tate et dans l'étude des groupes de Barsotti–Tate.

Rappelons cependant qu'à l'extension $\overline{K}/K$ sont associées toutes sortes d'anneaux topologiques contenant $\mathbb{Q}_p$, sur lesquels G_K opère continûment, munis éventuellement de structures supplémentaires. Tensorisant chacun de ces anneaux avec une représentation p-adique V de G_K et prenant les éléments fixes sous G_K, on obtient un objet "algébrique" associé à V qui reflète une partie des propriétés de V et qui permet même dans certains cas de reconstruire V.

2.1.2. Rappelons succinctement ce que sont ces anneaux et les faits les concernant que nous allons utiliser.

Le premier de ces anneaux est le corps $\mathbb{C}_p$, complété de $\overline{K}$ pour la topologie p-adique usuelle.

Pour tout monoïde associatif et commutatif U noté multiplicativement, notons $V_p(U)$ le monoïde $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], U)$; un élément de ce monoïde peut se voir comme une suite $u = (u^{(n)})_{n\in\mathbb{N}}$ avec les $u^{(n)} \in U$ vérifiant $(u^{(n+1)})^p = u^{(n)}$ pour tout n. On pose $R = V_p(\mathscr{O}_{\mathbb{C}_p})$ (où $\mathscr{O}_{\mathbb{C}_p}$ est l'anneau des entiers de $\mathbb{C}_p$), qui est muni d'une structure naturelle d'anneau local parfait de caractéristique p. Si $\mathscr{O}_{\overline{K}}$ désigne l'anneau des entiers de $\overline{K}$, l'anneau $W(R)$ des vecteurs de Witt à coefficients dans R s'identifie aussi à la limite projective des $W_n(\mathscr{O}_{\overline{K}}/p)$, l'application de transition de $W_{n+1}(\mathscr{O}_{\overline{K}}/p)$ dans $W_n(\mathscr{O}_{\overline{K}}/p)$ étant $(a_0, a_1, \ldots, a_{n-1}, a_n) \mapsto (a_0^p, a_1^p, \ldots, a_{n-1}^p)$; on munit $W(R)$ de la topologie de la limite projective, avec la topologie discrète sur chaque $W_n(\mathscr{O}_{\overline{K}}/p)$. Le corps résiduel de R s'identifie à $\overline{k}$ et $W(R)$ est une $W(\overline{k})$-algèbre. On pose $W_{K_0}(R) = \mathbb{Q}_p' \otimes_{W(\overline{k})} W(R) = K_0 \otimes_{W(k)} W(R) = W(R)[1/p]$. Comme $W(R)$ est sans p-torsion, il s'injecte dans $W_{K_0}(R)$. L'application

$$\theta\colon W(R) \to \mathscr{O}_{\mathbb{C}_p},$$

qui envoie $(u_0, u_1, \ldots, u_n, \ldots)$ sur $\sum p^n u_n^{(n)}$ est un homomorphisme continu surjectif de $W(\overline{k})$-algèbres topologiques dont le noyau est un idéal principal. Si ξ est un générateur de cet idéal, on note A_{cris} le séparé complété pour la topologie p-adique de la sous-$W(R)$-algèbre A^0_{cris} de $W_{K_0}(R)$ engendrée par les $\xi^m/m!$, pour $m \in \mathbb{N}$. On pose $B^+_{\mathrm{cris}} = K_0 \otimes_{W(k)} A_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$. L'anneau A^0_{cris} est stable sous les actions naturelles de G_K et du Frobenius φ sur $W_{K_0}(R)$ et ces actions s'étendent à A_{cris} et B^+_{cris}.

On note encore $\theta\colon W_{K_0}(R) \to \mathbb{C}_p$ l'homomorphisme de K_0-algèbres qui prolonge l'application définie plus haut. Son noyau est un idéal maximal. On note B^+_{dR} le séparé complété de $W_{K_0}(R)$ pour la topologie $(\mathrm{Ker}\,\theta)$-adique; c'est donc un anneau de valuation discrète de corps résiduel $\mathbb{C}_p$. La *topologie naturelle* de B^+_{dR} est celle de la limite projective, la topologie sur $W_{K_0}(R)/(\mathrm{Ker}\,\theta)^i$ étant la topologie induite par celle de $W_{K_0}(R)$ (celle-ci est donc moins fine que la topologie définie par la valuation).

On note encore θ son prolongement à B^+_{dR}. On identifie B^+_{cris} à un sous-anneau de B^+_{dR} (via l'homomorphisme continu qui prolonge l'inclusion de A^0_{cris} dans $W_{K_0}(R)$). L'anneau B^+_{cris} est dense dans B^+_{dR} muni de sa topologie d'anneau de valuation discrète (et donc a fortiori pour la topologie naturelle).

2.1.3. Soient $\mathscr{M}_p$ l'idéal maximal de l'anneau des entiers de $\mathbb{C}_p$ et $U_p = 1 + \mathscr{M}_p$. Alors $V_p(U_p)$ est de manière naturelle un $\mathbb{Q}_p$-espace vectoriel muni d'une action de G_K et l'application $u \mapsto \log(u^{(0)})$ induit une suite exacte

$$0 \to \mathbb{Q}_p(1) \to V_p(U_p) \to \mathbb{C}_p \to 0.$$

On voit que $V_p(U_p)$ s'identifie au groupe des unités congrues à 1 modulo l'idéal maximal de R et s'envoie dans le groupe des unités de $W(R)$ via l'application $x \mapsto [x] = (x, 0, 0, \dots, 0, \dots)$. Pour tout $x \in V_p(U_p)$, la série $\sum_{n\geq 1}(-1)^{n-1}([x]-1)^n/n$ converge vers un élément $\log([x]) \in B^+_{\text{cris}}$ et on a $\theta(\log([x])) = \log(x^{(0)})$. On sait (cf. **[Bu,** exp. II, n°3.1]) que l'application $x \mapsto \log([x])$ identifie $V_p(U_p)$ au sous-$\mathbb{Q}_p$-espace vectoriel $B^+_{\text{cris},\varphi=p}$ de B^+_{cris} sur lequel φ opère par la multiplication par p.

Choisissons un élément $t = \log([\varepsilon])$, avec $\varepsilon^{(0)} = 1$ et $\varepsilon^{(1)} \neq 1$. On voit que t peut être considéré comme un générateur du $\mathbb{Z}_p$-module $\mathbb{Z}_p(1) \subset \mathbb{Q}_p(1)$. C'est aussi un générateur de l'idéal maximal de B^+_{dR}, ce qui fait que le corps des fractions B_{dR} de B^+_{dR} s'identifie à $B^+_{\text{dR}}[t^{-1}]$. Il est muni d'une filtration naturelle par les puissances de l'idéal maximal de B^+_{dR} : on a donc $\text{Fil}^i B_{\text{dR}} = B^+_{\text{dR}} \cdot t^i$, pour tout $i \in \mathbb{Z}$. L'anneau gradué associé est la $\mathbb{C}_p$-algèbre $B_{\text{HT}} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$.

On pose aussi $B_{\text{cris}} = B^+_{\text{cris}}[t^{-1}]$. C'est un sous-anneau de B_{dR} stable par G_K. L'action de φ sur B^+_{cris} se prolonge sur B_{cris} en posant $\varphi(t^{-1}) = p^{-1}t^{-1}$. L'application

$$\varphi - 1 : B_{\text{cris}} \to B_{\text{cris}}$$

est surjective et son noyau $B_{\text{cris},\varphi=1}$ est une sous-$\mathbb{Q}_p$-algèbre de B_{cris}, stable par G_K. On a $\mathbb{Q}_p = B^+_{\text{dR}} \cap B_{\text{cris},\varphi=1}$.

2.1.4. Pour tout groupe abélien Γ, notons, pour tout entier $n \geq 1$, Γ_n le noyau de la multiplication par n et, pour tout nombre premier ℓ, $V^0_\ell(\Gamma) = \mathbb{Q}_\ell \otimes \text{lim. proj.}_{r\in\mathbb{N}} \Gamma_{\ell^n}$. Si $K^*_\ell = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} (\text{lim. proj.}_{r\in\mathbb{N}} K^*/(K^*)^{\ell^r})$, on a une suite exacte

$$0 \to \mathbb{Q}_\ell(1) \to V^0_\ell(\overline{K}^*/K^*) \to K^*_\ell \to 0$$

qui fait de $V^0_\ell(\overline{K}^*/K^*)$ une représentation ℓ-adique de G_K extension de la représentation triviale K^*_ℓ (de dimension 1 si $\ell \neq p$ et $[K : \mathbb{Q}_p] + 1$ si $\ell = p$) par $\mathbb{Q}_\ell(1)$. (On remarque que la représentation $V^0_\ell(\overline{K}^*/K^*)$ peut être caractérisée comme solution d'un problème universel: si V est une représentation ℓ-adique de G_K extension d'une représentation V' sur laquelle G_K opère trivialement par $\mathbb{Q}_\ell(1)$, il existe une et une seule application linéaire G_K-équivariante de V dans $V^0_\ell(\overline{K}^*/K^*)$ qui induit l'identité sur $\mathbb{Q}_\ell(1)$).

2.1.5. La représentation $V^0_\ell(\overline{K}^*/K^*)(-1)$ contient $\mathbb{Q}_\ell$. Notons B_ℓ la $\mathbb{Q}_\ell$-algèbre qui est le quotient de $\text{Sym}_{\mathbb{Q}_\ell}(V^0_\ell(\overline{K}^*/K^*)(-1))$ obtenu en identifiant $1 \in \mathbb{Q}_\ell = \text{Sym}^0_{\mathbb{Q}_\ell}(V^0_\ell(\overline{K}^*/K^*)(-1))$ à $1 \in \mathbb{Q}_\ell \subset V^0_\ell(\overline{K}^*/K^*)(-1) = \text{Sym}^1_{\mathbb{Q}_\ell}(V^0_\ell(\overline{K}^*/K^*)(-1))$. Si $\ell \neq p$ et si b_0 est un élément de $V^0_\ell(\overline{K}^*/K^*)(-1)$ qui n'est pas dans $\mathbb{Q}_\ell$, B_ℓ est l'algèbre des polynômes en b_0 à coefficients dans $\mathbb{Q}_\ell$.

Si $\ell = p$, $V_p^0(U_p/U_p \cap K^*)$ est un hyperplan de $V_p^0(\overline{K}^*/K^*)$; notons $B_{p,f}$ la sous-$\mathbb{Q}_p$-algèbre de B_p engendrée par $V_p^0(U_p/U_p \cap K^*)(-1)$. Choisissons $b_0, b_1, \dots, b_d \in V_p^0(\overline{K}^*/K^*)(-1)$, avec $b_i \in V_p^0(U_p/U_p \cap K^*)(-1)$ si et seulement si $i \neq 0$, de manière que leurs images dans $K_p^*(-1)$ forment une base de ce $\mathbb{Q}_p$-espace vectoriel. Alors $B_{p,f}$ est l'algèbre des polynômes en $b_1, b_2, \dots, b_d$ à coefficients dans $\mathbb{Q}_p$ tandis que B_p est l'algèbre des polynômes en b_0 à coefficients dans $B_{p,f}$.

Par ailleurs, $V_p^0(U_p/U_p \cap K^*)$ s'identifie à un sous-$\mathbb{Q}_p$-espace vectoriel de $V_p(U_p) = B^+_{\mathrm{cris}, \varphi=p}$ et la multiplication par t^{-1} permet d'identifier $V_p^0(U_p/U_p \cap K^*)(-1)$ à un sous-$\mathbb{Q}_p$-espace vectoriel de $B_{\mathrm{cris}, \varphi=1}$. Comme cette identification envoie 1 sur 1, elle se prolonge de manière unique en un homomorphisme (de $\mathbb{Q}_p$-algèbres) de $B_{p,f}$ dans $B_{\mathrm{cris}, \varphi=1}$ et celui-ci commute à l'action de G_K.

2.1.6. Lemme. *L'homomorphisme défini ci-dessus est injectif.*

Preuve. Soient $b_1, b_2, \dots, b_d$ comme ci-dessus. Il s'agit de vérifier que l'on ne peut pas trouver $P \in \mathbb{Q}_p[X_1, X_2, \dots, X_d]$ non nul tel que $P(b_1, b_2, \dots, b_d) = 0$. Il suffit de vérifier que, pour $1 \leq i \leq d$, s'il existe un tel $P \in \mathbb{Q}_p[X_1, X_2, \dots, X_i]$ de degré $r \geq 1$ en b_i, il en existe un autre de degré $r-1$ en b_i. Mais il est facile de voir que l'on peut trouver $g \in G_K$ tel que $g(b_i) = b_i + 1$ et $g(b_j) = b_j$ si $j \neq i$. Mais $g(P(b_1, b_2, \dots, b_i)) - P(b_1, b_2, \dots, b_i) = 0$ et le polynôme

$$P(X_1, X_2, \dots, X_{i-1}, X_i + 1) - P(X_1, X_2, \dots, X_{i-1}, X_i)$$

convient.

2.1.7. Dans la suite, on utilise cet homomorphisme pour identifer $B_{p,f}$ à un sous-anneau de $B_{\mathrm{cris}, \varphi=1}$. On pose $B_{st,p} = B_{\mathrm{cris}} \otimes_{B_{p,f}} B_p$ (c'est l'anneau noté B_{st} dans [FI]) et on prolonge φ à $B_{st,p}$ en convenant que $\varphi b = b$ pour tout $b \in B_p$.

Avec des conventions évidentes, $B_{st,p}$ est engendrée en tant que $\mathbb{Q}_p$-algèbre par les $x \otimes t^{-1}$, pour $x = (x^{(n)})_{n \in \mathbb{N}}$, avec $(x^{(n+1)})^p = x^{(n)}$ et $x^{(0)} \in K^*$. Un tel $x \otimes t^{-1}$ est dans $B_{p,f}$ si et seulement si $x^{(0)} \in U_K$, groupe des unités de l'anneau des entiers de K, auquel cas son image dans B_{dR} est

$$\begin{aligned} & t^{-1}\left(\sum_{n \geq 1} (-1)^{n-1}([x]-1)^n/n\right) \\ & \quad = t^{-1}\left(\sum_{n \geq 1} (-1)^{n-1}(([x]/x^{(0)})-1)^n/n + \log(x^{(0)})\right). \end{aligned}$$

On voit que, même si $x^{(0)} \notin U_K$, la série $\sum_{n\geq 1}(-1)^{n-1}(([x]/x^{(0)})-1)^n/n$ converge dans B_{dR}. Si l'on *choisit* un prolongement à K^*,

$$\log\colon K^* \to K$$

du logarithme p-adique usuel, il existe alors un et seul homomorphisme de B_{cris}-algèbres qui envoie $x\otimes t^{-1}$ sur

$$t^{-1}\left(\sum_{n\geq 1}(-1)^{n-1}(([x]/x^{(0)})-1)^n/n+\log(x^{(0)})\right).$$

Cette application est G_K-équivariante et on montre ([**Bu**, exp. II, n° 4.2.4 et n°4.3]) qu'elle est injective; nous l'utilisons dans la suite pour identifier $B_{st,p}$ à un sous-anneau de B_{dR} (prendre garde que cette identification *dépend du choix* du prolongement du logarithme). L'homomorphisme de $\overline{K}$-algèbres $\overline{K}\otimes_{\mathbb{Q}_p^{nr}} B_{st,p}\to B_{\mathrm{dR}}$ que l'on en déduit par extension des scalaires est encore injectif (loc.cit.).

On a $(B_{\mathrm{dR}})^{G_K}=K$ tandis que $(B_{\mathrm{dR}})^{I_K}$ est le complété de l'extension maximale non ramifiée de K contenue dans $\overline{K}$. En outre $(B_{\mathrm{cris}})^{G_K}=(B_{st,p})^{G_K}=K_0$ tandis que, pour tout sous-groupe ouvert H de I_K, $(B_{\mathrm{cris}})^H=(B_{st,p})^H=\mathbb{Q}_p'$.

Pour $\ell\neq p$, on écrit aussi $B_{st,\ell}=B_\ell$. Si b_0 est comme en 2.1.5, $B_{st,\ell}$ est l'anneau des polynômes en b_0 à coefficients dans $\mathbb{Q}_\ell$ (resp. B_{cris}) si $\ell\neq p$ (resp. $\ell=p$). Pour $\ell\neq p$ et tout sous-groupe H de G_K tel que $H\cap I_K$ est ouvert dans I_K, $(B_{st,\ell})^H=\mathbb{Q}_\ell$.

2.1.8. Soit v_K la valuation de K normalisée par $v_K(K^*)=\mathbb{Z}$. Elle se prolonge par continuité en un homomorphisme de K_ℓ^* dans $\mathbb{Z}_\ell\subset\mathbb{Q}_\ell$. Si $\ell=p$ (resp. $\neq p$), on note

$$N=N_K\colon B_{st,\ell}\to B_{st,\ell}(-1)$$

l'unique $\mathbb{Q}_\ell$-dérivation (resp. B_{cris}-dérivation) de $B_{st,\ell}$ telle que, si $t\in\mathbb{Q}_\ell(1)$ est non nul, si $u\in V_p^0(\overline{K}^*/K^*)$ et si $\overline{u}$ est son image dans K_ℓ^*, alors $N(u\otimes t^{-1})=v_K(\overline{u})\otimes t^{-1}$. On voit que N commute à l'action de G_K, est surjective et que son noyau est $\mathbb{Q}_\ell$ (resp. B_{cris}) si $\ell\neq p$ (resp. $\ell=p$). Si $\ell=p$, N commute à l'action de φ (agissant sur $B_{st,p}(-1)=B_{st,p}\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(-1)$ par $\varphi(b\otimes t^{-1})=\varphi b\otimes t^{-1}$). Attention que, si l'on utilise l'application $b\otimes t^{-1}\mapsto bt^{-1}$ pour identifier $B_{st,p}(-1)$ à $B_{st,p}$, alors $N\varphi=p\varphi N$.

2.2. Représentations potentiellement semi-stables. Dans tout le n°2.2, V est une représentation ℓ-adique de G_K. On pose $b(V) = \dim_{\mathbb{Q}_\ell} V$.

2.2.1. Si $\ell = p$, rappelons [Se67] que le K-espace vectoriel gradué

$$D_{\mathrm{HT}}(V) = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}^i D_{\mathrm{HT}}(V), \text{ avec } \mathrm{gr}^i D_{\mathrm{HT}}(V) = (\mathbb{C}_p(i) \otimes V)^{G_K}$$

est de dimension finie $\leq b(V)$. On introduit les K-espaces vectoriels

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \quad \text{et} \quad t_V = ((B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Ils sont munis d'une filtration décroissante naturelle

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) = (\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \quad \text{pour tout } i \in \mathbb{Z},$$

et

$$\mathrm{Fil}^i t_V = ((\mathrm{Fil}^i B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes_{\mathbb{Q}_p} V)^{G_K} \quad \text{pour } i \text{ entier } \leq 0.$$

On a des injections naturelles

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \to \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}^i D_{\mathrm{HT}}(V), \qquad \mathrm{gr}(\mathrm{Fil}^0 D_{\mathrm{dR}}(V)) \to \bigoplus_{i \geq 0} \mathrm{gr}^i D_{\mathrm{HT}}(V),$$

et

$$\mathrm{gr}(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)) \to \mathrm{gr}(t_V) \to \bigoplus_{i<0} \mathrm{gr}^i D_{\mathrm{HT}}(V).$$

On en déduit que $\dim_K D_{\mathrm{dR}}(V) \leq b(V)$ et on dit que V *est de de Rham* si l'on a l'égalité. On voit qu'alors toutes ces injections sont des isomorphismes et la proposition suivante est immédiate.

2.2.2. PROPOSITION. i) *On a* $\dim_K t_V \leq b(V)$.

ii) *Si* V *est de de Rham, l'application* $D_{\mathrm{dR}}(V) \to t_V$ *induite par la projection* $B_{\mathrm{dR}} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ *est surjective et de noyau égal à* $\mathrm{Fil}^0 D(V)$.

iii) *Si*

$$0 \to V' \to V \to V'' \to 0$$

est une suite exacte de représentations p-adiques et si V *est de de Rham, il en est de même de* V' *et* V'' *et les suites*

$$0 \to D_{\mathrm{dR}}(V') \to D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V'') \to 0 \quad \textit{et} \quad 0 \to t_{V'} \to t_V \to t_{V''} \to 0$$

sont exactes.

REMARQUE. On appelle t_V *l'espace tangent de* V. Il est parfois commode de le voir comme un groupe vectoriel sur K, autrement dit, de poser, pour toute K-algèbre R, $t_V(R) = R \otimes_K t_V$.

2.2.3. Notons $\mathcal{I}_K$ l'ensemble des sous-groupes ouverts de I_K. Rappelons que l'on a noté $\mathbb{Q}'_p$ le complété de $\mathbb{Q}_p^{nr}$; si $\ell \neq p$, posons $\mathbb{Q}'_\ell = \mathbb{Q}_\ell$. Remarquons que, pour tout ℓ (y compris $\ell = p$) et tout $H \in \mathcal{I}_K$, $(B_{st,\ell})^H = \mathbb{Q}'_\ell$.

On pose

$$\widehat{D}_{pst}(V) = \text{lim.ind.}_{H \in \mathcal{I}_K} (B_{st,\ell} \otimes_{\mathbb{Q}_\ell} V)^H .$$

On montre facilement ([**Bu,** Exp. VIII]) que $\widehat{D}_{pst}(V)$ est un $\mathbb{Q}'_\ell$-espace vectoriel de dimension finie $\leq b(V)$, avec l'égalité si et seulement si l'application naturelle

$$B_{st,\ell} \otimes_{\mathbb{Q}'_\ell} \widehat{D}_{pst}(V) \to B_{st,\ell} \otimes_{\mathbb{Q}_\ell} V$$

est un isomorphisme.

On voit aussi que $\widehat{D}_{pst}(V)$ est muni d'une action semi-linéaire de G_K, l'action de I_K étant linéaire et discrète, i.e., le noyau de l'action de I_K est ouvert dans I_K. Si t est encore un générateur de $\mathbb{Q}_\ell(1)$, et si l'on pose $Nb = N_t b \otimes t^{-1}$ pour tout $b \in B_{st,\ell}$, on étend l'opérateur de monodromie en un opérateur

$$N \colon B_{st,\ell} \otimes_{\mathbb{Q}_\ell} V \to B_{st,\ell} \otimes_{\mathbb{Q}_\ell} V(-1)$$

en posant $N(b \otimes v) = N_t b \otimes (v \otimes t^{-1})$. Le résultat est indépendent du choix de t et induit un opérateur $\mathbb{Q}'_\ell$-linéaire, G_K-équivariant

$$N \colon \widehat{D}_{pst}(V) \to \widehat{D}_{pst}(V(-1)).$$

Introduisons les sous-$\mathbb{Q}'_\ell$-espaces vectoriels de $\widehat{D}_{pst}(V)$:

$$\widehat{D}_{st}(V) = (\widehat{D}_{pst}(V))^{I_K} = (B_{st,\ell} \otimes V)^{I_K}, \qquad \widehat{D}_{pbr}(V) = (\widehat{D}_{pst}(V))_{N=0},$$

et

$$\widehat{D}_{br}(V) = \widehat{D}_{st}(V) \cap \widehat{D}_{pbr}(V) = (\widehat{D}_{st}(V))_{N=0}.$$

On dit que V est *potentiellement semi-stable* (resp. est *semi-stable*, resp. a *potentiellement bonne réduction*, resp. a *bonne réduction*) si la dimension de $\widehat{D}_{pst}(V)$ (resp. $\widehat{D}_{st}(V)$, resp. $\widehat{D}_{pbr}(V)$, resp. $\widehat{D}_{br}(V)$) est $b(V)$.

2.2.4. Supposons $\ell \neq p$. Il résulte facilement du théorème de monodromie ℓ-adique de Grothendieck [**ST68,** App.] que V est toujours potentiellement semi-stable. On voit que V est semi-stable si et seulement si l'action de I_K est unipotente. On a $\widehat{D}_{br}(V) = V^{I_K}$, ce qui fait que V a bonne réduction si et seulement si V est non ramifiée. Enfin $\widehat{D}_{pbr}(V) = \text{lim.ind.}_{H \in \mathcal{I}_K} V^H$ et V a potentiellement bonne réduction si et seulement si I_K opère à travers un quotient fini (auquel cas $\widehat{D}_{pbr}(V) = V$).

2.2.5. Supposons $\ell = p$. Notons $\mathcal{G}_K$ l'ensemble des sous-groupes ouverts de G_K, et posons

$$D_{pst}(V) = \text{lim.ind.}_{H \in \mathcal{G}_K} (B_{st,p} \otimes_{\mathbb{Q}_p} V)^H \quad \text{et} \quad D_{st}(V) = (B_{st,p} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Il est facile de voir que les applications naturelles $\mathbb{Q}'_p \otimes_{\mathbb{Q}_p^{nr}} D_{pst}(V) \to \widehat{D}_{pst}(V)$ et $\mathbb{Q}'_p \otimes_{K_0} D_{st}(V) \to \widehat{D}_{st}(V)$ sont des isomorphismes. En particulier, la notion de représentation p-adique (potentiellement) semi-stable coïncide avec celle qui est donnée dans **[Bu,** Exp. III, §5].

Rappelons (op.cit.) que la dimension du K-espace vectoriel $D_{pst,K}(V) = (\overline{K} \otimes_{\mathbb{Q}_p^{nr}} \widehat{D}_{pst}(V))^{G_K} = (\overline{K} \otimes_{\mathbb{Q}_p^{nr}} D_{pst}(V))^{G_K}$ est égale à la $\mathbb{Q}'_p$-dimension de $\widehat{D}_{pst}(V)$ et l'inclusion de $\overline{K} \otimes_{\mathbb{Q}_p^{nr}} B_{st,p}$ dans B_{dR} identifie $D_{pst,K}(V)$ à un sous-K-espace vectoriel de $D_{\mathrm{dR}}(V)$. En particulier, si V est potentiellement semi-stable, V est de de Rham.[2]

Rappelons (op.cit.) par ailleurs que l'on note $D_{\mathrm{cris}}(V)$ le K_0-espace vectoriel $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. L'application naturelle $\mathbb{Q}'_p \otimes_{K_0} D_{\mathrm{cris}}(V) \to \widehat{D}_{br}(V)$ est un isomorphisme; en particulier, $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}'_p} \widehat{D}_{br}(V)$ et dire que V a bonne réduction signifie qu'elle est *cristalline.*

2.2.6. Si $\ell \neq p$, posons aussi $D_{pst}(V) = \widehat{D}_{pst}(V)$. Dans tous les cas ($\ell =$ ou $\neq p$), $D_{pst}(V)$ a une structure naturelle d'objet de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$: si $\Delta = D_{pst}(V)$, alors

i) le sous-$\mathbb{Q}_\ell$-espace vectoriel Δ de $B_{st,\ell} \otimes_{\mathbb{Q}_\ell} V$ est stable par G_K et $\rho_{sl}\colon W_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(\Delta)$ est la restriction de l'action naturelle de G_K;
ii) l'opérateur $N\colon \Delta \to \Delta\{-1\}$ est induit par l'opérateur du même nom défini au n°2.2.3 une fois que l'on a remarqué que $\Delta\{-1\}$ s'identifie à $D_{pst}(V(-1))$;
iii) enfin, si $\ell = p$, l'action de φ sur Δ est la restriction de $\varphi \otimes \mathrm{id}$ sur $B_{st,p} \otimes_{\mathbb{Q}_p} V$.

2.2.7. Utilisant la "hiérarchie des représentations du groupe de Weil-Deligne" (n°1.2), on peut raffiner celle des représentations ℓ-adiques de G_K. Pour $\nu \in \{e, f, h, g, pe, pf, ph, pg\}$, on peut définir la sous-catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell,\nu}(G_K)$ comme étant la sous-catégorie pleine de $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K)$ dont les objets sont les représentations V qui sont potentiellement semi-stables et telles que $\widehat{D}_{st}(V)$ vu comme un objet de $\mathbf{Rep}_{\mathbb{Q}'_\ell}('W_K)$ est en fait dans $\mathbf{Rep}_{\mathbb{Q}'_\ell,\nu}('W_K)$.

Ce sont des sous-catégories tannakiennes de $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K)$. La catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell,pg}(G_K)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_\ell,g}(G_K)$, resp. $\mathbf{Rep}_{\mathbb{Q}_\ell,pf}(G_K)$, resp. $\mathbf{Rep}_{\mathbb{Q}_\ell,f}(G_K)$) est celle des représentations potentiellement semi-stables (resp. semi-stables, resp. ayant potentiellement bonne réduction, resp. ayant bonne réduction).

2.2.8. Remarques (cf. **[Bu,** exp. VIII)]. a) Supposons $\ell \neq p$. La façon dont nous avons associé à la représentation V l'objet $D_{pst}(V)$ de

[2]On peut se demander s'il existe un théorème de monodromie p-adique arithmétique, i.e. s'il est vrai que toute représentation de de Rham est potentiellement semi-stable.

$\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$ diffère de la construction usuelle: choisissons un 1-cocycle non trivial $c\colon G_K \to \mathbb{Q}_\ell(1)$ (on peut par example choisir un relèvement f_K de f_k dans G_K ainsi qu'un homomorphisme non trivial, G_K-équivariant, $\pi\colon I_K \to \mathbb{Q}_\ell(1)$, poser $g = \alpha(g)\cdot f_K^{\nu(g)}$, avec $\nu(g) \in \widehat{\mathbb{Z}}$ et $\alpha(g) \in I_K$ et prendre $c = \pi\circ\alpha$). Il existe alors une unique structure (V, ρ, N) d'objet de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$ sur le $\mathbb{Q}_\ell$-espace vectoriel sous-jacent à V telle que, si, pour tout $g \in G$, on note N_g l'endomorphisme nilpotent de V défini par $N_g(v) = Nv \otimes c(g)$, on ait

$$g(v) = (\rho(g)\circ\exp(N_g))(v) \text{ si } v \in V \text{ et } g \in W_K .$$

On obtient ainsi un $\otimes$-foncteur $D_{pst,c}\colon \mathbf{Rep}_{\mathbb{Q}_\ell}(G_K) \to \mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$. Il n'est pas difficile de voir que les foncteurs $D_{pst,c}$ et D_{pst} sont naturellement équivalents.

b) Encore pour $\ell \neq p$, on voit que le foncteur D_{pst} induit une $\otimes$-équivalence entre $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K) = \mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K)$ et la sous-catégorie pleine $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'G_K)$ de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$. En particulier, les résultats du n°1.4 permettent de calculer la cohomologie galoisienne de V.

c) Lorsque $\ell = p$, on peut considérer D_{pst} comme un foncteur additif $\mathbb{Q}_p$-linéaire de $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$ dans $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$; sa restriction à $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K)$ est un $\otimes$-foncteur exact et fidèle, dont l'image essentielle est contenue (et vraisemblablement égale à $\mathbf{Rep}_{\mathbb{Q}_p}(D'G_K)$).

d) Si Δ est un objet de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$, $\Delta_K = (\overline{K}\otimes_{\mathbb{Q}_p^{nr}}\Delta)^{W_K}$ est un espace vectoriel de dimension finie sur K égale à la dimension de Δ sur $\mathbb{Q}_p^{nr}$. Appelons *DWD-module filtré* la donnée d'un couple formé un objet Δ de $\mathbf{Rep}_{\mathbb{Q}_p}(D'W_K)$ et d'une filtration décroissante $(\mathrm{Fil}^i\Delta_K)_{i\in\mathbb{Z}}$ de Δ_K par des sous-K-espaces vectoriels vérifiant $\mathrm{Fil}^i\Delta_K = \Delta_K$ si $i \ll 0$ et $= 0$ si $i \gg 0$. Les DWD-filtrés forment de manière évidente une catégorie $\mathbb{Q}_p$-linéaire, qui n'est pas abélienne. Si $\ell \neq p$ et si la représentation V est potentiellement semi-stable, $D_{pst}(V)_K$ s'identifie à $D_{\mathrm{dR}}(V)$ et $D_{pst}(V)$ a donc une structure naturelle de DWD-module filtré. On en déduit que D_{pst} induit un foncteur $\mathbb{Q}_p$-linéaire de la catégorie des représentations p-adiques potentiellement semi-stables de G_K dans celle des DWD-modules filtrés et on montre que ce foncteur est pleinement fidèle.

3. Cohomologie galoisienne

3.1. Quelques suites exactes. Dans la suite, on prendra garde que, comme on l'a déjà remarqué, l'application $b\otimes t^{-1} \mapsto bt^{-1}$ permet d'identifier $B_{st,p}(-1)$ à $B_{st,p}$ mais ne commute pas à φ (elle envoie $\varphi b\otimes t^{-1}$ sur $p\varphi(bt^{-1})$).

3.1.1. PROPOSITION. *Soit* $\lambda\colon B_{st,p} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ *le composé de l'inclusion de* $B_{st,p}$ *dans* B_{dR} *avec la projection de* B_{dR} *sur* $B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$. *Les suites*

$$(S_e) \qquad 0 \to \mathbb{Q}_p \to B_{\mathrm{cris},\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$
$$(\textit{avec } a \mapsto a;\ b \mapsto \lambda(b)),$$

$$(S_f) \qquad 0 \to \mathbb{Q}_p \to B_{\mathrm{cris}} \to B_{\mathrm{cris}} \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$
$$(\textit{avec } a \mapsto a;\ b \mapsto (\varphi b - b, \lambda(b))),$$

$$(S_h) \qquad 0 \to \mathbb{Q}_p \to B_{st,p,\varphi=1} \to B_{st,p}(-1)_{\varphi=1} \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$
$$(\textit{avec } a \mapsto a;\ b \mapsto (Nb, \lambda(b))),$$

$$(S_g) \quad 0 \to \mathbb{Q}_p \to B_{st,p} \to B_{st,p} \oplus B_{st,p}(-1) \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to B_{st,p}(-1) \to 0$$
$$(\textit{avec } a \mapsto a;\ b \mapsto (\varphi b - b, Nb, \lambda(b));\ (b_1, b_2, c) \mapsto (Nb_1 - \varphi b_2 + b_2))$$

sont exactes. Elles sont scindées en tant que suites exactes de $\mathbb{Q}_p$*-espaces vectoriels topologiques.*

3.1.2. PREUVE DE L'EXACTITUDE. L'exactitude en $\mathbb{Q}_p$ de chacune de ces quatre suites est claire. L'exactitude en le second terme résulte, pour (S_e) et (S_f), de ce que $\mathbb{Q}_p = B_{\mathrm{cris},\varphi=1} \cap B_{\mathrm{dR}}^+$ (n° 2.1.3) et, pour (S_g) et (S_h), de ce qu'en outre, $B_{\mathrm{cris}} = B_{st,p,N=0}$ (n°2.1.8).

Pour achever la preuve de *l'exactitude de* (S_e), il reste donc à vérifier que $B_{\mathrm{dR}} = B_{\mathrm{cris},\varphi=1} + B_{\mathrm{dR}}^+$. Soit t un générateur de $\mathbb{Q}_p(1)$; c'est un élément de $B_{\mathrm{cris},\varphi=p}^+$ vérifiant $\varphi(t) = pt$ et c'est aussi un générateur de l'idéal maximal de B_{dR}^+. Comme, pour tout $c \in B_{\mathrm{dR}}$, il existe $r \in \mathbb{N}$ tel que $t^r c \in B_{\mathrm{dR}}^+$, il suffit de vérifier que pour tout $r \in \mathbb{N}$, $B_{\mathrm{dR}}^+ \subset B_{\mathrm{cris},\varphi=p^r}^+ + \mathrm{Fil}^r B_{\mathrm{dR}}$. De proche en proche, il suffit de s'assurer que $B_{\mathrm{dR}}^+ \subset B_{\mathrm{cris},\varphi=p^r}^+ + \mathrm{Fil}^1 B_{\mathrm{dR}}$. Or on sait (cf. n°2.1.3) que la suite

$$0 \to \mathbb{Q}_p(1) \to B_{\mathrm{cris},\varphi=p}^+ \to \mathbb{C}_p \to 0$$

est exacte. On peut donc, si $b \in B_{\mathrm{dR}}^+$, trouver $b_1, b_2, \dots, b_r \in B_{\mathrm{cris},\varphi=p}^+$ tels que $\theta(b_1, b_2 \cdots b_r) = \theta(b_1) \cdot \theta(b_2) \cdots \theta(b_r) = \theta(b)$. On a bien $b_1 b_2 \cdots b_r \in B_{\mathrm{cris},\varphi=p^r}^+$, tandis que $b - b_1 b_2 \cdots b_r \in \mathrm{Fil}^1 B_{\mathrm{dR}}$.

Pour achever la preuve de l'*exactitude de* (S_f), il reste à vérifier que, si $b \in B_{\mathrm{cris}}$ et $c \in B_{\mathrm{dR}}$, alors il existe $x \in B_{\mathrm{cris}}$ et $y \in B_{\mathrm{dR}}^+$ tels que $b = \varphi x - x$ et $c = b + y$. Comme $\varphi - 1$ est surjectif sur B_{cris} (n°2.1.3), il existe $x_0 \in B_{\mathrm{cris}}$ tel que $\varphi x_0 - x_0 = b$; il suffit alors de prendre $x = x_0 + x_1$ où $x_1 \in B_{\mathrm{cris},\varphi=1}$ et $y \in B_{\mathrm{dR}}^+$ sont tels que $c = x_1 + y$, ce qui est possible d'après l'exactitude de (S_e).

Pour *l'exactitude de* (S_h), il reste à vérifier la surjectivité de $B_{st,p,\varphi=1} \to B_{st,p}(-1)_{\varphi=1} \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$: elle résulte de ce que $N\colon B_{st,p} \to B_{st,p}(-1)$ est surjective et envoie $B_{st,p,\varphi=1}$ sur $B_{st,p}(-1)_{\varphi=1}$ et de ce que le noyau de N sur $B_{st,p,\varphi=1}$ est $B_{\mathrm{cris},\varphi=1}$ qui s'envoie surjectivement sur $B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$.

Enfin pour *l'exactitude de* (S_g), il reste à vérifier

a) l'exactitude en $X = B_{st,p} \oplus B_{st,p}(-1) \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$: il est clair que l'application composée $B_{st,p} \to X \to B_{st,p}(-1)$ est nulle; comme $N: B_{st,p} \to B_{st,p}(-1)$ est surjective, il suffit de vérifier que si $(b_1, 0, c) \in X$ s'envoie sur 0 dans $B_{st,p}(-1)$, i.e., si $b_1 \in B_{\mathrm{cris}}$, alors $(b_1, 0, c)$ est dans l'image de $B_{st,p}$, ce qui est l'exactitude de (S_f) en $B_{\mathrm{cris}} \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$;

b) la surjectivité de $X \to B_{st,p}(-1)$ qui est immédiate puisque $N: B_{st,p} \to B_{st,p}(-1)$ est déjà surjective.

3.1.3. Il est clair que les applications qui interviennent dans les suites exactes ci-dessus sont continues. *Prouvons l'existence de sections* $\mathbb{Q}_p$*-linéaires continues.* On vérifie facilement qu'il suffit de prouver que l'application $B_{\mathrm{cris},\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ admet une section continue s_1, que $\varphi - 1: B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ admet une section continue s_2 à valeurs dans B_{cris}^+ et que $N: B_{st,p} \to B_{st,p}$ admet une section continue s_3 à valeurs dans $B_{st,p}$ commutant à φ. Il résulte de 2.1.7 que l'on peut trouver $y = x \otimes t^{-1} \in B_{st,p,\varphi=1}$ tel que $B_{st,p}$ est une algèbre de polynômes en y à coefficients dans B_{cris}; si $b = \sum b_n \cdot y^n$ avec $b_n \in B_{\mathrm{cris}}$, on prend $s_3(b) = \sum b_n \cdot y^{n+1}/(n+1)$.

On déduit l'existence de s_2 des suites exactes de [**Bu**, Exp. II, n°5.3.7].

Montrons rapidement l'existence de s_1 [**BK90**, 1.18]: Soit

$$W_n = \{\log[w_1] \cdots \log[w_n] \text{ avec } w_i \in V_p(U_p)\} \subset B_{\mathrm{cris},\varphi=p^n}.$$

On vient de voir que l'application $W_n \to B_{\mathrm{dR}}^+/t^n B_{\mathrm{dR}}^+$ est surjective. Comme le noyau de la projection de $B_{\mathrm{cris},\varphi=p^n}$ sur $B_{\mathrm{dR}}^+/t^n B_{\mathrm{dR}}^+$ est $\mathbb{Q}_p \cdot t^n \subset W_n$, on a $W_n = B_{\mathrm{cris},\varphi=p^n}$. On sait que, pour tout $n \geq 1$, la suite

$$0 \to W_{n-1} \cdot t \to W_n \to \mathbb{C}_p \to 0$$

est exacte. Posons

$$\mathscr{W}_n = \{\log[w_1] \cdots \log[w_n] \quad \text{avec } w_i \in V_p(1 + p\mathscr{O}_{\mathbb{C}_p})\} \subset W_n$$

où $\mathscr{O}_{\mathbb{C}_p}$ est l'anneau des entiers de $\mathbb{C}_p$. On vérifie que la flèche naturelle $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathscr{W}_n \to W_n$ est un isomorphisme et que $\mathscr{W}_n$ est contenu dans A_{cris}. On en déduit que $\mathscr{W}_n$ est séparé et complet pour la topologie p-adique et que la topologie de W_n est la topologie induite. Comme l'image de $\mathscr{W}_n$ par la projection de W_n sur $\mathbb{C}_p$ est fermée dans $\mathscr{O}_{\mathbb{C}_p}$, cette projection admet une section continue $\sigma_n: \mathbb{C}_p \to W_n$. On peut donc écrire $W_n = W_{n-1} \cdot t \oplus S_n$ avec $S_n = \sigma_n(\mathbb{C}_p)$. Posons aussi $S_0 = \mathbb{Q}_p = W_0$. On a $B_{\mathrm{cris},\varphi=1} = \bigcup_{n \in \mathbb{N}} W_n t^{-n} = \bigoplus_{n \in \mathbb{N}} S_n \cdot t^{-n}$ tandis que $B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ s'identifie à $B_{\mathrm{cris},\varphi=1}/\mathbb{Q}_p = B_{\mathrm{cris},\varphi=1}/S_0$ et, si $a \in B_{\mathrm{cris},\varphi=1}/\mathbb{Q}_p$ est l'image de $\sum_{n \in \mathbb{N}} a_n t^{-n}$, avec $a_n \in S_n$, il suffit de prendre $s_1(a) = \sum_{n \geq 1} a_n t^{-n}$.

3.2. La cohomologie galoisienne classique.

3.2.1. Soient T une représentation $\mathbb{Z}_\ell$-adique de G_K et $i \in \mathbb{Z}$. Si T est un groupe fini, on a $H^i(K, T) = H^i(G_K, T) = H^i_{\text{cont}}(G_K, T)$. Dans le cas général, on pose

$$H^i(K, T) = \text{lim.proj.}_{n \in \mathbb{N}} H^i(G_K, T/\ell^n T).$$

Les $H^i(K, -)$ forment de manière naturelle un foncteur cohomologique et, pour tout T et tout i, la flèche canonique

$$H^i(K, T) \to H^i_{\text{cont}}(G_K, T)$$

est un isomorphisme. Nous l'utilisons pour identifier ces deux $\mathbb{Z}_\ell$-modules de type fini.

3.2.2. Soit V une représentation ℓ-adique de G_K. On peut toujours trouver une représentation $\mathbb{Z}_\ell$-adique T de G_K munie d'un isomorphisme de $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T$ sur V (par exemple un réseau de V stable par G_K) et, pour tout i, le $\mathbb{Q}_\ell$-espace vectoriel $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^i(K, T)$ est indépendant de T et on le note $H^i(K, V)$.

Les $H^i(K, -)$ forment un foncteur cohomologique. Pour tout V et tout i, la flèche canonique $H^i(K, V) \to H^i_{\text{cont}}(G_K, V)$ est un isomorphisme et nous l'utilisons pour identifier ces deux $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie.

Pour tout i et tout V, on dispose aussi d'une application naturelle

$$H^i(K, V) \to H^i(G_K, V) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K), V)$$

(n°01). On vérifie que c'est un isomorphisme et nous l'utilisons pour identifier ces deux $\mathbb{Q}_\ell$-espaces vectoriels.

3.2.3. Soit T un groupe abélien fini d'ordre une puissance de ℓ. Rappelons (cf., par exemple, [**Mi86**, chap. I]) que les $H^i(K, T)$ sont des groupes finis, nuls si $i \neq 0, 1, 2$ et que l'on a

$$\prod_{i \in \mathbb{Z}} \#(H^i(K, T))^{(-1)^i} = \begin{cases} 1 & \text{si } \ell \neq p, \\ p^{-[K:\mathbb{Q}_p]} & \text{si } \ell = p. \end{cases}$$

Rappelons aussi (loc. cit.) que $H^2(K, \mathbb{Q}_\ell(1)/\mathbb{Z}_\ell(1))$ s'identifie à $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ et que, si $T\hat{} = \text{Hom}(T, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, le cup-produit

$$H^i(K, T) \times H^{2-i}(K, T\hat{}(1)) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

est un accouplement parfait.

3.2.4. De même, si V est une représentation ℓ-adique de G_K, les $H^i(K, V)$ sont des $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie, nuls si $i \neq 0, 1, 2$; on a

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}_\ell} H^i(K, V) = \begin{cases} 0 & \text{si } \ell \neq p, \\ -[K:\mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_\ell} V & \text{si } \ell = p. \end{cases}$$

En outre, $H^2(K, \mathbb{Q}_\ell(1))$ s'identifie à $\mathbb{Q}_\ell$ et le cup-produit

$$H^i(K, V) \times H^{2-i}(K, V^*(1)) \to \mathbb{Q}_\ell$$

(où V^* désigne le dual de V) est un accouplement parfait.

REMARQUE. Lorsque $\ell \neq p$, ces résultats classiques peuvent aussi se déduire des techniques développées au n°1.4 (cf. remarque (b) du n°2.2.8).

3.3. Les H^i_e, H^i_f, H^i_g et H^i_h. Dans tout le n°3.3, V est une représentation ℓ-adique de G_K.

3.3.1. Rappelons que $D_{pst}(V)$ est un objet de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$.

Si $\ell \neq p$ et si $\nu \in \{e, f, h, g\}$, on note $C_\nu(K, V)$ le complexe $C_\nu(K, D_{pst}(V))$ (cf. n°1.4).

Si $\ell = p$, l'inclusion de $B_{st,p}$ dans B_{dR} induit une inclusion de $D_{st}(V) = (D_{pst}(V))^{G_K}$ dans $D_{\mathrm{dR}}(V)$ et on note $\lambda_V : D_{st}(V) \to t_V$ le composé de cette inclusion avec la projection de $D_{\mathrm{dR}}(V)$ dans t_V. Pour $\nu \in \{e, f, h, g\}$, $C^0_\nu(K, D_{pst}(V)) \subset D_{st}(V)$ et λ_V induit un morphisme du complexe $C_\nu(K, D_{pst}(V))$ dans $t_V[0]$; on note $C_\nu(K, V)$ le décalé de 1 vers la droite du cône de ce morphisme, de sorte que l'on a une suite exacte de complexes

$$0 \to C_\nu(K, V) \to C_\nu(K, D_{pst}(V)) \to t_V[0] \to 0.$$

3.3.2. Pour $\nu \in \{e, f, h, g\}$, définissons $D_\nu(V)$ ainsi:

- si $\ell \neq p$,

$$D_e(V) = V^{G_K}, \quad D_f(V) = V^{I_K},$$
$$D_h(V) = (B_{st,\ell} \otimes V)^{G_K} \quad \text{et} \quad D_g(V) = (B_{st,\ell} \otimes V)^{I_K};$$

- si $\ell = p$, $D_\nu(V) = (B_\nu \otimes_{\mathbb{Q}_p} V)^{G_K}$, avec

$$B_e = B_{\mathrm{cris},\varphi=1}, \quad B_f = B_{\mathrm{cris}}, \quad B_h = (B_{st,p})_{\varphi=1}, \quad \text{et} \quad B_g = B_{st,p}.$$

Si l'on pose $t_V = 0$ lorsque $\ell \neq p$, on voit que les complexes $C_\nu(K, V)$ s'écrivent (on renvoie au n°1.4 pour la définition des flèches autre que λ_V):

$C_e(K, V) : D_e(V) \to t_V \to 0 \to 0 \to \cdots$;
$C_f(K, V) : D_f(V) \to D_f(V) \oplus t_V \to 0 \to 0 \to \cdots$,
$C_h(K, V) : D_h(V) \to D_h(V(-1)) \oplus t_V \to 0 \to 0 \to \cdots$,
$C_g(K, V) : D_g(V) \to D_g(V) \oplus D_g(V(-1)) \oplus t_V \to D_g(V(-1)) \to 0 \to \cdots$.

3.3.3. Pour $\nu \in \{e, f, h, g\}$, on note $H^i_\nu(K, V)$ les groupes de cohomologie de $C_\nu(K, V)$. Remarquons que ce sont des $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie, que $H^i_\nu(K, V) = 0$ pour $i \notin \{0, 1, 2\}$, que $H^2_\nu(K, V) = 0$ pour $\nu = e, f, h$. On a $H^i_\nu(K, V) = H^i_\nu(K, D_{pst}(V))$ si $i \neq 0, 1$ et l'on dispose de suites exactes

$$0 \to H^0_\nu(K, V) \to H^0_\nu(K, D_{pst}(V)) \to t_V \to H^1_\nu(K, V) \to H^1_\nu(K, D_{pst}(V)) \to 0$$

et

$$0 \to H^1_e(K, V) \to H^1_\nu(K, V) \to H^1_\nu(K, D_{pst}(V)) \to 0.$$

Si $\ell \neq p$, on a aussi $H^1_e(K, V) = 0$ et $H^1_\nu(K, V) = H^i_\nu(K, D_{pst}(V))$ pour tout i et tout ν.

REMARQUE. Dans **[BK90]**, Bloch et Kato définissent aussi des groupes $H^1_\nu(K, V)$, pour $\nu \in \{e, f, g\}$. Pour $\ell \neq p$, nos définitions redonnent les leurs. Pour $\ell = p$, ils ne définissent ces groupes que lorsque V est de de Rham; pour $\nu = e$ ou f, nos définitions redonnent les leurs, ainsi que pour $\nu = g$ si V est potentiellement semi-stable (cf. *infra*, remarque (i) du n°3.3.10). Pour une représentation p-adique V qui serait de de Rham sans être potentiellement semi-stable, le $H^1_g(K, V)$ de Bloch et Kato pourrait être différent du nôtre. Nous ne savons pas si une telle représentation existe mais on s'attend à ce que ce ne soit jamais le cas pour la réalisation p-adique d'un motif. C'est pourquoi nous avons fait le choix, discutable, de ne pas introduire une notation nouvelle.

3.3.4. Compte-tenu de la remarque (b) du n°2.2.8, la proposition 1.4.5 implique que:

PROPOSITION. *Supposons $\ell \neq p$. Pour $\nu \in \{e, f, h, g\}$ et $i \in \mathbb{Z}$, soit V_ν (resp. $V_{p\nu}$) la plus grande sous-représentation de V qui est dans $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(G_K)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(G_K)$). Alors $H^i_\nu(K, V)$ s'identifie canoniquement et fonctoriellement à $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(G_K), V_\nu)$ et à $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(G_K), V_{p\nu})$. En particulier, $H^i(K, V) = H^i_g(K, V)$.*

3.3.5. Si maintenant $\ell = p$, en tensorisant la suite exacte S_ν de la proposition 3.1.1 avec V, on obtient une résolution $S_\nu(V)$ de V par des $\mathbb{Q}_p$-espaces vectoriels topologiques avec action linéaire et continue de G_K (et existence de sections continues si l'on oublie l'action de G_K). Le complexe que l'on obtient en prenant les invariants sous G_K n'est autre que $C_\nu(K, V)$ et on obtient donc encore une application canonique

$$H^i_\nu(K, V) \to H^i(K, V).$$

3.3.6. Revenons au cas général. Remarquons que, pour $\nu \in \{e, f, h, g\}$, D_ν, vu comme foncteur additif de $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K)$ dans la catégorie des $\mathbb{Q}_\ell$-espaces vectoriels si $\ell \neq p$ (resp. K_0-espaces vectoriels si $\ell = p$), est exact à gauche et que $D_\nu(\mathbb{Q}_\ell) = \mathbb{Q}_\ell$ (resp. K_0).

Disons qu'une suite exacte courte de $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K)$ de la forme

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

est une *ν-suite exacte* si la suite

$$0 \to D_\nu(V_1) \to D_\nu(V_2) \to D_\nu(V_3) \to 0$$

est exacte. Remarquons que, lorsque V_1 et V_3 sont des objets de $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(G_K)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(G_K)$), cela revient à demander que V_2 aussi.

3.3.7 **Proposition.** *Soit* $\nu \in \{e, f, h, g\}$.

i) *L'application* $H^0_\nu(K, V) \to H^0(K, V)$ *est bijective.*

ii) *L'application* $H^1_\nu(K, V) \to H^1(K, V)$ *est injective et identifie* $H^1_\nu(K, V)$ *au sous-*$\mathbb{Q}_\ell$*-espace vectoriel de* $H^1(K, V)$ *classifiant les extensions* W *de* $\mathbb{Q}_\ell$ *par* V *telles que la suite*

$$0 \to V \to W \to \mathbb{Q}_\ell \to 0$$

soit une ν*-suite exacte.*

iii) *Si* V *est un objet de* $\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(G_K)$ *(resp.* $\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(G_K)$*),* $H^1_\nu(K, V)$ *s'identifie à* $H^1(\mathbf{Rep}_{\mathbb{Q}_\ell, \nu}(G_K), V)$ *(resp.* $H^1(\mathbf{Rep}_{\mathbb{Q}_\ell, p\nu}(G_K), V)$*).*

Preuve. Lorsque $\ell \neq p$, c'est une conséquence immédiate de la proposition 3.3.4.

Lorsque $\ell = p$, commençons par observer que, si

$$(\beta) \qquad 0 \to V \to W \to \mathbb{Q}_p \to 0$$

est une suite exacte courte et si X désigne la somme amalgamée de $B_\nu \otimes V$ et de W au dessus de V (V étant identifiée à un sous-module de $B_\nu \otimes V$ via l'application $v \mapsto 1 \otimes v$), on a un diagramme commutatif

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & W & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B_\nu \otimes V & \longrightarrow & X & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_\nu \otimes V & \longrightarrow & B_\nu \otimes W & \longrightarrow & B_\nu & \longrightarrow & 0
\end{array}$$

dont les lignes sont exactes et toutes les flèches verticales sont injectives. Si (β) correspond à un élément $x \in H^1(K, V)$, on voit alors que dire que (β) est une ν-suite exacte revient à dire que x appartient au noyau $N_\nu H^1(K, V)$ de l'application $H^1(K, V) \to H^1(K, B_\nu \otimes V)$ induite par l'inclusion de V dans $B_\nu \otimes V$.

Mais le complexe $S_\nu(V)$ (n°3.3.5) fournit une suite exacte courte

$$0 \to V \to B_\nu \otimes V \to C_\nu \otimes V \to 0,$$

où B_ν est comme au n°3.3.2 et C_ν est un $\mathbb{Q}_p$-espace vectoriel topologique avec action linéaire et continue de G_K convenable, l'application $B_\nu \otimes V \to C_\nu \otimes V$ admettant une section continue. On a $(B_\nu \otimes V)^{G_K} = D_\nu(V) = C^0_\nu(K, V)$ tandis que $(C_\nu \otimes V)^{G_K}$ s'identifie à $Z^1_\nu(K, V)$. En prenant les éléments fixes par G_K, on obtient une suite exacte

$$0 \to H^0(K, V) \to C^0_\nu(K, V) \to Z^1_\nu(K, V) \to H^1(K, V) \to H^1(K, B_\nu \otimes V)$$

ou encore

$$0 \to H^0(K, V) \to C_\nu^0(K, V) \to Z_\nu^1(K, V) \to N_\nu H^1(K, V) \to 0$$

d'où (i) et (ii). L'assertion (iii) est maintenant immédiate.

3.3.8. Rappelons que $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K)$ est la catégorie des représentations ℓ-adiques potentiellement semi-stables (et que c'est aussi $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_K)$ si $\ell \neq p$).

Proposition. *Si V est potentiellement semi-stable, pour tout $i \in \mathbb{Z}$,*

$$H_g^i(K, V) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K), V).$$

Preuve. Pour $\ell \neq p$, ce résultat est contenu dans la proposition 3.3.4. Supposons donc $\ell = p$. Toute suite exacte courte de $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K)$ induit une suite exacte courte des complexes $C_g(K, -)$ correspondant et $H_g^*(K, -)$ est donc un foncteur cohomologique. Comme $H^0(\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K), -)$ s'identifie à $H_g^0(K, -)$, on a des flèches naturelles $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K), -) \to H_g^i(K, -)$. D'après la proposition précédente, ce sont des isomorphismes pour $i = 0$ et 1. Comme $H_g^i(K, V) = 0$ pour $i \geq 3$, on en déduit que $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_K), -) \to H_g^i(K, -)$ est un isomorphisme pour $i \neq 2$ et est injective pour $i = 2$. En raisonnant comme à la fin de la preuve de la proposition 1.4.5, on se ramène à vérifier que, pour toute représentation potentiellement semi-stable V et tout $x \in D_e(V)$, il existe une représentation potentiellement semi-stable W et un épimorphisme de W sur V tels que x n'est pas dans l'image de $D_e(W)$. On se ramène facilement au cas où V est semi-stable. C'est alors un exercice sur les représentations semi-stables que nous laissons au lecteur (utiliser la suite exacte (S_g)).

3.3.9. Si V_1 et V_2 sont deux représentations ℓ-adiques de G_K, la structure d'anneau sur $B_{st,\ell}$ induit une application bilinéaire

$$D_{pst}(V_1) \times D_{pst}(V_2) \to D_{pst}(V_1 \otimes V_2).$$

Lorsque l'on prend $V_1 = V^*(1)$ et $V_2 = V$, si l'on compose avec $D_{pst}(\eta)$ où $\eta\colon V^*(1) \otimes V \to \mathbb{Q}_\ell(1)$ est la projection canonique, on en déduit en particulier une application bilinéaire

$$D_{pst}(V^*(1)) \times D_{pst}(V) \to D_{pst}(\mathbb{Q}_\ell(1)) = Ta$$

(où Ta est l'objet de Tate de la catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$, cf. n°1.4.3). On en déduit un morphisme de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$,

$$D_{pst}(V^*(1)) \to D_{pst}(V)^*(Ta)$$

qui est un isomorphisme lorsque V est potentiellement semi-stable.

Par ailleurs, on sait que $H_g^2(K, \mathbb{Q}_\ell(1)) = H_g^2(K, Ta)$ et $H^2(K, \mathbb{Q}_\ell(1))$ sont tous deux naturellement isomorphes à $\mathbb{Q}_\ell$ (n°1.4.7 et 3.2.4).

Proposition. i) *Pour tout* $i \in \mathbb{Z}$, *le diagramme*

$$\begin{array}{ccc} H^i_g(K, V^*(1)) \times H^{2-i}_g(K, V) & \longrightarrow & H^i(K, V^*(1)) \times H^{2-i}(K, V) \\ \downarrow & & \\ H^i_g(K, D_{pst}(V^*(1))) \times H^{2-i}_g(K, D_{pst}(V)) & & \downarrow \\ \downarrow & & \\ H^i_g(K, D_{pst}(V)^*(Ta)) \times H^{2-i}_g(K, D_{pst}(V)) & \longrightarrow & \mathbb{Q}_\ell \end{array}$$

(*où les flèches sont soient les flèches naturelles soient les flèches induites par le cup-produit*) *est commutatif.*

ii) *Soient* $x \in H^1(K, V^*(1))$ *et* $y \in H^1(K, V)$; *si* $x \in H^1_e(K, V^*(1))$ (*resp.* $H^1_f(K, V^*(1))$, $H^1_h(K, V^*(1))$, $H^1_g(K, V^*(1))$) *tandis que* $y \in H^1_g(K, V)$ (*resp.* $H^1_f(K, V)$, $H^1_h(K, V)$, $H^1_e(K, V)$), *alors l'image dans* $\mathbb{Q}_\ell$ *du cup-produit de* x *avec* y *est nulle.*

iii) *Si* V *est potentiellement semi-stable* (*en particuler, si* $\ell \neq p$), *dans la dualité entre* $H^1(K, V^*(1))$ *et* $H^1(K, V)$ *l'orthogonal de* $H^1_e(K, V^*(1))$ *est* $H^1_g(K, V)$, *celui de* $H^1_f(K, V^*(1))$ *est* $H^1_f(K, V)$, *celui de* $H^1_h(K, V^*(1))$ *est* $H^1_h(K, V)$ *et celui de* $H^1_g(K, V^*(1))$ *est* $H^1_e(K, V)$.

iv) *Lorsque l'action de* f_k *sur* $D_f(V)$ *est semi-simple, on a*

$$H^1_f(K, V)/H^1_e(K, V) \simeq H^1_g(K, V)/H^1_h(K, V).$$

Preuve. Pour vérifier (i), on commence par vérifier la commutativité du diagramme

$$\begin{array}{ccc} H^2_g(K, Ta) & \longrightarrow & H^2(K, \mathbb{Q}_\ell(1)) \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell & = & \mathbb{Q}_\ell, \end{array}$$

ce qui se fait en calculant explicitement l'image inverse de 1 dans $H^2_g(K, Ta)$ et dans $H^2(K, \mathbb{Q}_\ell(1))$. On termine alors comme pour la preuve de l'assertion (i) de la Proposition 1.4.7.

On montre (ii) par un calcul explicite (laissé au lecteur) de la forme bilinéaire à l'aide du complexe $C_g(K, V)$. On démontre (iii) par un argument de dimensions: on a en utilisant les suites exactes du n°3.3.3

$$\dim H^1_e(K, V) = \dim t_V - \dim H^0_g(K, D_{pst}(V)) + \dim H^0(K, V),$$

$$\dim H^1_g(K, V^*(1)) = \dim t_{V^*(1)} - \dim H^0_g(K, D_{pst}(V^*(1))) + \dim H^0(K, V^*(1)) + \dim H^1_g(K, D_{pst}(V^*(1)));$$

d'où, puisque $\dim H^0_g(K, D_{pst}(V^*(1))) = \dim H^2_g(K, D_{pst}(V))$ (on utilise ici le fait que V est potentiellement semi-stable) et en utilisant le n°3.2.4,

$$\dim H^1_e(K, V) + \dim H^1_g(K, V^*(1)) = \dim H^1(K, V).$$

On a de même pour $\nu \in \{f, h\}$

$$\begin{aligned}\dim H^1_\nu(K, V) &+ \dim H^1_\nu(K, V^*(1)) \\ &= \dim t_V + \dim t_{V^*(1)} + \dim H^0(K, V) + \dim H^0(K, V^*(1)) \\ &= \dim H^1(K, V),\end{aligned}$$

d'où iii).

On déduit aussi des suites exactes du n°3.3.3 que $H^1_f(K, V)/H^1_e(K, V) = H^1_f(K, D_{pst}(V))$ et $H^1_g(K, V)/H^1_h(K, V) = H^1_g(K, D_{pst}(V))/H^1_h(K, D_{pst}(V))$. L'assertion iv) se déduit alors de la proposition 1.4.7, iii).

3.3.10. Remarques. i) Supposons $\ell = p$ et V de de Rham. L'argument de dimension est encore valable pour $\nu \in \{f, h\}$. On en déduit que l'orthogonal de $H^1_f(K, V)$ (resp. $H^1_h(K, V)$) est encore égal à $H^1_f(K, V^*(1))$ (resp. $H^1_h(K, V^*(1))$). Pour $\nu = e$, il est montré dans **[BK90]** que l'orthogonal du noyau de l'application $H^1(K, V) \to H^1(K, B_{\mathrm{dR}} \otimes V)$ est $H^1_e(K, V^*(1))$. En particulier, on en déduit lorsque V est potentiellement semi-stable l'égalité de ce noyau avec $H^1_g(K, V)$ (voir **[H, N]**).

ii) Lorsque l'action de f_k sur $D_f(V)$ est semi-simple, on voit que $H^1_f(K, V) + H^1_h(K, V) = H^1_g(K, V)$ tandis que $H^1_f(K, V) \cap H^1_h(K, V) = H^1_e(K, V)$.

3.3.11. Il est commode de poser $H^i_{/f}(K, V) = H^i(K, V)/H^i_f(K, V)$ pour tout $i \in \mathbb{Z}$; en particulier, $H^2_{/f}(K, V) = H^2(K, V)$ et $H^i_{/f}(K, V) = 0$ si $i \neq 1, 2$.

Il résulte de la définition des H^i_f que

$$\sum (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H^i_f(K, V) = \begin{cases} 0 & \text{si } \ell \neq p, \\ -[K : \mathbb{Q}_p] \cdot \dim_K t_V & \text{si } \ell = p. \end{cases}$$

Compte-tenu des résultats rappelés au n°3.2.4, on a

$$\sum (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H^i_{/f}(K, V) = \begin{cases} 0 & \text{si } \ell \neq p, \\ -[K : \mathbb{Q}_p] \cdot (\dim_{\mathbb{Q}_p} V - \dim_K t_V) & \text{si } \ell = p. \end{cases}$$

Lorsque V est de de Rham, cette somme est aussi égale à

$$-[K : \mathbb{Q}_p] \cdot \dim_K \mathrm{Fil}^0 D_{\mathrm{dR}}(V) = -[K : \mathbb{Q}_p] \cdot \dim_K t_{V^*(1)}.$$

4. Nombres de Tamagawa locaux

Dans tout ce paragraphe, T est un $\mathbb{Z}_\ell$-module de type fini muni d'une action linéaire et continue de G_K et $V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T$.

4.1. Généralités.

4.1.1. Rappelons (n°3.3.2) que $t_V = 0$ et $D_f(V) = V^{I_K}$ si $\ell \neq p$, tandis que $t_V = ((B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes V)^{G_K}$ et $D_f(V) = D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes V)^{G_K}$ si $\ell = p$.

Considérons la $\mathbb{Q}_\ell$-droite

$$L_f(K, V) = \det_{\mathbb{Q}_\ell} H^0(K, V) \otimes (\det_{\mathbb{Q}_\ell} H_f^1(K, V))^{-1}.$$

Le complexe $C_f(K, V)$ induit une suite exacte

$$(s_{f,K}(V)) \quad 0 \to H^0(K, V) \to D_f(V) \overset{\varphi-1}{\to} D_f(V) \oplus t_V \to H_f^1(K, V) \to 0,$$

qui nous fournit un isomorphisme

$$\iota_V \colon L_f(K, V) \to (\det_{\mathbb{Q}_\ell} t_V)^{-1}$$

(si $\ell \neq p$, ι_V est donc un isomorphisme de $L_f(K, V)$ sur $\mathbb{Q}_\ell$).

4.1.2. On définit $H_f^1(K, T)$ comme l'image inverse dans $H^1(K, T)$ de $H_f^1(K, V) \subset H^1(K, V)$. Posons

$$L_f(K, T) = \det_{\mathbb{Z}_\ell} H^0(K, T) \otimes (\det_{\mathbb{Z}_\ell} H_f^1(K, T))^{-1}$$

(cf. 05). C'est un $\mathbb{Z}_\ell$-module libre de rang 1 qui s'identifie à un réseau de $L_f(K, V)$. Si ω est un élément non nul de $\det_{\mathbb{Q}_\ell} t_V$, on appelle *nombre de Tamagawa de* T relatif à ω, et on note $\mathrm{Tam}^0_{K,\omega}(T)$ l'unique puissance de ℓ telle que

$$\iota_V(L_f(K, T)) = \mathbb{Z}_\ell \cdot \mathrm{Tam}^0_{K,\omega}(T) \cdot \omega^{-1}.$$

Pour tout $\alpha \in \mathbb{Q}_\ell^*$, on a $\mathrm{Tam}^0_{K,\alpha\omega}(T) = (|\alpha|_\ell)^{-1} \mathrm{Tam}^0_{K,\omega}(T)$.

Lorsque $\ell \neq p$, ou, plus généralement lorsque $t_V = 0$, on pose aussi $\mathrm{Tam}^0_K(T) = \mathrm{Tam}^0_{K,1}(T)$. On a alors

$$\iota_V(L_f(K, T)) = \mathbb{Z}_\ell \cdot \mathrm{Tam}^0_K(T).$$

4.2. Calculs de nombres de Tamagawa.

4.2.1. PROPOSITION. *Si T est de torsion (autrement dit, si $V = 0$, auquel cas $t_V = 0$), alors*

$$\mathrm{Tam}^0_{K,1}(T) = \# H^1(K, T)/\# H^0(K, T).$$

PREUVE. C'est clair puisqu'alors $H_f^1(K, T) = H^1(K, T)$.

4.2.2. PROPOSITION. *Supposons $\ell \neq p$, et T sans torsion. Alors,*

i) *si V est non ramifiée, on a*

$$\mathrm{Tam}^0_{K,1}(T) = 1;$$

ii) *dans le cas général,*

$$\mathrm{Tam}^0_{K,1}(T) = \#(H^1(I_K, T)^{G_K})_{\mathrm{tor}}.$$

Preuve. Montrons (i): Si V est non ramifiée, $H^0(K, V) = H^0(k, V)$ tandis que $H^1_f(K, V) = H^1(k, V)$ classifie les représentations extensions de $\mathbb{Q}_\ell$ par V qui sont non ramifiées. On voit que T est aussi non ramifiée, en particulier $H^0(K, T) = H^0(k, T)$, et qu'une extension E de $\mathbb{Z}_\ell$ par T est telle que $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} E$ est encore non ramifiée si et seulement si E l'est; autrement dit, $H^1_f(K, T) = H^1(k, T)$. Mais on a un diagramme commutatif

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(k, T) & \longrightarrow & T & \xrightarrow{f_k - 1} & T & \longrightarrow & H^1(k, T) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(k, V) & \longrightarrow & V & \xrightarrow{f_k - 1} & V & \longrightarrow & H^1(k, V) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & H^0(K, V) & \longrightarrow & V^{I_K} & \xrightarrow{\varphi - 1} & V^{I_K} & \longrightarrow & H^1_f(K, V) & \longrightarrow & 0
\end{array}$$

dont les lignes sont exactes. Par conséquent, l'isomorphisme ι_V provient par extension des scalaires de l'isomorphisme de $L_f(K, T)$ sur $\mathbb{Z}_\ell$ fourni par la première ligne de ce diagramme, ce qui signifie que $\mathrm{Tam}^0_{K,1}(T) = 1$.

Montrons (ii): Comme $H^2(k, T^{I_K}) = H^2(k, V^{I_K}) = 0$, dans le diagramme commutatif

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(k, T^{I_K}) & \longrightarrow & H^1(K, T) & \longrightarrow & H^1(I_K, T)^{G_K} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(k, V^{I_K}) & \longrightarrow & H^1(K, V) & \longrightarrow & H^1(I_K, V)^{G_K} & \longrightarrow & 0,
\end{array}$$

les lignes sont exactes. Comme $H^1(I_K, V)^{G_K} = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^1(I_K, T)^{G_K}$, le noyau de $H^1(I_K, T)^{G_K} \to H^1(I_K, V)^{G_K}$ est son sous-groupe de torsion; comme $H^1_f(K, T) = H^1(k, V^{I_K}) \times_{H^1(K,V)} H^1(K, T)$, on a une suite exacte

$$0 \to H^1(k, T^{I_K}) \to H^1_f(K, T) \to ((H^1(I_K, T)^{G_K})_{\mathrm{tor}} \to 0.$$

En raisonnant comme dans (i), mais en remplaçant T par T^{I_K} et V par V^{I_K}, on voit que $\iota_V(L_f(K, T)) = \#((H^1(I_K, T)^{I_K})_{\mathrm{tor}} \cdot \mathbb{Z}_\ell$, et on a bien $\mathrm{Tam}^0_{K,1}(T) = \#((H^1(I_K, T)^{I_K})_{\mathrm{tor}}$.

4.2.3. Notons t un générateur de $\mathbb{Z}_p(1)$. Rappelons (n°2.1.3) que $\mathbb{Z}_p(1)$ se plonge dans $A_{\mathrm{cris}} \subset B_{\mathrm{cris}}$ et que t est inversible dans B_{cris}. On peut alors, pour tout $n \in \mathbb{Z}$, identifier $A_{\mathrm{cris}}(n) = A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$ au sous-A_{cris}-module libre de rang 1 de B_{cris} engendré par t^n en posant $a \otimes t^n = at^n$ (attention toutefois que, si $a \in A_{\mathrm{cris}}$, $\varphi(at^n) = p^n \varphi a \cdot t^n$).

PROPOSITION. *Supposons* $K/\mathbb{Q}_p$ *non ramifiée,* $\ell = p$, T *sans torsion et* V *cristalline (de sorte que* $D_{\mathrm{dR}}(V) = D_{\mathrm{cris}}(V) = D(V)$ *et que* t_V *est un quotient de* $D(V)$*). Supposons aussi qu'il existe un entier* r *tel que* $0 \leq r < p-1$ *tel que* $\mathrm{Fil}^{-r} D_{\mathrm{dR}}(V) = D(V)$, $\mathrm{Fil}^{-r+p} D_{\mathrm{dR}}(V) = 0$ *et que* V *n'a pas de sous-objet* V' *non trivial tel que* $\mathrm{Fil}^{-r+p-1} D_{\mathrm{dR}}(V') = D_{\mathrm{dR}}(V')$*. Alors* $M = (A_{\mathrm{cris}}(-r) \otimes_{\mathbb{Z}_p} T)^{G_K}$ *est un réseau de* $D(V)$*; si* $\overline{M}$ *désigne le réseau de* t_V *qui est l'image de* M *et si* ω *est un élément non nul de* $\det_{\mathbb{Q}_\ell} t_V$*, alors*

$$\det_{\mathbb{Z}_\ell} \overline{M} = \mathbb{Z}_\ell \cdot \mathrm{Tam}^0_{K,\omega}(T)^{-1} \cdot \omega .$$

PREUVE. Il suffit de montrer que l'on a une suite exacte

$$0 \to H^0(K, T) \to M \to M \oplus \overline{M} \to H^1_f(K, T) \to 0,$$

où la flèche $M \to M \oplus \overline{M}$ est donnée par $x \mapsto ((\varphi - 1)(x)$, projection de $x)$.

Pour cela, si a et b sont des entiers vérifiant $a < b$, notons $\mathbf{Rep}_{\mathbb{Q}_p, \mathrm{cris}, [a,b[}(G_K)$ la sous-catégorie pleine de la catégorie des représentations cristallines de G_K dont les objets sont les U tels que $\mathrm{Fil}^a D_{\mathrm{dR}}(V) = D(V)$, $\mathrm{Fil}^{b+1} D_{\mathrm{dR}}(V) = 0$ et que U n'a pas de sous-objet U' non trivial tel que $\mathrm{Fil}^b D_{\mathrm{dR}}(U') = D_{\mathrm{dR}}(U')$. Notons $\mathbf{Rep}_{\mathbb{Z}_p, \mathrm{cris}, [a,b[}(G_K)$ la sous-catégorie pleine de la catégorie des représentations $\mathbb{Z}_p$-adiques de G_K dont les objets sont les T qui sont isomorphes à un sous-quotient d'un objet de $\mathbf{Rep}_{\mathbb{Q}_p, \mathrm{cris}, [a,b[}(G_K)$.

Soit W l'anneau des entiers de $K = K_0$ (c'est aussi l'anneau des vecteurs de Witt à coefficients dans k). Appelons φ-module filtré faiblement admissible sur W la donnée d'un W-module de type fini M muni

i) d'une filtration décroissante $(\mathrm{Fil}^i M)_{i \in \mathbb{Z}}$ par des sous-W-modules facteurs directs de M, telle que $\mathrm{Fil}^i M = M$ si $i \ll 0$ et $= 0$ si $i \gg 0$,
ii) d'une famille $(\varphi^i)_{i \in \mathbb{Z}}$ d'applications σ-semi-linéaires $\varphi^i \colon \mathrm{Fil}^i M \to M$ telles que $\varphi^i(x) = p\varphi^{i+1}(x)$ pour tout $x \in \mathrm{Fil}^{i+1} M$ et que $M = \sum_{i \in \mathbb{Z}} \mathrm{Im}\, \varphi^i$.

Ces φ-modules filtrés faiblement admissibles forment de manière évidente une catégorie additive $\mathbb{Z}_p$-linéaire et on démontre **[FL82]** qu'elle est abélienne. Notons $MF_{W[a,b[}$ la sous-catégorie pleine de celle-ci formée des objets M tels que $\mathrm{Fil}^a M = M$, $\mathrm{Fil}^{b+1} M = 0$, et M n'a pas de sous-objet M' non trivial tel que $\mathrm{Fil}^b M' = M'$.

Pour $0 \leq i \leq p-1$, si $x \in \mathrm{Fil}^i A_{\mathrm{cris}} = A_{\mathrm{cris}} \cap \mathrm{Fil}^i B_{\mathrm{dR}}$, on a $\varphi(x) \in p^i A_{\mathrm{cris}}$, ce qui permet de définir $\varphi^i \colon \mathrm{Fil}^i A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ par $\varphi^i(x) = p^{-i} \varphi(x)$. Ceci permet, si Λ est un objet de $\mathbf{Rep}_{\mathbb{Z}_p, \mathrm{cris}, [0,p-1[}(G_K)$, de munir $M_{\mathrm{cris}}(\Lambda) = (A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \Lambda)^{G_K}$ d'une structure d'objet de $MF_{W,[0,p-1[}$. On démontre **[FL82, Fo83]** que le foncteur M_{cris} induit une équivalence entre ces deux catégories.

En tordant par $\mathbb{Z}_p(-r)$, on en déduit un foncteur $M_{\mathrm{cris},r}$ qui est une équivalence entre $\mathbf{Rep}_{\mathbb{Z}_p,\mathrm{cris},[-r,-r+p-1[}(G_K)$ et $MF_{W,[-r,-r+p-1[}$.

Si maintenant T est comme dans la proposition, T et $\mathbb{Z}_p$ sont tous deux des objets de $\mathbf{Rep}_{\mathbb{Z}_p,\mathrm{cris},[-r,-r+p-1[}(G_K)$ et on voit que $H^1_f(K, T)$ classifie les extensions de $\mathbb{Z}_p$ par T qui sont encore dans $\mathbf{Rep}_{\mathbb{Z}_p,\mathrm{cris},[-r,-r+p-1[}(G_K)$. Alors le W-module sous-jacent à $M_{\mathrm{cris},r}(\mathbb{Z}_p)$ (resp. $M_{\mathrm{cris},r}(T)$) s'identifie à W (resp. M). Avec des notations évidentes, la proposition résulte alors facilement du calcul explicite de $\mathrm{Ext}^1_{MF_{W,[-r,-r+p-1[}}(W, M)$.

4.3. Comportement par induction. Dans ce numéro, on se donne une extension finie K' de K contenue dans $\overline{K}$ et un $\mathbb{Z}_\ell$-module de type fini T', muni d'une action linéaire et continue de G_K. On suppose que $T = \mathbb{Z}_\ell[G_K]\otimes_{\mathbb{Z}_\ell[G_{K'}]} T'$ est la représentation de G_K induite par T'. Si $V' = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T'$, il est immédiat que t_V s'identifie au K-espace vectoriel sous-jacent au K'-espace vectoriel $t_{V'}$. En particulier $\det_{\mathbb{Q}_\ell} t_V = \det_{\mathbb{Q}_\ell} t_{V'}$.

Proposition. *Avec les hypothèses et les notations qui précèdent, si ω est une base de* $\det_{\mathbb{Q}_\ell} t_{V'}$, *on a*

$$\mathrm{Tam}^0_{K',\omega}(T') = \mathrm{Tam}^0_{K,\omega}(T).$$

Preuve. On dispose d'isomorphismes canoniques

$$H^0(K', V') \simeq H^0(K, V), \qquad H^1(K', V') \simeq H^1(K, V), \\ H^1_f(K', V') \simeq H^1_f(K, V):$$

lorsque $\ell \neq p$, le troisième isomorphisme se déduit des égalités $H^1_f(K, V) = H^1(k, V^{I_K})$, $H^1_f(K', V') = H^1(k', V'^{I_{K'}})$ et de l'isomorphisme naturel $V^{I_K} = \mathbb{Z}_\ell[G_k]\otimes_{\mathbb{Z}_\ell[G_{k'}]}(V'^{I_{K'}})$; lorsque $\ell = p$, on remarque que le complexe $C_f(K, V)$ s'identifie au complexe $C_f(K', V')$.

Comme d'autre part les deux premiers isomorphismes proviennent par extension des scalaires de $\mathbb{Z}_\ell$ à $\mathbb{Q}_\ell$ d'isomorphismes $H^0(K', T') \simeq H^0(K, T)$ et $H^1(K', T') \simeq H^1(K, T)$, il est clair que $H^1_f(K', V') \simeq H^1_f(K, V)$ provient par extension des scalaires d'un isomorphisme $H^1_f(K', T') \simeq H^1_f(K, T)$. D'où la proposition.

CHAPITRE II

REPRÉSENTATIONS ℓ-ADIQUES DES CORPS DE NOMBRES

1. Cohomologie galoisienne

1.1. Généralités.

1.1.1. Dans tout ce chapitre, F est une extension finie de $\mathbb{Q}$. Pour fixer les idées (plus exactement pour pouvoir parler en termes de cohomologie galoisienne plutôt qu'en termes de cohomologie étale), on choisit une clôture algébrique $\overline{F}$ de F et, pour chaque place $\mathfrak{p}$ de F, une clôture algébrique $\overline{F}_{\mathfrak{p}}$ de $F_{\mathfrak{p}}$ et un plongement de $\overline{F}$ dans $\overline{F}_{\mathfrak{p}}$; on pose $G_F = \mathrm{Gal}(\overline{F}/F)$ et $G_{\mathfrak{p}} = \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \subset G_F$.

On note $S(F)$ (resp. $S_f(F)$) l'ensemble des places (resp. places finies) de F. Si P est l'ensemble des nombres premiers, on identifie $S(\mathbb{Q})$ à $P \cup \{\infty\}$. Pour tout $p \in S(\mathbb{Q})$, on note $S_p(F)$ l'ensembles des places de F au-dessus de p. Si Σ est un ensemble de places de F, on pose $\Sigma_f = \Sigma \cap S_f(F)$ et, pour tout $p \in S(\mathbb{Q})$, $\Sigma_p = \Sigma \cap S_p(F)$.

1.1.2. Dans toute la suite, on fixe un nombre premier ℓ. On fixe aussi un ensemble fini S de places de F tel que $S_\infty = S_\infty(F)$ (on sera parfois amené à regarder ce qui se passe lorsque l'on remplace S par un ensemble fini plus grand S'). Dans les paragraphes 1, 2, et 4, on suppose que S contient $S_\ell(F)$.

On note F_S la plus grande extension galoisienne de F, contenue dans $\overline{F}$, non ramifiée en dehors de S et on pose $G_{F,S} = \mathrm{Gal}(F_S/F)$. Lorsque S contient $S_\ell(F)$, on note $\mathbf{Rep}_{\mathbb{Q}_\ell,S}(G_F)$ la catégorie des représentations ℓ-adiques de $G_{F,S}$. C'est une sous-catégorie tannakienne de la catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_F)$ des représentations ℓ-adiques de G_F qui contient $\mathbb{Q}_\ell(1)$.

1.1.3. Pour tout groupe abélien fini M muni d'une action linéaire continue de $G_{F,S}$ et pour tout $i \in \mathbb{Z}$, on pose $H^i(U_S, M) = H^i(G_{F,S}, M)$ (cette notation est suggérée par le fait que, si $\mathcal{O}_F$ désigne l'anneau des entiers de F et U_S l'ouvert de $\mathrm{Spec}\,\mathcal{O}_F$ compémentaire de S_f, et que si l'on note encore M l'unique faisceau localement constant sur $(U_S)_{\text{ét}}$ qui prolonge M, on a $H^i(U_S, M) = H^i((U_S)_{\text{ét}}, M)$).

De même, si T est un $\mathbb{Z}_\ell$-module de type fini muni d'une action linéaire continue de $G_{F,S}$, on pose $H^i(U_S, T) = \text{lim.proj.} H^i(U_S, T/\ell^n T)$; si V est une représentation ℓ-adique de $G_{F,S}$, et si l'on choisit T tel que $V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T$, $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^i(U_S, T)$ ne dépend pas du choix de T et on le note $H^i(U_S, V)$. Rappelons (cf. par exemple **[Mi86]** par passage à la limite de 1.4.20) que les $H^i(U_S, V)$ sont des $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie, nuls si $i \neq 0, 1, 2$.

1.1.4. Rappelons (cf. **[Ta76]**) que le complexe des cochaînes continues calcule les $H^i(U_S, V)$. Autrement dit, si pour tout $i \in \mathbb{Z}$, on note $C^i(U_S, V)$ (resp. $Z^i(U_S, V)$, $B^i(U_S, V)$) le groupe des i-cochaînes (resp. cocycles, cobords) continu(e)s de $G_{F,S}$ à valeurs dans V (on convient que ces groupes sont nuls pour $i < 0$), $H^i(U_S, V)$ s'identifie à $Z^i(U_S, V)/B^i(U_S, V)$. On vérifie facilement en outre que, pour tout i et tout V, la flèche naturelle de $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, S}(G_F), V)$ dans $H^i(U_S, V)$ est un isomorphisme.

Pour toute place $\mathfrak{p}$ de F, il en est de même de la cohomologie de V vue comme représentation de $G_\mathfrak{p}$. Autrement dit, si l'on note $C^i(F_\mathfrak{p}, V)$ (resp. $Z^i(F_\mathfrak{p}, V)$, $B^i(F_\mathfrak{p}, V)$) le groupe des i-cochaînes (resp. cocycles, cobords) continu(e)s de $G_\mathfrak{p}$ à valeurs dans V, $H^i(F_\mathfrak{p}, V)$ s'identifie à $Z^i(F_\mathfrak{p}, V)/B^i(F_\mathfrak{p}, V)$ et aussi à $H^i(\mathbf{Rep}_{\mathbb{Q}_\ell}(G_\mathfrak{p}), V)$.

1.1.5. Nous faisons la convention suivante qui est abusive mais commode: une *représentation ℓ-adiques de* G_F est une représentation ℓ-adique V non ramifiée en dehors d'un nombre fini de places (autrement dit telle que l'on peut trouver un S fini tel que V est un objet de $\mathbf{Rep}_{\mathbb{Q}_\ell, S}(G_F)$. Pour une telle représentation V, et pour $i \in \mathbb{Z}$, on note $H^i(F, V)$ la limite inductive des $H^i(U_S, V)$ pour S parcourant tous les ensembles finis de places de F contenant $S_\infty(F)$, $S_\ell(F)$ et toutes les places où V est ramifiée.

1.2. Théorèmes de dualité globale. Commençons par quelques rappels et quelques conséquences faciles des théorèmes de dualité globale de Poitou–Tate **[Ta62, Po67, Mi86]**.

1.2.1. Pour tout objet V de $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_{F,S})$, et pour $i \in \mathbb{Z}$, notons $H^i_0(U_S, V)$ le noyau de l'homomorphisme de restriction

$$H^i(U_S, V) \to \bigoplus_{\mathfrak{p}\in S} H^i(F_\mathfrak{p}, V),$$

qui est donc nul si $i \neq 1, 2$. Rappelons **[Mi86,** 1.4] que, pour $i \in \{0, 1, 2\}$, dans la dualité

$$\bigoplus_{\mathfrak{p}\in S_f} H^i(F_\mathfrak{p}, V) \times \bigoplus_{\mathfrak{p}\in S_f} H^{2-i}(F_\mathfrak{p}, V^*(1)) \to \mathbb{Q}_\ell,$$

l'orthogonal de l'image de $H^i(U_S, V)$ est l'image de $H^{2-i}(U_S, V^*(1))$. En particulier, on a une suite exacte

(1.1)
$$0 \to H^i_0(U_S, V) \to H^i(U_S, V) \to \bigoplus_{\mathfrak{p}\in S_f} H^i(F_\mathfrak{p}, V) \to (H^{2-i}(U_S, V^*(1)))^*$$
$$\to (H^{2-i}_0(U_S, V^*(1)))^* \to 0.$$

1.2.2. Si V et W sont deux objets de $\mathbf{Rep}_{\mathbb{Q}_\ell,S}(G_F)$, on dispose d'un homomorphisme canonique et fonctoriel

$$H_0^1(U_S, V) \times H_0^2(U_S, W) \to (((V \otimes W)^*(1))^{G_F})^*,$$

que nous notons $(a, b) \mapsto [a, b]$. Il s'obtient ainsi (pour tout $\rho \in C^i(U_S, -)$, on note $\rho_\mathfrak{p}$ sa restriction à $C^i(F_\mathfrak{p}, -)$) : On choisit $\alpha \in Z^1(U_S, V)$ représentant $a \in H_0^1(U_S, V)$ et $\beta \in Z^2(U_S, W)$ représentant $b \in H_0^2(U_S, W)$; le cup-produit $\alpha \cup \beta \in Z^3(U_S, V) = B^3(U_S, V)$ et on choisit $\varepsilon \in C^2(U_S, V \otimes W)$ tel que $d\varepsilon = \alpha \cup \beta$. Pour tout $\mathfrak{p} \in S_f$, on a $\beta_\mathfrak{p} \in B^2(F_\mathfrak{p}, W)$ et on choisit $\gamma_\mathfrak{p} \in C^1(F_\mathfrak{p}, W)$ tel que $d\gamma_\mathfrak{p} = \beta_\mathfrak{p}$; alors $\varepsilon_\mathfrak{p} + \alpha_\mathfrak{p} \cup \gamma_\mathfrak{p} \in Z^2(F_\mathfrak{p}, V \otimes W)$ et définit un élément $c_\mathfrak{p} \in H^2(F_\mathfrak{p}, V \otimes W)$; on vérifie que, si l'on change les choix faits, $(c_\mathfrak{p})_{\mathfrak{p} \in S_f} \in \prod_{\mathfrak{p} \in S_f} H^2(F_\mathfrak{p}, V \otimes W)$ ne change que par un élément qui est dans l'image de $H^2(U_S, V \otimes W)$; en appliquant (1.1) à $V \otimes W$, on voit alors que l'image $[a, b]$ de $(c_\mathfrak{p})_{\mathfrak{p} \in S_f}$ dans $H^0(U_S, (V \otimes W)^*(1))^* = (((V \otimes W)^*(1))^{G_F})^*$ ne dépend que de a et b.

Dans le cas particulier où $W = V^*(1)$, on a $((V \otimes W)^*(1))^{G_F} = \mathrm{End}_{G_F}(V)$; si l'on compose alors $[\cdot, \cdot]$ avec la transposée de l'injection canonique de $\mathbb{Q}_\ell$ dans $\mathrm{End}_{G_F}(V)$, on obtient un accouplement

$$H_0^1(U_S, V) \times H_0^2(U_S, W) \to \mathbb{Q}_\ell.$$

qui est une *dualité parfaite* [**Mi86,** 1.4, Theorem 4.10].

Remarque. Dans la construction qui précède, on peut échanger les rôles de α et β : si l'on choisit $\delta_\mathfrak{p} \in C^0(F_\mathfrak{p}, W)$ tel que $d\delta_\mathfrak{p} = \alpha_\mathfrak{p}$, on a $d(\delta_\mathfrak{p} \cup \gamma_\mathfrak{p}) = \alpha_\mathfrak{p} \cup \gamma_\mathfrak{p} + \delta_\mathfrak{p} \cup \beta_\mathfrak{p}$; alors $\varepsilon_\mathfrak{p} + \alpha_\mathfrak{p} \cup \gamma_\mathfrak{p}$ et $\varepsilon_\mathfrak{p} - \delta_\mathfrak{p} \cup \beta_\mathfrak{p}$ définissent le même élément $c_\mathfrak{p}$ dans $H^2(F_\mathfrak{p}, V \otimes W)$.

1.2.3. L'énoncé suivant est alors évident:

Théorème. *Soit V une représentation ℓ-adique de $G_{F,S}$. La suite*

$$(1.2) \qquad \cdots \to H^i(U_S, V) \to \bigoplus_{\mathfrak{p} \in S_f} H^i(F_\mathfrak{p}, V) \to (H^{2-i}(U_S, V^*(1)))^* \to \cdots$$

(où les flèches sont les flèches évidentes, compte-tenu des dualités que l'on vient de rappeler) est exacte et fonctorielle en V. La suite exacte transposée s'identifie à la suite exacte associée à $V^*(1)$.

1.2.4. Remarque. Soit V une représentation ℓ-adique de $G_{F,S}$. On a alors

$$\begin{aligned}\sum(-1)^i \dim_{\mathbb{Q}_\ell}(H^i(U_S, V)) &= \sum_{\mathfrak{p} \in S_\infty} \dim_{\mathbb{Q}_\ell} V^{G_{F_\mathfrak{p}}} - [F : \mathbb{Q}] \dim_{\mathbb{Q}_\ell} V \\ &= -\sum_{\mathfrak{p} \in S_\infty} \dim_{\mathbb{Q}_\ell} V^*(1)^{G_{F_\mathfrak{p}}}\end{aligned}$$

(cela peut se déduire par passage à la limite des formules de caractéristique d'Euler–Poincaré pour les modules finis, ([**Mi86,** 1.5, Theorem 5.1] par exemple)).

1.3. Les groupes $H^1_{f,\Sigma}(F, V)$ et $H^1_f(F, V)$.

1.3.1. Soit V une représentation ℓ-adique de G_F. Pour tout $i \in \mathbb{N}$, toute place $\mathfrak{p}$ de F et tout $x \in H^i(F, V)$, notons $x_\mathfrak{p}$ l'image de x dans $H^i(F_\mathfrak{p}, V)$. On pose

$$H^1_g(F, V) = \{x \in H^1(F, V) \mid x_\mathfrak{p} \in H^1_g(F_\mathfrak{p}, V) \text{ si } \mathfrak{p} \in S_\ell(F)\}.$$

Remarquons que le théorème de monodromie ℓ-adique implique que, si p est un nombre premier $\neq \ell$ et si $\mathfrak{p} \in S_p(F)$, alors pour tout $x \in H^1(F, V)$, $x_\mathfrak{p} \in H^1_g(F_\mathfrak{p}, V)$, ce qui fait que $H^1_g(F, V)$ est en fait formé des $x \in H^1(F, V)$ tels que $x_\mathfrak{p} \in H^1_g(F_\mathfrak{p}, V)$ pour tout $\mathfrak{p} \in S_f(F)$.

Si Σ est un ensemble fini de places de F contenant les places à l'infini, on pose

$$H^1_{f,\Sigma}(F, V) = \{x \in H^1_g(F, V) \mid x_\mathfrak{p} \in H^1_f(F_\mathfrak{p}, V) \text{ si } \mathfrak{p} \notin \Sigma\}.$$

On pose aussi $H^1_f(F,V) = H^1_{f,S_\infty(F)}(F,V)$. Pour $i \in \mathbb{Z}$, on définit $\widetilde{H}^i_f(F,V)$ par

$$\widetilde{H}^0_f(F, V) = H^0(F, V), \qquad \widetilde{H}^1_f(F, V) = H^1_f(F, V),$$
$$\widetilde{H}^2_f(F, V) = (\widetilde{H}^1_f(F, V^*(1)))^*,$$
$$\widetilde{H}^3_f(F, V) = (H^0(F, V^*(1)))^* \quad \text{et} \quad \widetilde{H}^i_f(F, V) = 0 \text{ si } i \notin \{0, 1, 2, 3\}.$$

Remarque. C'est sans doute Jannsen [**Ja89**] qui le premier a compris l'intérêt des groupes de ce genre. Le groupe $H^1_{f,\Sigma}(F, V)$ a été introduit, pour les représentations que nous appelons pseudo-géométriques (cf. infra 2.1), par Bloch et Kato [**BK90,** def. 5.1] qui le notent $H^1_{f,U}(F, V)$, où $U = U_\Sigma$.

1.3.2. Si V est non ramifiée en dehors de S et si $\Sigma \subset S$, $H^1_{f,\Sigma}(F, V)$ est le noyau de

$$H^1(U_S, V) \to \left(\bigoplus_{\mathfrak{p}\in S-\Sigma} H^1_{/f}(F_\mathfrak{p}, V)\right) \oplus \left(\bigoplus_{\mathfrak{p}\in\Sigma_\ell} H^1_{/g}(F_\mathfrak{p}, V)\right)$$

(où $H^1_{/f}(F_\mathfrak{p}, V) = H^1(F_\mathfrak{p}, V)/H^1_f(F_\mathfrak{p}, V)$ et $H^1_{/g}(F_\mathfrak{p}, V) = H^1(F_\mathfrak{p}, V)/H^1_g(F_\mathfrak{p}, V)$).

En particulier, $H^1_f(F, V)$ est le noyau de

$$H^1(U_S, V) \to \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, V).$$

1.3.3. Soient

$$(\beta) \qquad 0 \to V' \to V \to V'' \to 0$$

une suite exacte courte de représentations ℓ-adiques de $G_{F,S}$ et $u_V \in \mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{Q}_\ell,S}(G_F)}(V'', V') = H^1(U_S, V''^* \otimes V')$ la classe de cette extension. Si $\mathfrak{p}$ est une place finie de F, on dit que *la suite exacte* (β) *a bonne réduction en* $\mathfrak{p}$ si l'image de u_V dans $H^1(F_\mathfrak{p}, V''^* \otimes V')$ appartient à $H^1_f(F_\mathfrak{p}, V''^* \otimes V')$. C'est bien sûr le cas si $\mathfrak{p} \notin S$. Remarquons que la représentation V elle-même a bonne réduction en $\mathfrak{p}$ si et seulement si V', V'' et la suite exacte (β) ont toutes les trois bonne réduction en $\mathfrak{p}$.

On appelle *f-suite exacte courte de représentations ℓ-adiques de $G_{F,S}$* toute suite exacte courte (β) qui a bonne réduction en toutes les places finies de F ; cela revient donc à dire que $u_V \in H^1_f(F, V''^* \otimes V')$.

1.4. L'application de connexion.

1.4.1. Soit

$$(\beta) \qquad 0 \to V' \to V \to V'' \to 0$$

une f-suite exacte de représentations ℓ-adiques de $G_{F,S}$. Le but de ce paragraphe est de construire un homomorphisme de connexion

$$\delta = \delta_{u_V} : H^1_f(F, V'') \to H^1_f(V'^*(1))^* .$$

1.4.2. Pour toute place finie $\mathfrak{p}$ de F et pour toute représentation ℓ-adique V de $G_\mathfrak{p}$, notons $Z^1_f(F_\mathfrak{p}, V)$ l'image inverse de $H^1_f(F_\mathfrak{p}, V)$ dans $Z^1(F_\mathfrak{p}, V)$. Dans le diagramme commutatif rectangulaire tordu

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(F_\mathfrak{p}, V') & \to & H^0(F_\mathfrak{p}, V) & \to & H^0(F_\mathfrak{p}, V'') & \to & \cdots (1) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V' & \to & V & \to & V'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z^1_f(F_\mathfrak{p}, V') & \to & Z^1_f(F_\mathfrak{p}, V) & \to & Z^1_f(F_\mathfrak{p}, V'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(1)\cdots & \to & H^1_f(F_\mathfrak{p}, V') & \to & H^1_f(F_\mathfrak{p}, V) & \to & H^1_f(F_\mathfrak{p}, V'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

toutes les colonnes et toutes les lignes, sauf peut-être la troisième, sont exactes. Comme cette dernière est un complexe, elle est encore exacte (n°06).

1.4.3. Si H est un sous-groupe fermé de $G_{F,S}$, si $\lambda \in Z^i(H, V')$ et $\mu \in Z^j(H, V'^*(1))$, on note $\lambda \cup \mu \in Z^{i+j}(H, \mathbb{Q}_\ell(1))$ l'image du cup-produit. Soient $x \in H^1_f(F, V'')$ et $y \in H^1_f(F, V'^*(1))$. Choisissons $\alpha \in Z^1(U_S, V'')$

représentant x et $\beta \in Z^1(U_S, V'^*(1))$ représentant y. Soit $\alpha_1 \in C^1(U_S, V)$ relevant α. Alors $d\alpha_1 \in Z^2(U_S, V')$ (sa classe dans $H^2(U_S, V')$ est l'image de x par l'homomorphisme de connexion). Comme $H^3(U_S, \mathbb{Q}_\ell(1)) = 0$, il existe $\varepsilon \in C^2(U_S, \mathbb{Q}_\ell(1))$ tel que $d\alpha_1 \cup \beta = d\varepsilon$. Pour tout $\mathfrak{p} \in S_f$, l'image $\alpha_\mathfrak{p}$ de α dans $Z^1(F_\mathfrak{p}, V'')$ appartient à $Z^1_f(F_\mathfrak{p}, V'')$. D'après 1.4.2, il existe donc $\alpha_{\mathfrak{p},1} \in Z^1_f(F_\mathfrak{p}, V)$ dont l'image dans V'' est $\alpha_\mathfrak{p}$. Donc, $\alpha_{\mathfrak{p},1} - \alpha_{1,\mathfrak{p}} \in C^1(F_\mathfrak{p}, V')$. On vérifie facilement que

$$(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_\mathfrak{p} - \varepsilon_\mathfrak{p} \in Z^1(F_\mathfrak{p}, \mathbb{Q}_\ell(1)).$$

Si $\mathrm{inv}_\mathfrak{p} \colon H^2(F_\mathfrak{p}, \mathbb{Q}_\ell(1)) \to \mathbb{Q}_\ell$ désigne l'isomorphisme canonique et si l'on note $[\gamma]$ l'image dans $H^1(F_\mathfrak{p}, \mathbb{Q}_\ell(1))$ de $\gamma \in Z^1(F_\mathfrak{p}, \mathbb{Q}_\ell(1))$, on définit alors

$$[x, y]_{u_V} = \sum_{\mathfrak{p} \in S_f} \mathrm{inv}_\mathfrak{p}[(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_\mathfrak{p} - \varepsilon_\mathfrak{p}].$$

C'est un élément de $\mathbb{Q}_\ell$ indépendant des choix faits. On a ainsi obtenu une forme bilinéaire, d'où une application linéaire

$$H^1_f(F, V'') \to (H^1_f(F, V'^*(1)))^*,$$

qui, avec les notations du n°1.3.1, peut se voir comme une application

$$\widetilde{H}^1_f(F, V'') \to \widetilde{H}^2_f(F, V').$$

1.4.4. Remarque. Supposons que $x \in H^1_f(F, V'')$ soit l'image d'un élément $\hat{x} \in H^1(U_S, V)$. Le calcul de $[x, y]_{u_V}$ est alors plus facile: remarquons que, pour toute place finie $\mathfrak{p}$ de F, on dispose d'une suite exacte courte

$$0 \to H^1_{/f}(F_\mathfrak{p}, V') \to H^1_{/f}(F_\mathfrak{p}, V) \to H^1_{/f}(F_\mathfrak{p}, V'') \to 0.$$

Si l'on désigne par $\hat{x}_\mathfrak{p}$ l'image de $\hat{x}$ dans $H^1_{/f}(F_\mathfrak{p}, V)$, $\hat{x}_\mathfrak{p}$ s'identifie à un élément de $H^1_{/f}(F_\mathfrak{p}, V')$, espace vectoriel qui est en dualité avec $H^1_f(F_\mathfrak{p}, V'^*(1))$. Si $y_\mathfrak{p}$ est l'image de y dans $H^1_f(F_\mathfrak{p}, V'^*(1))$, on vérifie que

$$[x, y]_{u_V} = \sum_{\mathfrak{p} \in S_f} \langle \hat{x}_\mathfrak{p}, y_\mathfrak{p} \rangle.$$

1.4.5. Proposition. *Soient* $x \in H^1_f(F, V'')$ *et* $y \in H^1_f(F, V'^*(1))$. *Alors*

$$[x, y]_{u_V} = [y, x]_{u_{V^*(1)}}.$$

Preuve. Avec des notations évidentes, les classes dans $H^2(F_\mathfrak{p}, \mathbb{Q}_\ell(1))$ de

$$(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta \quad \text{et} \quad (\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_{\mathfrak{p},1}$$

sont égales. Comme $\alpha_{\mathfrak{p},1}$ (resp. $\beta_{\mathfrak{p},1}$) est un élément de $Z_f^1(F_\mathfrak{p}, V)$ (resp. $Z_f^1(F_\mathfrak{p}, V^*(1))$, le cup-produit de $\alpha_{\mathfrak{p},1}$ et de $\beta_{\mathfrak{p},1}$ est un cobord. On peut donc écrire

$$(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_{\mathfrak{p},1} - \varepsilon_\mathfrak{p} \equiv \alpha_{1,\mathfrak{p}} \cup \beta_{\mathfrak{p},1} - \varepsilon_\mathfrak{p} \quad \text{dans } H^2(F_\mathfrak{p}, \mathbb{Q}_\ell(1)).$$

On a d'autre part

$$d(\alpha_1 \cup \beta_1) = d\alpha_1 \cup \beta_1 - \alpha_1 \cup d\beta_1 = d\alpha_1 \cup \beta_1 - d\beta_1 \cup \alpha_1 .$$

On a donc $d\beta_1 \cup \alpha = d\beta_1 \cup \alpha_1 = d(-\alpha_1 \cup \beta_1 + \varepsilon) = d\varepsilon'$ avec $\varepsilon' = -\alpha_1 \cup \beta_1 + \varepsilon$. Donc,

$$\begin{aligned}\alpha_{1,\mathfrak{p}} \cup \beta_{\mathfrak{p},1} - \varepsilon_\mathfrak{p} &= \alpha_{1,\mathfrak{p}} \cup \beta_{\mathfrak{p},1} - \varepsilon'_\mathfrak{p} - \alpha_{1,\mathfrak{p}} \cup \beta_{1,\mathfrak{p}} \\ &= (\beta_{1,\mathfrak{p}} - \beta_{\mathfrak{p},1}) \cup \alpha_{1,\mathfrak{p}} - \varepsilon'_\mathfrak{p} .\end{aligned}$$

Enfin, comme $\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}$ appartient à $C^1(F_\mathfrak{p}, V')$ et que $\beta_{1,\mathfrak{p}} - \beta_{\mathfrak{p},1}$ appartient à $C^1(F_\mathfrak{p}, V''^*(1))$, la classe de $(\beta_{1,\mathfrak{p}} - \beta_{\mathfrak{p},1}) \cup \alpha_{1,\mathfrak{p}}$ dans $H^2(F_\mathfrak{p}, \mathbb{Q}_\ell(1))$ est égale à celle de $(\beta_{1,\mathfrak{p}} - \beta_{\mathfrak{p},1}) \cup \alpha_{\mathfrak{p},1}$. On en déduit la proposition.

1.4.6. Remarque. On peut se demander si la suite

$$\cdots \to \widetilde{H}_f^i(F, V') \to \widetilde{H}_f^i(F, V) \to \widetilde{H}_f^i(F, V'') \to \cdots$$

est exacte. Nous verrons au paragraphe suivant que c'est le cas lorsque la représentation V est "pseudo-géométrique". Sans hypothèse, on vérifie que:

i) *la suite ci-dessus est exacte, sauf peut-être en* $\widetilde{H}_f^1(F,V'')$ *et* $\widetilde{H}_f^2(F,V')$;

ii) *le composé* $\widetilde{H}_f^1(F, V) \to \widetilde{H}_f^1(F, V'') \to \widetilde{H}_f^2(F, V')$ *est nul*;

iii) *le composé* $\widetilde{H}_f^1(F, V'') \to \widetilde{H}_f^2(F, V') \to \widetilde{H}_f^2(F, V)$ *est nul.*

En effet, (i) est facile. Il suffit ensuite de démontrer (ii) car (iii) est l'énoncé analogue pour la suite exacte "duale tordue". Soient $x \in H_f^1(F, V)$ et y son image dans $H_f^1(F, V'')$. Dans la définition de l'image de y dans $\widetilde{H}_f^2(F, V') = H_f^1(F, V'^*(1))^*$, l'élément ε de $Z^2(U_S, \mathbb{Q}_\ell(1))$ peut être pris nul et $\alpha'_\mathfrak{p}$ peut être pris égal à $x_\mathfrak{p}$. On a alors pour $z \in H_f^1(F, V'^*(1))$

$$[y, z]_{u_V} = \pm \sum_{\mathfrak{p} \in S_f} \mathrm{inv}_\mathfrak{p}(x_\mathfrak{p} \cup z'_\mathfrak{p})$$

où z' est un relèvement de z de $H^1(U_S, V^*(1))$, ce qui est nul comme somme d'invariants d'un élément global.

2. Représentations pseudo-géométriques

2.1. Généralités.

2.1.1. Nous disons qu'une représentation ℓ-adique V de G_F est *pseudo-géométrique* si elle est non ramifiée en dehors d'un nombre fini de places et de de Rham en toutes les places divisant ℓ. Ces représentations forment une sous-catégorie tannakienne, contenant $\mathbb{Q}_\ell(1)$, de la catégorie de toutes les représentations ℓ-adiques de G_F.

Remarquons que, si V est pseudo-géométrique et si Σ un ensemble fini de places de F contenant $S_\infty(F)$, le groupe $H^1_{f,\Sigma}(F, V)$ classifie les extensions U de $\mathbb{Q}_\ell$ par V qui sont pseudo-géométriques et telles que la suite exacte

$$0 \to V \to U \to \mathbb{Q}_\ell \to 0$$

a bonne réduction en dehors de Σ.

2.1.2. Si V est une représentation ℓ-adique pseudo-géométrique, pour toute place $\mathfrak{p}$ divisant ℓ, on note $t_{V,\mathfrak{p}}$ (resp. $t'_{V,\mathfrak{p}}$) l'espace tangent de la restriction de V (resp. $V^*(1)$) à $G_\mathfrak{p}$ (cf. I, 2.2.1). Avec des notations évidentes, le $F_\mathfrak{p}$-espace vectoriel $D_{\mathrm{dR},\mathfrak{p}}(V^*(1))$ s'identifie au dual de $D_{\mathrm{dR},\mathfrak{p}}(V)$, $\mathrm{Fil}^0 D_{\mathrm{dR},\mathfrak{p}}(V^*(1))$ étant l'orthogonal de $\mathrm{Fil}^0 D_{\mathrm{dR},\mathfrak{p}}(V)$. Donc $t'_{V,\mathfrak{p}}$ s'identifie au dual de $\mathrm{Fil}^0 D_{\mathrm{dR},\mathfrak{p}}(V)$ et $\dim_{F_\mathfrak{p}} t_{V,\mathfrak{p}} + \dim_{F_\mathfrak{p}} t'_{V,\mathfrak{p}} = \dim_{\mathbb{Q}_\ell} V$.

On pose $t_V = \bigoplus_{\mathfrak{p}|\ell} t_{V,\mathfrak{p}}$ et $t'_V = \bigoplus_{\mathfrak{p}|\ell} t'_{V,\mathfrak{p}}$. On voit que

$$\dim_{\mathbb{Q}_\ell} t_V + \dim_{\mathbb{Q}_\ell} t'_V = [F : \mathbb{Q}] \cdot \dim_{\mathbb{Q}_\ell} V.$$

2.2. Deux suites exactes.

2.2.1. Soit V une représentation non ramifiée en dehors de S. Pour $i \in \{2, 3\}$, $\widetilde{H}^i_f(F, V)$ a été défini comme le dual de $H^{3-i}_f(F, V^*(1))$; la dualité locale entre $H^{3-i}(F_\mathfrak{p}, V)$ et $H^{i-1}(F_\mathfrak{p}, V^*(1))$ induit une dualité entre $H^{3-i}_f(F_\mathfrak{p}, V)$ et $H^{i-1}_{/f}(F_\mathfrak{p}, V^*(1))$ (cf. I, n°3.3.9 et n°3.3.10) et la transposée de l'application naturelle de $H^{3-i}_f(F, V^*(1))$ dans

$$\bigoplus_{\mathfrak{p}\in S_f} H^{3-i}_f(F_\mathfrak{p}, V^*(1))$$

peut être considérée comme une application de $\bigoplus_{\mathfrak{p}\in S_f} H^{i-1}_{/f}(F_\mathfrak{p}, V^*(1))$ dans $\widetilde{H}^i_f(F, V)$.

Par ailleurs, la dualité de Poitou–Tate permet d'identifier $H^1_0(U_S, V^*(1))^*$ à $H^2_0(U_S, V)$ (cf. n°1.2.2) et le composé de la projection de $H^1_f(F, V^*(1))^*$ sur $H^1_0(U_S, V^*(1))^*$ avec l'inclusion de $H^2_0(U_S, V)$ dans $H^2(U_S, V)$ fournit une application de $H^1_f(F, V^*(1))^*$ dans $H^2(U_S, V)$.

Proposition. *Soit V une représentation ℓ-adique pseudo-géométrique non ramifiée en dehors de S. La suite*

$$(s_{f,S}(V)) \qquad \cdots \to \widetilde{H}^i_f(F, V) \to H^i(U_S, V) \to \bigoplus_{\mathfrak{p}\in S_f} H^i_{/f}(F_\mathfrak{p}, V) \to \cdots$$

(où toutes les flèches, autre que celles que l'on vient de décrire, sont évidentes) est exacte.

PREUVE. Il s'agit de vérifier l'exactitude de la suite

$$0 \to H^1_f(F, V) \to H^1(U_S, V) \to \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, V) \to H^1_f(F, V^*(1))^* \\ \to H^2(U_S, V) \to \bigoplus_{\mathfrak{p}\in S_f} H^2(F_\mathfrak{p}, V) \to (V^*(1))^{G_F})^* \to 0.$$

L'exactitude en $H^1_f(F, V)$, $H^1(U_S, V)$, $H^2(U_S, V)$ et $(V^*(1)^{G_F})^*$ est immédiate. L'exactitude en $\bigoplus_{\mathfrak{p}\in S_f} H^2(F_\mathfrak{p}, V)$ résulte du n°1.2.1. L'exactitude en $\bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, V)$ et $H^1_f(F, V^*(1))^*$ enfin résulte de ce que, dans le diagramme commutatif

$$\begin{array}{ccccccc}
 & & \bigoplus_{\mathfrak{p}\in S_f} H^1_f(F_\mathfrak{p}, V) & \cong & \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, V^*(1))^* & & \\
 & & \downarrow & & \downarrow & & \\
H^1(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^1(F_\mathfrak{p}, V) & \to & H^1(U_S, V^*(1))^* & \to & H^2(U_S, V) \\
\| & & \downarrow & & \downarrow & & \| \\
H^1(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, V) & \to & H^1_f(F, V^*(1))^* & \to & H^2(U_S, V) \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

la deuxième ligne (cf. 1.2.3) et les colonnes sont exactes.

2.2.2. REMARQUE. On déduit de cette proposition, de la remarque 1.2.4 et de I, n°3.3.11 que

$$\sum_i (-1)^i \dim_{\mathbf{Q}_\ell}(\widetilde{H}^i_f(F, V)) = -\dim_{\mathbf{Q}_\ell} t_V + \sum_{\mathfrak{p}\in S_\infty} \dim_{\mathbf{Q}_\ell}(H^0(F_\mathfrak{p}, V)).$$

2.2.3. PROPOSITION. *Soit*

$$(\beta) \qquad 0 \to V' \to V \to V'' \to 0$$

une f-suite exacte courte de représentations ℓ-adiques pseudo-géométriques de $G_{F,S}$. Alors

i) *la suite*

$$(s_f(\beta)) \qquad \cdots \to \widetilde{H}^i_f(F, V') \to \widetilde{H}^i_f(F, V) \to \widetilde{H}^i_f(F, V'') \to \cdots$$

est exacte;

ii) *le diagramme rectangulaire tordu*

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \widetilde{H}^i_f(F, V') & \to & H^i(U_S, V') & \to & \bigoplus_{\mathfrak{p}\in S_f} H^i_{/f}(F_\mathfrak{p}, V') & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \widetilde{H}^i_f(F, V) & \to & H^i(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^i_{/f}(F_\mathfrak{p}, V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \widetilde{H}^i_f(f, V'') & \to & H^i(U_S, V'') & \to & \bigoplus_{\mathfrak{p}\in S_f} H^i_{/f}(F_\mathfrak{p}, V'') & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

est commutatif et ses trois lignes et ses trois colonnes sont exactes.

PREUVE. On commence par vérifier la commutativité du diagramme. Le seul point délicat est la commutativité du carré

$$\begin{array}{ccc}
H^1_f(F, V'') & \longrightarrow & H^1(U_S, V'') \\
\downarrow & & \downarrow \\
H^1_f(F, V'^*(1))^* & \longrightarrow & H^2(U_S, V').
\end{array}$$

Remarquons que l'image des flèches dans $H^2(U_S, V')$ est en fait contenue dans $H^2_0(U_S, V')$. Il s'agit donc de vérifier la compatibilité de la flèche $H^1_f(F, V'') \to H^1_f(F, V'^*(1))^*$ avec l'isomorphisme de $H^2_0(U_S, V')$ sur $H^1_0(U_S, V'^*(1))^*$ c'est-à-dire la commutativité du triangle

$$\begin{array}{ccc}
H^1_f(F, V'^*(1)) \times H^1_f(F, V'') & & \\
 & \searrow & \\
\downarrow & & \mathbb{Q}_\ell \\
 & \nearrow & \\
H^1_0(U_S, V'^*(1)) \times H^2_0(U_S, V') & &
\end{array}$$

ce qui se voit en comparant la définition des deux accouplements.

L'exactitude des lignes et des colonnes du diagramme sauf peut-être la première colonne (qui est en fait la suite $s_f(\beta)$) est claire. Comme la suite $s_f(\beta)$ est un complexe (n°1.4.6), elle est exacte (n°06).

3. Représentations géométriques

On ne suppose plus dans ce paragraphe que S contient les places divisant ℓ.

3.1. Généralités.

3.1.1. Une représentation ℓ-adique V de G_F est dite *S-géométrique* si

i) elle a bonne réduction en dehors de S;
ii) elle est potentiellement semi-stable en toutes les places finies de F (condition qui, d'après le théorème de monodromie ℓ-adique, est automatiquement réalisée en les places ne divisant pas ℓ).

Une *représentation géométrique* est une représentation qui est S-géométrique pour un S convenable.

On note $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_F)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_\ell, pg, S}(G_F)$) la sous-catégorie tannakienne de $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_F)$ formée des représentations géométriques (resp. S-géométriques). C'est donc la catégorie des représentations ℓ-adiques de G_F formée des V qui, pour toute place finie $\mathfrak{p}$ de F, sont dans $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_\mathfrak{p})$ (resp. et dans $\mathbf{Rep}_{\mathbb{Q}_\ell, f}(G_\mathfrak{p})$ si $\mathfrak{p} \notin S$). La catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_F)$ est donc la limite inductive, sur tous les S, des $\mathbf{Rep}_{\mathbb{Q}_\ell, pg, S}(G_F)$.

3.1.2. Pour tout objet V de $\mathbf{Rep}_{\mathbb{Q}_\ell, pg, S}(G_F)$ et tout $i \in \mathbb{Z}$, on pose

$$H^i_g(U_S, V) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, pg, S}(G_F), V)$$

et

$$H^i_g(F, V) = H^i(\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_F), V) = \text{lim.ind.}_S H^i_g(U_S, V).$$

On a donc $H^0_g(U_S, V) = H^0_g(F, V) = H^0(F, V)$ et, si $\widetilde{S} = S \cup S_\ell(F)$,

$$H^1_g(U_S, V) = \{x \in H^1(U_{\widetilde{S}}, V) \mid x_\mathfrak{p} \in H^1_f(F_\mathfrak{p}, V) \text{ si } \mathfrak{p} \in \widetilde{S} - S,$$
$$x_\mathfrak{p} \in H^1_g(F_\mathfrak{p}, V) \text{ si } \mathfrak{p} \in S_f\}$$

et

$$H^1_g(F, V) = \{x \in H^1(F, V) \mid x_\mathfrak{p} \in H^1_g(F_\mathfrak{p}, V) \text{ pour tout } \mathfrak{p} \in S_f(F),$$
$$x_\mathfrak{p} \in H^1_f(F_\mathfrak{p}, V) \text{ pour presque tout } \mathfrak{p} \in S_f(F)\}.$$

3.1.3. Soit V une représentation ℓ-adique S-géométrique de G_F. Si Σ est un ensemble fini de places vérifiant $S_\infty \subset \Sigma \subset S$, le groupe $H^1_{f,\Sigma}(F, V)$ classifie les extensions U de $\mathbb{Q}_\ell$ par V dans $\mathbf{Rep}_{\mathbb{Q}_\ell, pg, S}(G_F)$ telles que

$$0 \to V \to U \to \mathbb{Q}_\ell \to 0$$

a bonne réduction en dehors de Σ; la suite

$$0 \to H^1_{f,\Sigma}(F, V) \to H^1_g(U_S, V) \to \prod_{\mathfrak{p} \in S - \Sigma} H^1_{g/f}(F_\mathfrak{p}, V)$$

est exacte. En particulier, $H^1_{f,S}(F, V) = H^1_g(U_S, V)$.

Plus généralement, si Σ et Σ' sont deux ensembles finis de places de F vérifiant $S_\infty \subset \Sigma \subset \Sigma'$, la suite

$$0 \to H^1_{f,\Sigma}(F, V) \to H^1_{f,\Sigma'}(F, V) \to \prod_{\mathfrak{p}\in\Sigma'-\Sigma} H^1_{g/f}(F_\mathfrak{p}, V)$$

est exacte.

3.2. Représentations strictement géométriques.

3.2.1. Si V est une représentation ℓ-adique géométrique de G_F et $\mathfrak{p}$ une place finie de F, on note $D_{pst,\mathfrak{p}}(V)$ l'objet de $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_{F_\mathfrak{p}})$ associé à la restriction de V à $G_\mathfrak{p}$ (cf. I, 2.2.5 et 2.2.6).

DÉFINITION. On dit qu'une représentation ℓ-adique géométrique V de G_F est *F-semi-simple* si, pour toute place finie $\mathfrak{p}$ de F, $D_{pst,\mathfrak{p}}(V)$ est F-semi-simple (I, 1.1.5).

3.2.2. CONJECTURE $C_{S,sg}(V)$. *Soit V une représentation S-géométrique. Alors,*

i) *pour toute place finie $\mathfrak{p}$ de F, $D_{pst,\mathfrak{p}}(V)$ est F-semi-simple;*

ii) *pour tout quotient V'' de V, l'application naturelle*

$$H^1_g(U_S, V) \to H^1_g(U_S, V'')$$

est surjective.

La seconde propriété signifie que la catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell,pg,S}(G_F)$ est de dimension cohomologique 1.

3.2.3. Faute de savoir prouver cette conjecture, il est commode d'introduire la catégorie des *représentations ℓ-adiques strictement S-géométriques de G_F* : c'est la plus grande sous-catégorie pleine de $\mathbf{Rep}_{\mathbb{Q}_\ell,pg,S}(G_F)$, telle que

- tout objet V satisfait $C_{S,sg}(V)$,
- si V est strictement S-géométrique, $V^*(1)$ l'est aussi,
- si W, extension de $\mathbb{Q}_\ell$ par une représentation strictement S-géométrique, est S-géométrique, alors W est strictement S-géométrique.

Comme sous-catégorie pleine de $\mathbf{Rep}_{\mathbb{Q}_\ell,pg,S}(G_F)$, cette catégorie est stable par sous-objet, quotient, somme directe et est donc abélienne.

EXEMPLES. On conjecture **[Li72, Li73, Sc79, Ja89]** que, pour tout entier $i \neq 1$, $H^2(U_S, \mathbb{Q}_\ell(i)) = 0$ (on le sait grâce à Soulé [So79] lorsque $i \geq 2$). Si cette conjecture est vraie pour tout corps de nombres, on vérifie facilement que toute représentation S-géométrique dont tous les sous-quotients simples sont de dimension 1 est strictement S-géométrique. De même, si X est une variété abélienne ayant bonne réduction en dehors de S dont la ℓ-composante du groupe de Shafarevich–Tate est finie et si $T_\ell(X)$ est son module de Tate, alors $V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(X)$ est strictement S-géométrique.

3.3. Encore des suites exactes.

3.3.1. On munit l'ensemble $\{e, f, g, h\}$ de la relation d'ordre partielle définie par $e \leq f \leq g$ et $e \leq h \leq g$. Si $\nu, \nu' \in \{e, f, g, h\}$ vérifient $\nu \leq \nu'$, on pose

$$H^1_{\nu'/\nu}(F_{\mathfrak{p}}, V) = H^1_{\nu'}(F_{\mathfrak{p}}, V)/H^1_{\nu}(F_{\mathfrak{p}}, V)$$

et

$$H^1_{/\nu}(F_{\mathfrak{p}}, V) = H^1(F_{\mathfrak{p}}, V)/H^1_{\nu}(F_{\mathfrak{p}}, V).$$

PROPOSITION. *Soient V une représentation ℓ-adique strictement S-géométrique de G_F, et Σ et Σ' des ensembles finis de places de F vérifiant $S_\infty(F) \subset \Sigma \subset \Sigma' \subset S$; alors les suites*

(a) $$0 \to H^1_{f,\Sigma}(F, V) \to H^1_{f,\Sigma'}(F, V) \to \bigoplus_{\mathfrak{p}\in\Sigma'-\Sigma} H^1_{g/e}(F_{\mathfrak{p}}, V) \to H^1_{f,\Sigma'}(F, V^*(1))^* \to H^1_{f,\Sigma}(F, V^*(1))^* \to 0$$

et

(b) $$0 \to H^1_{f,\Sigma}(F, V) \to H^1_{f,\Sigma'}(F, V) \to \bigoplus_{\mathfrak{p}\in\Sigma'-\Sigma} H^1_{g/f}(F_{\mathfrak{p}}, V) \to 0$$

sont exactes.

PREUVE. Pour toute représentation ℓ-adique W de G_F, posons

$$H^1_{f,e,\Sigma}(F, W) = \{x \in H^1_f(F, W) \mid x_{\mathfrak{p}} \in H^1_e(F_{\mathfrak{p}}, W) \text{ si } \mathfrak{p} \in \Sigma_f\}.$$

En procédant comme au n°2.2.1 (en utilisant la dualité entre $H^1_{/g}(F_{\mathfrak{p}}, V)$ et $H^1_e(F_{\mathfrak{p}}, V^*(1))$, cf. I, n°3.3.9), le fait que V soit pseudo-géométrique permet déjà de lui associer une suite exacte

$$0 \to H^1_{f,\Sigma}(F, V) \to H^1(U_S, V) \to \left(\bigoplus_{\mathfrak{p}\in S-\Sigma} H^1_{/f}(F_{\mathfrak{p}}, V)\right) \oplus \left(\bigoplus_{\mathfrak{p}\in\Sigma_f} H^1_{/g}(F_{\mathfrak{p}}, V)\right) \to H^1_{f,e,\Sigma}(F, V^*(1))^* \to H^2(U_S, V)$$

et on a une suite exacte analogue lorsque l'on remplace Σ par Σ'.

Comme $V^*(1)$ est strictement géométrique, si $x \in H^1_{f,\Sigma}(F, V^*(1))$, pour toute place finie $\mathfrak{p}$ de F, $x_{\mathfrak{p}} \in H^1_f(F_{\mathfrak{p}}, V^*(1))$ équivaut à $x_{\mathfrak{p}} \in H^1_e(F_{\mathfrak{p}}, V^*(1))$. On en déduit que $H^1_{f,e,\Sigma}(F, V^*(1)) = H^1_f(F, V^*(1))$. De même, $H^1_{f,e,\Sigma'}(F, V^*(1)) = H^1_f(F, V^*(1))$.

La suite exacte (b) résulte alors de la commutativité du diagramme

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \bigoplus_{\Sigma'-\Sigma} H^1_{g/f}(F_\mathfrak{p}, V) & & & & \\
 & & \downarrow & & & & \\
H^1(U_S, V) & \to & \left(\bigoplus_{S-\Sigma} H^1_{/f}(F_\mathfrak{p}, V)\right) \oplus \left(\bigoplus_{\Sigma_f} H^1_{/g}(F_\mathfrak{p}, V\right) & \to & H^1_{f,e,\Sigma}(F, V^*(1))^* & \to & H^2(U_S, V) \\
\| & & \downarrow & & \| & & \| \\
H^1(U_S, V) & \to & \left(\bigoplus_{S-\Sigma'} H^1_{/f}(F_\mathfrak{p}, V\right) \oplus \left(\bigoplus_{\Sigma'_f} H^1_{/g}(F_\mathfrak{p}, V)\right) & \to & H^1_{f,e,\Sigma'}(F, V^*(1))^* & \to & H^2(U_S, V) \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

dont les lignes et les colonnes sont exactes.

Rappelons (I, n°3.3.9) que, pour toute place finie $\mathfrak{p}$ de F, le fait que $D_{pst,\mathfrak{p}}(V)$ soit F-semi-simple implique que l'application naturelle de $H^1_{h/e}(F_\mathfrak{p}, V)$ dans $H^1_{g/f}(F_\mathfrak{p}, V)$ est un isomorphisme. De plus, dans la dualité

$$H^1_{g/e}(F_\mathfrak{p}, V) \times H^1_{g/e}(F_\mathfrak{p}, V^*(1)) \to \mathbb{Q}_\ell ,$$

l'orthogonal de $H^1_{h/e}(F_\mathfrak{p}, V)$ est $H^1_{h/e}(F_\mathfrak{p}, V^*(1))$ (cf. I, n°3.3.9). D'où des isomorphismes

$$H^1_{g/h}(F_\mathfrak{p}, V) \simeq H^1_{h/e}(F_\mathfrak{p}, V^*(1))^* \simeq H^1_{g/f}(F_\mathfrak{p}, V^*(1))^* .$$

En utilisant tous ces isomorphismes, la suite exacte

$$0 \to H^1_{h/e}(F_\mathfrak{p}, V) \to H^1_{g/e}(F_\mathfrak{p}, V) \to H^1_{g/h}(F_\mathfrak{p}, V) \to 0$$

devient la suite exacte

$$0 \to H^1_{g/f}(F_\mathfrak{p}, V) \to H^1_{g/e}(F_\mathfrak{p}, V) \to H^1_{g/f}(F_\mathfrak{p}, V^*(1))^* \to 0 .$$

En la combinant avec la suite exacte (b) et avec la transposée de l'analogue de (b) pour $V^*(1)$, on obtient (a).

Remarque. Le lecteur soigneux vérifiera que la proposition est vraie sous des hypothèses plus faibles. On n'utilise en effet que le fait que la représentation *V est géométrique et que, pour toute représentation S-géométrique U,* extension de $\mathbb{Q}_\ell$ par V , l'action de F sur $D_{pst,\mathfrak{p}}(U)$ est semi-simple en 1.

3.3.2. Proposition. *Soient Σ un sous-ensemble de S contenant S_∞ et*

$$(\beta) \qquad 0 \to V' \to V \to V'' \to 0$$

une suite exacte courte de représentations ℓ-adiques de $G_{F,S}$. Si V est strictement S-géométrique et si (β) a bonne réduction en dehors de Σ, la suite exacte de cohomologie associée à (β) induit une suite exacte

$$(s_{g,\Sigma}(\beta)) \qquad \begin{aligned} & 0 \to H^0(F, V') \to H^0(F, V) \to H^0(F, V'') \\ & \to H^1_{f,\Sigma}(F, V') \to H^1_{f,\Sigma}(F, V) \to H^1_{f,\Sigma}(F, V'') \to 0. \end{aligned}$$

PREUVE. En toute place $\mathfrak{p}$ n'appartenant pas à Σ, on a la suite exacte

$$0 \to H^1_{g/f}(F_\mathfrak{p}, V') \to H^1_{g/f}(F_\mathfrak{p}, V) \to H^1_{g/f}(F_\mathfrak{p}, V'');$$

en fait la dernière flèche est surjective, puisque l'on peut partout remplacer g/f par h/e et que, comme V en tant que représentation de $G_\mathfrak{p}$ est dans $\mathbf{Rep}_{\mathbb{Q}_\ell, ph}(G_\mathfrak{p})$, l'application $H^1_h(F_\mathfrak{p}, V) \to H^1_h(F_\mathfrak{p}, V'')$ est surjective (I, n°3.3). On a donc le diagramme commutatif dont les lignes et les deux colonnes de droite sont exactes

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 \to H^1_{f,\Sigma}(F, V') & \to & H^1_g(U_S, V') & \to & \bigoplus_{\mathfrak{p}\in S-\Sigma} H^1_{g/f}(F_\mathfrak{p}, V') & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to H^1_{f,\Sigma}(F, V) & \to & H^1_g(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S-\Sigma} H^1_{g/f}(F_\mathfrak{p}, V) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to H^1_{f,\Sigma}(F, V'') & \to & H^1_g(U_S, V'') & \to & \bigoplus_{\mathfrak{p}\in S-\Sigma} H^1_{g/f}(F_\mathfrak{p}, V'') & \to & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

On en déduit la suite exacte

$$H^1_{f,\Sigma}(F, V') \to H^1_{f,\Sigma}(F, V) \to H^1_{f,\Sigma}(F, V'') \to 0.$$

L'image de $H^0(F, V'')$ étant clairement dans $H^1_{f,\Sigma}(F, V')$, l'exactitude de la suite $s_{g,\Sigma}(\beta)$ ne pose plus de problèmes.

REMARQUE. De nouveau, le lecteur soigneux vérifiera que la proposition est vraie sous des hypothèses plus faibles: on n'utilise que la F-simplicité en 1 et la surjectivité de l'application $H^1_g(U_S, V) \to H^1_g(U_S, V'')$.

3.4. La fonction L d'une représentation géométrique.

3.4.1. De façon à simplifier l'exposition, on suppose choisi un plongement de $\mathbb{Q}_\ell$ dans $\mathbb{C}$. Dans les applications que l'on a en vue, on s'attend à ce que les polynômes à coefficients dans $\mathbb{Q}_\ell$ que l'on va rencontrer soient en fait à coefficients algébriques sur $\mathbb{Q}$, et on ne devrait avoir besoin de plonger dans $\mathbb{C}$ que la fermeture algébrique de $\mathbb{Q}$ dans $\mathbb{Q}_\ell$, ce qui est quand même plus raisonnable!

3.4.2. Pour toute représentation ℓ-adique V de G_F et toute place finie $\mathfrak{p}$ de F, on sait définir le facteur local

$$L_\mathfrak{p}(V, s) = P(D_{pst,\mathfrak{p}}(V), (q_\mathfrak{p})^s)^{-1}$$

(cf. I, n°1.2.1, ici $q_\mathfrak{p}$ désigne le nombre d'éléments du corps résiduel de $F_\mathfrak{p}$; si $\mathfrak{p}$ divise ℓ, $P(D_{pst,\mathfrak{p}}(V), u) \in \mathbb{Q}_\ell[u]$, d'après la remarque (ii) de I, 1.3.3). Cette notion est surtout raisonnable lorsque V est géométrique, auquel cas la dimension de $D_{pst,\mathfrak{p}}(V)$ est égale à celle de V sur $\mathbb{Q}_\ell$.

Pour tout ensemble fini Σ de places de F contenant $S_\infty(F)$, on pose

$$L_\Sigma(V, s) = \prod_{\mathfrak{p} \notin \Sigma} L_\mathfrak{p}(V, s);$$

on pose aussi $L(V, s) = L_{S_\infty(F)}(V, s)$.

3.4.3. Conjecture $C_{\text{gal}, L}(V)$ ([**Bu**, exp. VIII]). *Soit V une représentation ℓ-adique géométrique de G_F. La série $L(V, s)$ converge pour* $\mathrm{Re}(s) \gg 0$ *et admet un prolongement analytique méromorphe dans tout le plan complexe.*

3.4.4. Remarques. i) Si Σ est un ensemble fini de places de F contenant $S_\infty(F)$, $C_{\text{gal}, L}$ est vraie pour V si et seulement si la conjecture obtenue en remplaçant $L(V, s)$ par $L_\Sigma(V, s)$ est vraie.

ii) Si

$$0 \to V' \to V \to V'' \to 0$$

est une suite exacte courte de représentations géométriques et si la conjecture est vraie pour deux de ces trois représentations, elle l'est aussi pour la troisième: en effet, si l'on choisit Σ tel que V, et donc aussi V' et V'', ont bonne réduction en dehors de Σ, on a $L_\Sigma(V, s) = L_\Sigma(V', s) \cdot L_\Sigma(V'', s)$.

En particulier, il "suffit" de vérifier cette conjecture pour les représentations géométriques simples. Nous renvoyons à l'exposé VIII de [**Bu**] pour des formes plus précises de cette conjecture.

iii) Dans les applications, on peut rencontrer une représentation ℓ-adique V dont on ne sait peut-être pas qu'elle est géométrique et encore moins qu'elle satisfait $C_{\text{gal}, L}$. Nous dirons que *la fonction L d'une représentation ℓ-adique V de G_F existe en* 0 (sous-entendu relativement au plongement de $\mathbb{Q}_\ell$ dans $\mathbb{C}$ choisi!) si la série $L(V, s)$ converge pour $\mathrm{Re}(s) \gg 0$ et admet un prolongement analytique méromorphe dans un ouvert connexe contenant 0. S'il en est ainsi, on peut alors parler de *l'ordre de $L(V, s)$ en* 0 noté $\mathrm{ord}_{s=0} L(V, s)$ et on note $L^*(V, 0)$ le coefficient dominant de son développement de Laurent en $s = 0$.

3.4.5. Nous allons maintenant préciser l'ordre conjectural ($\in \mathbb{Z}$) du zéro éventuel en $s = 0$ de ces fonctions L.

Conjectures. *Soit V une représentation ℓ-adique géométrique de G_F dont la fonction L existe en* 0.

i) $C_{\text{ord}}(S_\infty(F), V)$: *On a*

$$\mathrm{ord}_{s=0} L(V, s) = \dim_{\mathbb{Q}_\ell} H^1_f(F, V^*(1)) - \dim_{\mathbb{Q}_\ell} H^0(F, V^*(1));$$

ii) $C_{\text{ord}}(\Sigma, V)$: *Pour tout ensemble fini Σ de places de F contenant* $S_\infty(F)$,

$$\mathrm{ord}_{s=0} L_\Sigma(V, s) = \dim_{\mathbb{Q}_\ell} H^1_{f,\Sigma}(F, V^*(1)) - \dim_{\mathbb{Q}_\ell} H^0(F, V^*(1)).$$

3.4.6. PROPOSITION. *Soit* V *une représentation strictement* S*-géométrique, ayant bonne réduction en dehors de* S *et dont la fonction* L *existe en* 0. *Soit* Σ *vérifiant* $S_\infty(F) \subset \Sigma \subset S$.

i) *Les conjectures* $C_{\mathrm{ord}}(S_\infty(F), V)$ *et* $C_{\mathrm{ord}}(\Sigma, V)$ *sont équivalentes.*

ii) *Soit* V' *une sous-représentation de* V *dont la fonction* L *existe en* 0 *et posons* $V'' = V/V'$. *Alors si deux des trois conjectures* $C_{\mathrm{ord}}(\Sigma, V')$, $C_{\mathrm{ord}}(\Sigma, V)$, $C_{\mathrm{ord}}(\Sigma, V'')$ *sont vraies, la troisième l'est aussi.*

PREUVE. Remarquons que, comme la flèche naturelle

$$H^1_{g/f}(F_\mathfrak{p}, V^*(1)) \to H^1_g(F_\mathfrak{p}, (D_{pst,\mathfrak{p}}(V))^*(Ta))/H^1_f(F_\mathfrak{p}, (D_{pst,\mathfrak{p}}(V))^*(Ta))$$

est un isomorphisme, il résulte de la F-semi-simplicité de $D_{pst,\mathfrak{p}}(V)$ et de la proposition 3.3.9 que

$$\dim_{\mathbf{Q}_\ell} H^1_{g/f}(F_\mathfrak{p}, V^*(1)) = \mathrm{ord}_{u=1} P_\mathfrak{p}(V, u) = -\mathrm{ord}_{s=0} L_\mathfrak{p}(V, s).$$

L'assertion (i) résulte alors de l'exactitude de la suite exacte (b) du n°3.3.1 pour $V^*(1)$.

Pour prouver (ii), remarquons que (i) nous permet de se ramener au cas où Σ contient l'ensemble des places de mauvaise réduction de l'extension. On a alors $L_\Sigma(V, s) = L_\Sigma(V', s) L_\Sigma(V'', s)$. Il suffit donc de montrer que

$$\dim_{\mathbf{Q}_\ell} H^1_{f,\Sigma}(F, V^*(1)) - \dim_{\mathbf{Q}_\ell} H^0(F, V^*(1)) = \dim_{\mathbf{Q}_\ell} H^1_{f,\Sigma}(F, V'^*(1))$$
$$- \dim_{\mathbf{Q}_\ell} H^0(F, V'^*(1)) + \dim_{\mathbf{Q}_\ell} H^1_{f,\Sigma}(F, V''^*(1)) - \dim_{\mathbf{Q}_\ell} H^0(F, V''^*(1)),$$

ce qui résulte de la proposition 3.3.2.

4. Caractéristique d'Euler-Poincaré

Sauf mention explicite, nous ne supposerons pas dans ce paragraphe les conjectures du paragraphe 3 vérifiées. On suppose de nouveau que S est un ensemble fini de places de F contenant $S_\infty(F)$ et $S_\ell(F)$.

4.1. Droite d'Euler-Poincaré.

4.1.1. Pour toute représentation ℓ-adique V de $G_{F,S}$, on pose

$$\Delta_\mathfrak{p}(V) = \bigotimes_{0 \le i \le 2} (\det_{\mathbf{Q}_\ell} H^i(F_\mathfrak{p}, V))^{(-1)^i} \text{ pour toute place } \mathfrak{p} \text{ de } F,$$

$$\Delta_{S,\mathrm{glob}}(V) = \bigotimes_{0 \le i \le 2} (\det_{\mathbf{Q}_\ell} H^i(G_S, V))^{(-1)^i}$$

et

$$\Delta_S(V) = \Delta_{S,\mathrm{glob}}(V) \otimes \left(\bigotimes_{\mathfrak{p} \in S} \Delta_\mathfrak{p}(V) \right)^{-1}$$

4.1.2. De même, si T est une représentation $\mathbb{Z}_\ell$-adique de $G_{F,S}$ (n°0.2), on pose

$$\Delta_{\mathfrak{p}}(T) = \bigotimes_{0\leq i\leq 2} (\det_{\mathbb{Z}_\ell} H^i(F_{\mathfrak{p}}, T))^{(-1)^i} \quad \text{pour toute place } \mathfrak{p} \text{ de } F,$$

$$\Delta_{S,\mathrm{glob}}(T) = \bigotimes_{0\leq i\leq 2} (\det_{\mathbb{Z}_\ell} H^i(U_S, T))^{(-1)^i}$$

et

$$\Delta_S(T) = \Delta_{S,\mathrm{glob}}(T) \otimes \left(\bigotimes_{\mathfrak{p}\in S} \Delta_{\mathfrak{p}}(T)\right)^{-1}.$$

Si $\ell = 2$, pour $\mathfrak{p}$ place archimédienne, il ne faut pas oublier de compter les groupes finis $H^1(F_{\mathfrak{p}}, T)$ et $H^2(F_{\mathfrak{p}}, T)$.

Si $V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T$, le $\mathbb{Z}_\ell$-module $\Delta_S(T)$ est libre de rang un et s'identifie à un réseau de $\Delta_S(V)$.

4.1.3. Soit

$$0 \to V' \to V \to V'' \to 0$$

une suite exacte courte de représentations ℓ-adiques de $G_{F,S}$. Les suites exactes de cohomologie

$$\cdots \to H^i(U_S, V') \to H^i(U_S, V) \to H^i(U_S, V'') \to \cdots$$

et

$$\cdots \to H^i(F_{\mathfrak{p}}, V') \to H^i(F_{\mathfrak{p}}, V) \to H^i(F_{\mathfrak{p}}, V'') \to \cdots$$

induisent un isomorphisme naturel

$$\Delta_S(V') \otimes_{\mathbb{Q}_\ell} \Delta_S(V'') \simeq \Delta_S(V).$$

Le résultat suivant est évident:

Proposition. *Soit*

$$0 \to T' \to T \to T'' \to 0$$

une suite exacte courte de représentations $\mathbb{Z}_\ell$-adiques de $G_{F,S}$. Posons $V' = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T'$, $V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_l} T$, et $V'' = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T''$. Dans l'isomorphisme ci-dessus, l'image de $\Delta_S(T') \otimes_{\mathbb{Z}_\ell} \Delta_S(T'')$ est $\Delta_S(T)$.

4.1.4. Proposition. *Soient T_1 et T_2 deux représentations $\mathbb{Z}_\ell$-adiques de $G_{F,S}$ telles que $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_1 = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_2 = V$. Alors, dans $\Delta_S(V)$, $\Delta_S(T_1) = \Delta_S(T_2)$.*

Preuve. Par dévissage, on se ramène au cas où l'isomorphisme $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_1 \cong \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_2$ est induit par un morphisme de T_1 dans T_2 que l'on peut supposer injectif ou surjectif.

Dans le premier cas, on a donc une suite exacte

$$0 \to T_1 \to T_2 \to T'' \to 0,$$

où T'' est un groupe fini d'ordre une puissance de ℓ. On peut appliquer la proposition ci-dessus avec $T' = T_1$, $T = T_2$, $V' = V$, $V'' = 0$ et on a

$$\Delta_S(T_2) = \Delta_S(T_1) \cdot \chi(T''),$$

où

$$\chi(T'') = \prod_{0\leq i\leq 2} \#(H^i(U_S, T''))^{(-1)^i} / \prod_{\mathfrak{p}\in S}\left(\prod_{0\leq i\leq 2} \#(H^i(F_\mathfrak{p}, T''))^{(-1)^i}\right) = 1$$

car **[Mi86**, 1.5.1]

$$\prod_{0\leq i\leq 2} \#(H^i(U_S, T''))^{(-1)^i} = \#(T'')^{-[F:\mathbb{Q}]} \cdot \prod_{\mathfrak{p}\in S_\infty} \#(H^0(F_\mathfrak{p}, T'')),$$

tandis que **[Mi86**, 1.2.8]

$$\prod_{\mathfrak{p}\in S_f}\left(\prod_{0\leq i\leq 2} \#(H^i(F_\mathfrak{p}, T''))^{(-1)^i}\right) = \#(T'')^{-[F:\mathbb{Q}]},$$

et que, pour $\mathfrak{p} \in S_\infty(F)$, comme $G_\mathfrak{p}$ est cyclique, $\#(H^1(F_\mathfrak{p}, T'') = \#(H^2(F_\mathfrak{p}, T''))$.

Le second cas se traite de la même manière.

4.1.5. Si V est une représentation ℓ-adique de $G_{F,S}$, le sous-$\mathbb{Z}_\ell$-module $\Delta_S(T)$ de $\Delta_S(V)$ est donc le même pour n'importe quel réseau T de V stable par $G_{F,S}$. *On le note* $\Delta_{S,\mathbb{Z}_\ell}(V)$. On appelle *norme canonique* sur $\Delta_S(V)$ l'unique norme $|\cdot|_{\mathrm{can},S}$ telle que, si ω est une base de $\Delta_{S,\mathbb{Z}_\ell}(V)$, alors $|\omega|_{\mathrm{can},S} = 1$.

La proposition 4.1.3 nous dit alors que, avec des conventions évidentes,

$$\Delta_{S,\mathbb{Z}_\ell}(V) \simeq \Delta_{S,\mathbb{Z}_\ell}(V') \otimes_{\mathbb{Z}_\ell} \Delta_{S,\mathbb{Z}_\ell}(V''),$$

ou encore que l'isomorphisme $\Delta_S(V') \otimes_{\mathbb{Q}_\ell} \Delta_S(V'') \simeq \Delta_S(V)$ est un isomorphisme d'espaces vectoriels normés.

4.1.6. LEMME. *Soit V une représentation ℓ-adique de $G_{F,S}$ et soit S' un ensemble fini de places de F contenant S. Alors, la suite*

$$(s_{S,S'}(V)) \qquad \cdots \to H^i(U_S, V) \to H^i(U_{S'}, V) \to \bigoplus_{\mathfrak{p}\in S'-S} H^i_{/f}(F_\mathfrak{p}, V) \to \cdots$$

(*où les applications sont définies comme pour* $s_{f,S}(V)$, *cf.* n°2.2.1) *est exacte.*

PREUVE. On peut montrer l'exactitude de cette suite directement en raisonnant comme pour la suite $s_{f,S}(V)$. On peut aussi la déduire de l'exactitude de $s_{f,S}(V)$ en remarquant que, dans le diagramme commutatif rectangulaire tordu

$$\begin{array}{ccccccc}
 & 0 & & \cdots & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & \widetilde{H}^i_f(F, V) & \to & H^i(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^i_{/f}(F_\mathfrak{p}, V) & \to & \widetilde{H}^{i+1}_f(F, V) \to \cdots \\
 & \| & & \downarrow & & \downarrow & & \| \\
\cdots \to & \widetilde{H}^i_f(F, V) & \to & H^i(U_{S'}, V) & \to & \bigoplus_{\mathfrak{p}\in S'_f} H^i_{/f}(F_\mathfrak{p}, V) & \to & \widetilde{H}^{i+1}_f(F, V) \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \to & \bigoplus_{\mathfrak{p}\in S'-S} H^i_{/f}(F_\mathfrak{p}, V) & \to & \bigoplus_{\mathfrak{p}\in S'-S} H^i_{/f}(F_\mathfrak{p}, V) & \to & 0 \\
 & & & \downarrow & & \downarrow & & \\
 & & & \cdots & & 0 & &
\end{array}$$

toutes les lignes et les colonnes, sauf peut-être la deuxième colonne, sont exactes et que cette dernière est un complexe; et que, par conséquent, elle est exacte aussi.

4.1.7. On dispose alors d'un s-isomorphisme (i.e. un isomorphisme défini au signe près) naturel

$$i_{S,S'}(V) : \Delta_{S'}(V) \to \Delta_S(V).$$

La suite exacte précédente fournit en effet un s-isomorphisme

$$\begin{aligned}
\Delta_{S',\mathrm{glob}}(V) &\simeq \Delta_{S,\mathrm{glob}}(V) \otimes \left(\bigotimes_{i,\mathfrak{p}\in S'-S} (\det_{\mathbb{Q}_\ell} H^i_{/f}(F_\mathfrak{p}, V))^{(-1)^i} \right) \\
&\simeq \Delta_{S,\mathrm{glob}}(V) \otimes \left(\bigotimes_{i,\mathfrak{p}\in S'-S} \det_{\mathbb{Q}_\ell}(H^i(F_\mathfrak{p}, V))^{(-1)^i} \right) \\
&\otimes \left(\bigotimes_{\mathfrak{p}\in S'-S} L_f(F_\mathfrak{p}, V)^{-1} \right)
\end{aligned}$$

(cf. I, 4.1.1 pour la définition de $L_f(F_\mathfrak{p}, V)$), ou encore

$$\Delta_{S'}(V) \simeq \Delta_S(V) \otimes \left(\bigotimes_{\mathfrak{p}\in S'-S} L_f(F_\mathfrak{p}, V) \right)^{-1}.$$

Comme $\mathfrak{p} \in S' - S$ implique que $\mathfrak{p}$ ne divise pas ℓ, $L_f(F_\mathfrak{p}, V)$ s'identifie à $\mathbb{Q}_\ell$ (cf. I, 4.1.1) et on obtient l'isomorphisme cherché. On utilise $i_{S,S'}(V)$ pour identifier $\Delta_{S'}(V)$ et $\Delta_S(V)$.

Proposition. *Dans l'identification précédente on a* $\Delta_{S',\mathbb{Z}_\ell}(V) \simeq \Delta_{S,\mathbb{Z}_\ell}(V)$ *et* $|\cdot|_{\mathrm{can},S'} = |\cdot|_{\mathrm{can},S}$.

Preuve. Choisissons un réseau T de V stable par $G_{F,S}$. Pour $i \in \mathbb{Z}$, posons aussi $H^i_f(F_\mathfrak{p}, T) = 0$ si $i \neq 0, 1$ et

$$H^i_{/f}(F_\mathfrak{p}, T) = H^i(F_\mathfrak{p}, T)/H^i_f(F_\mathfrak{p}, T)$$

pour tout i. Avec des conventions évidentes, on a

$$L(F_\mathfrak{p}, T) = L_f(F_\mathfrak{p}, T) \otimes L_{/f}(F_\mathfrak{p}, T)$$

et on voit que, lorsque l'on identifie $L_f(F_\mathfrak{p}, V)$ à $\mathbb{Q}_\ell$, $L_f(F_\mathfrak{p}, T)$ s'identifie à $\mathbb{Z}_\ell$ (I, 4.1.2 et 4.2.1). La proposition résulte alors du lemme suivant:

4.1.8 Lemme. *La suite exacte* $s_{S,S'}(V)$ *provient, par extension des scalaires, d'une suite exacte de la forme*

$$(s_{S,S'}(T)) \quad \cdots \to H^i(U_S, T) \to H^i(U_{S'}, T) \to \prod_{\mathfrak{p}\in S'-S} H^i_{/f}(F_\mathfrak{p}, T) \to \cdots .$$

Ce lemme se démontre par des techniques analogues à celles utilisées au n°2.2.1 et nous en laissons la vérification au lecteur.

4.1.9. Ceci nous permet de définir (au signe près) la droite $\Delta_{\mathrm{EP}}(V)$ comme étant la droite $\Delta_S(V)$ pour n'importe quel ensemble fini S assez grand. On l'appelle *la droite d'Euler-Poincaré de* V. Elle est munie d'une *norme canonique* $|\cdot|_{\mathrm{can}} (= |\cdot|_{\mathrm{can},S}$, pour S assez grand) ou, ce qui revient au même, d'un sous-$\mathbb{Z}_\ell$-module canonique $\Delta_{\mathrm{EP},\mathbb{Z}_\ell}(V)$: on a $|\omega|_{\mathrm{can}} = 1$ si ω est une base de $\Delta_{\mathrm{EP},\mathbb{Z}_\ell}(V)$.

4.2. La droite fondamentale normée $\Delta_f(V)$.

4.2.1. A toute représentation ℓ-adique pseudo-géométrique V de G_F, on associe la $\mathbb{Q}_\ell$-droite

$$\Delta_f(V) = \left(\bigotimes_{0\le i\le 3} (\det_{\mathbb{Q}_\ell} \widetilde{H}^i_f(F, V))^{(-1)^i}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty(F)} \det_{\mathbb{Q}_\ell} H^0(F_\mathfrak{p}, V)\right)^{-1} \otimes \det_{\mathbb{Q}_\ell} t_V .$$

4.2.2. Supposons que V soit en outre non ramifiée en dehors de S. La suite exacte $s_{f,S}(V)$ du n°2.2.1 nous fournit un s-isomorphisme

$$\begin{aligned}\Delta_{S,\mathrm{glob}}(V) &\simeq \left(\bigotimes_i (\det_{\mathbb{Q}_\ell} \widetilde{H}^i_f(F, V))^{(-1)^i}\right) \otimes \left(\bigotimes_{i,\mathfrak{p}\in S_f} (\det_{\mathbb{Q}_\ell} H^i_{/f}(F_\mathfrak{p}, V))^{(-1)^i}\right)\\ &\simeq \left(\bigotimes_i (\det_{\mathbb{Q}_\ell} \widetilde{H}^i_f(F, V))^{(-1)^i}\right) \otimes \left(\bigotimes_{i,\mathfrak{p}\in S_f} \det_{\mathbb{Q}_\ell} (H^i(F_\mathfrak{p}, V))^{(-1)^i}\right)\\ &\qquad \otimes \left(\bigotimes_{\mathfrak{p}\in S_f} L_f(F_\mathfrak{p}, V)\right)^{-1},\end{aligned}$$

ou encore, puisque $H^i(F_\mathfrak{p}, V) = 0$ si $\mathfrak{p} \in S_\infty$ et $i \neq 0$ et puisque $L_f(F_\mathfrak{p}, V) \simeq \mathbb{Q}_\ell$ si $\mathfrak{p}$ ne devise pas ℓ tandis que $L_f(F_\mathfrak{p}, V) \simeq$

$(\det_{\mathbb{Q}_\ell} t_{V,\mathfrak{p}})^{-1}$ si $\mathfrak{p}$ divise ℓ,

$$\Delta_S(V) \simeq \left(\bigotimes_i \det_{\mathbb{Q}_\ell}(\widetilde{H}^i_f(F, V))^{(-1)^i}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty} \det_{\mathbb{Q}_\ell}(H^0(F_\mathfrak{p}, V))^{-1}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\ell} \det_{\mathbb{Q}_\ell} t_{V,\mathfrak{p}}\right)$$
$$= \Delta_f(V).$$

D'où un isomorphisme $i_S(V) : \Delta_f(V) \simeq \Delta_S(V)$. Il est immédiat que, lorsque l'on identifie $\Delta_S(V)$ à $\Delta(V)$, $i_S(V)$ est indépendant (au signe près) de S ; on obtient ainsi un s-isomorphisme

$$i(V) : \Delta_f(V) \to \Delta_{\mathrm{EP}}(V).$$

Nous notons $|\cdot|_{\mathrm{EP},V}$ la norme sur $\Delta_f(V)$ déduite de $|\cdot|_{\mathrm{can}}$ via $i(V)$ (on a donc $|\omega|_{\mathrm{EP},V} = |i(V)(\omega)|_{\mathrm{can}}$, si $\omega \in \Delta_f(V)$) ; nous notons $\Delta_{f,\mathbb{Z}_\ell}(V)$ le sous-$\mathbb{Z}_\ell$-module de $\Delta_f(V)$ déduit de $\Delta_{\mathrm{EP},\mathbb{Z}_\ell}(V)$ via $i(V)$ (on a donc $|\omega|_{\mathrm{EP},V} = 1$ si et seulement si ω est une base du $\mathbb{Z}_\ell$-module $\Delta_{f,\mathbb{Z}_\ell}(V)$.

5. Compléments

Ici encore, S contient $S_\infty(F)$ et $S_\ell(F)$.

5.1. Multiplicativité.

5.1.1. Soit

$$0 \to V' \to V \to V'' \to 0$$

une f-suite exacte courte de représentations ℓ-adiques pseudo-géométriques de $G_{F,S}$. Les suites exactes

$$\cdots \to \widetilde{H}^i_f(F, V') \to \widetilde{H}^i_f(F, V) \to \widetilde{H}^i_f(F, V'') \to \cdots,$$
$$0 \to H^0(F_\mathfrak{p}, V') \to H^0(F_\mathfrak{p}, V) \to H^0(F_\mathfrak{p}, V'') \to 0 \quad \text{pour } \mathfrak{p} \in S_\infty,$$

et

$$0 \to t_{V',\mathfrak{p}} \to t_{V,\mathfrak{p}} \to t_{V'',\mathfrak{p}} \to 0 \quad \text{pour } \mathfrak{p} \in S_\ell$$

permettent de définir un isomorphisme

$$\Delta_f(V') \otimes_{\mathbb{Q}_\ell} \Delta_f(V'') \simeq \Delta_f(V).$$

5.1.2. Proposition. *L'isomorphisme ci-dessus est un isomorphisme d'espaces vectoriels normés, autrement dit, il induit un isomorphisme*

$$\Delta_{f,\mathbb{Z}_\ell}(V') \otimes_{\mathbb{Z}_\ell} \Delta_{f,\mathbb{Z}_\ell}(V'') \simeq \Delta_{f,\mathbb{Z}_\ell}(V).$$

Preuve. On dispose (cf. n°4.1.3) d'un isomorphisme canonique $\Delta_S(V') \otimes_{\mathbb{Q}_\ell} \Delta_S(V'') \simeq \Delta_S(V)$ d'espaces vectoriels normés. Il suffit alors de vérifier la commutativité du diagramme

$$\begin{array}{ccc} \Delta_f(V') \otimes_{\mathbb{Q}_\ell} \Delta_f(V'') & \longrightarrow & \Delta_f(V) \\ {\scriptstyle i_S(V')\otimes i_S(V'')}\downarrow & & \downarrow{\scriptstyle i_S(V)} \\ \Delta_S(V') \otimes_{\mathbb{Q}_\ell} \Delta_S(V'') & \longrightarrow & \Delta_S(V). \end{array}$$

Cela résulte formellement de la commutativité des diagrammes

$$\begin{array}{ccccccccc} & & \cdots & & \cdots & & \cdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H^i_f(F_\mathfrak{p}, V') & \to & H^i(F_\mathfrak{p}, V') & \to & H^i_{/f}(F_\mathfrak{p}, V') & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H^i_f(F_\mathfrak{p}, V) & \to & H^i(F_\mathfrak{p}, V) & \to & H^i_{/f}(F_\mathfrak{p}, V) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H^i_f(F_\mathfrak{p}, V'') & \to & H^i(F_\mathfrak{p}, V'') & \to & H^i_{/f}(F_\mathfrak{p}, V'') & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \cdots & & \cdots & & \cdots & & \end{array}$$

et

$$\begin{array}{ccccccccc} & & \cdots & & \cdots & & \cdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \widetilde{H}^i_f(F, V') & \to & H^i(U_S, V') & \to & \bigoplus_{\mathfrak{p}\in S} H^i_{/f}(F_\mathfrak{p}, V') & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \widetilde{H}^i_f(F, V) & \to & H^i(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S} H^i_{/f}(F_\mathfrak{p}, V) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \widetilde{H}^i_f(F, V'') & \to & H^i(U_S, V'') & \to & \bigoplus_{\mathfrak{p}\in S} H^i_{/f}(F_\mathfrak{p}, V'') & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \cdots & & \cdots & & \cdots & & \end{array}$$

5.2. Comportement par induction.

5.2.1. Soient F' une extension finie de F contenue dans $\overline{F}$, V' une représentation ℓ-adique de $G_{F'}$ et $V = \mathrm{Ind}_{F'/F} V'$ la représentation de G_F induite par V. Soit $S_{F'}$ l'ensemble des places de F' qui sont au dessus d'une place de S. On suppose que V est non ramifiée en dehors de S, ce qui implique que V' est non ramifiée en dehors de $S_{F'}$.

Avec des conventions évidentes, on a des isomorphismes canoniques

$$H^i(U_S, V) \simeq H^i(U_{S_{F'}}, V') \quad \text{et} \quad H^i(F_\mathfrak{p}, V) \simeq \bigoplus_{\mathfrak{p}'|\mathfrak{p}} H^i(F'_{\mathfrak{p}'}, V'),$$

d'où un isomorphisme $\Delta_S(V) \simeq \Delta_{S_{F'}}(V')$. On a aussi des isomorphismes canoniques

$$H^i_f(F_\mathfrak{p}, V) \simeq \bigoplus_{\mathfrak{p}'|\mathfrak{p}} H^i_f(F'_{\mathfrak{p}'}, V') \quad \text{et, pour } \mathfrak{p} \in S_\ell(F), \quad t_{V,\mathfrak{p}} \simeq \bigoplus_{\mathfrak{p}'|\mathfrak{p}} t_{V',\mathfrak{p}'}.$$

Des premiers on déduit des isomorphismes $\widetilde{H}^i_f(F, V) \simeq \widetilde{H}^i_f(F', V')$ pour $i \in \{0, 1, 2, 3\}$. Joints aux derniers, ils fournissent un isomorphisme

$$\Delta_f(V') \simeq \Delta_f(V).$$

5.2.2. PROPOSITION. *Les isomorphismes* $\Delta_S(V) \simeq \Delta_{S_{F'}}(V')$ *et* $\Delta_f(V) \simeq \Delta_f(V')$ *définis ci-dessus sont des isomorphismes d'espaces vectoriels normés.*

PREUVE. Laissée au lecteur.

5.3. Groupes de Shafarevich–Tate et nombres de Tamagawa.

5.3.1. Dans tout le n°5.3, V est une représentation ℓ-adique pseudo-géométrique de $G_{F,S}$ et T est un réseau de V stable par $G_{F,S}$. On choisit une base ω de $\det_{\mathbb{Q}_\ell} t_V$.

Rappelons que si W est un $\mathbb{Z}_\ell$-module de type fini (resp. un $\mathbb{Q}_\ell$-espace vectoriel de dimension finie), on pose $W^* = \mathrm{Hom}_{\mathbb{Z}_\ell}(W, \mathbb{Z}_\ell)$ (resp. $W^* = \mathrm{Hom}_{\mathbb{Q}_\ell}(W, \mathbb{Q}_\ell)$). Si W est un $\mathbb{Z}_\ell$-module, on note $W^\frown = \mathrm{Hom}_{\mathbb{Z}_\ell}(W, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ son dual de Pontryagin.

On voit que $T^\frown(1)$ s'identifie à $V^*(1)/T^*(1)$ et $T^*(1)^\frown(1)$ à V/T.

5.3.2. Posons

$$\Delta'_f(V) = \left(\bigotimes_{0\le i\le 3} (\det_{\mathbb{Q}_\ell} \widetilde{H}^i_f(F, V))^{(-1)^i}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty(F)} (\det_{\mathbb{Q}_\ell} H^0(F_\mathfrak{p}, V))^{-1}\right),$$

de sorte que $\Delta_f(V) = \Delta'_f(V) \otimes \det_{\mathbb{Q}_\ell} t_V$.

Posons $H^0_f(F, T) = H^0(F, T)$, notons $H^1_f(F, T)$ l'image inverse de $H^1_f(F, V)$ dans $H^1(F, T)$ et définissons de la même manière $H^i_f(F, T^*(1))$ pour $i = 0, 1$.

On obtient un réseau $\Delta'_f(T)$ de $\Delta'_f(V)$ en posant

$$\Delta'_f(T) = \left(\bigotimes_{0\le i\le 1} (\det_{\mathbb{Z}_\ell} H^i_f(F, T))^{(-1)^i}\right) \otimes \left(\bigotimes_{0\le i\le 1} (\det_{\mathbb{Z}_\ell} H^i_f(F, T^*(1)))^{(-1)^i}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty} \det_{\mathbb{Z}_\ell} H^0(F_\mathfrak{p}, T)\right)^{-1}.$$

Si l'on choisit une base ω'_T de $\Delta'_f(T)$, $\omega'_T \otimes \omega$ est une base de $\Delta_f(V)$ et l'objet essentiel de ce paragraphe est de calculer $|\omega'_T \otimes \omega|_{\mathrm{EP},V}$ (qui ne dépend que de T et du réseau de $\det_{\mathbb{Q}_\ell} t_V$ engendré par ω, mais pas du choix des bases ω'_T et ω). Ce calcul fait intervenir le *nombre de Tamagawa* et le *groupe de Shafarevich–Tate de* T, que nous allons définir maintenant.

5.3.3. Choisissons, pour chaque $\mathfrak{p} \in S_\ell(F)$, une base $\omega_\mathfrak{p}$ de $\det_{\mathbb{Q}_\ell} t_{V,\mathfrak{p}}$ de manière que, "au signe près", $\omega = \bigotimes_{\mathfrak{p}|\ell} \omega_\mathfrak{p}$. Pour toute place finie $\mathfrak{p}$ de

F, on pose $\mathrm{Tam}^0_{\mathfrak{p}}(T) = \mathrm{Tam}^0_{F_{\mathfrak{p}}}(T)$ (resp. $\mathrm{Tam}^0_{\mathfrak{p},\omega_{\mathfrak{p}}}(T) = \mathrm{Tam}^0_{F_{\mathfrak{p}},\omega_{\mathfrak{p}}}(T)$) si $\mathfrak{p} \notin S_\ell(F)$ (resp. $\mathfrak{p} \in S_\ell(F)$) (cf. I, 4.1.2). Si $\mathfrak{p}$ est une place infinie, on pose

$$\mathrm{Tam}^0_{\mathfrak{p}}(T) = \#(H^1(F_{\mathfrak{p}}, T)).$$

On pose

$$\mathrm{Tam}^0_{\omega}(T) = \prod_{\mathfrak{p}\in S-S_\ell} \mathrm{Tam}^0_{\mathfrak{p}}(T) \times \prod_{\mathfrak{p}\in S_\ell} \mathrm{Tam}^0_{\mathfrak{p},\omega_{\mathfrak{p}}}(T).$$

C'est une puissance de ℓ (positive, négative, ou nulle) qui dépend de T et du réseau de $\det_{\mathbb{Q}_\ell} t_V$ engendré par ω mais qui ne dépend pas des autres choix. Comme $\mathrm{Tam}^0_{\mathfrak{p}}(T) = 1$ si $\mathfrak{p} \notin S$, $\mathrm{Tam}^0_{\omega}(T)$ ne change pas non plus si l'on remplace S par un ensemble fini de places plus grand. Nous l'appelons *le nombre de Tamagawa de T relatif à ω*.

5.3.4. Soit $\mathfrak{p}$ une place de F. Si $\mathfrak{p} \in S_\infty(F)$, on pose $H^1_f(F_{\mathfrak{p}}, V) = 0$; dans tous les cas, on note $H^1_f(F_{\mathfrak{p}}, V/T)$ l'image de $H^1_f(F_{\mathfrak{p}}, V)$ dans $H^1(F_{\mathfrak{p}}, V/T)$. Appelons *groupe de Selmer de T* le sous-groupe $H^1_f(F, V/T)$ de $H^1(F, V/T)$ formé des x tels que, pour toute place $\mathfrak{p}$ de F, l'image de x dans $H^1(F_{\mathfrak{p}}, V/T)$ appartient à $H^1_f(F_{\mathfrak{p}}, V/T)$. On définit alors le *groupe de Shafarevich–Tate* $Ш(T)$ de T comme le quotient du groupe de Selmer par l'image de $H^1_f(F, V)$. Par exemple, si T est le module de Tate d'une variété abélienne X sur F, $Ш(T)$ est le quotient de la partie ℓ-primaire du groupe de Shafarevich–Tate de X par sa partie ℓ-divisible.

5.3.5. PROPOSITION. *Le groupe $Ш(T)$ est fini.*

PREUVE. Remarquons d'abord que, comme $H^1_{/f}(F_{\mathfrak{p}}, T)$ est un $\mathbb{Z}_\ell$-module sans torsion, on a $\mathbb{Q}_\ell/\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} H^1_f(F_{\mathfrak{p}}, T) = H^1_f(F_{\mathfrak{p}}, V/T)$. D'autre part, $H^1_f(F, V/T)$ est contenu dans $H^1(U_S, V/T)$: en effet, $H^1(U_S, V/T)$ est l'ensemble des éléments de $H^1(F, V/T)$ dont l'image dans $H^1(F_{\mathfrak{p}}, V/T)$ pour $\mathfrak{p}$ n'appartenant pas à S appartient à $H^1(k_{\mathfrak{p}}, V/T)$ où $k_{\mathfrak{p}}$ désigne le corps résiduel de $F_{\mathfrak{p}}$ et pour un tel $\mathfrak{p}$,

$$\begin{aligned} H^1(k_{\mathfrak{p}}, V/T) &= \mathbb{Q}_\ell/\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} H^1(k_{\mathfrak{p}}, T) \\ &= \mathbb{Q}_\ell/\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} H^1_f(F_{\mathfrak{p}}, T) = H^1_f(F_{\mathfrak{p}}, V/T). \end{aligned}$$

Ainsi, $H^1_f(F, V/T)$ peut être défini comme l'ensemble des éléments x de $H^1(U_S, V/T)$ tels que, pour toute place $\mathfrak{p}$ de S, l'image de x dans $H^1(F_{\mathfrak{p}}, V/T)$ appartient à $H^1_f(F_{\mathfrak{p}}, V/T)$.

Le quotient de $H^1(U_S, V/T)$ par l'image de $H^1(U_S, V)$ est fini car il s'identifie au sous-groupe de torsion du $\mathbb{Z}_\ell$-module de type fini $H^2(U_S, T)$.

On a d'autre part le diagramme commutatif

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & H^1(U_S, T) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, T) \\ & & & & \downarrow & & \downarrow \\ 0 \to & H^1_f(F, V) & \to & & H^1(U_S, V) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, V) \\ & & & & \downarrow & & \downarrow \\ 0 \to & H^1_f(F, V/T) & \to & & H^1(U_S, V/T) & \to & \bigoplus_{\mathfrak{p}\in S_f} H^1(F_\mathfrak{p}, V/T)/H^1_f(F_\mathfrak{p}, V/T) \end{array}$$

dont les lignes et les colonnes sont exactes (l'injectivité de $H^1_{/f}(F_\mathfrak{p}, T)$ dans $H^1_{/f}(F_\mathfrak{p}, V)$ vient de ce que $H^1_f(F_\mathfrak{p}, T)$ est défini comme l'image réciproque de $H^1_f(F_\mathfrak{p}, V)$ dans $H^1(F_\mathfrak{p}, T)$). On en déduit par le lemme du serpent une injection

$$(H^1_f(F, V/T) \cap \operatorname{Im} H^1(U_S, V))/\operatorname{Im} H^1_f(F, V) \\ \to \bigoplus_{\mathfrak{p}\in S_f} H^1_{/f}(F_\mathfrak{p}, T)/\operatorname{Im} H^1(U_S, T).$$

Le second module étant de type fini, le premier qui est de torsion est fini. La proposition s'en déduit.

5.3.6. THÉORÈME. *Avec les hypothèses et les notations qui précèdent, on a*

$$(|\omega'_T \otimes \omega|_{\mathrm{EP}, V})^{-1} = \mathrm{Tam}^0_\omega(T) \cdot \# Ш(T^*(1)).$$

En vue de *prouver* ce théorème, définissons les $\mathbb{Z}_\ell$-modules $\widetilde{H}^i_f(F, T)$, pour $i \in \mathbb{Z}$, en posant

$$\widetilde{H}^0_f(F, T) = H^0(F, T), \qquad \widetilde{H}^1_f(F, T) = H^1_f(F, T),$$
$$\widetilde{H}^2_f(F, T) = H^1_f(F, (V^*/T^*)(1))^{\wedge},$$
$$\widetilde{H}^3_f(F, T) = H^0(F, (V^*/T^*)(1))^{\wedge} \quad \text{et} \quad \widetilde{H}^i_f(F, T) = 0 \quad \text{si } i \neq 0, 1, 2, 3.$$

Si $V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T$, on voit que, pour tout i, $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \widetilde{H}^i_f(F, T)$ s'identifie à $\widetilde{H}^i_f(F, V)$ ce qui fait que

$$\widetilde{\Delta}'_f(T) = \left(\bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{Z}_\ell} \widetilde{H}^i_f(F, T))^{(-1)^i}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty(F)} \det_{\mathbb{Z}_\ell} H^0(F_\mathfrak{p}, T)\right)^{-1}$$

est encore un réseau de $\Delta'_f(V)$.

5.3.7. Si $\mathfrak{p} \in S_\infty(F)$, on pose $H^1_f(F_\mathfrak{p}, T) = H^1(F_\mathfrak{p}, T)$. Pour toute place $\mathfrak{p}$ de F, on pose toujours $H^0_f(F_\mathfrak{p}, T) = H^0(F_\mathfrak{p}, T)$ et $H^i_f(F_\mathfrak{p}, T) = 0$ pour $i \neq 0, 1$; pour tout $i \in \mathbb{Z}$, $H^i_{/f}(F_\mathfrak{p}, T) = H^i(F_\mathfrak{p}, T)/H^i_f(F_\mathfrak{p}, T)$.

PROPOSITION. *Il existe une suite exacte*

$$(s_{f,S}(T)) \qquad \cdots \to \widetilde{H}^i_f(F, T) \to H^i(U_S, T) \to \bigoplus_{\mathfrak{p}\in S} H^i_{/f}(F_\mathfrak{p}, T) \to \cdots$$

induisant, par extension des scalaires de $\mathbb{Z}_\ell$ *à* $\mathbb{Q}_\ell$, *la suite exacte* $s_{f,S}(V)$.

PREUVE. La construction des flèches et la démonstration de cette proposition est très similaire à celle de la proposition 2.2.1 et nous en laissons les détails au lecteur (utiliser le fait que $H^1_{/f}(F_\mathfrak{p}, T) = H^1_f(F_\mathfrak{p}, T^\frown(1))^\frown$ et la suite exacte de Poitou–Tate

$$\cdots \to H^i(U_S, T) \to \prod_{\mathfrak{p}\in S} \widehat{H}^i(F_\mathfrak{p}, T) \to H^{2-i}(U_S, T^\frown(1))^\frown \to \cdots$$

(cf. [**Mi,** I, Theorem 4.10]; ici $\widehat{H}^i(F_\mathfrak{p}, T) = H^i(F_\mathfrak{p}, T)$, sauf si $i = 0$ et $\mathfrak{p} \in S_\infty$, auquel cas $\widehat{H}^0(F_\mathfrak{p}, T) = 0$ si $\mathfrak{p}$ est complexe et $\widehat{H}^0(\mathbb{R}, T) = T^\tau/(1+\tau)T$ si τ est la conjugaison complexe).

5.3.8. Le théorème 5.3.6 résulte alors de la proposition suivante:

PROPOSITION. *Si* $\tilde{\omega}'_T$ *est une base de* $\widetilde{\Delta}'_f(T)$, *on a*

i)

$$(|\tilde{\omega}'_T \otimes \omega|_{\mathrm{EP}, V})^{-1} = \mathrm{Tam}^0_\omega(T);$$

ii)

$$\Delta'_f(T) = \#\text{Ш}(T^*(1)) \cdot \widetilde{\Delta}'_f(T).$$

PREUVE. L'isomorphisme de $\mathbb{Q}_\ell$-espaces vectoriels $i_S(V) : \Delta_f(V) \simeq \Delta_S(V)$ se factorise de la manière suivante

$$\Delta_f(V) = \widetilde{\Delta}'_f(V) \otimes \bigotimes_{\mathfrak{p}\in S_f} \det_{\mathbb{Q}_\ell} t_{V,\mathfrak{p}} \simeq \widetilde{\Delta}'_f(V) \otimes \bigotimes_{\mathfrak{p}\in S_f} L_f(F_\mathfrak{p}, V)^{-1} \simeq \Delta_S(V).$$

La suite exacte $s_{f,S}(T)$ implique que le deuxième isomorphisme provient d'un isomorphisme de $\mathbb{Z}_\ell$-modules

$$\widetilde{\Delta}'_f(T) \otimes \left(\bigotimes_{\mathfrak{p}\in S_f} L_f(F_\mathfrak{p}, T)^{-1}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty} H^1(F_\mathfrak{p}, T)\right) \simeq \Delta_S(T).$$

Par définition de $\mathrm{Tam}^0_\omega(T)$, i) s'en déduit. Pour montrer ii), on remarque que l'exactitude des deux suites suivantes

$$0 \to (H^1_f(F, T^*(1))_{\mathrm{tor}})^\frown \to \widetilde{H}^3_f(F, T) \to H^0(F, T^*(1))^* \to 0$$

(son dual de Pontryagin est

$$0 \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} H^0(F, T^*(1)) \to H^0(F, V^*/T^*(1)) \to H^1_f(F, T^*(1))_{\mathrm{tor}} \to 0);$$

$$0 \to \text{Ш}(T^*(1))^\frown \to \widetilde{H}^2_f(F, T) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(H^1_f(F, T^*(1)), \mathbb{Z}_\ell) \to 0$$

(son dual de Pontryagin est

$$0 \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} H^1_f(F, T^*(1)) \to H^1_f(F, (V^*/T^*)(1)) \to Ш(T^*(1)) \to 0).$$

On en déduit que

$$\begin{aligned}\widetilde{\Delta}'_f(T) &\simeq (\# Ш(T^*(1)))^{-1} \cdot \left(\bigotimes_{0\le i\le 1} (\det_{\mathbb{Z}_\ell} H^i_f(F, T))^{(-1)^i}\right) \\ &\quad \otimes \left(\bigotimes_{0\le i\le 1} (\det_{\mathbb{Z}_\ell} H^i_f(F, T^*(1)))^{(-1)^i}\right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_\infty(F)} \det_{\mathbb{Z}_\ell} H^0(F_\mathfrak{p}, T)\right)^{-1} \\ &= (\# Ш(T^*(1)))^{-1} \Delta'_f(T).\end{aligned}$$

5.4. La dualité entre $Ш(T)$ et $Ш(T^*(1))$. On conserve les hypothèses et notations du numéro précédent.

5.4.1. On se propose de construire une application $\delta_T \colon H^1_f(F, V/T) \to H^1_f(F, (V^*/T^*)(1))^\wedge$. Soient $x \in H^1_f(F, V/T)$ et $y \in H^1_f(F, (V^*/T^*)(1))$. Choisissons $\alpha \in Z^1(U_S, V/T)$ et $\beta \in Z^1(U_S, (V^*/T^*)(1))$ représentant x et y respectivement. Choisissons $\alpha_1 \in C^1(U_S, V)$ relevant α. Alors $d\alpha_1$ appartient à $Z^2(U_S, T)$ (sa classe dans $H^2(U_S, V/T)$ est l'image de x par l'homomorphisme de connexion). Notons $d\alpha_1 \cup \beta$ l'image du cup-produit de $d\alpha_1$ et de β dans $Z^3(U_S, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(1))$. Le groupe $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(1)$ se plonge dans le groupe $\mathscr{E}_S$ des S-unités de F_S. Comme $H^3(U_S, \mathscr{E}_S) = 0$ **[Mi86,** I, Corollary 4.18], il existe $\varepsilon \in C^2(U_S, \mathscr{E}_S)$ tel que $d\alpha_1 \cup \beta = d\varepsilon$. Pour tout $\mathfrak{p} \in S$, soit $\alpha_{\mathfrak{p},1}$ un relèvement dans $Z^1_f(F_\mathfrak{p}, V)$ de la restriction $\alpha_\mathfrak{p}$ de α à $Z^1(F_\mathfrak{p}, V/T)$ (cela existe puisque l'image $x_\mathfrak{p}$ de x dans $H^1(F_\mathfrak{p}, V/T)$ appartient à l'image de $H^1_f(F_\mathfrak{p}, V)$. Si $\alpha_{1,\mathfrak{p}}$ désigne la restriction de α_1 à $C^1(F_\mathfrak{p}, V)$, $\alpha_{\mathfrak{p},1} - \alpha_{1,\mathfrak{p}}$ appartient à $Z^1(F_\mathfrak{p}, T)$. On vérifie que, avec des notations évidentes, $(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_\mathfrak{p} - \varepsilon_\mathfrak{p} \in Z^2(F_\mathfrak{p}, \overline{F}^*_\mathfrak{p})$. En utilisant l'isomorphisme canonique $\mathrm{inv}_\mathfrak{p} \colon H^2(F_\mathfrak{p}, \overline{F}^*_\mathfrak{p}) \to \mathbb{Q}/\mathbb{Z}$, on définit ainsi un élément de $\mathbb{Q}_\ell/\mathbb{Z}_\ell$

$$\sum_{\mathfrak{p}\in S} \mathrm{inv}_\mathfrak{p}[(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_\mathfrak{p} - \varepsilon_\mathfrak{p}]$$

qui ne dépend pas des choix faits. L'homomorphisme

$$y \mapsto \sum_{\mathfrak{p}\in S} \mathrm{inv}_\mathfrak{p}[(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_\mathfrak{p} - \varepsilon_\mathfrak{p}]$$

définit donc un élément de $H^1_f(F, (V^*/T^*)(1))^\wedge$ que l'on note $\delta_T(x)$.

5.4.2. **Proposition [Fl90].** *Soient V une représentation ℓ-adique pseudo-géométrique et T un $\mathbb{Z}_\ell$-réseau de V. La suite*

$$H^1_f(F, T) \to H^1_f(F, V) \to H^1_f(F, V/T) \xrightarrow{\delta_T} H^1_f(F, (V^*/T^*)(1))^\wedge \\ \to H^1_f(F, V^*(1))^* \to H^1_f(F, T^*(1))^*$$

est exacte. L'homomorphisme δ_T induit un isomorphisme entre $\text{Ш}(T)$ et $\text{Ш}(T^*(1))^\wedge$, c'est-à-dire une dualité parfaite

$$C_T\colon \text{Ш}(T) \times \text{Ш}(T^*(1)) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell .$$

En particulier, $\text{Ш}(T)$ et $\text{Ш}(T^*(1))$ ont même cardinal.

Preuve. La deuxième affirmation résulte de la première. Prouvons celle-ci: On montre comme en 1.4.6 et 2.2.3 que l'image de $H^1_f(F, V)$ dans $H^1_f(F, V/T)$ est égale au noyau de δ_T et que l'image de $H^1_f(F, V/T)$ est contenue dans le noyau de $H^1_f(F, (V^*/T^*)(1))^\wedge \to H^1_f(F, V^*(1))^*$. On en déduit un homomorphisme injectif de $\text{Ш}(T)$ dans $\text{Ш}(T^*(1))$. En échangeant le rôle de T et de $T^*(1)$, on en déduit que $\text{Ш}(T)$ et $\text{Ш}(T^*(1))$ ont même cardinal et donc que l'application de $\text{Ш}(T)$ dans $\text{Ш}(T^*(1))$ induite par δ_T est un isomorphisme.

5.4.3. La forme bilinéaire C_T est une généralisation de la forme de Cassels –Tate pour les variétés abéliennes **[Ca62]**.

Proposition. *Si $x \in \text{Ш}(T)$ et $y \in \text{Ш}(T^*(1))$, on a*

$$C_T(x, y) = C_{T^*(1)}(y, x).$$

Preuve. On reprend les notations de 5.4.1 et on définit $\beta_{\mathfrak{p},1}$ et $\beta_{1,\mathfrak{p}}$ comme on pense. Les classes dans $H^2(F_\mathfrak{p}, \overline{F}^*_\mathfrak{p})$ de

$$(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta \quad \text{et} \quad (\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_{\mathfrak{p},1}$$

sont égales. On remarque ensuite que, comme $\alpha_{\mathfrak{p},1}$ (resp. $\beta_{\mathfrak{p},1}$) est un élément de $Z^1_f(F_\mathfrak{p}, V)$ (resp. $Z^1_f(F_\mathfrak{p}, V^*(1))$, le cup-produit de $\alpha_{\mathfrak{p},1}$ et de $\beta_{\mathfrak{p},1}$ est un cobord. On peut donc écrire

$$(\alpha_{1,\mathfrak{p}} - \alpha_{\mathfrak{p},1}) \cup \beta_{\mathfrak{p},1} - \varepsilon_\mathfrak{p} \equiv \alpha_{1,\mathfrak{p}} \cup \beta_{\mathfrak{p},1} - \varepsilon_\mathfrak{p} \quad \text{dans } H^2(F_\mathfrak{p}, \overline{F}^*_\mathfrak{p}).$$

On a d'autre part

$$d(\alpha_1 \cup \beta_1) = d\alpha_1 \cup \beta_1 - \alpha_1 \cup d\beta_1 = d\alpha_1 \cup \beta_1 - d\beta_1 \cup \alpha_1 .$$

On a donc $d\beta_1 \cup \alpha = d\beta_1 \cup \alpha_1 = d(\alpha_1 \cup \beta_1 + \varepsilon)$. Il est facile d'en déduire la proposition comme en 1.4.5.

CHAPITRE III

STRUCTURES MOTIVIQUES

1. Structures de Hodge mixtes sur $\mathbb{R}$ et sur $\mathbb{C}$

Il ne s'agit pas de notions nouvelles: une structure de Hodge mixte (sous-entendu réelle) sur $\mathbb{C}$ est ce que Deligne appelle une structure de Hodge mixte réelle **[De71]**; pour obtenir une structure sur $\mathbb{R}$, il faut rajouter une action de $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

1.1. Structures de Hodge mixte sur $\mathbb{C}$.

1.1.1. DÉFINITION. Une structure de Hodge mixte (sous-entendu réelle) (V, W, Fil) sur $\mathbb{C}$ consiste en

a) un $\mathbb{R}$-espace vectoriel de dimension finie V appelé *réalisation tautologique* ou $\mathbb{R}$-*réalisation Betti*; le $\mathbb{C}$-espace vectoriel $\mathbb{C} \otimes_{\mathbb{R}} V$ est noté indifféremment $D_{\mathrm{dR}}(V)$, V_{dR} ou $V_{\mathbb{C}}$ et est appelé *réalisation de Rham*;
b) une filtration finie croissante $(W_n V)_{n\in\mathbb{Z}}$ par des sous-$\mathbb{R}$-espaces vectoriels de V, appelée *filtration par le poids*;
c) une filtration finie décroissante $(\mathrm{Fil}^i V_{\mathrm{dR}})_{i\in\mathbb{Z}}$ par des sous-$\mathbb{C}$-espaces vectoriels de V_{dR}, appelée *filtration de Hodge*;

On exige que, sur $V_{\mathbb{C}}$, $(W, \mathrm{Fil}, \overline{\mathrm{Fil}})$ forment un système de trois filtrations opposées.

[Rappelons ce que cela signifie: Si f désigne l'automorphisme $\mathbb{R}$-linéaire de $V_{\mathbb{C}}$ qui est la conjugaison complexe sur $\mathbb{C}$ et l'identité sur V, la *filtration conjuguée* est la filtration $\overline{\mathrm{Fil}}$ de $V_{\mathbb{C}}$ définie par $\overline{\mathrm{Fil}}^i V_{\mathbb{C}} = f(\mathrm{Fil}^i V_{\mathbb{C}})$. Si l'on note encore W la filtration sur $V_{\mathbb{C}}$ déduite de W par extension des scalaires, on demande que pour tout $m \in \mathbb{Z}$, les filtrations induites par Fil et $\overline{\mathrm{Fil}}$ sur $\mathrm{gr}^m V_{\mathbb{C}}$ soient *m-opposées*, i.e., que, si l'on note encore Fil et $\overline{\mathrm{Fil}}$ ces filtrations induites, alors, pour tout $i \in \mathbb{Z}$, l'application naturelle

$$\mathrm{Fil}^i \mathrm{gr}^m V_{\mathbb{C}} \oplus \overline{\mathrm{Fil}}^{m+1-i} \mathrm{gr}^m V_{\mathbb{C}} \to \mathrm{gr}^m V_{\mathbb{C}}$$

soit un isomorphisme].

Dans la suite, on parlera abusivement de la structure de Hodge mixte V, la filtration par le poids et la filtration de Hodge étant sous-entendues.

1.1.2. Si V est une structure de Hodge mixte sur $\mathbb{C}$, on appelle *espace tangent* de V le $\mathbb{C}$-espace vectoriel $t_V = \mathbf{t}_V(\mathbb{C}) = V_{\mathrm{dR}}/\mathrm{Fil}^0 V_{\mathrm{dR}}$. On identifie V de manière naturelle à un sous-$\mathbb{R}$-espace vectoriel de $V_{\mathbb{C}}$ et on note α_V l'application composée

$$\alpha_V : V \hookrightarrow V_{\mathbb{C}} \to t_V .$$

1.1.3. *Un morphisme* de structures de Hodge mixtes sur $\mathbb{C}$ est un morphisme des $\mathbb{R}$-espaces vectoriels sous-jacents, compatible aux filtrations W et Fil. Il est bien connu (cf. **[De71,** n°2.3]) que tout morphisme de la catégorie $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$ des structures de Hodge mixtes sur $\mathbb{C}$ est strictement compatible aux filtrations. En particulier, si

$$0 \to V' \to V \to V'' \to 0$$

est une suite exacte de structures de Hodge mixtes sur $\mathbb{C}$, la suite

$$0 \to t_{V'} \to t_V \to t_{V''} \to 0$$

est exacte.

La catégorie $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$ est de manière évidente une catégorie tannakienne neutre sur $\mathbb{R}$, dont l'objet-unité, noté $\mathbf{1}_{\mathbb{C}}$ ou $\mathbf{1}$ s'il n'y a pas de risque de confusion, s'identifie à $\mathbb{R}$ (avec $W_{-1}\mathbb{R} = 0$, $W_0\mathbb{R} = \mathbb{R}$, $\mathrm{Fil}^0\mathbb{R} = \mathbb{R}$, $\mathrm{Fil}^1\mathbb{R} = 0$).

1.1.4. On dit qu'une structure de Hodge mixte V sur $\mathbb{C}$, non réduite à 0, est une structure de Hodge *pure* s'il existe un entier $m \in \mathbb{Z}$, appelé *le poids de* V, tel que $\mathrm{gr}_n^W(V) = 0$ pour $n \neq m$.

La filtration par le poids définit en fait une filtration, que l'on note encore W, de tout objet V de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$ par des sous-objets. Pour tout $n \in \mathbb{Z}$, $\mathrm{gr}_n^W V$ est pur de poids n.

Un objet V de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$ est semi-simple si et seulement si c'est une somme directe de structures de Hodge pures; en particulier, pour tout V, $\bigoplus_{n\in\mathbb{Z}} \mathrm{gr}_n^W V$ est semi-simple.

Enfin on définit la *structure de Hodge de Tate sur* $\mathbb{C}$, notée $\mathbf{1}_{\mathbb{C}}(1)$ ou $\mathbf{1}(1)$ comme l'unique structure de Hodge pure de poids -2 portée par le sous-$\mathbb{R}$-espace vectoriel $\mathbb{R}(1)$ de $\mathbb{C}$ engendré par $2\pi i$. Si $r \in \mathbb{N}$, on pose aussi $\mathbf{1}_{\mathbb{C}}(r) = \mathbf{1}(r) = \mathrm{Sym}^r\mathbf{1}(1)$ et $\mathbf{1}_{\mathbb{C}}(-r) = \mathbf{1}(-r) = \mathbf{1}(r)^*$; si V est un objet de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$ et si $r \in \mathbb{Z}$, on pose $V(r) = V \otimes \mathbf{1}(r)$.

Soit V un objet simple de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$. Alors il est pur; si n est son poids, ou bien n est pair, $\dim_{\mathbb{R}} V = 1$ et $V \simeq \mathbf{1}(-n/2)$, ou bien $\dim_{\mathbb{R}} V = 2$ et il existe des entiers $p < q$ vérifiant $p + q = n$ et une base $\{e_1, e_2\}$ de V telle que

$$\mathrm{Fil}^r V_{\mathbb{C}} = \begin{cases} \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \text{si } r \le p, \\ \mathbb{C}(e_1 \oplus ie_2) & \text{si } p < r \le q, \\ 0 & \text{si } r > q. \end{cases}$$

1.1.5. Pour toute structure de Hodge mixte V sur K, pour tout $i \in \mathbb{Z}$, on note $\gamma^i V$ le plus grand sous-$\mathbb{R}$-espace vectoriel de V contenu dans $\mathrm{Fil}^i V_{\mathbb{C}}$ et $\delta^i V$ le plus petit sous-$\mathbb{R}$-espace vectoriel de V tel que $\mathbb{C} \otimes \delta^i V$ contient $\mathrm{Fil}^i V_{\mathbb{C}}$. On obtient ainsi deux filtrations finies décroissantes $(\gamma^i V)_{i\in\mathbb{Z}}$ et $(\delta^i V)_{i\in\mathbb{Z}}$ de V ; on a

$$\gamma^i V = V \cap \mathrm{Fil}^i V_{\mathbb{C}} \cap \overline{\mathrm{Fil}}^i V_{\mathbb{C}} \quad \text{et} \quad \delta^i V = V \cap (\mathrm{Fil}^i V_{\mathbb{C}} + \overline{\mathrm{Fil}}^i V_{\mathbb{C}})$$

et les applications naturelles

$$\mathbb{C} \otimes_{\mathbb{R}} \gamma^i V \to \mathrm{Fil}^i V_{\mathbb{C}} \cap \overline{\mathrm{Fil}}^i V_{\mathbb{C}} \quad \text{et} \quad \mathbb{C} \otimes_{\mathbb{R}} \delta^i V \to \mathrm{Fil}^i V_{\mathbb{C}} + \overline{\mathrm{Fil}}^i V_{\mathbb{C}}$$

sont des isomorphismes. En outre, dans la dualité

$$V^*(1) \times V \to \mathbb{R}(1),$$

l'orthogonal de $\gamma^i V$ (resp. $\delta^i V$) est $\delta^{-i} V^*(1)$, i.e. $\delta^{-i}(V^*(1))$ (resp. $\gamma^{-i} V^*(1)$, i.e. $\gamma^{-i}(V^*(1))$).

Si $\eta: V_1 \to V_2$ est un morphisme de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$, l'application induite de V_1 dans V_2 est compatible aux filtrations γ et δ, mais elle n'est en général pas strictement compatible (i.e., par exemple, l'inclusion $\eta(\gamma^i V_1) \subset \gamma^i V_2$ peut être stricte).

1.1.6. Pour toute structure de Hodge mixte V sur $\mathbb{C}$, et tout $i \in \mathbb{Z}$, on pose

$$H^i(\mathbb{C}, V) = H^i_g(\mathbb{C}, V) = H^i(\mathbf{SH}_{\mathbb{C}}(\mathbb{R}), V) = \mathrm{Ext}^i_{\mathbf{SH}_{\mathbb{C}}(\mathbb{R})}(\mathbf{1}, V).$$

On pose $H^1_e(\mathbb{C}, V) = 0$ et on note $H^1_f(\mathbb{C}, V)$ le sous-ensemble de $H^1(\mathbb{C}, V)$ formé des éléments qui sont la classe d'une extension

$$0 \to V \to U \to \mathbf{1} \to 0$$

telle que la projection $U \to \mathbb{R}$ est strictement compatible à la filtration γ (i.e. telle que l'application $\gamma^0 U \to \mathbb{R}$ est surjective).

Proposition. *Soient V une structure de Hodge mixte sur $\mathbb{C}$, $V_0 = W_0 V$ et $\overline{V} = V/W_0 V$.*

i) *On a $H^i(\mathbb{C}, V) = 0$ si $i \geq 2$, $H^1_f(\mathbb{C}, V)$ est un sous-$\mathbb{R}$-espace vectoriel de $H^1(\mathbb{C}, V)$ et on a un diagramme commutatif de $\mathbb{R}$-espaces vectoriels, dont les lignes et les colonnes sont exactes*

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & H^1_f(\mathbb{C}, V) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathbb{C}, V) & \to & V_0 & \xrightarrow{\alpha_{V_0}} & t_{V_0} & \to & H^1(\mathbb{C}, V) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}\,\alpha_V & \to & V & \xrightarrow{\alpha_V} & t_V & \to & \mathrm{Coker}\,\alpha_V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}\,\alpha_{\overline{V}} & \to & \overline{V} & \xrightarrow{\alpha_{\overline{V}}} & t_{\overline{V}} & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & H^1_f(\mathbb{C}, V) & & 0 & & 0 & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

ii) *Si V est semi-simple, on a $H^1_f(\mathbb{C}, V) = 0$.*

iii) *La dualité* $V^*(1) \times V \to \mathbb{R}(1)$ *induit des dualités parfaites pour tout* $p \in \mathbb{Z}$

$$(V^*(1)/\delta^{-p}V^*(1)) \times \gamma^p V \to \mathbb{R}(1).$$

On a $\gamma^0 V = \operatorname{Ker}\alpha_V$, *l'application composée*

$$V(1) \subset V_{\mathbb{C}} = V_{\mathrm{dR}} \to t_V \to \operatorname{Coker}\alpha_V$$

est surjective et a pour noyau $(\delta^0 V)(1)$, *d'où des isomorphismes canoniques de* $\mathbb{R}$*-espaces vectoriels* $(V/\delta^0 V)(1) \simeq \operatorname{Coker}\alpha_V$ (*où* $(V/\delta^0 V)(1) = \mathbb{R}(1) \otimes (V/\delta^0 V)$), *et* $(\operatorname{Ker}\alpha_V)^* = \operatorname{Coker}\alpha_{V^*(1)}$.

iv) *On a* $\dim_{\mathbb{R}} \operatorname{Coker}\alpha_V = \dim_{\mathbb{R}} H^1_g(\mathbb{C}, V)/H^1_f(\mathbb{C}, V) = \dim_{\mathbb{R}} \operatorname{Ker}\alpha_{V^*(1)}$ *et* $\dim_{\mathbb{R}} \operatorname{Ker}\alpha_V = \dim_{\mathbb{R}} H^1_g(\mathbb{C}, V^*(1))/H^1_f(\mathbb{C}, V^*(1)) = \dim_{\mathbb{R}} \operatorname{Coker}\alpha_{V^*(1)}$.

PREUVE. i) Se donner $\eta \in H^0(\mathbb{C}, V) = \operatorname{Hom}(\mathbb{1}, V)$ revient à se donner l'image a de 1 dans V : inversement un tel $a \in V$ définit un morphisme si et seulement si $a \in W_0 V \cap \operatorname{Fil}^0 V_{\mathbb{C}} = V_0 \cap \operatorname{Fil}^0(V_0)_{\mathbb{C}} = \operatorname{Ker}\alpha_{V_0}$.

Soit U une extension de $\mathbb{1}$ par V ; on doit avoir une suite exacte de $\mathbb{R}$-espaces vectoriels

$$0 \to W_0 V \to W_0 U \to \mathbb{R} \to 0$$

et des suites exactes de $\mathbb{C}$-espaces vectoriels

$$0 \to W_0 V_{\mathbb{C}} \to W_0 U_{\mathbb{C}} \to \mathbb{C} \to 0$$

et

$$0 \to W_0 \operatorname{Fil}^0 V_{\mathbb{C}} \to W_0 \operatorname{Fil}^0 U_{\mathbb{C}} \to \mathbb{C} \to 0;$$

si ε désigne un relèvement de 1 dans $W_0 U$, il doit exister $\lambda \in W_0 V_{\mathbb{C}} = (V_0)_{\mathbb{C}}$ tel que $\varepsilon + \lambda \in W_0 \operatorname{Fil}^0 U_{\mathbb{C}}$. On voit que l'image de λ dans $\operatorname{Coker}\alpha_{V_0}$ ne dépend pas des choix faits et que l'on définit ainsi un isomorphisme entre $H^1(\mathbb{C}, V)$ et $\operatorname{Coker}\alpha_{V_0}$. Dire que l'image de λ est 0 dans $\operatorname{Coker}\alpha_V$ signifie que $\lambda \in \operatorname{Fil}^0 V_{\mathbb{C}} + V$, ce qui revient à dire qu'il existe un relèvement ε' de 1 dans U tel que $\varepsilon' \in \operatorname{Fil}^0 U_{\mathbb{C}} \cap U = \gamma^0 U$ ou encore que la classe de l'extension appartient à $H^1_f(\mathbb{C}, V)$.

L'exactitude de la deuxième ligne est triviale et celle de la troisième revient à prouver que $\operatorname{Coker}\alpha_{\overline{V}} = 0$ ou encore que $\operatorname{Coker}\alpha_V = 0$ si $W_0 V = 0$; si

$$0 \to V' \to V \to V'' \to 0$$

est une suite exacte courte de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$, la suite

$$\operatorname{Coker}\alpha_{V'} \to \operatorname{Coker}\alpha_V \to \operatorname{Coker}\alpha_{V''} \to 0$$

est exacte, ce qui permet par dévissage de se ramener au cas où V est un objet simple de poids ≥ 1 et c'est alors une vérification immédiate (cf. la liste des objets simples au n°1.1.4).

Le reste des assertions concernant le diagramme est alors clair. Le fait que $H^i(\mathbb{C}, V) = 0$ pour $i \geq 2$ résulte de ce que, pour tout épimorphisme $V_2 \to V_1$, l'application $H^1(\mathbb{C}, V_2) \to H^1(\mathbb{C}, V_1)$ est surjective.

Il suffit de prouver l'assertion (ii) lorsque V est simple. Dans ce cas, ou bien $W_0 V = V$, donc $H^1(\mathbb{C}, V) \to \operatorname{Coker} \alpha_V$ est un isomorphisme, donc $H^1_f(\mathbb{C}, V) = 0$; ou bien $W_0 V = 0$ et $H^1(\mathbb{C}, V) = 0$, donc a fortiori $H^1_f(\mathbb{C}, V)$.

Prouvons (iii): La première partie est immédiate. On a $\operatorname{Ker} \alpha_V = V \cap \operatorname{Fil}^0 V_{\mathbb{C}} = \gamma^0 V$. On a $V_{\mathrm{dR}} = V \oplus V(1)$, ce qui fait que l'inclusion de $V(1)$ dans V_{dR} induit bien, par passage au quotient, une application surjective $V(1) \to \operatorname{Coker} \alpha_V$. Si $v \in V$ dire que $v \otimes (2\pi i)$ appartient au noyau signifie qu'il existe $w \in V$ et $z \in \operatorname{Fil}^0 V_{\mathbb{C}}$ tels que $iv = w + z$; on a alors $-iv = w + \overline{z}$, d'où $2iv = z - \overline{z}$ et $v \in \delta^0 V$; réciproquement, si $v \in \delta^0 V$, il existe $z_1, z_2 \in \operatorname{Fil}^0 V_{\mathbb{C}}$ tels que $v = z_1 + \overline{z}_2 = \overline{z}_1 = z_2$; on a aussi $2iv = i(z_1 + z_2) + i(\overline{z}_1 + \overline{z}_2)$ et $iv = w + z$, avec $z = i(z_1 + z_2) \in \operatorname{Fil}^0 V_{\mathbb{C}}$ et $w = (z + \overline{z})/2 \in V$, et $v \otimes (2\pi i) \in \operatorname{Ker}(V(1) \to \operatorname{Coker} \alpha_V)$.

La dernière partie de (iii) et l'assertion (iv) sont maintenant immédiates.

1.1.7. Par analogie avec les représentations ℓ-adiques d'un corps de nombres, il est commode de poser, pour toute structure de Hodge mixte V sur $\mathbb{C}$, $H^0_e(\mathbb{C}, V) = H^0_f(\mathbb{C}, V) = H^0(\mathbb{C}, V)$, $H^i_e(\mathbb{C}, V) = 0$ si $i \neq 0$, $H^i_f(\mathbb{C}, V) = 0$ si $i \neq 0, 1$ et, pour tout $i \in \mathbb{Z}$,

$$H^i_{/f}(\mathbb{C}, V) = H^i_{g/f}(\mathbb{C}, V) = H^i_g(\mathbb{C}, V)/H^i_f(\mathbb{C}, V).$$

En particulier, $H^i_{/f}(\mathbb{C}, V) = 0$ si $i \neq 1$, tandis que le noyau et le conoyau de α_V se réinterprètent

$$\operatorname{Coker} \alpha_V = H^1_{/f}(\mathbb{C}, V) \quad \text{et} \quad \operatorname{Ker} \alpha_V = H^1_{/f}(\mathbb{C}, V^*(1))^*.$$

1.1.8. Rappelons **[De79]** que l'on pose

$$\Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s} \cdot \Gamma(s).$$

Si V est une structure de Hodge mixte sur $\mathbb{C}$, et si, pour tout $p \in \mathbb{Z}$, $n_p = \dim_{\mathbf{R}}(\gamma^p V / \gamma^{p+1} V)$, on pose

$$L(V, s) = \prod_{p \in \mathbf{Z}} \Gamma_{\mathbb{C}}(s - p)^{n_p}.$$

Lorsque V est pur, ou plus généralement semi-simple, on voit que, avec des conventions évidentes,

$$\gamma^i V = V \cap \left(\bigoplus_{p, q \geq i} H^{p,q} \right),$$

ce qui fait que $n_p = h^{p,p} + 2 \cdot \sum_{q > p} H^{p,q}$. On retrouve donc la définition usuelle **[Se70, De79]**.

1.1.9. **Proposition.** *Soit V une structure de Hodge mixte sur $\mathbb{C}$.*

i) *On a $L(V(1), s) = L(V, s+1)$.*

ii) *L'ordre du pôle en $s = 0$ de $L(V, s)$ est égal à* $\dim_{\mathbb{R}} \operatorname{Ker} \alpha_V = \dim_{\mathbb{R}} H^1_{g/f}(\mathbb{C}, V^*(1))$.

iii) *Soient*

$$0 \to V' \to V \to V'' \to 0$$

une suite exacte courte de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$,

$$\nu \in \operatorname{Ext}^1(V'', V') = H^1(\mathbb{C}, V''^* \otimes V')$$

la classe de cette extension, $L(\nu, s) = L(V, s)/L(V', s) \cdot L(V'', s)$. Alors, on a $L(\nu, s) = 1$ si et seulement si $\nu \in H^1_f(\mathbb{C}, V''^* \otimes V')$.

Preuve. (i) est clair. Montrons (ii). L'ordre du pôle en $s = 0$ est égal à $\sum_{p \geq 0} n_p = \dim_{\mathbb{R}} \gamma^0 V$; mais $\gamma^0 V = V \cap \operatorname{Fil}^0 V_{\mathbb{C}} = \operatorname{Ker} \alpha_V$ et la dimension de cet espace vectoriel est bien égale à celle de $H^1_{g/f}(\mathbb{C}, V^*(1))$. Montrons (iii). Pour tout $p \in \mathbb{Z}$, la suite

$$0 \to \gamma^p V' \to \gamma^p V \to \gamma^p V''$$

est exacte et il est immédiat que $L(\nu, s) = 1$ si et seulement si $\gamma^p V \to \gamma^p V''$ est surjective pour tout p. L'image de ν dans $H^1(\mathbb{C}, V''^* \otimes V')$ est représentée par une extension U de $\mathbb{R}$ par $V''^* \otimes V'$.

Montrons que la condition est suffisante. On a des diagrammes commutatifs

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V''^* \otimes V' & \longrightarrow & V''^* \otimes V & \longrightarrow & V''^* \otimes V'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V''^* \otimes V' & \longrightarrow & U & \longrightarrow & \mathbb{R} & \longrightarrow & 0 \end{array}$$

et, pour tout p,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \gamma^p(V''^* \otimes V') & \longrightarrow & \gamma^p(V''^* \otimes V) & \longrightarrow & \gamma^p(V''^* \otimes V'') \\ & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \gamma^p(V''^* \otimes V') & \longrightarrow & \gamma^p U & \longrightarrow & \gamma^p \mathbf{1}. \end{array}$$

Le morphisme $V''^* \otimes V'' \to \mathbb{R}$ est l'application trace et il est clair que $\gamma^0(V''^* \otimes V'') \to \mathbb{R}$ est surjectif. La surjectivité de $\gamma^p V \to \gamma^p V''$ pour tout p implique la surjectivité de $\gamma^0(V''^* \otimes V) \to \gamma^0(V''^* \otimes V'')$ et donc celle de $\gamma^0 U \to \mathbb{R}$. L'image de ν dans $H^1(\mathbb{C}, V''^* \otimes V')$ appartient donc à $H^1_f(\mathbb{C}, V''^* \otimes V')$.

Réciproquement, on a des diagrammes commutatifs

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V'' \otimes V''^* \otimes V' & \longrightarrow & V'' \otimes U & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \end{array}$$

et, pour tout p,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \gamma^p(V''\otimes V''^*\otimes V') & \longrightarrow & \gamma^p(V''\otimes U) & \longrightarrow & \gamma^p V'' \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \gamma^p V' & \longrightarrow & \gamma^p V & \longrightarrow & \gamma^p V''. \end{array}$$

La surjectivité de $\gamma^0 U \to \mathbb{R}$ implique la surjectivité de $\gamma^p(V''\otimes U)\to\gamma^p V''$ pour tout p et donc la surjectivité de $\gamma^p V \to \gamma^p V''$.

1.1.10. Pour toute structure de Hodge mixte V sur $\mathbb{C}$, on pose

$$\varepsilon_0(V) = (-1)^{\sum_{p\in\mathbb{Z}} p(h_p - n_p)}$$

où $h_p = \dim_{\mathbb{C}}(\mathrm{Fil}^p V_{\mathbb{C}}/\mathrm{Fil}^{p+1} V_{\mathbb{C}})$ et $n_p = \dim_{\mathbb{R}}(\gamma^p V/\gamma^{p+1} V)$.

Soit $\psi_0 : \mathbb{C} \to \mathbb{C}^*$ le caractère additif défini par $\psi_0(z) = \exp(2i\pi\cdot\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(z))$. Si $\mu = 2a\cdot dx\cdot dy$ avec $a \neq 0$ est une mesure de Haar sur $\mathbb{C}$, on pose

$$\varepsilon(V, \psi_0, \mu) = a^{\dim V}\cdot\varepsilon_0(V).$$

On vérifie facilement que, lorsque V est semi-simple, on retrouve la définition usuelle du facteur ε **[De73, Ta79]**.

1.2. Structures de Hodge mixtes sur $\mathbb{R}$. On note c la conjugaison complexe. Pour tout groupe abélien A sur lequel $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ opère, on pose

$$A^+ = \{a\in A \mid c(a) = a\} \quad \text{et} \quad A^- = \{a \in A \mid c(a) = -a\}.$$

1.2.1. DÉFINITION. Une structure de Hodge mixte (sous-entendu réelle) V sur $\mathbb{R}$ consiste en la donnée d'un couple formé d'une structure de Hodge mixte (réelle) sur $\mathbb{C}$ et d'une action de $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ sur le $\mathbb{R}$-espace vectoriel sous-jacent V compatible avec la filtration par le poids et telle que, si l'on étend l'action de $G_{\mathbb{R}}$ à $V_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}} V$ par semi-linéarité, la filtration de Hodge soit stable par cette action. On appelle *réalisation de Rham* de V le $\mathbb{R}$-espace vectoriel $V_{\mathrm{dR}} = D_{\mathrm{dR}}(V) = (V_{\mathbb{C}})^{G_{\mathbb{R}}}$.

Se donner une structure de Hodge mixte sur $\mathbb{R}$ revient donc à se donner

- un $\mathbb{R}$-espace vectoriel de dimension finie V, muni d'une filtration finie croissante $(W_i V)_{i\in\mathbb{Z}}$ et d'une involution notée c telle que $c(W_i V) = W_i V$ pour tout i;
- une filtration finie décroissante $(\mathrm{Fil}^i V_{\mathrm{dR}})_{i\in\mathbb{Z}}$ du $\mathbb{R}$-espace vectoriel $V_{\mathrm{dR}}(= D_{\mathrm{dR}}(V)) = (\mathbb{C}\otimes_{\mathbb{R}} V)^{c=1}$ (où $c(z\otimes v) = \bar{z}\otimes c(z))$, *la filtration de Hodge de la réalisation de Rham*, telle que V, muni de la filtration par le poids et de la filtration de $V_{\mathbb{C}}$ déduite de la filtration de Hodge par extension des scalaires, soit une structure de Hodge mixte sur $\mathbb{C}$.

On appelle *espace tangent* de la structure de Hodge mixte V sur $\mathbb{R}$ le $\mathbb{R}$-espace vectoriel $t_V = t_V(\mathbb{R}) = V_{\mathrm{dR}}/\mathrm{Fil}^0 V_{\mathrm{dR}}$, de sorte que l'espace tangent de la structure de Hodge mixte sur $\mathbb{C}$ sous-jacente à V est $t_V(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} t_V$.

Pour toute structure de Hodge mixte V sur $\mathbb{R}$, on a une décomposition canonique $V = V^+ \oplus V^-$; l'application α_V induit, en prenant la partie fixe par c, une application

$$\alpha_V^+ : V^+ \to t_V .$$

1.2.2. Les structures de Hodge mixtes sur $\mathbb{R}$ forment, de manière évidente, une catégorie tannakienne $\mathbb{R}$-neutre $\mathbf{SH}_{\mathbb{R}}(\mathbb{R})$; la filtration par le poids définit encore une filtration de tout objet V de $\mathbf{SH}_{\mathbb{R}}(\mathbb{R})$ par des sous-objets. On note $\mathbb{1}$ ou $\mathbb{1}_{\mathbb{R}}$ l'objet-unité. *La structure de Hodge de Tate* $\mathbb{1}(1)$ est de façon naturelle un objet de cette catégorie, que l'on note parfois $\mathbb{1}_{\mathbb{R}}(1)$ (la conjugaison complexe est la multiplication par -1 sur le $\mathbb{R}$-espace vectoriel sous-jacent). Pour tout objet V de $\mathbf{SH}_{\mathbb{R}}(\mathbb{R})$ et tout $r \in \mathbb{Z}$, on définit $V(r)$ comme on pense.

Le foncteur *extension des scalaires*: $\mathbf{SH}_{\mathbb{R}}(\mathbb{R}) \to \mathbf{SH}_{\mathbb{C}}(\mathbb{R})$, noté $V \mapsto V_{/\mathbb{C}}$, est l'oubli de la conjugaison complexe. C'est un $\otimes$-foncteur exact et fidèle, qui admet un adjoint à gauche, la *restriction des scalaires* $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} : \mathbf{SH}_{\mathbb{C}}(\mathbb{R}) \to \mathbf{SH}_{\mathbb{R}}(\mathbb{R})$ définie ainsi: soit $\mathbb{R}[c] = \mathbb{R}[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$; si V est un objet de $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$, le $\mathbb{R}$-espace vectoriel sous-jacent à $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(M)$ est $U = \mathbb{R}[c] \otimes_{\mathbb{R}} V$, avec action de c par multiplication à gauche et filtration par le poids déduite de celle de V par extension des scalaires de $\mathbb{R}$ à $\mathbb{R}[c]$; l'application $z \mapsto z \otimes 1 + \overline{z} \otimes c$ est un isomorphisme $\mathbb{R}$-linéaire de $\mathbb{C}$ sur $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[c])^c$ donc aussi de $V_{\mathrm{dR}} = \mathbb{C} \otimes_{\mathbb{R}} V$ sur $U_{\mathrm{dR}} = (\mathbb{C} \otimes_{\mathbb{R}} U)^c = (\mathbb{C} \otimes \mathbb{R}[c] \otimes V)^c$, ce qui nous permet d'identifier les deux réalisations de Rham; enfin on conserve la filtration de Hodge.

1.2.3. Pour toute structure de Hodge mixte V sur $\mathbb{R}$ et tout $i \in \mathbb{Z}$, on pose

$$H^i(\mathbb{R}, V) = H_g^i(\mathbb{R}, V) = H^i(\mathbf{SH}_{\mathbb{R}}(\mathbb{R}), V) = \mathrm{Ext}^i_{\mathbf{SH}_{\mathbb{R}}(\mathbb{R})}(\mathbb{1}, V).$$

Pour $i \in \mathbb{Z}$ et $\nu \in \{e, f\}$, on note $H_\nu^i(\mathbb{R}, V)$ l'image inverse de $H_\nu^i(\mathbb{C}, V)$ dans $H^i(\mathbb{R}, V)$. On définit encore $H^i_{/f}(\mathbb{R}, V) = H^i_{g/f}(\mathbb{R}, V) = H_g^i(\mathbb{R}, V)/H_f^i(\mathbb{R}, V)$.

Proposition. *Soient V une structure de Hodge mixte sur $\mathbb{R}$, $V_0 = W_0 V$ et $\overline{V} = V/W_0 V$.*

i) *Pour tout i, $H^i(\mathbb{R}, V) = H^i(\mathbb{C}, V)^c$ et le même résultat vaut pour les $H_j^1(\mathbb{R}, V)$, avec $j \in \{e, f, g\}$.*

ii) *On a $H^i(\mathbb{R}, V) = 0$ si $i \geq 2$, $H_e^1(\mathbb{R}, V) = 0$, $H^i_{/f}(\mathbb{R}, V) = 0$ si $i \neq 1$, $H^1_{/f}(\mathbb{R}, V) \simeq \mathrm{Coker}\, \alpha_V^+$ et on a un diagramme commutatif de $\mathbb{R}$-*

espaces vectoriels, dont les lignes et les colonnes sont exactes

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & H^1_f(\mathbb{R}, V) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathbb{R}, V) & \to & V_0^+ & \overset{\alpha^+_{V_0}}{\to} & t_{V_0}(\mathbb{R}) & \to & H^1(\mathbb{R}, V) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Ker}\alpha^+_V & \to & V^+ & \overset{\alpha^+_V}{\to} & t_V(\mathbb{R}) & \to & \operatorname{Coker}\alpha^+_V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Ker}\alpha^+_{\overline{V}} & \to & \overline{V} & \overset{\alpha^+_{\overline{V}}}{\to} & t_{\overline{V}}(\mathbb{R}) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & H^1_f(\mathbb{R}, V) & & 0 & & 0 & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

iii) *Si* V *est semi-simple, on a* $H^1_f(\mathbb{R}, V) = 0$.

iv) *Pour tout* $i \in \mathbb{Z}$, *les sous-$\mathbb{R}$-espaces vectoriels* $\gamma^i V$ *et* $\delta^i V$ *sont stables par* c; *la dualité* $V^*(1) \times V \to \mathbb{R}(1)$ *induit des dualités parfaites*

$$(V^*(1)/\delta^{-i}V^*(1))^+ \times (\gamma^i V)^- \to \mathbb{R}(1) \quad \text{et} \quad (V^*(1)/\delta^{-i}V^*(1))^- \times (\gamma^i V)^+ \to \mathbb{R}(1).$$

On a des isomorphismes canoniques de $\mathbb{R}$*-espaces vectoriels*

$$(\gamma^0 V)^+ \simeq \operatorname{Ker}\alpha^+_V \quad \textit{et} \quad (V/\delta^0 V)^-(1) \simeq \operatorname{Coker}\alpha^+_V\,;$$

en outre $\dim_{\mathbb{R}} \operatorname{Coker}\alpha^+_V = \dim_{\mathbb{R}} H^1_{/f}(\mathbb{R}, V) = \dim_{\mathbb{R}} \operatorname{Ker}\alpha^+_{V^*(1)}$ *et* $\dim_{\mathbb{R}} \operatorname{Ker}\alpha^+_V = \dim_{\mathbb{R}} H^1_{/f}(\mathbb{R}, V^*(1)) = \dim_{\mathbb{R}} \operatorname{Coker}\alpha^+_{V^*(1)}$.

PREUVE. Sur la catégorie des $\mathbb{R}$-espaces vectoriels munis d'une involution, le foncteur $V \mapsto V^+ = V^{c=1}$ est exact et les assertions résultent facilement de la proposition 1.1.6.

REMARQUE. On voit qu'ici encore $\operatorname{Ker}\alpha^+_V$ s'identifie au dual de $\operatorname{Coker}\alpha^+_{V^*(1)}$ donc aussi à $H^1_{/f}(\mathbb{R}, V^*(1))^*$.

1.2.4. Rappelons [De79] que l'on pose $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \cdot \Gamma(s/2)$.

Si V est une structure de Hodge mixte sur $\mathbb{R}$, on pose

$$n_p^+ = \dim_{\mathbb{R}}(\gamma^p V/\gamma^{p+1} V)^+, \; n_p^- = \dim_{\mathbb{R}}(\gamma^p V/\gamma^{p+1} V)^-,$$

pour tout $p \in \mathbb{Z}$ (on a $(\gamma^p V/\gamma^{p+1} V)^+ = (\gamma^p V)^+/(\gamma^{p+1} V)^+$ et $(\gamma^p V/\gamma^{p+1} V)^- = (\gamma^p V)^-/(\gamma^{p+1} V)^-$) et

$$L(V, s) = \prod_{p \in \mathbb{Z}} \Gamma_{\mathbb{R}}(s + \varepsilon_p - p)^{n_p^+} \cdot \Gamma_{\mathbb{R}}(s + 1 - \varepsilon_p - p)^{n_p^-},$$

où $\varepsilon_p \in \{0, 1\}$ est défini par $\varepsilon_p \equiv p \pmod 2$.

1.2.5. Proposition. *Soit V une structure de Hodge mixte sur $\mathbb{R}$.*

i) *Si V est semi-simple, la définition de $L(V, s)$ que l'on vient de donner coïncide avec la définition usuelle* **[Se70, De79]**;

ii) *on a $L(V(1), s) = L(V, s+1)$;*

iii) *l'ordre du pôle en $s = 0$ de $L(V, s)$ est égal à la dimension du $\mathbb{R}$-espace vectoriel $H^1_{g/f}(\mathbb{R}, V^*(1))$.*

iv) *si U est une structure de Hodge mixte sur $\mathbb{C}$, $L(\mathrm{Res}_{\mathbf{C}/\mathbf{R}} U, s) = L(U, s)$.*

v) *Soient*

$$0 \to V' \to V \to V'' \to 0$$

une suite exacte courte de $\mathbf{SH}_{\mathbf{R}}(\mathbb{R})$,

$$\nu \in \mathrm{Ext}^1(V'', V') = H^1(\mathbb{R}, V''^* \otimes V')$$

la classe de cette extension, $L(\nu, s) = L(V, s)/L(V', s) \cdot L(V'', s)$. Alors, on a $L(\nu, s) = 1$ si et seulement si $\nu \in H^1_f(\mathbb{R}, V''^* \otimes V')$.

Preuve. Il suffit de *vérifier* (i) lorsque V est simple. On est alors dans l'un des cas suivants:

- ou bien $V_{\mathbf{C}} \simeq \mathbf{1}_{\mathbf{C}}(-r)$, pour un $r \in \mathbb{Z}$ convenable, et $V = V^+$ (resp. V^-); tous les n_p^+ et les n_p^- sont nuls sauf n_r^+ (resp. n_r^-) qui vaut 1, donc $L(V, s) = \Gamma_{\mathbf{R}}(s+\varepsilon_r - r)$ (resp. $\Gamma_{\mathbf{R}}(s+1-\varepsilon_r - r)) = \Gamma_{\mathbf{R}}(s+\varepsilon - r)$, avec $\varepsilon \in \{0, 1\}$ si c agit par multiplication par $(-1)^{r+\varepsilon}$ sur V;
- ou bien V est "de type (r, s)", avec $r < s$; tous les n_p^+ et les n_p^- sont nuls sauf $n_r^+ = n_r^- = 1$, et

$$\begin{aligned} L(V, s) &= \Gamma_{\mathbf{R}}(s + \varepsilon_p - p) \cdot \Gamma_{\mathbf{R}}(s + 1 - \varepsilon_p - p) \\ &= \Gamma_{\mathbf{R}}(s - p) \cdot \Gamma_{\mathbf{R}}(s + 1 - p) = \Gamma_{\mathbf{C}}(s - p). \end{aligned}$$

L'*assertion* (ii) est immédiate. *Prouvons* (iii): l'ordre du pôle en $s = 0$ de $L(V, s)$ est $\sum_{p \geq 0} n_p^+ = \dim_{\mathbf{R}} \gamma^0 V^+ = \dim_{\mathbf{R}} H^1_g(\mathbb{R}, V^*(1))/H^1_f(\mathbb{R}, V^*(1))$, d'après la proposition 1.2.3.

Prouvons (iv): Si $n_p = \dim_{\mathbf{R}} \gamma^p U/\gamma^{p+1} U$, si $V = \mathrm{Res}_{\mathbf{C}/\mathbf{R}} U$, $n_p^+ = \dim_{\mathbf{R}}(\gamma^p V/\gamma^{p+1} V)^+$ et $n_p^- = \dim_{\mathbf{R}}(\gamma^p V/\gamma^{p+1} V)^-$, on a $n_p = n_p^+ = n_p^-$ et la formule résulte de ce que $\Gamma_{\mathbf{R}}(s-p) \cdot \Gamma_{\mathbf{R}}(s+1-p) = \Gamma_{\mathbf{C}}(s-p)$. Enfin, l'assertion v) se démontre comme dans la proposition 1.1.9.

1.2.6. Remarque. On ne change pas les "bonnes propriétés" des facteurs locaux $L(V, s)$ pour une structure de Hodge mixte V sur $\mathbb{R}$ ou sur $\mathbb{C}$ si l'on remplace $\Gamma_{\mathbf{R}}(s)$ par $a \cdot \Gamma_{\mathbf{R}}(s)$ et $\Gamma_{\mathbf{C}}(s)$ par $a^2 \cdot \Gamma_{\mathbf{C}}(s)$, où a est une constante non nulle arbitraire. Nous avons choisi ces normalisations afin de retrouver les facteurs locaux usuels dans le cas des structures de Hodge pures. Les travaux de Deninger **[Den91]** suggèrent que $\Gamma_{\mathbf{C}}(s) = (2\pi)^{-s} \cdot \Gamma(s)$ et $\Gamma_{\mathbf{R}}(s) = (2)^{-1/2} \cdot \pi^{-s/2} \cdot \Gamma(s/2)$ est un meilleur choix.

1.2.7. Pour toute structure de Hodge mixte V sur $\mathbb{R}$, on pose

$$\varepsilon_0(V) = i^{\sum_{p\in\mathbb{Z}} p(h_p - n_p) + \varepsilon_p n_p^+ + (1-\varepsilon_p) n_p^-}$$

où ε_p, n_p, n_p^+, n_p^- sont comme au n°1.2.4 et $h_p = \dim_{\mathbb{R}} \mathrm{Fil}^p V/\mathrm{Fil}^{p+1} V$.

Soit $\psi_0 \colon \mathbb{R} \to \mathbb{C}^*$ le caractère additif défini par $\psi_0(x) = \exp(2i\pi x)$. Si $\mu = a \cdot dx$, avec $a \neq 0$, est une mesure de Haar sur $\mathbb{R}$, on pose

$$\varepsilon(V, \psi_0, \mu) = a \cdot \varepsilon_0(V).$$

On vérifie facilement que, lorsque V est semi-simple, on retrouve la définition usuelle du facteur ε **[De73, Ta79]**.

1.2.8. Soit $K = \mathbb{R}$ ou $\mathbb{C}$. Soit

$$(\beta) \qquad 0 \to V' \to V \to V'' \to 0$$

une suite exacte courte de $\mathbf{SH}_K(\mathbb{R})$ et $\beta^*(1)$ la suite exacte

$$(\beta^*(1)) \qquad 0 \to V''^*(1) \to V^*(1) \to V'^*(1) \to 0.$$

Posons

$$L(\beta, s) = L(V, s)/L(V', s) \cdot L(V'', s)$$

et

$$\varepsilon(\beta) = \varepsilon(V, \psi_0, \mu)/\varepsilon(V', \psi_0, \mu) \cdot \varepsilon(V'', \psi_0, \mu)$$

(il est immédiat que $\varepsilon(\beta)$ est indépendant du choix de la mesure de Haar μ sur K).

Proposition. *Soient $K = \mathbb{R}$ ou $\mathbb{C}$ et (β) une suite exacte de* $\mathbf{SH}_K(\mathbb{R})$. *Alors $\varepsilon(\beta) \in \{-1, 1\}$ et $L(\beta, s) = \varepsilon(\beta) \cdot L(\beta^*(1), -s)$.*

Preuve. Soit $p \in \mathbb{Z}$. Si $x \in \gamma^p V''$ et si $y \in \mathrm{Fil}^p V_{\mathbb{C}}$ relève x, $y - \overline{y} \in 2i\pi \cdot V'$ et sa projection $z \in (2i\pi \cdot V')/(2i\pi \cdot \delta^p V')$ ne dépend pas du choix de y. Comme $V'/\delta^p V'$ s'identifie à $\mathrm{Hom}_{\mathbb{R}}(\gamma^{-p} V'^*(1), 2i\pi \cdot \mathbb{R})$ (n°1.1.6), $2i\pi \cdot V'/(2i\pi \cdot \delta^p V')$ s'identifie au dual de $\gamma^{-p} V'^*(1)$ et on obtient ainsi une application $\mathbb{R}$-linéaire

$$\gamma^p V'' \to (\gamma^{-p} V'^*(1))^*.$$

On vérifie que la suite

$$0 \to \gamma^p V' \to \gamma^p V \to \gamma^p V'' \to (\gamma^{-p} V'^*(1))^* \to (\gamma^{-p} V^*(1))^* \to (\gamma^{-p} V''^*(1))^* \to 0$$

est exacte.

Pour tout $p \in \mathbb{Z}$, posons $n'_p = \dim_{\mathbb{R}} \gamma^p V'/\gamma^{p+1} V'$, $n_p = \dim_{\mathbb{R}} \gamma^p V/\gamma^{p+1} V$ et $n''_p = \dim_{\mathbb{R}} \gamma^p V''/\gamma^{p+1} V''$. Posons $n_p(\beta) = n_p - n'_p - n''_p$ et définissons $n_p(\beta^*(1))$ de manière similaire. On a alors

$$\sum_p n_p(\beta) = 0 \quad \text{et} \quad n_p(\beta^*(1)) + n_{-1-p}(\beta) = 0.$$

Prenons d'abord $K = \mathbb{C}$. On a $L(\beta, s) = \prod_p \Gamma_{\mathbb{C}}(s-p)^{n_p(\beta)}$ et

$$L(\beta^*(1), -s) = \prod_p \Gamma_{\mathbb{C}}(-s-p)^{n_p(\beta^*(1))} = \prod_p \Gamma_{\mathbb{C}}(-s-p)^{-n_{-p-1}(\beta)}$$
$$= \prod_p \Gamma_{\mathbb{C}}(1-s+p)^{-n_p(\beta)}.$$

Mais

$$\Gamma_{\mathbb{C}}(1-s+p)\cdot\Gamma_{\mathbb{C}}(s-p) = 4\cdot(2\pi)^{-1}\cdot\pi/\sin(\pi(s-p)) = (-1)^p\cdot 4\cdot(2\pi)^{-1}\cdot\pi/\sin(\pi s).$$

Donc

$$L(\beta^*(1), -s)/L(\beta, s) = [4\cdot(2\pi)^{-1}\cdot\pi/\sin(\pi s)]^{-\sum_p n_p(\beta)}\cdot(-1)^{\sum_p p\cdot n_p(\beta)}$$
$$= (-1)^{\sum_p p\cdot n_p(\beta)} = \varepsilon(\beta).$$

Pour $K = \mathbb{R}$, on pose $n_p^{\pm}(\beta) = n_p^{\pm} - n_p'^{\pm} - n_p''^{\pm}$ (avec des notations évidentes). On a

$$\sum_p n_p^{\pm}(\beta) = 0 \quad \text{et} \quad n_p^{\pm}(\beta^*(1)) + n_{-1-p}^{\pm}(\beta) = 0.$$

De plus, par définition de $L(\beta, s)$, on a

$$L(\beta, s) = \prod_p \Gamma_{\mathbb{R}}(s+\varepsilon_p - p)^{n_p^+(\beta)}\cdot\Gamma_{\mathbb{R}}(s+1-\varepsilon_p-p)^{n_p^-(\beta)}.$$

On utilise l'équation fonctionnelle de Γ sous la forme

$$\Gamma_{\mathbb{R}}(2-s+p)\cdot\Gamma_{\mathbb{R}}(s-p) = 1/\sin(\pi\cdot(s-p)/2).$$

On a donc

$$\begin{aligned} L(\beta, s)/L(\beta^*(1), -s) &= (-1)^{\nu}\cdot\prod_p \Gamma_{\mathbb{R}}(s+\varepsilon_p-p)^{n_p^+(\beta)+n_{-1-p}^+(\beta^*(1))} \\ &\quad\cdot\Gamma_{\mathbb{R}}(s+1-\varepsilon_p-p)^{n_p^-(\beta)+n_{-1-p}^-(\beta^*(1))} \\ &\qquad\cdot\sin(\pi\cdot s/2)^{\sum_p n_{-1-p}^{\pm}(\beta^*(1))}\cdot\sin(\pi\cdot(s-1)/2)^{\sum_p n_{-1-p}^{\pm}(\beta^*(1))} \end{aligned}$$

avec

$$\nu \equiv \sum_p (p-\varepsilon_p)\cdot n_p^+(\beta^*(1))/2 + \sum_p (p+\varepsilon_p)\cdot n_p^-(\beta^*(1))/2 \mod 2$$
$$\equiv \sum_p (p-\varepsilon_p)\cdot n_p^+(\beta)/2 + \sum_p (p+\varepsilon_p)\cdot n_p^-(\beta)/2 \mod 2.$$

D'autre part, comme $\sum_p n_p^-(\beta) = 0$, le calcul montre que

$$\varepsilon(\beta) = i^{-\sum_p (p-\varepsilon_p)\cdot n_p^+(\beta)-(p+\varepsilon_p)\cdot n_p^-(\beta)} = (-1)^{\nu}.$$

D'où la proposition.

2. Structures pré-motiviques sur un corps de nombres

2.1. La catégorie des structures pré-motiviques. Les structures pré-motiviques que nous allons considérer dans ce paragraphe sont des structures à coefficients dans $\mathbb{Q}$.

Dans toute la suite de ce chapitre, on reprend les hypothèses et les notations du chapitre II, n°1.1.1. Si $\mathfrak{p} \in S_\infty(F)$, on choisit en outre une identification de $\overline{F}_\mathfrak{p}$ à $\mathbb{C}$ et on pose $B_{\mathrm{dR},\mathfrak{p}} = \mathbb{C}$; si $\mathfrak{p} \in S_f(F)$, on note $B_{\mathrm{dR},\mathfrak{p}}$ le corps B_{dR} relatif à $\overline{F}_\mathfrak{p}/F_\mathfrak{p}$ (cf. I, n°2.1.2).

2.1.1. Commençons par introduire une catégorie auxiliaire, la catégorie $\mathbf{PSPM}_F(\mathbb{Q})$.

Un objet M de cette catégorie consiste en la donnée

- d'une part,
  - i) d'un F-espace vectoriel noté M_{dR}, de dimension finie, appelée *réalisation de Rham de* M, muni d'une filtration finie décroissante par des sous-F-espaces vectoriels $(\mathrm{Fil}^i M_{\mathrm{dR}})_{i\in\mathbb{Z}}$, appelée *la filtration de Hodge*;
  - ii) pour chaque $\mathfrak{p} \in S_\infty(F)$, d'un $\mathbb{Q}$-espace vectoriel de dimension finie, noté $M_{B,\mathfrak{p}}$, muni d'une action linéaire de $G_\mathfrak{p}$ (i.e. d'une involution $c_\mathfrak{p}$ si $F_\mathfrak{p} = \mathbb{R}$, de "rien du tout" si $F_\mathfrak{p} = \mathbb{C}$), appelé $\mathfrak{p}$-*réalisation Betti de* M;
  - iii) pour chaque nombre premier ℓ, une représentation ℓ-adique pseudo-géométrique de G_F, notée M_ℓ et appelée *réalisation ℓ-adique de* M;
- d'autre part d'isomorphismes de comparaison, i.e.,
  - i) pour chaque couple $(\ell, \mathfrak{p})$ formé d'un nombre premier ℓ et de $\mathfrak{p} \in S_\infty(F)$, d'un isomorphisme
  $$\iota_{\ell,\mathfrak{p}} : \mathbb{Q}_\ell \otimes_\mathbb{Q} M_{B,\mathfrak{p}} \simeq M_\ell$$
  de $\mathbb{Q}_\ell$-espaces vectoriels, compatible avec l'action de $G_\mathfrak{p}$;
  - ii) pour chaque $\mathfrak{p} \in S_\infty(F)$ d'un isomorphisme de $\mathbb{C}$-espaces vectoriels
  $$\iota_\mathfrak{p} : \mathbb{C} \otimes_\mathbb{Q} M_{B,\mathfrak{p}} \simeq \mathbb{C} \otimes_F M_{\mathrm{dR}},$$
  compatible avec l'action naturelle de $G_\mathfrak{p}$ (il revient au même de se donner un isomorphisme du $F_\mathfrak{p}$-espace vectoriel $(\mathbb{C} \otimes_\mathbb{Q} M_{B,\mathfrak{p}})^{G_\mathfrak{p}}$ sur $F_\mathfrak{p} \otimes_F M_{\mathrm{dR}}$);
  - iii) pour chaque place finie $\mathfrak{p}$ au-dessus d'un nombre premier p, d'un isomorphisme de $B_{\mathrm{dR},\mathfrak{p}}$-espaces vectoriels
  $$\iota_\mathfrak{p} : B_{\mathrm{dR},\mathfrak{p}} \otimes_{\mathbb{Q}_p} M_p \simeq B_{\mathrm{dR},\mathfrak{p}} \otimes_{F_\mathfrak{p}} M_{\mathrm{dR}},$$
  compatible avec l'action naturelle de $G_\mathfrak{p}$ et avec la filtration de Hodge (il revient au même de se donner un isomorphisme du $F_\mathfrak{p}$-espace vectoriel filtré
  $$D_{\mathrm{dR},\mathfrak{p}}(M_p) = (B_{\mathrm{dR},\mathfrak{p}} \otimes_{\mathbb{Q}_p} M_p)^{G_\mathfrak{p}} \quad \text{sur } F_\mathfrak{p} \otimes_F M_{\mathrm{dR}}).$$

Dans la suite, on utilise les applications $i_{\ell,\mathfrak{p}}$, $i_{\mathfrak{p}}$ pour identifier les espaces vectoriels entre lesquels ces applications définissent des isomorphismes.

Un morphisme $f=(f_{\mathrm{dR}}, (f_{B,\mathfrak{p}})_{\mathfrak{p}\in S_\infty}, (f_\ell)_{\ell \text{ premier}}) : M \to N$ *de* $\mathbf{PSPM}_F(\mathbb{Q})$ est la donnée d'un morphisme pour chacune des réalisations, ces morphismes étant compatibles avec les isomorphismes de comparaison.

2.1.2. Il est immédiat que la catégorie $\mathbf{PSPM}_F(\mathbb{Q})$ est additive, $\mathbb{Q}$-linéaire. En fait elle est *abélienne*: cela résulte de ce que, pour tout morphisme f, la réalisation f_{dR} est strictement compatible aux filtrations (pour le voir on peut étendre les scalaires à un $F_{\mathfrak{p}}$ pour $\mathfrak{p}$ place finie, et c'est alors une conséquence des propriétés des représentations de de Rham de $G_{\mathfrak{p}}$).

2.1.3. Soit M un objet de $\mathbf{PSPM}_F(\mathbb{Q})$. Pour tout couple $(\ell, \mathfrak{p})$ formé d'un nombre premier ℓ et d'une place $\mathfrak{p}$ de F, on note $M_{\ell,\mathfrak{p}}$ la représentation ℓ-adique de $G_{\mathfrak{p}}$ obtenue par restriction à partir de M_ℓ.

Si $\mathfrak{p} \in S(F)$, on définit $M_{(\mathfrak{p})}$ comme étant $M_{p,\mathfrak{p}}$ si $\mathfrak{p} \in S_p(F)$, avec p premier, et $\mathbb{R} \otimes_{\mathbb{Q}} M_{B,\mathfrak{p}}$ si $\mathfrak{p} \in S_\infty(F)$; on convient aussi que $\mathbb{Q}_\infty = \mathbb{R}$. On remarque qu'avec ces notations, pour tout $\mathfrak{p} \in S(F)$, l'isomorphisme de comparaison $\iota_{\mathfrak{p}}$ identifie $(B_{\mathrm{dR},\mathfrak{p}} \otimes_{\mathbb{Q}_p} M_{(\mathfrak{p})})^{G_{\mathfrak{p}}}$ à $F_{\mathfrak{p}} \otimes_F M_{\mathrm{dR}}$.

2.1.4. Définition. *Une structure pré-motivique sur* F est un couple formé d'un objet M de $\mathbf{PSPM}_F(\mathbb{Q})$ et d'une filtration finie croissante $(W_nM)_{n\in\mathbf{Z}}$, appelée *filtration par le poids*, tels que, pour tout $\mathfrak{p} \in S_\infty$, le $\mathbb{R}$-espace vectoriel $M_{(\mathfrak{p})}$, muni de la filtration induite par la filtration par le poids, de la filtration induite par la filtration de Hodge via l'isomorphisme de comparaison, et, lorsque $F_{\mathfrak{p}} = \mathbb{R}$, de l'action de la conjugaison complexe, soit une structure de Hodge mixte sur $F_{\mathfrak{p}}$.

On parle abusivement de la structure pré-motivique M, la filtration par le poids étant sous-entendue.

On dit qu'une structure pré-motivique M non nulle est *pure de poids* r si $\mathrm{gr}_n^W M = 0$ pour $n \neq r$. Plus généralement, les *poids d'une structure prémotivique* M sont les entiers n tels que $\mathrm{gr}_n^W M \neq 0$.

Un morphisme de structures pré-motiviques sur F est un morphisme des objets de $\mathbf{PSPM}_F(\mathbb{Q})$ sous-jacents qui est compatible avec la filtration par le poids. On vérifie facilement que la catégorie $\mathbf{SPM}_F(\mathbb{Q})$ des structures pré-motiviques sur F est abélienne.

Cette catégorie a *un objet-unité*, noté $\mathbf{1}$, qui est la structure pré-motivique associée à $\operatorname{Spec} F$ (elle est de poids 0, avec $M_{\mathrm{dR}} = F$, $M_{B,\mathfrak{p}} = \mathbb{Q}$ pour tout $\mathfrak{p} \in S_\infty(F)$, $M_\ell = \mathbb{Q}_\ell$ pour tout nombre premier ℓ; les actions des groupes de Galois sont toutes triviales et les isomorphismes de comparaison sont les isomorphismes évidents). On a, de façon évidente, des notions de *produit tensoriel*, de *hom interne* qui font de cette catégorie une *catégorie tannakienne neutre* (chaque réalisation est un foncteur fibre; elle est neutre parce qu'il y a une réalisation dans les $\mathbb{Q}$-espaces vectoriels: pour chaque $\mathfrak{p} \in S_\infty(F)$, le foncteur $M \mapsto M_{B,\mathfrak{p}}$ en est une).

Pour chaque nombre premier ℓ, la correspondance $M \mapsto M_\ell$ définit un $\otimes$-foncteur exact et fidèle de $\mathbf{SPM}_F(\mathbb{Q})$ dans la catégorie $\mathbf{Rep}_{\mathbb{Q}_\ell, pg}(G_F)$ des représentations ℓ-adiques pseudo-géométriques de G_F. Pour chaque $\mathfrak{p} \in S_\infty(F)$, la correspondance $M \mapsto M_{(\mathfrak{p})}$ définit un $\otimes$-foncteur exact et fidèle de $\mathbf{SPM}_F(\mathbb{Q})$ dans la catégorie $\mathbf{SH}_{F_\mathfrak{p}}(\mathbb{R})$ des structures de Hodge mixtes sur $F_\mathfrak{p}$.

2.1.5. Soit M une structure pré-motivique sur F. Pour tout couple $(\ell, \mathfrak{p})$ formé d'un nombre premier ℓ et d'une place finie $\mathfrak{p}$ de F, on note $M_{\ell,\mathfrak{p}}$ la restriction de la représentation M_ℓ de G_F à $G_\mathfrak{p}$. On sait lui associer (cf. I, n°2.2.5 et 2.2.6) une représentation $D_{pst}(M_{\ell,\mathfrak{p}})$ du groupe de Weil-Deligne $'W_\mathfrak{p} = 'W_{F_\mathfrak{p}}$ de $F_\mathfrak{p}$. Si $\mathfrak{p}$ est une place finie, on dit que

i) M *a bonne réduction en* $\mathfrak{p}$ si $M_{\ell,\mathfrak{p}}$ a bonne réduction pour tout ℓ;

ii) M est *L-admissible en* $\mathfrak{p}$ si les représentations $D_{pst}(M_{\ell,\mathfrak{p}})^{ss}$, pour ℓ premier, forment un système compatible de représentations $\mathbb{Q}$-rationnelles de $'W_\mathfrak{p}$ (I, n°1.2.5); on dit que M est *strictement L-admissible en* $\mathfrak{p}$ si, en outre, les $D_{pst}(M_{\ell,\mathfrak{p}})$ sont en fait F-semi-simples (auquel cas les $D_{pst}(M_{\ell,\mathfrak{p}})$ forment donc un système compatible de représentations $\mathbb{Q}$-rationnelles de $'W_\mathfrak{p}$).

Si $\mathfrak{p} \in S_\infty(F)$, on convient que M est strictement *L-admissible en* $\mathfrak{p}$; on dit que M *a bonne réduction en* $\mathfrak{p}$ si $M_{(\mathfrak{p})}$ est un objet semi-simple de $\mathbf{SH}_{F_\mathfrak{p}}(\mathbb{R})$.

Si Σ est un ensemble fini de places de F, on dit que M est *L-admissible* (resp. *strictement admissible*) *en dehors de* Σ si elle est admissible (resp. strictement admissible) pour toute place $\mathfrak{p} \notin \Sigma$. S'il en est ainsi, il existe un ensemble fini de places de F tel que M a bonne réduction en dehors de cet ensemble; il suffit de choisir un nombre premier ℓ et de prendre l'union de $\Sigma \cup S_\infty(F)$ avec l'ensemble des places finies où M_ℓ a mauvaise réduction.

On dit que M est *L-admissible* s'il existe un ensemble fini Σ de places de F tel que M soit L-admissible en dehors de Σ. On dit que M est *L-admissible partout* si M est L-admissible en toute place $\mathfrak{p}$ de F.

Remarques. a) On prendra garde que la sous-catégorie pleine de la catégorie des structures pré-motiviques formée des objets L-admissibles n'est pas abélienne.

b) On s'attend à ce que les réalisations d'un "vrai" motif correspondent à une structure pré-motivique qui est strictement L-admissible en toute place. Pour un motif donné, on ne sait pas toujours prouver la F-semi-simplicité et on ne sait souvent prouver la L-admissibilité qu'en dehors d'un nombre fini de places. C'est pourquoi la notion de structure pré-motivique L-admissible est commode.

C'est pour la même raison qu'on n'a pas demandé que, pour toute structure pré-motivique M et tout nombre premier ℓ, la représentation M_ℓ soit géométrique, mais seulement qu'elle soit pseudo-géométrique. On remarque toutefois que, si M est L-admissible, M_ℓ est géométrique pour presque tout ℓ.

2.1.6. Soit $\mathfrak{p} \in S(F)$. Si $\mathfrak{p} \in S_\infty(F)$ (resp. $S_f(F)$), la correspondance $M \mapsto M_{(\mathfrak{p})}$ définit un $\otimes$-foncteur exact et fidèle de la catégorie des structures pré-motiviques sur F dans celle des structures de Hodge mixtes sur $F_\mathfrak{p}$ (resp. des représentations de de Rham de $G_\mathfrak{p}$) ; l'isomorphisme de comparaison $\iota_\mathfrak{p}$ permet d'identifier $t_{M_{(\mathfrak{p})}}$ et $\mathbf{t}_M(F_\mathfrak{p}) := F_\mathfrak{p} \otimes_F t_M$.

2.1.7. Si F' est une extension finie de F, on a des foncteurs exacts et fidèles

$$\mathrm{Ext}_{F/F'} : \mathbf{SPM}_F(\mathbb{Q}) \to \mathbf{SPM}_{F'}(\mathbb{Q}) \quad \text{et} \quad \mathrm{Res}_{F'/F} : \mathbf{SPM}_{F'}(\mathbb{Q}) \to \mathbf{SPM}_F(\mathbb{Q}),$$

dont la description est aussi évidente que fastidieuse, $\mathrm{Res}_{F'/F}$ étant l'adjoint à gauche de $\mathrm{Ext}_{F/F'}$. On observera toutefois que l'extension des scalaires correspond, sur la réalisation ℓ-adique, à la restriction de la représentation, alors que la restriction des scalaires correspond à l'induction. On voit que ces foncteurs transforment une structure pré-motivique L-admissible en une structure pré-motivique L-admissible.

2.1.8. La *structure pré-motivique de Tate* $\mathbb{1}_F(1)$ sur F (ou $\mathbb{1}(1)$ s'il n'y a pas de risque de confusion) est la structure pré-motivique, pure de poids -2, que l'on associe naturellement à G_m, ou si l'on préfère la duale de la structure pré-motivique associée à $H^2(P^1)$: c'est l'extension des scalaires de $\mathbb{Q}$ à F de $\mathbb{1}_\mathbb{Q}(1)$: on a $\mathbb{1}_\mathbb{Q}(1)_{\mathrm{dR}} = \mathbb{R}$, $\mathbb{1}_\mathbb{Q}(1)_B = (2\pi i)\cdot\mathbb{Q}$, $\mathbb{1}_\mathbb{Q}(1)_\ell = \mathbb{Q}_\ell(1)$.
Si $r \in \mathbb{N}$, $\mathbb{1}(r)$ est la puissance symétrique r-ième de $\mathbb{1}(1)$ et $\mathbb{1}(-r) = \mathbb{1}(r)^*$; si M est une structure pré-motivique sur F et si $r \in \mathbb{Z}$, $M(r) = M \otimes \mathbb{1}(r)$.

2.2. La fonction L d'une structure pré-motivique. Pour tout nombre premier ℓ, on suppose choisi un plongement de $\mathbb{Q}_\ell$ dans $\mathbb{C}$ (cf. II, n°3.4.1).

2.2.1. Soit M une structure pré-motivique sur F que l'on suppose L-admissible de sorte que l'ensemble Σ_M des places $\mathfrak{p}$ de F où M n'est pas L-admissible est fini. Si $\mathfrak{p} \notin \Sigma_M$, on pose

$$L_\mathfrak{p}(M, s) = L(M_{(\mathfrak{p})}, s).$$

Pour tout ensemble fini Σ de places de F contenant Σ_M, on pose

$$L_\Sigma(M, s) = \prod_{\mathfrak{p} \notin \Sigma} L_\mathfrak{p}(M, s).$$

En particulier, si M est L-admissible partout, *la fonction* L *de* M est $L(M, s) = L_{S_\infty(F)}(M, s)$ et *la fonction* L *complète de* M est $L_\varnothing(M, s)$.

2.2.2. Conjecture $C_{\mathrm{pr.an}}(M)$. *Si M est une structure pré-motivique L-admissible sur F, le produit $L_{\Sigma_M}(M, s)$ converge absolument pour* $\mathrm{Re}(s) \gg 0$ *et admet un prolongement analytique méromorphe dans tout le plan complexe.*

2.2.3. REMARQUES. a) Choisissons un ensemble fini Σ de places de F contenant Σ_M. Cette conjecture équivaut à celle que l'on obtient en remplaçant $L_{\Sigma_M}(M, s)$ par $L_\Sigma(M, s)$.

b) Soit ℓ un nombre premier. La fonction $L_{\Sigma_M}(M, s)$ ne diffère de la fonction $L(M_\ell, s)$ qu'en un nombre fini de facteurs et on peut dans l'énoncé de la conjecture remplacer $L_{\Sigma_M}(M, s)$ par $L(M_\ell, s)$. Si l'on choisit ℓ tel que M_ℓ soit géométrique (par example si l'on choisit ℓ tel que M ait bonne réduction en toutes les places divisant ℓ), la conjecture $C_{\mathrm{gal}, L}(M_\ell)$ (II, n°3.4.3) implique $C_{\mathrm{pr.an}}(M)$.

c) Si $\mathfrak{p} \notin \Sigma_M$, $P_\mathfrak{p}(M, u) \in \mathbb{Q}[u]$. En particulier, pour tout $\Sigma \supset \Sigma_M$, la fonction $L_\Sigma(M, s)$ est indépendante des plongements des $\mathbb{Q}_\ell$ dans $\mathbb{C}$ choisis.

d) Soit

$$0 \to M' \to M \to M'' \to 0$$

une suite exacte courte de structures pré-motiviques; supposons que M' et M'' sont L-admissibles; il en est alors de même de M; en outre, si deux des trois conjectures $C_{\mathrm{pr.an}}(M')$, $C_{\mathrm{pr.an}}(M)$ et $C_{\mathrm{pr.an}}(M'')$ sont vraies, il en est de même de la troisième: cela résulte de ce que, pour presque toute place $\mathfrak{p}$ de F, M', M, et M'' sont L-admissibles et $L_\mathfrak{p}(M, s) = L_\mathfrak{p}(M', s) \cdot L_\mathfrak{p}(M'', s)$.

3. Catégories pré-motiviques admissibles

3.1. Généralités.

3.1.1. Nous appelons *sous-catégorie pré-motivique de* $\mathbf{SPM}_F(\mathbb{Q})$ toute sous-catégorie strictement pleine stable par somme directe, sous-objet et quotient (et donc, en particulier, abélienne), contenant $M^*(1)$ dès qu'elle contient M et contenant $\mathbf{1}$.

Une sous-catégorie tannakienne de $\mathbf{SPM}_F(\mathbb{Q})$ est donc pré-motivique si et seulement si elle contient $\mathbf{1}(1)$.

Dans toute la suite du §3, $\mathbb{M}$ *est une sous-catégorie pré-motivique de* $\mathbf{SPM}_F(\mathbb{Q})$.

3.1.2. Pour tout objet M de $\mathbb{M}$ de tout $i \in \mathbb{Z}$, on pose $H^i_\mathbb{M}(F, M) = \mathrm{Ext}^i_\mathbb{M}(\mathbf{1}, M)$. Le $\mathbb{Q}$-espace vectoriel $H^0_\mathbb{M}(F, M)$ est indépendant du choix de la sous-catégorie pleine $\mathbb{M}$ de $\mathbf{SPM}_F(\mathbb{Q})$ contenant M et on le note aussi $H^0(F, M)$.

Si S est un ensemble fini de places de F, la sous-catégorie pleine $\mathbb{M}_S$ de $\mathbb{M}$ dont les objets sont ceux qui ont bonne réduction en dehors de S est encore une sous-catégorie pré-motivique. Par analogie avec les représentations ℓ-adiques, pour tout $i \in \mathbb{Z}$ et tout objet M de $\mathbb{M}_S$, on pose

$$H^i_\mathbb{M}(U_S, M) = H^i_{\mathbb{M}_S}(F, M).$$

3.1.3. Soit S un ensemble fini de places de F contenant $S_\infty(F)$ et M un objet de $\mathbb{M}_S$.

Si $x \in H^1_{\mathbf{M}}(U_S, M)$, pour tout nombre premier ℓ, on note x_ℓ l'image de x dans $H^1(U_S, M_\ell)$, qui appartient donc à $H^1_g(U_S, M_\ell)$; si $\mathfrak{p}$ est une place finie de F, on note $x_{\ell,\mathfrak{p}}$ l'image de x_ℓ dans $H^1_g(F_\mathfrak{p}, M_{\ell,\mathfrak{p}}) \subset H^1(F_\mathfrak{p}, M_{\ell,\mathfrak{p}})$.

Pour tout ensemble Σ de places de F, *vérifiant* $S_\infty \subset \Sigma \subset S$, on pose

$$\begin{aligned} H^1_{\mathbf{M},f,\Sigma}(F, M) &= \{x \in H^1_{\mathbf{M}}(U_S, M) \mid \text{pour tout } \ell,\ x_\ell \in H^1_{f,\Sigma}(F, M_\ell)\} \\ &= \{x \in H^1_{\mathbf{M}}(U_S, M) \mid \text{pour tout } \ell,\ x_{\ell,\mathfrak{p}} \in H^1_f(F_\mathfrak{p}, M_{\ell,\mathfrak{p}}) \\ &\qquad \text{si } \mathfrak{p} \notin \Sigma\} \end{aligned}$$

(remarquer que, pour tout $x \in H^1_{\mathbf{M}}(U_S, M)$, si $\mathfrak{p} \notin S$, $x_{\ell,\mathfrak{p}} \in H^1_f(F_\mathfrak{p}, M_{\ell,\mathfrak{p}})$ pour tout ℓ ; et aussi que $H^1_{\mathbf{M},f,\Sigma}(F, M)$ est indépendant du choix de S qui peut être n'importe quel ensemble fini de places de F contenant Σ et tel que M a bonne réduction en dehors de S). On a $H^1_{\mathbf{M}}(U_S, M) = H^1_{\mathbf{M},f,S}(F, M)$.

On pose aussi $H^0_{\mathbf{M},f,\Sigma}(F, M) = H^0(F, M)$, $H^i_{\mathbf{M},f,\Sigma}(F, M) = 0$ pour $i \notin \{0, 1\}$ et $H^i_{\mathbf{M},f}(F, M) = H^i_{\mathbf{M},f,S_\infty}(F, M)$ pour tout i.

3.1.4. Soit

$$(\beta) \qquad 0 \to M' \to M \to M'' \to 0$$

une suite exacte courte de $\mathbb{M}$. C'est aussi une suite exacte courte de $\mathbf{SPM}_F(\mathbb{Q})$ et elle définit un élément u de

$$\mathrm{Ext}^1_{\mathbf{SPM}_F(\mathbb{Q})}(M'', M') = H^1_{\mathbf{SPM}_F(\mathbb{Q})}(F, M''^* \otimes M').$$

On dit que c'est une *f-suite exacte courte de* $\mathbb{M}$ si

$$u \in H^1_{\mathbf{SPM}_F(\mathbb{Q}),f}(F, M''^* \otimes M').$$

Si, pour tout objet M de $\mathbb{M}$, $M''^* \otimes M$ est encore un objet de $\mathbb{M}$ (c'est en particulier le cas lorsque $M'' = \mathbb{1}$), $u \in H^1_{\mathbf{M}}(F, M''^* \otimes M')$ et cela revient à demander que $u \in H^1_{\mathbf{M},f}(F, M''^* \otimes M')$. *Une telle f-suite exacte courte donne naissance à la suite exacte longue*

$$\begin{aligned} 0 &\to H^0(F, M') \to H^0(F, M) \to H^0(F, M'') \\ &\to H^1_{\mathbf{M},f}(F, M') \to H^1_{\mathbf{M},f}(F, M) \to H^1_{\mathbf{M},f}(F, M''). \end{aligned}$$

On voit en effet que, pour $N = M'$, M, ou M'', si, pour tout nombre premier ℓ et toute place finie $\mathfrak{p}$ de F, on note $x_{\ell,\mathfrak{p}}$ l'image de $x \in H^1_{\mathbf{M},f}(F, N)$ dans $H^1(F_\mathfrak{p}, N_\ell)$, alors

$$\begin{aligned} &H^1_{\mathbf{M},f}(F, N) \\ &\quad = \{x \in H^1_{\mathbf{M},f}(F, N) \mid x_{\ell,\mathfrak{p}} \in H^1_f(F_\mathfrak{p}, N_\ell) \text{ pour tout } \ell \text{ et tout } \mathfrak{p}\}; \end{aligned}$$

comme, par ailleurs, la suite

$$0 \to H^0(F, M') \to H^0(F, M) \to H^0(F, M'')$$
$$\to H^1_{\mathbf{M}}(F, M') \to H^1_{\mathbf{M}}(F, M) \to H^1_{\mathbf{M}}(F, M'')$$

est exacte, il s'agit de vérifier

i) que l'image de $H^0(F, M'')$ dans $H^1_{\mathbf{M}}(F, M')$ est dans $H^1_{\mathbf{M},f}(F, M')$;

ii) que si $x \in H^1_{\mathbf{M},f}(F, M) \subset H^1_{\mathbf{M}}(F, M)$ s'envoie sur 0 dans $H^1_{\mathbf{M}}(F, M'')$, alors x appartient à l'image de $H^1_{\mathbf{M},f}(F, M')$.

Le fait que (β) soit une f-suite exacte implique que, pour tout ℓ et tout $\mathfrak{p}$, la suite

$$0 \to H^0(F_{\mathfrak{p}}, M'_\ell) \to H^0(F_{\mathfrak{p}}, M_\ell) \to H^0(F_{\mathfrak{p}}, M''_\ell)$$
$$\to H^1_f(F_{\mathfrak{p}}, M'_\ell) \to H^1_f(F_{\mathfrak{p}}, M_\ell) \to H^1_f(F_{\mathfrak{p}}, M''_\ell) \to 0$$

est exacte. En particulier l'image de $H^0(F_{\mathfrak{p}}, M''_\ell)$ dans $H^1(F_{\mathfrak{p}}, M'_\ell)$ est contenue dans $H^1_f(F_{\mathfrak{p}}, M'_\ell)$, ce qui implique (i). Si x est comme dans (ii), il provient d'un $x' \in H^1_{\mathbf{M}}(F, M')$. Le fait que, pour tout ℓ et tout $\mathfrak{p}$, $x_{\ell,\mathfrak{p}} \in H^1_f(F_{\mathfrak{p}}, M_\ell)$ joint au fait que, dans le diagramme commutatif

$$\begin{array}{ccccc} H^0(F_{\mathfrak{p}}, M''_\ell) & \longrightarrow & H^1_f(F_{\mathfrak{p}}, M'_\ell) & \longrightarrow & H^1_f(F_{\mathfrak{p}}, M_\ell) \\ \| & & \downarrow & & \downarrow \\ H^0(F_{\mathfrak{p}}, M''_\ell) & \longrightarrow & H^1(F_{\mathfrak{p}}, M'_\ell) & \longrightarrow & H^1(F_{\mathfrak{p}}, M_\ell), \end{array}$$

les lignes sont exactes et les flèches verticales injectives implique que $x_{\ell,\mathfrak{p}} \in H^1_f(F_{\mathfrak{p}}, M'_\ell)$, d'où (ii).

3.1.5. Soient M un objet de $\mathbb{M}$ et $\mathfrak{p} \in S_\infty(F)$. Si $F_{\mathfrak{p}} = \mathbb{C}$, on pose $M^+_{(\mathfrak{p})} = M_{(\mathfrak{p})}$ et on note $\alpha_{M,\mathfrak{p}}: M^+_{(\mathfrak{p})} \to t_{M_{(\mathfrak{p})}} = F_{\mathfrak{p}} \otimes_F t_M$ l'application notée $\alpha_{M_{(\mathfrak{p})}}$ au n°1.1.2; si $F_{\mathfrak{p}} = \mathbb{R}$, on pose $M^+_{(\mathfrak{p})} = (M_{(\mathfrak{p})})^+$ et on note $\alpha_{M,\mathfrak{p}}: M^+_{(\mathfrak{p})} \to t_{M_{(\mathfrak{p})}} = F_{\mathfrak{p}} \otimes_F t_M$ l'application notée $\alpha^+_{M_{(\mathfrak{p})}}$ au n°1.2.1. On pose aussi

$$\alpha_M = \bigoplus_{\mathfrak{p}\in S_\infty(F)} \alpha_{M,\mathfrak{p}}: \bigoplus_{\mathfrak{p}\in S_\infty} M^+_{(\mathfrak{p})} \to \bigoplus_{\mathfrak{p}\in S_\infty(F)} t_{M_{(\mathfrak{p})}} = \mathbb{R} \otimes_{\mathbb{Q}} t_M .$$

Si Σ est un ensemble fini de places de F et si $i \in \mathbb{Z}$, on définit des $\mathbb{R}$-espaces vectoriels $H^i_{\mathbf{M},f,\Sigma}(F, M)_{\mathbb{R}}$ de la façon suivante:

- *si* $S_\infty(F) \subset \Sigma$, $H^i_{\mathbf{M},f,\Sigma}(F, M)_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} H^i_{\mathbf{M},f,\Sigma}(F, M)$;
- si $S_\infty(F) \not\subset \Sigma$, $H^0_{\mathbf{M},f,\Sigma}(F, M)_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} H^0(F, M)$; $H^1_{\mathbf{M},f,\Sigma}(F, M)_{\mathbb{R}}$ (resp. $H^2_{\mathbf{M},f,\Sigma}(F, M)_{\mathbb{R}}$) est le noyau (resp. le conoyau) de l'homo-

morphisme naturel

$$\mathbb{R} \otimes_{\mathbb{Q}} H^1_{\mathbf{M},f,\Sigma\cup S_\infty(F)}(F, M) \to \bigoplus_{\mathfrak{p}\in S_\infty(F)-\Sigma_\infty} H^1_{/f}(F_\mathfrak{p}, M_{(\mathfrak{p})})$$

(rappelons que $H^1_{/f}(F_\mathfrak{p}, M_{(\mathfrak{p})}) = \operatorname{Coker} \alpha_{M,\mathfrak{p}}$) et $H^i_{\mathbf{M},f,\Sigma}(F, M)_\mathbb{R} = 0$ si $i \notin \{0, 1, 2\}$.

3.1.6. Soit $\mathfrak{p} \in S_\infty(F)$. Le foncteur qui, à un objet M de $\mathbb{M}$, associe la structure de Hodge mixte $M_{(\mathfrak{p})}$, induit, pour tout $i \in \mathbb{N}$, une application $\mathbb{Q}$-linéaire $H^i_\mathbf{M}(F, M) \to H^i(F_\mathfrak{p}, M_{(\mathfrak{p})})$, qui s'étend par linéarité en une application $\mathbb{R}$-linéaire $H^i_\mathbf{M}(F, M)_\mathbb{R} \to H^i(F_\mathfrak{p}, M_{(\mathfrak{p})})$; pour $i = 0$ (resp. 1), en composant avec la flèche de $H^i(F_\mathfrak{p}, M_{(\mathfrak{p})})$ dans $\operatorname{Ker} \alpha_{M,\mathfrak{p}}$ (resp. $\operatorname{Coker} \alpha_{M,\mathfrak{p}}$) définie dans les propositions 1.1.6 et 1.2.3, et en faisant la somme directe sur tous les $\mathfrak{p}$, on obtient des applications

$$u_M : H^0(F, M)_\mathbb{R} \to \operatorname{Ker} \alpha_M$$

et

$$v_M : H^1_\mathbf{M}(F, M)_\mathbb{R} \to \operatorname{Coker} \alpha_M;$$

on note encore

$$v_M : H^1_{\mathbf{M},f}(F, M)_\mathbb{R} \to \operatorname{Coker} \alpha_M$$

l'application obtenue en composant avec l'inclusion de $H^1_{\mathbf{M},f}(F, M)_\mathbb{R}$ dans $H^1_\mathbf{M}(F, M)_\mathbb{R}$.

Par ailleurs on sait (propositions 1.1.6 et 1.2.3) que, pour tout $\mathfrak{p} \in S_\infty(F)$, le dual de $\operatorname{Ker} \alpha_{M,\mathfrak{p}}$ s'identifie à $\operatorname{Coker} \alpha_{M^*(1),\mathfrak{p}}$ et celui de $\operatorname{Coker} \alpha_{M,\mathfrak{p}}$ à $\operatorname{Ker} \alpha_{M^*(1),\mathfrak{p}}$. On note

$$v'_M: \operatorname{Ker} \alpha_M \to H^1_{\mathbf{M},f}(F, M^*(1))^*_\mathbb{R}$$

la transposée de $v_{M^*(1)}$ et

$$u'_M: \operatorname{Coker} \alpha_M \to H^0(F, M^*(1))^*_\mathbb{R}$$

la transposée de $u_{M^*(1)}$.

3.2. Catégories pré-motiviques f-admissibles. Dans tout le n°3.2, $\mathbb{M}$ une sous-catégorie pré-motivique de $\mathbf{SPM}_F(\mathbb{Q})$.

3.2.1. Si ℓ est un nombre premier, on dit que *la propriété* $C_\ell(\mathbb{M})$ *est vraie* si tout objet M de $\mathbb{M}$ vérifie $C_\ell(M, \mathbb{M})$, c'est-à-dire est tel que, pour $i \in \{0, 1\}$, l'application naturelle

$$\mathbb{Q}_\ell \otimes_\mathbb{Q} H^i_{\mathbf{M},f}(F, M) \to H^i_f(F, M_\ell)$$

est un isomorphisme.

On dit que *la propriété* $C_\infty(\mathbb{M})$ *est vraie* si tout objet M de $\mathbb{M}$ tel que $H^i_{\mathbb{M},f}(F, M^*(1)) = 0$ vérifie $C_\infty(M, \mathbb{M})$, c'est-à-dire est tel que

$$u_M\colon H^0(F, M)_{\mathbb{R}} \to \operatorname{Ker}\alpha_M$$

est un isomorphisme et que la suite

$$0 \to H^1_{\mathbb{M},f}(F, M)_{\mathbb{R}} \xrightarrow{v_M} \operatorname{Coker}\alpha_M \xrightarrow{u'_M} H^0(F, M^*(1))^*_{\mathbb{R}} \to 0$$

est exacte.

On dit que $\mathbb{M}$ est *f-admissible* si $C_\ell(\mathbb{M})$ est vraie pour tout $\ell \in S(\mathbb{Q})$.

3.2.2. On dit que $\mathbb{M}$ est *f-admissible à l'infini*[3] si elle satisfait $C_\infty(\mathbb{M})$ et vérifie en outre

i) on a $\dim_{\mathbb{Q}} H^1_{\mathbb{M},f}(F, M) < +\infty$ pour tout objet M de $\mathbb{M}$;

ii) on a $H^1_{\mathbb{M}}(F, \mathbb{1}) = 0$;

iii) pour toute f-suite exacte courte de $\mathbb{M}$ de la forme

$$0 \to M \to N \to \mathbb{1} \to 0,$$

l'application

$$H^1_{\mathbb{M},f}(F, N^*(1)) \to H^1_{\mathbb{M},f}(F, M^*(1))$$

est surjective.

3.2.3. Proposition. *Si* $\mathbb{M}$ *est f-admissible, elle est f-admissible à l'infini.*

Preuve. Pour prouver que (i), (ii), et (iii) ont vérifiés, il suffit de choisir un nombre premier ℓ et grâce à $C_\ell(\mathbb{M})$ de vérifier que les propriétés analogues pour M_ℓ sont vraies: (i) est clair; (ii) vient de ce que $H^1_g(F, \mathbb{Q}_\ell) = H^1_f(F, \mathbb{Q}_\ell) = 0$ par la finitude du groupe des classes de F; pour montrer (iii), c'est-à-dire la sujectivité de $H^1_f(F, N^*(1)_\ell) \to H^1_f(F, M^*(1)_\ell)$, on utilise la suite exacte s_f de la proposition II, n°2.2.3 et la nullité de $H^1_f(F, \mathbb{Q}_\ell)$.

3.2.4. Pour toute f-suite exacte de $\mathbb{M}$ de la forme

$$0 \to M \to N \to \mathbb{1} \to 0,$$

et pour $x \in H^1_{\mathbb{M},f}(F, M^*(1))^*_{\mathbb{R}}$, notons x_N son image dans $H^1_{\mathbb{M},f}(F, N^*(1))^*_{\mathbb{R}}$. Soit

$$\Lambda(M, N) = \{x \in H^1_{\mathbb{M},f}(F, M^*(1))^*_{\mathbb{R}} \mid x_N \in \operatorname{Im} v'_N\}.$$

Si $x \in \Lambda(M, N)$ et si $x_N = v'_N(y)$, l'image de y dans $\operatorname{Ker}\alpha_{\mathbb{1}}$ est l'image d'un élément $z \in H^0(F, \mathbb{1})_{\mathbb{R}}$ et l'image $\delta_{M,N}(x)$ de z dans $H^1_{\mathbb{M},f}(F, M)_{\mathbb{R}}$ ne dépend pas des choix faits.

[3] La terminologie diffère de **[FP91c]** où l'on avait appelé f-admissible ce qui s'appelle ici f-admissible à l'infini.

3.2.5. **Proposition.** *Supposons que* $\mathbb{M}$ *est* f*-admissible à l'infini.*

i) *Il existe une unique transformation naturelle*

$$\delta = (\delta_M)_{M\in \mathrm{Ob}(\mathbf{M})}, \text{ avec } \delta_M : (H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}} \to H^1_{\mathbf{M},f}(F, M)_{\mathbb{R}}$$

telle que, pour toute suite exacte comme ci-dessus et tout $x \in \Lambda(M, N)$, $\delta_M(x) = \delta_{M,N}(x)$;

ii) *pour tout objet* M *de* $\mathbb{M}$, *la suite*

$$s_f(M) \qquad \begin{aligned} &0 \to H^0(F, M)_{\mathbb{R}} \to \operatorname{Ker}\alpha_M \to H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}} \\ &\to H^1_{\mathbf{M},f}(F, M)_{\mathbb{R}} \to \operatorname{Coker}\alpha_M \to H^0(F, M^*(1))^*_{\mathbb{R}} \to 0 \end{aligned}$$

est exacte.

La *preuve* sera donnée au n°3.2.7.

3.2.6. Commençons par un lemme "standard" d'algèbre homologique que nous allons énoncer dans le cadre des groupes abéliens bien que l'on puisse remplacer la catégorie qu'ils forment par n'importe quelle catégorie abélienne. Soit

$$\begin{array}{ccccccc} & & & & & & A'' & & \\ & & & & & & \downarrow & & \\ & & B' & \to & B & \to & B'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' & \to & C & \to & C'' & & \\ & & \downarrow & & & & & & \\ & & D' & & & & & & \end{array}$$

un diagramme commutatif de groupes abéliens, dont les lignes et les colonnes sont exactes. Appelons *homomorphisme de connexion relatif à ce diagramme* l'homomorphisme

$$\delta : A'' \to D'$$

que l'on obtient par chasse au diagramme (si $a'' \in A''$ s'envoie sur $b'' \in B''$, si $b \in B$ est un relèvement de b'', il existe $c' \in C'$ dont l'image dans C est la même que celle de b et $\delta a''$ est l'image de c' dans D').

Lemme. *Soit*

$$\begin{array}{ccccccccccccccc} & & 0 & & 0 & & 0 & & (1) & & (2) & & (3) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & A & \to & A'' & & D' & \to & D & \to & D'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & E' & \to & E & \to & E'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' & \to & C & \to & C'' & \to & F' & \to & F & \to & F'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & (1) & & (2) & & (3) & & 0 & & 0 & & 0 & & \end{array}$$

un diagramme commutatif dont les deux lignes inférieures et les trois colonnes sont exactes. On suppose les flèches $B'' \to E'$ *et* $C'' \to F'$ *nulles. Le* *diagramme obtenu en rajoutant l'homomorphisme de connexion $A'' \to D'$ est encore commutatif et la suite*

$$0 \to A' \to A \to A'' \xrightarrow{\delta} D' \to D \to D'' \to 0$$

est exacte.

PREUVE. Chasse au diagramme, longue mais facile.

3.2.7. Prouvons alors la Proposition 3.2.5. Soit $r = \dim_{\mathbb{Q}} H^1_{\mathbf{M},f}(F, M)$. Il existe une f-suite exacte courte dans $\mathbb{M}$

$$0 \to M \to M_u \to \mathbb{1}^r \to 0$$

avec $H^1_{\mathbf{M},f}(F, M_u) = 0$: par exemple, on choisit une base (u_i) du $\mathbb{Q}$-espace vectoriel $H^1_{\mathbf{M},f}(F, M)$, chaque u_i est représenté par une extension M_i de $\mathbb{1}$ par M dans $\mathbb{M}$ telle que la suite exacte courte $0 \to M \to M_i \to \mathbb{1} \to 0$ soit f. On en déduit une f-suite exacte courte dans $\mathbb{M}$

$$0 \to M \to M_u \to \mathbf{1}^r \to 0;$$

l'application $H^0(F, \mathbf{1}^r) \to H^1_{\mathbf{M},f}(F, M)$ est un isomorphisme et la nullité de $H^1_{\mathbf{M},f}(F, \mathbf{1}^r)$ implique que $H^1_{\mathbf{M},f}(F, M_u) = 0$. L'extension M_u de $\mathbf{1}^r$ par M est définie à isomorphisme près. On applique (iii) de 3.2.2 à $M_u^*(1)$, la transposée de la suite exacte obtenue est la suite exacte

$$0 \to H^0(F, M_u) \to \operatorname{Ker} \alpha_{M_u} \to H^1_{\mathbf{M},f}(F, M_u^*(1))^* \to 0.$$

Si $\Lambda(M, M_u) = \{x \in H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}} \mid x_{M_u} \in \operatorname{Im} v'_{M_u}\}$, on a donc

$$\Lambda(M, M_u) = H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}}.$$

On définit alors comme en 3.2.4 une application

$$\delta_{M,M_u} : H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}} \to H^1_{\mathbf{M},f}(F, M)^*_{\mathbb{R}}.$$

Il est facile de vérifier qu'elle ne dépend pas du choix de M_u et qu'elle coincide avec $\delta_{M,N}$ sur $\Lambda(M, N)$ pour toute f-suite exacte de $\mathbb{M}$ de la forme

$$0 \to M \to N \to \mathbb{1} \to 0.$$

On en déduit (i). Pour (ii), considérons les deux diagrammes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(F, M)_{\mathbb{R}} & \to & \operatorname{Ker}\alpha_M & \to & H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(F, M_u)_{\mathbb{R}} & \to & \operatorname{Ker}\alpha_{M_u} & \to & H^1_{\mathbf{M},f}(F, M_u^*(1))^*_{\mathbb{R}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(F, \mathbf{1}^r)_{\mathbb{R}} & \to & \operatorname{Ker}\alpha_{\mathbf{1}^r} & \to & H^1_{\mathbf{M},f}(F, \mathbf{1}^r(1))^*_{\mathbb{R}} & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & H^1_{\mathbf{M},f}(F, M)_{\mathbb{R}} & & & & & &
\end{array}$$

et

$$\begin{array}{ccccccc}
H^1_{\mathbf{M},f}(F, M)_{\mathbb{R}} & \to & \operatorname{Coker}\alpha_M & \to & H^0(F, M^*(1))^*_{\mathbb{R}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Coker}\alpha_{M_u} & \to & H^0(F, M^*_u(1))^*_{\mathbb{R}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Coker}\alpha_{\mathbb{1}^r} & \to & H^0(F, \mathbf{1}^r)^*_{\mathbb{R}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

L'application $\Delta_{M, M_u} = \delta_M$ est l'application de connexion du premier diagramme au sens du paragraphe 3.2.6. En appliquant le lemme 3.2.6 aux deux diagrammes juxtaposés, on en déduit la suite exacte $s_f(M)$.

3.2.8. PROPOSITION. *Si* $\mathbb{M}$ *est* f*-admissible à l'infini et si*

$$0 \to M' \to M \to M'' \to 0$$

est une f*-suite exacte courte de* $\mathbb{M}$, *le diagramme rectangulaire tordu*

$$\begin{array}{ccccccccccccc}
 & 0 & & 0 & & & & (1) & & (2) & & (3) & \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(F, M')_{\mathbb{R}} & \to & \operatorname{Ker}\alpha_{M'} & \to & H^1_{\mathbf{M},f}(F, M'^*(1))^*_{\mathbb{R}} & \to & H^1_{\mathbf{M},f}(F, M')_{\mathbb{R}} & \to & \operatorname{Coker}\alpha_{M'} & \to & H^0(F, M'^*(1))^*_{\mathbb{R}} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
0 \to & H^0(F, M)_{\mathbb{R}} & \to & \operatorname{Ker}\alpha_{M} & \to & H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}} & \to & H^1_{\mathbf{M},f}(F, M)_{\mathbb{R}} & \to & \operatorname{Coker}\alpha_{M} & \to & H^0(F, M^*(1))^*_{\mathbb{R}} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
0 \to & H^0(F, M'')_{\mathbb{R}} & \to & \operatorname{Ker}\alpha_{M''} & \to & H^1_{\mathbf{M},f}(F, M''^*(1))^*_{\mathbb{R}} & \to & H^1_{\mathbf{M},f}(F, M'')_{\mathbb{R}} & \to & \operatorname{Coker}\alpha_{M''} & \to & H^0(F, M''^*(1))^*_{\mathbb{R}} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
 & (1) & & (2) & & (3) & & & & 0 & & 0 &
\end{array}$$

est commutatif et a ses lignes et colonnes exactes.

PREUVE. Les lignes sont les suites exactes $s_f(M')$, $s_f(M)$, $s_f(M'')$, les colonnes sont exactes par 3.1.4; pour la commutativité du diagramme, le seul point délicat se déduit de la fonctorialité de la transformation δ_M.

3.2.9. REMARQUES. i) L'existence et l'exactitude de $s_f(M)$ reviennent pour tout objet M de $\mathbb{M}$ à l'existence de dualités parfaites, fonctorielles en M,

$$H^i_{\mathbf{M},f,\varnothing}(F, M)_{\mathbb{R}} \times H^j_{\mathbf{M},f,\varnothing}(F, M^*(1))_{\mathbb{R}} \to \mathbb{R} \quad \text{pour } i + j = 2.$$

En effet,

- si $i \neq 0, 1, 2$, les deux groupes sont nuls;
- si $i = 2$, $H^2_{\mathbf{M},f,\varnothing}(F, M)_{\mathbb{R}}$ est par définition le conoyau de la flèche
$$H^1_{\mathbf{M},f}(F, M)_{\mathbb{R}} \to \bigoplus_{\mathscr{P} \in S_\infty(F)} \operatorname{Coker}\alpha_{M,\mathscr{P}} = \operatorname{Coker}\alpha_M$$
et $s_f(M)$ dit que ce conoyau s'identifie au dual de $H^0_f(F, M^*(1))_{\mathbb{R}} = H^0_{\mathbf{M},f,\varnothing}(F, M^*(1))_{\mathbb{R}}$;
- de même si $i = 0$, le dual de $H^2_{\mathbf{M},f,\varnothing}(F, M^*(1))_{\mathbb{R}}$ est le noyau de la flèche
$$(\operatorname{Coker}\alpha_{M^*(1)})^* = \operatorname{Ker}\alpha_M \to H^1_{\mathbf{M},f}(F, M^*(1))^*_{\mathbb{R}}$$

et $s_f(M)$ dit que ce noyau s'identifie à $H^0_f(F,M)_{\mathbb{R}} = H^0_{\mathbb{M},f,\varnothing}(F,M)_{\mathbb{R}}$;

- enfin si $i = 1$, par définition, $H^i_{\mathbb{M},f,\varnothing}(F, M)_{\mathbb{R}}$ est le noyau de

$$H^1_{\mathbb{M},f}(F, M)_{\mathbb{R}} \to \operatorname{Coker}\alpha_M$$

tandis que $H^1_{\mathbb{M},f,\varnothing}(F, M^*(1))^*_{\mathbb{R}}$ est le conoyau de

$$\operatorname{Ker}\alpha_M \to (H^1_{\mathbb{M},f}(F, M^*(1))^*_{\mathbb{R}} ;$$

et $s_f(M)$ nous donne un isomorphisme

$$H^1_{\mathbb{M},f,\varnothing}(F, M^*(1))^*_{\mathbb{R}} \to H^1_{\mathbb{M},f,\varnothing}(F, M)_{\mathbb{R}}$$

ce qui est la même chose qu'une dualité parfaite entre $H^1_{\mathbb{M},f,\varnothing}(F, M)_{\mathbb{R}}$ et $H^1_{\mathbb{M},f,\varnothing}(F, M^*(1))_{\mathbb{R}}$.

ii) On peut voir l'accouplement

$$H^1_{\mathbb{M},f,\varnothing}(F, M)_{\mathbb{R}} \times H^1_{\mathbb{M},f,\varnothing}(F, M^*(1))_{\mathbb{R}} \to \mathbb{R}$$

comme une généralisation de l'accouplement de Bloch et Beilinson [**Be87**, **Bl84**] correspondant au cas où $M = H^i(X)((1+i)/2)$ (avec i impair), qui lui-même généralise la hauteur de Néron–Tate (cas où M "=" $H^1(X)(1)$, avec X variété abélienne, [**Ne65**]).

iii) On devrait pouvoir vérifier que l'application $\delta_{M^*(1)}$ s'identifie à l'opposé de la transposée de δ_M (autrement dit l'accouplement ci-dessus est le même qu'on le définisse à partir de M ou à partir de $M^*(1)$) ; la suite exacte $s_f(M^*(1))$ est alors au signe près la suite exacte transposée de $s_f(M)$.

3.3. Catégories pré-motiviques S-admissibles. Dans ce n°, on fixe un ensemble fini S de places de F contenant $S_\infty(F)$ et on suppose que tout objet de $\mathbb{M}$ a bonne réduction en dehors de S.

3.3.1. Pour tout nombre premier ℓ, on dit que *la propriété* $C_{\ell,S}(\mathbb{M})$ *est vraie* si, pour tout objet M de $\mathbb{M}$, la représentation M_ℓ est strictement S-géométrique (II, n°3.2.3), les applications naturelles

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^0(F, M) \to H^0(F, M_\ell) \quad \text{et} \quad \mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^1_{\mathbb{M}}(F, M) \to H^1_g(U_S, M_\ell)$$

sont des isomorphismes et si cette dernière application induit, pour tout sous-ensemble Σ de S contenant $S_\infty(F)$, un isomorphisme

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^1_{\mathbb{M},f,\Sigma}(F, M) \to H^1_{f,\Sigma}(F, M_\ell).$$

On dit que $\mathbb{M}$ est *S-admissible* si elle vérifie $C_\infty(\mathbb{M})$ et $C_{\ell,S}(\mathbb{M})$ pour tout nombre premier ℓ. On voit que si $\mathbb{M}$ est S-admissible, $\mathbb{M}$ est f-admissible (prendre $\Sigma = S_\infty(F)$).

3.3.2. Supposons que $\mathbb{M}$ soit S-admissible. Elle a alors un certain nombre de propriétés très agréables. Par exemple:

a) Elle est de *dimension cohomologique* 1, autrement dit toute suite exacte courte

$$0 \to M' \to M \to M'' \to 0$$

de $\mathbb{M}$ donne naissance à une suite exacte longue

$$\begin{aligned} 0 \to H^0(F, M') \to H^0(F, M) \to H^0(F, M'') \\ \to H^1_{\mathbf{M}}(F, M') \to H^1_{\mathbf{M}}(F, M) \to H^1_{\mathbf{M}}(F, M'') \to 0. \end{aligned}$$

En effet, l'application naturelle

$$H^1_{\mathbf{M}}(F, M) \to H^1_{\mathbf{M}}(F, M'')$$

est surjective, puisque si ℓ est un nombre premier, après tensorisation par $\mathbb{Q}_\ell$, cette application s'identifie à l'application $H^1_g(U_S, M_\ell) \to H^1_g(U_S, M''_\ell)$ qui est surjective puisque M_ℓ est strictement S-géométrique.

b) De même, si

$$0 \to M' \to M \to M'' \to 0$$

est une f-suite exacte courte de $\mathbb{M}$, *la suite*

$$\begin{aligned} 0 \to H^0(F, M') \to H^0(F, M) \to H^0(F, M'') \\ \to H^1_{\mathbf{M},f}(F, M') \to H^1_{\mathbf{M},f}(F, M) \to H^1_{\mathbf{M},f}(F, M'') \to 0 \end{aligned}$$

est exacte. Il suffit en effet de choisir un nombre premier ℓ et de remarquer que, après tensorisation par $\mathbb{Q}_\ell$, cette suite s'identifie à la suite exacte de la proposition II, n°3.3.2.

c) Soient $\Sigma \subset \Sigma'$ deux sous-ensembles de S vérifiant $\Sigma_\infty = \Sigma'_\infty = S_\infty(F)$. Pour tout objet M de $\mathbb{M}$ et tout nombre premier ℓ, *on a une suite exacte*

$$\begin{aligned} 0 \to \mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^1_{\mathbf{M},f,\Sigma}(F, M) \to \mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^1_{\mathbf{M},f,\Sigma'}(F, M) \\ \to \bigoplus_{\mathfrak{p}\in\Sigma'-\Sigma} H^1_{g/f}(F_{\mathfrak{p}}, M_\ell) \to 0 \end{aligned}$$

(ce n'est rien d'autre que la suite exacte, cf. II, n°3.3.1,

$$0 \to H^1_{f,\Sigma}(F, M_\ell) \to H^1_{f,\Sigma'}(F, M_\ell) \to \bigoplus_{\mathfrak{p}\in\Sigma'-\Sigma} H^1_{g/f}(F_{\mathfrak{p}}, M_\ell) \to 0).$$

d) De même, si $\Sigma \subset \Sigma'$ sont deux sous-ensembles de S vérifiant $\Sigma_f = \Sigma'_f$, *on a une suite exacte*

$$\begin{aligned} 0 \to H^1_{\mathbf{M},f,\Sigma}(F, M)_{\mathbf{R}} \to H^1_{\mathbf{M},f,\Sigma'}(F, M)_{\mathbf{R}} \to \bigoplus_{\mathfrak{p}\in\Sigma'-\Sigma} H^1_{/f}(F_{\mathfrak{p}}, M_{(\mathfrak{p})}) \\ \to H^2_{\mathbf{M},f,\Sigma}(F, M)_{\mathbf{R}} \to H^2_{\mathbf{M},f,\Sigma'}(F, M)_{\mathbf{R}} \to 0, \end{aligned}$$

comme on le voit par chasse au diagramme à partir du diagramme commutatif

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \bigoplus_{\mathfrak{p}\in\Sigma'_\infty-\Sigma_\infty} \operatorname{Coker}\alpha_{M,\mathfrak{p}} & & \\
 & & & & & & \downarrow & & \\
0 \to & H^1_{\mathbb{M},f,\Sigma}(F,M)_{\mathbb{R}} & \to & H^1_{\mathbb{M},f,\Sigma\cup S_\infty}(F,M)_{\mathbb{R}} & \to & & \bigoplus_{\mathfrak{p}\in S_\infty-\Sigma_\infty} \operatorname{Coker}\alpha_{M,\mathfrak{p}} & \to & H^2_{\mathbb{M},f,\Sigma}(F,M)_{\mathbb{R}} \to 0 \\
 & \downarrow & & \| & & & \downarrow & & \downarrow \\
0 \to & H^1_{\mathbb{M},f,\Sigma'}(F,M)_{\mathbb{R}} & \to & H^1_{\mathbb{M},f,\Sigma'\cup S_\infty}(F,M)_{\mathbb{R}} & \to & & \bigoplus_{\mathfrak{p}\in S_\infty-\Sigma'_\infty} \operatorname{Coker}\alpha_{M,\mathfrak{p}} & \to & H^2_{\mathbb{M},f,\Sigma'}(F,M)_{\mathbb{R}} \to 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

dont les lignes et les colonnes sont exactes.

4. Structures motiviques et valeurs de fonctions L

4.1. Généralités.

4.1.1. Soit X une variété propre et lisse sur F. Si $i \in \mathbb{N}$, les différentes théories de cohomologie, jointes aux différents théorèmes de comparaison (voir **[Il90]** pour une revue des résultats sur les isomorphismes de comparaison p-adiques) permettent d'associer au couple (X, i) une structure pré-motivique $M = H^i(X)$ (avec des conventions évidentes, $M_{\mathrm{dR}} = H^i_{\mathrm{dR}}(X)$, $M_{B,\mathfrak{p}} = H^i_B(X_{\mathfrak{p}}(\mathbb{C}), \mathbb{Q})$, $M_\ell = H^i_{\text{ét}}(X \times \overline{F}, \mathbb{Q}_\ell))$. Notons $\mathbf{SM}_{F,\mathrm{pl}}(\mathbb{Q})$ la sous-catégorie tannakienne de $\mathbf{SPM}_F(\mathbb{Q})$ engendrée par les $H^i(X)$, pour $i \in \mathbb{N}$ et X propre et lisse sur F. Cette catégorie contient en particulier les $H^i(X)(r)$, pour $i \in \mathbb{N}$, X propre et lisse sur F et $r \in \mathbb{Z}$. La structure pré-motivique $H^i(X)(r)$ est pure de poids $i - 2r$.

4.1.2. Dans la suite de ce paragraphe, on fait semblant de savoir ce qu'est un motif mixte sur F. On admet savoir associer à un tel motif une structure pré-motivique, que l'on appelle *sa réalisation.* On note $\mathbf{SM}_F(\mathbb{Q})$ la sous-catégorie tannakienne de $\mathbf{SPM}_F(\mathbb{Q})$ engendrée par les réalisations des motifs mixtes sur F. Les objets de cette catégorie sont appelés les *structures motiviques.*

La catégorie $\mathbf{SM}_{F,\mathrm{pl}}(\mathbb{Q})$ est une sous-catégorie tannakienne de $\mathbf{SM}_F(\mathbb{Q})$ (on s'attend à ce que $\mathbf{SM}_{F,\mathrm{pl}}(\mathbb{Q})$ soit la sous-catégorie pleine de $\mathbf{SM}_F(\mathbb{Q})$ dont les objets sont ceux qui sont semi-simples).

Il existe des définitions conjecturales de la notion de motif mixte sur F (cf. notamment Jannsen **[Ja90]**, autres textes de ce volume); pour notre propos, il serait satisfaisant de pouvoir choisir une telle définition, mais ce n'est pas essentiel: le but est de donner une description conjecturale "harmonieuse" de quelques propriétés de $\mathbf{SM}_F(\mathbb{Q})$ que l'on espère indépendante des ajustements qui pourraient s'avérer nécessaires pour la définitions des motifs mixtes. Cette description se fera via des "M-conjectures" (i.e. des "méta-conjectures" dépendant de la définition précise de $\mathbf{SM}_F(\mathbb{Q})$).

Pour tout ensemble fini S de places de F, on note $\mathbf{SM}_{F,S}(\mathbb{Q})$ la sous-catégorie tannakienne de $\mathbf{SM}_F(\mathbb{Q})$ dont les objets sont ceux qui ont bonne réduction en dehors de S. Cette catégorie contient les $H^i(X)(r)$, pour X propre et lisse sur F ayant bonne réduction en dehors de S (on convient qu'une variété propre et lisse sur F a bonne réduction en toutes les places infinies de F).

Enfin, on appelle *catégorie motivique* toute sous-catégorie pré-motivique de $\mathbf{SM}_F(\mathbb{Q})$.

4.1.3. Soient S un ensemble fini de places contenant $S_\infty(F)$ et M un objet de $\mathbf{SM}_{F,S}(\mathbb{Q})$. Pour tout $i \in \mathbb{Z}$, on pose

$$H^i_g(F, M) = H^i_{\mathbf{SM}_F(\mathbb{Q})}(F, M) \quad \text{et} \quad H^i_g(U_S, M) = H^i_{\mathbf{SM}_{F,S}(\mathbb{Q})}(F, M).$$

Soient Σ un sous-ensemble de S et $i \in \mathbb{Z}$. Si $\Sigma \supset S_\infty(F)$, on pose $H^i_{f,\Sigma}(F, M) = H^i_{\mathbf{SM}_{F,S}(\mathbb{Q}),f,\Sigma}(F, M)$; dans tous les cas on pose $H^i_{f,\Sigma}(F, M)_{\mathbb{R}} = H^i_{\mathbf{SM}_{F,S}(\mathbb{Q}),f,\Sigma}(F, M)_{\mathbb{R}}$. On pose aussi $H^i_f(F, M) = H^i_{g,S_\infty}(F, M)$.

Tous ces groupes sont indépendants du choix de S. Remarquons que pour calculer l'un d'entre eux, il n'est pas nécessaire de connaître toute la catégorie $\mathbf{SM}_{F,S}(\mathbb{Q})$. Par exemple, étant donné M, pour calculer $H^0_f(F, M) = H^0(F, M)$, il suffit de connaître M en tant que structure pré-motivique, pour calculer $H^1_f(F, M)$, il "suffit" de connaître une sous-catégorie motivique contenant M et stable par f-suite exacte courte de la forme

$$0 \to N' \to N \to \mathbf{1} \to 0.$$

4.1.4. Remarque. Des variantes "mixtes" des conjectures standards, il devrait résulter que le foncteur "réalisations" est une équivalence entre la catégorie des motifs mixtes et celle des structures motiviques. Les conjectures qui vont suivre ne nécessitent pas que cela soit vrai. Plus précisément, pour ce qui va suivre, on a besoin de la notion de motif mixte pour produire suffisamment de structures motiviques (et pas trop); mais on n'a pas besoin de savoir ce qu'est un morphisme de motifs mixtes.

4.1.5. M-conjecture $C_{\mathrm{SM}}(F, S)$. *Soit S un ensemble fini de places de F contenant $S_\infty(F)$. La catégorie $\mathbf{SM}_{F,S}(\mathbb{Q})$ est S-admissible. En outre, chaque objet M de $\mathbf{SM}_{F,S}(\mathbb{Q})$ est strictement L-admissible.*

Cette M-conjecture paraît tout à fait hors d'atteinte pour le moment. Elle implique un formalisme permettant d'énoncer toutes sortes de conjectures sur les valeurs des fonctions L des structures motiviques et de vérifier certaines compatibilités entre ces conjectures. Pour l'énoncé de la conjecture sur la valeur en $s = 0$ d'une structure motivique donnée, on a besoin d'en savoir beaucoup moins!

4.2. L'ordre du zéro de la fonction L en $s = 0$. *Dans toute la fin, M est une structure motivique sur F et $\mathbb{M}$ une catégorie motivique contenant*

M. A chaque fois que l'on utilise l'un des groupes introduits au n°4.1.3, on suppose implicitement que ce groupe reste le même si, dans sa définition, on remplace $\mathbf{SM}_F(\mathbb{Q})$ par $\mathbb{M}$ (ce discours n'a comme seul but que de dire que, pour que chacune des conjectures qui nous intéresse, il n'est pas nécessaire de connaître toute la catégorie $\mathbf{SM}_F(\mathbb{Q})$ mais seulement une sous-catégorie motivique assez grosse $\mathbb{M}$ (cf. le cas des 1-motifs **[Fo92,** §8])).

4.2.1. Soit Σ un ensemble fini de places de F. Disons que la fonction $L_\Sigma(M, s)$ *existe en* $s = 0$ si M est L-admissible en dehors de Σ, si le produit infini $L_\Sigma(M, s)$ défini au n°2.2 converge absolument pour $\mathrm{Re}(s) \gg 0$ et si la fonction ainsi définie admet un prolongement analytique dans un ouvert connexe contenant 0. Lorsque $\Sigma = S_\infty(F)$, on dit que la fonction $L(M, s)$ *existe en* $s = 0$.

Les conjectures $C_{\mathrm{pr.an}}(M)$ du n°2.2.2 et $C_{SM}(F,S)$ du n°4.1.5 impliquent que pour tout M et tout Σ, la fonction $L_\Sigma(M, s)$ existe en 0. En outre, si $\Sigma \subset \Sigma'$ et si $L_\Sigma(M, s)$ existe en 0, la fonction $L_{\Sigma'}(M, s)$ aussi.

4.2.2. Enonçons quelques conjectures sur l'ordre du zéro de la fonction L d'une structure motivique en $s = 0$.

M-CONJECTURE $C_{\mathrm{ord}}(M)$. *La fonction* $L(M, s)$ *existe en* $s = 0$ *et*

$$\mathrm{ord}_{s=0} L(M, s) = \dim_{\mathbb{Q}} H^1_f(F, M^*(1)) - \dim_{\mathbb{Q}} H^0(F, M^*(1)).$$

C'est le cas particulier $\Sigma = S_\infty(F)$ de:

M-CONJECTURE $C_{\mathrm{ord}}(M, \Sigma)$. *Soit* Σ *un ensemble fini de places de* F *contenant* $S_\infty(F)$. *Alors la fonction* $L_\Sigma(M, s)$ *existe en* $s = 0$ *et*

$$\mathrm{ord}_{s=0} L_\Sigma(M, s) = \dim_{\mathbb{Q}} H^1_{f,\Sigma}(F, M^*(1)) - \dim_{\mathbb{Q}} H^0(F, M^*(1)).$$

M-CONJECTURE $C_{\mathrm{ord},\ell}(M, \Sigma)$. *Soient* ℓ *un nombre premier et* Σ *un ensemble fini de places de* F *contenant* $S_\infty(F)$. *Alors la fonction* $L_\Sigma(M, s)$ *existe en* $s = 0$ *et*

$$\mathrm{ord}_{s=0} L_\Sigma(M, s) = \dim_{\mathbb{Q}} H^1_{f,\Sigma}(F, M^*_\ell(1)) - \dim_{\mathbb{Q}} H^0(F, M^*_\ell(1)).$$

S'il existe un ensemble fini de places S de F contenant Σ et une catégorie motivique contenant M qui est S-admissible, $C_{\mathrm{ord},\ell}(M, \Sigma)$ équivaut à $C_{\mathrm{ord}}(M, \Sigma)$. S'il existe un catégorie motivique contenant M qui est f-admissible, $C_{\mathrm{ord},\ell}(M, S_\infty(F))$ équivaut à $C_{\mathrm{ord}}(M)$.

M-CONJECTURE $C'_{\mathrm{ord}}(M, \Sigma)$. *Soit* Σ *un ensemble fini de places de* F. *Alors la fonction* $L_\Sigma(M, s)$ *existe en* $s = 0$ *et*

$$\mathrm{ord}_{s=0} L_\Sigma(M, s) = -\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_{f,\Sigma}(F, M^*(1))_{\mathbb{R}}.$$

Dans le cas particulier où $\Sigma \supset S_\infty(F)$, la définition des $H^1_{f,\Sigma}(F, M^*(1))_{\mathbb{R}}$ implique que $C'_{\mathrm{ord}}(M, \Sigma)$ équivaut à $C_{\mathrm{ord}}(M, \Sigma)$.

4.2.3. PROPOSITION. *Soit* S *un ensemble fini de places de* F *contenant* $S_\infty(F)$. *On suppose* $C_{\mathrm{SM}}(F, S)$ *vraie et que* M *est un objet de* $\mathbf{SM}_{F,S}(\mathbb{Q})$.

i) *Pour tout sous-ensemble* Σ *de* S, $C_{\mathrm{ord}}(M, \Sigma)$ *et* $C_{\mathrm{ord}}(M)$ *sont équivalentes; si de plus* M *est* Σ*-admissible et si* ℓ *est un nombre premier,* $C_{\mathrm{ord}}(M, \Sigma)$ *équivaut à* $C_{\mathrm{ord}}(M_\ell, \Sigma)$ (*cf.* II n°3.4);

ii) *si* M *est comme ci-dessus, l'ordre conjectural, en* $s = 0$, *de* $L_\varnothing(M^*(1), s)$ *est égal à celui de* $L_\varnothing(M, s)$;

iii) *si*
$$0 \to M' \to M \to M'' \to 0$$
est une suite exacte de $\mathbf{SM}_{F,S}(\mathbb{Q})$, *et si* $\Sigma \subset S$, *si deux des trois* M*-conjectures* $C_{\mathrm{ord}}(M', \Sigma)$, $C_{\mathrm{ord}}(M, \Sigma)$, *et* $C_{\mathrm{ord}}(M'', \Sigma)$ *sont vraies, il en est de même de la troisième.*

PREUVE. Lorsque $\Sigma_\infty = S_\infty$, (i) se déduit de l'assertion (i) de la Proposition 3.2.2 (on a en fait $H^2_{f,\Sigma}(F, M) = 0$ dans ce cas) et de l'égalité
$$\dim_{\mathbb{Q}_\ell}(H^1_{g/f}(F_{\mathfrak{p}}, M_\ell)) = -\mathrm{ord}_{s=0}(L_{\mathfrak{p}}(M, s))$$
si $\mathfrak{p}$ est une place finie (I, n°3.4.6). Lorsque $\Sigma_\infty \neq S_\infty$, on se ramène au cas précédent en montrant que les M-conjectures $C_{\mathrm{ord}}(M, \Sigma)$ et $C_{\mathrm{ord}}(M, \Sigma \cup S_\infty)$ sont équivalentes. Pour cela, on utilise la suite exacte de la Proposition 3.2.2, (ii) et l'égalité
$$\dim_{\mathbb{R}}(H^1_{g/f}(F_{\mathfrak{p}}, M_\ell)) = -\mathrm{ord}_{s=0}(L_{\mathfrak{p}}(M, s))$$
lorsque $\mathfrak{p}$ est une place infinie (voir 1.1.9 et 1.2.5).

Lorsque M est Σ-admissible (pour Σ contenant S_∞) et ℓ un nombre premier, on a $L_\Sigma(M, s) = L_\Sigma(M_\ell, s)$. On en déduit par $C_{\ell,S}(\mathrm{M})$ que les M-conjectures $C_{\mathrm{ord}}(M_\ell, \Sigma)$ et $C_{\mathrm{ord}}(M, \Sigma)$ sont équivalentes.

L'assertion (ii) résulte de la dualité entre les $H^i_{f,\varnothing}(F, M^*(1))_{\mathbb{R}}$ et les $H^{2-i}_{f,\varnothing}(F, M)_{\mathbb{R}}$ (remarque (i) du n°3.2.9). Pour (iii), on peut choisir Σ suffisamment grand et contenant $S_\infty(F)$ de manière que M soit Σ-admissible et ait bonne réduction en dehors de Σ. Il suffit de montrer l'assertion (iii) pour cet ensemble Σ. On a alors
$$L_\Sigma(M, s) = L_\Sigma(M', s) \cdot L_\Sigma(M'', s)$$
et on utilise 3.3.2, (b).

4.3. L'équation fonctionnelle.

4.3.1. Soit $\mathbb{A}(F)$ l'anneau des adèles de F. Pour toute place $\mathfrak{p} \in S_f(F)$ (resp. $\in S_\infty(F)$), soit $\psi_{0,\mathfrak{p}}$ le cararatère de $F_{\mathfrak{p}}$ dans $\mathbb{C}^*$ défini par $\psi_{0,\mathfrak{p}}(x) = \exp(-2i\pi \cdot \tau_p(\mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(x)))$ où pour $y \in \mathbb{Q}_p$, $\tau_p(y)$ est un rationnel congru à y modulo $\mathbb{Z}_p$ (resp. $\psi_{0,\mathfrak{p}}(x) = \exp(2i\pi \cdot \mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{R}}(x))$). Soit ψ_0 le caractère additif non trivial de $\mathbb{A}(F) \to \mathbb{C}^*$ dont les composantes locales sont les $\psi_{0,\mathfrak{p}}$.

Son noyau contient F. Choisissons, pour chaque $\mathfrak{p}$, une mesure de Haar $\mu_{\mathfrak{p}}$ sur $F_{\mathfrak{p}}$ de façon que $\mu_{\mathfrak{p}}(\mathscr{O}_{F_{\mathfrak{p}}}) = 1$ pour presque toute place finie et que $\mu = \prod \mu_{\mathfrak{p}}$ soit la mesure de Tamagawa sur $\mathbb{A}(F)$ (c'est-à-dire l'unique mesure de Haar telle que $\mu(\mathbb{A}(F)/F) = 1$).

Supposons que M soit L-admissible partout (cf. 2.2.1). On peut alors définir la fonction L complète $L_{\varnothing}(M, s)$, produit des $L_{\mathfrak{p}}(M, s)$ pour toutes les places $\mathfrak{p}$ de F finies ou non.

Si $\mathfrak{p} \in S_f(F)$, la L-admissibilité en $\mathfrak{p}$ implique que le conducteur $a_{\mathfrak{p}}(D_{pst,\mathfrak{p}}(M_\ell))$ et le facteur $\varepsilon(D_{pst,\mathfrak{p}}(M_\ell), \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}})$ (cf. I, n°1.2.2) sont indépendants de ℓ et nous les notons $a_{\mathfrak{p}}(M)$ et $\varepsilon_{\mathfrak{p}}(M, \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}})$.

Si $\mathfrak{p} \in S_\infty(F)$, le $\mathbb{R}$-espace vectoriel $M_{(\mathfrak{p})}$ a une structure naturelle d'objet de $\mathbf{SH}_{F_{\mathfrak{p}}}(\mathbb{R})$ et on pose $\varepsilon_{\mathfrak{p}}(M, \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}}) = \varepsilon_{\mathfrak{p}}(M_{(\mathfrak{p})}, \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}})$ (cf. n°1.1.10 et 1.2.7).

Pour presque tout $\mathfrak{p}$, on a $a_{\mathfrak{p}}(M) = 0$ et $\varepsilon_{\mathfrak{p}}(M, \psi_{\mathfrak{p}}, \mu_{\mathfrak{p}}) = 1$. Si pour tout $\mathfrak{p} \in S_f(F)$, on note $q(\mathfrak{p})$ le nombre d'éléments du corps résiduel de $F_{\mathfrak{p}}$, la fonction de la variable complexe s,

$$\varepsilon(M, s) = \prod_{\mathfrak{p} \in S(F)} \varepsilon_{\mathfrak{p}}(M, \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}}) \times \prod_{\mathfrak{p} \in S_f(F)} q(\mathfrak{p})^{-a_{\mathfrak{p}}(M)\cdot s},$$

est bien définie et est indépendante du choix des $\mu_{\mathfrak{p}}$.

4.3.2. M-Conjecture $C_{\text{éq.fon}}(M)$. i) *Les structures motiviques M et $M^*(1)$ sont L-admissibles partout.*

ii) *Les produits infinis $L_{\varnothing}(M, s)$ et $L_{\varnothing}(M^*(1), s)$ convergent pour* $\operatorname{Re}(s) \gg 0$ *et admettent un prolongement analytique méromorphie dans tout le plan complexe;*

iii) *on a $L_{\varnothing}(M, s) = \varepsilon(M, s) \cdot L_{\varnothing}(M^*(1), -s)$.*

4.3.3. Pour les motifs purs, on retrouve la conjecture usuelle **[De79]**. La proposition suivante ramène le cas général au cas des motifs purs:

Proposition. *Soit*

$$(\beta) \qquad 0 \to M' \to M \to M'' \to 0$$

une suite exacte courte de structures pré-motiviques. Si deux des propriétés $C_{\text{éq.fon}}(M')$, $C_{\text{éq.fon}}(M)$, $C_{\text{éq.fon}}(M'')$ sont vraies, il en est de même de la troisième.

Preuve. Il est clair que la propriété $C_{\text{éq.fon}}(M)$, (i) est "additive". Posons

$$L_{\mathfrak{p}}(\beta, s) = L_{\mathfrak{p}}(M, s) \cdot L_{\mathfrak{p}}(M', s)^{-1} \cdot L_{\mathfrak{p}}(M'', s)^{-1},$$

$$L_{\mathfrak{p}}(\beta^*(1), s) = L_{\mathfrak{p}}(M^*(1), s) \cdot L_{\mathfrak{p}}(M'^*(1), s)^{-1} \cdot L_{\mathfrak{p}}(M''^*(1), s)^{-1},$$

$$\varepsilon_{\mathfrak{p}}(\beta) = \varepsilon_{\mathfrak{p}}(M, \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}}) \cdot \varepsilon_{\mathfrak{p}}(M', \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}})^{-1} \cdot \varepsilon_{\mathfrak{p}}(M'', \psi_{0,\mathfrak{p}}, \mu_{\mathfrak{p}})^{-1},$$

etc. Il existe un ensemble fini de places S de F (contenant les places à l'infini) tel que $L_{\mathfrak{p}}(\beta, s) = L_{\mathfrak{p}}(\beta^*(1), s) = 1$, $\varepsilon_{\mathfrak{p}}(\beta) = 1$, $a_{\mathfrak{p}}(\beta) = 1$ pour $\mathfrak{p} \notin S$. On en déduit que si deux des propriétés $C_{\text{éq.fon}}(M)$, (ii), $C_{\text{éq.fon}}(M')$, (ii), $C_{\text{éq.fon}}(M'')$, (ii) sont vraies, la troisième l'est. La propriété concernant $C_{\text{éq.fon}}$, (iii) se déduit alors des formules

$$L_{\mathfrak{p}}(\beta, s) = q(\mathfrak{p})^{-a_{\mathfrak{p}}(\beta)\cdot s} \cdot \varepsilon_{\mathfrak{p}}(\beta) \cdot L_{\mathfrak{p}}(\beta^*(1), -s)$$

(I, n°1.2.3 pour $\mathfrak{p} \in S_f$, n°1.2.8 pour $\mathfrak{p} \in S_\infty$).

4.4. La valeur de $L^*(M, 0)$ à un nombre rationnel près.

4.4.1. On introduit les $\mathbb{Q}$-droites

$$L_f(M) = \det_{\mathbb{Q}} H^0(F, M) \otimes (\det_{\mathbb{Q}} H^1_f(F, M))^{-1},$$

$$\Delta_f(M) = L_f(M) \otimes_{\mathbb{Q}} L_f(M^*(1)) \otimes_{\mathbb{Q}} \left(\bigotimes_{\mathfrak{p} \in S_\infty(F)} (\det_{\mathbb{Q}} H^0(F_{\mathfrak{p}}, M_{B,\mathfrak{p}}))^{-1} \right) \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} t_M.$$

Si E est une $\mathbb{Q}$-algèbre non nulle, on utilise l'application $b \mapsto 1 \otimes b$ pour plonger $\Delta_f(M)$ dans $E \otimes_{\mathbb{Q}} \Delta_f(M)$.

4.4.2. La conjecture $C_{\text{SM}}(F, S)$ du n°4.1.5 implique que l'on doit pouvoir choisir $\mathbb{M}$ f-admissible à l'infini (n°3.2.2). Supposons qu'il en est ainsi. La *suite exacte fondamentale à l'infini*, i.e., la suite exacte $s_f(M)$ du n°3.2.5 nous fournit un isomorphisme canonique

$$\mathbb{R} \otimes (L_f(M) \otimes L_f(M^*(1))) \to \det_{\mathbb{R}}(\operatorname{Ker} \alpha_M) \otimes (\det_{\mathbb{R}}(\operatorname{Coker} \alpha_M))^{-1}.$$

Mais la *suite exacte tautologique*

$$0 \to \operatorname{Ker} \alpha_M \to \bigoplus_{\mathfrak{p} \in S_\infty(F)} \mathbb{R} \otimes H^0(F_{\mathfrak{p}}, M_{B,\mathfrak{p}}) \to \mathbb{R} \otimes t_M \to \operatorname{Coker} \alpha_M \to 0$$

fournit un isomorphisme

$$\mathbb{R} \otimes \left(\bigotimes_{\mathfrak{p} \in S_\infty(F)} (\det_{\mathbb{Q}} H^0(F_{\mathfrak{p}}, M_{B,\mathfrak{p}}))^{-1} \right) \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} t_M) \to ((\det_{\mathbb{R}}(\operatorname{Ker} \alpha_M))^{-1} \otimes \det_{\mathbb{R}}(\operatorname{Coker} \alpha_M).$$

Le produit tensoriel de ces deux isomorphismes nous fournit un isomorphisme

$$\iota_M : \mathbb{R} \otimes_{\mathbb{Q}} \Delta_f(M) \to \mathbb{R}.$$

4.4.3. M-CONJECTURE $C_{\mathrm{DB}}(M)$. *Il existe une base b de $\Delta_f(M)$ sur $\mathbb{Q}$ telle que*

$$\iota_M(b) = 1/L^*(M, 0).$$

Bien sûr, si un tel b existe, il est unique. En outre, cette conjecture détermine $L^*(M, 0)$ à un nombre rationnel près.

4.4.4. REMARQUES. i) Modulo des interprétations convenables des groupes de K-théorie, des régulateurs et des hauteurs qui interviennent, la conjecture $C_{\mathrm{DB}}(M)$ pour les motifs purs redonne les conjectures classiques de Deligne et Beilinson (cf. **[Fo92]** et les articles de Jannsen, Nekovar, et Scholl dans ce volume).

ii) Choisissons un ensemble fini S de places de F tel que M a bonne réduction en dehors de S. Supposons $C_{\mathrm{SM}}(F, S)$ (cf. 4.1.5). Soit

$$(\beta) \qquad 0 \to M' \to M \to M'' \to 0$$

une f-suite exacte courte. Des suites exactes

$$\cdots \to H^i_f(F, M') \to H^i_f(F, M) \to H^i_f(F, M'') \to \cdots,$$

$$\cdots \to H^i_f(F, M''^*(1)) \to H^i_f(F, M^*(1)) \to H^i_f(F, M'^*(1)) \to \cdots,$$

$$0 \to t_{M'} \to t_M \to t_{M''} \to 0,$$

$$0 \to H^0(F_\mathfrak{p}, M'_{B,\mathfrak{p}}) \to H^0(F_\mathfrak{p}, M_{B,\mathfrak{p}}) \to H^0(F_\mathfrak{p}, M''_{B,\mathfrak{p}}) \to 0 \quad \text{pour } \mathfrak{p} \in S_\infty,$$

on déduit un isomorphisme canonique

$$\Delta_f(M') \otimes \Delta_f(M'') \simeq \Delta_f(M).$$

On déduit facilement de la proposition 3.2.8 et de 3.3.2, (b) que l'on a

$$\iota_M(\omega' \cdot \omega'') = \iota_{M'}(\omega') \cdot \iota_{M''}(\omega'')$$

si $\omega' \in \Delta_f(M')$, $\omega'' \in \Delta_f(M'')$, et $\omega' \cdot \omega''$ est l'image de $\omega' \otimes \omega''$ dans $\Delta_f(M)$. Il en résulte que si deux des conjectures $C_{\mathrm{DB}}(M)$, $C_{\mathrm{DB}}(M')$, $C_{\mathrm{DB}}(M'')$ sont vraies, la troisième l'est.

On doit pouvoir vérifier que le même résultat est vrai en supprimant l'hypothèse que la suite (β) est f.

iii) Soit F'/F une extension finie. Soit M' une structure motivique sur F' et $M = \mathrm{Res}_{F'/F} M'$ la restriction des scalaires de M de F' à F. On a alors des isomorphismes canoniques

$$H^i_f(F', M') \simeq H^i_f(F, M), \qquad t_M \simeq t_{M'}$$

et

$$\bigoplus_{\mathfrak{p} \in S_\infty(F)} H^0(F_\mathfrak{p}, M_{B,\mathfrak{p}}) \simeq \bigoplus_{\mathfrak{p}' \in S_\infty(F')} H^0(F_{\mathfrak{p}'}, M'_{B,\mathfrak{p}'}).$$

D'où un isomorphisme $\mathrm{Res}_{F'/F}: \Delta_f(M') \simeq \Delta_f(\mathrm{Res}_{F'/F}M')$. Il est de nouveau facile de montrer que si $\omega' \in \Delta_f(M')$, on a $\imath_M(\mathrm{Res}_{F'/F}(\omega')) = \imath_{M'}(\omega')$. On en déduit encore que $C_{\mathrm{DB}}(M)$ est vraie si et seulement si $C_{\mathrm{DB}}(M')$ l'est.

iv) Enfin, on voit par un calcul analogue à celui de **[De79]**, théorème 5.6 (cf. aussi l'exposé de Jannsen dans **[RSS88]**) et en utilisant la remarque 3.2.9, (iii) que, si l'on suppose que $\det(M)$ est un motif d'Artin tordu et que $C_{\text{éq.fon}}(M)$ est vraie, la conjecture $C_{\mathrm{DB}}(M)$ est vraie si et seulement si $C_{\mathrm{DB}}(M^*(1))$ l'est.

4.5. La valeur exacte de $L^*(M, 0)$.

4.5.1. Soit ℓ un nombre premier. La conjecture $C_{\mathrm{SM}}(F, S)$ du n°4.1.5 implique que l'on doit pouvoir choisir $\mathbb{M}$ vérifiant $C_\infty(\mathbb{M})$ et $C_\ell(\mathbb{M})$. Supposons qu'il en soit ainsi. On voit alors que $\mathbb{M}$ est f-admissible à l'infini mais aussi que $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \Delta_f(M)$ s'identifie à $\Delta_f(M_\ell)$ et est donc muni d'une norme canonique $|\cdot|_{\ell, \mathrm{EP}}$.

4.5.2. M-CONJECTURE $C_{\mathrm{BK},\ell}(M)$. *La conjecture $C_{\mathrm{DB}}(M)$ est vraie et si b est l'élément de $\Delta_f(M)$ tel que $\imath_M(b) = 1/L^*(M, 0)$, on a*

$$|b|_{\ell, \mathrm{EP}} = 1 .$$

4.5.3. REMARQUES. i) Soit $c \in \Delta_f(M)$ non nul. Si $C_{\mathrm{BK},\ell}(M)$ est vrai pour tout ℓ, on doit avoir $|c|_{\ell,\mathrm{EP}} = 1$ pour presque tout ℓ. Si c'est le cas, il existe $b' \in \Delta_f(M)$, unique au signe près, tel que $|b'|_{\ell,\mathrm{EP}} = 1$ pour tout ℓ et on doit avoir $L^*(M, 0) = \pm 1/\imath_M(b')$. Ces conjectures $C_{\mathrm{BK},\ell}(M)$ déterminent donc $L^*(M, 0)$ au signe près.

On a aussi une formule conjecturale pour le signe qui s'obtient en conjecturant que $L(M, s)$ n'a ni zéro ni pôle sur la droite réelle en dehors des entiers naturels, en observant que $L^*(M, s) > 0$ pour s réel suffisamment grand, et que la conjecture $C_{\mathrm{ord}}(M(r))$ nous donne l'ordre du zéro (ou pôle) éventuel de $L(M, s)$ en $s = r$.

ii) C'est un exercice facile de déduire des résultats de II, n°5.3 une formulation plus classique des conjectures $C_{\mathrm{BK},\ell}(M)$ en termes du groupe de Shafarevich–Tate et des nombres de Tamagawa (cf. **[Fo92]**).

iii) On déduit de II, n°5.1 et de 4.4.4, (ii) que si $C_{\mathrm{SM}}(F, S)$ est vraie pour un S convenable et si

$$0 \to M' \to M \to M'' \to 0$$

est une f-suite exacte courte, si deux des conjectures $C_{\mathrm{BK}}(M)$, $C_{\mathrm{BK}}(M')$, $C_{\mathrm{BK}}(M'')$ sont vraies, la troisième l'est. Le fait que le même résultat soit vrai pour une suite exacte quelconque (pas nécessairement f) ne résulte pas du formalisme développé jusqu'à présent. Il semble que cela conduise à introduire la notion de "réduction modulo une place finie de F d'une structure motivique semi-stable".

iv) On déduit de II, n°5.2 et de 4.4.4, (iii) que si F'/F est une extension finie et M' une structure motivique sur F', $C_{\mathrm{BK}}(M')$ est vraie si et seulement si $C_{\mathrm{BK}}(\mathrm{Res}_{F'/F}M')$ l'est.

4.5.4. La compatibilité de $C_{\mathrm{BK}}(M)$ à l'équation fonctionnelle est loin d'être évidente. Pour expliquer la situation, revenons d'abord rapidement au cas local.

Soit K comme dans le chapitre I et V une représentation p-adique potentiellement semi-stable de G_K. L'isomorphisme de comparaison induit une application $\mathbb{Q}_p$-linéaire non nulle ξ_V de

$$\Delta_K(V) = (\det_{\mathbb{Q}_p} D_{\mathrm{dR}}(V))^{-1} \otimes (\det_{\mathbb{Q}_p} V)^{\otimes[K:\mathbb{Q}_p]}$$

dans $\mathbb{C}_p \cdot t^r$ pour r convenable. Posons $\tilde{\xi}_V = t^{-r} \cdot \xi_V$. Choisissons un réseau T de V stable par G_K et une base ω de $\Delta_K(V)$. Comme $t_{V^*(1)}$ s'identifie au dual de $\mathrm{Fil}^0 D_{\mathrm{dR}}$, on a

$$\det_{\mathbb{Q}_p} D_{\mathrm{dR}}(V) = (\det_{\mathbb{Q}_p} t_{V^*(1)})^{-1} \otimes \det_{\mathbb{Q}_p} t_V .$$

Soient ω_T une base du sous-$\mathbb{Z}_p$-module $\det_{\mathbb{Z}_p} T$ de $\det_{\mathbb{Q}_p} V$, ω_1 (resp. ω_2) une base de $\det_{\mathbb{Q}_p} t_V$ (resp. $\det_{\mathbb{Q}_p} t_{V^*(1)}$) telles que $\omega = \omega_2 \otimes \omega_1^{-1} \otimes ((\omega_T)^{\otimes[K:\mathbb{Q}_p]})$. Soit d_K le discriminant de $K/\mathbb{Q}_p$ et $\mu_{0,K}$ la mesure de Haar sur K telle que $\mu_{0,K}(\mathscr{O}_K) = 1$. Enfin, si $j \in \mathbb{Z}$, posons $\Gamma^*(j) = \Gamma(j) = (j-1)!$ si $j > 0$; $\Gamma^*(j) = \lim_{s\to j}(s-j)\cdot\Gamma(s)$ si $j \leq 0$ (on a en fait dans ce dernier cas $\Gamma^*(j) = (-1)^j \cdot (-j)!)^{-1}$.

Posons $\eta_V(\omega) = |d_K|^{\dim(V)/2} \cdot \varepsilon(V, \psi_{0,K}, \mu_{0,K}) \cdot \tilde{\xi}_V(\omega)$ (on suppose choisi un plongement de $\overline{\mathbb{Q}}$, fermeture algébrique de $\mathbb{Q}$ dans $\mathbb{C}$, dans $\overline{\mathbb{Q}}_p$); on montre que $\eta_V(\omega)$ appartient à la complétion p-adique de $\mathbb{Q}_p^{nr}$.

Conjecture $C_{\mathrm{EP},K}(V)$. *Soit V une représentation p-adique potentiellement semi-stable de G_K. Avec les notations précédentes, si ω est un élément non nul de $\Delta_K(V)$, on a*

$$\mathrm{Tam}^0_{\omega_1}(T) \cdot \mathrm{Tam}^0_{\omega_2}(T^*(1))^{-1} = \left| \eta_V(\omega) \cdot \prod_j \Gamma^*(-j)^{h_j(V)\cdot[K:\mathbb{Q}_p]} \right|_p$$

où $h_j(V) = \dim_K \mathrm{Fil}^j D_{\mathrm{dR}}(V)/\mathrm{Fil}^{j+1} D_{\mathrm{dR}}(V)$.

Bloch et Kato [**BK90,** Theorem 4.2] ont montré $C_{\mathrm{EP},K}(\mathbb{Q}_p(r))$ pour $r \in \mathbb{Z}$ et $K/\mathbb{Q}_p$ non ramifiée.

Revenons à M. Il n'est pas difficile de montrer (en utilisant II, 5.4) que, si l'on suppose que $\det(M)$ est un motif tordu d'Artin, si les conjectures $C_{\mathrm{EP},F_\mathfrak{p}}(M_{(\mathfrak{p})})$ sont vraies pour toute place $\mathfrak{p} \in S_f(F)$, si la conjecture $C_{\text{éq.fon}}(M)$ est vraie, alors la conjecture $C_{\mathrm{BK}}(M)$ est vraie si et seulement si la conjecture $C_{\mathrm{BK}}(M^*(1))$ est vraie. Nous reviendrons sur ce point dans un autre article ainsi que sur une formulation plus fonctorielle des conjectures $C_{\mathrm{EP},K}(V)$.

Principales notations

01: $H^i(G, V)$.
02: $C^i(G, T)$, $H^i_{\mathrm{cont}}(G, T)$.
03: $\mathbb{Z}_\ell(1)$, $M(i)$.
04: $\det_E(C)$.
05: $\det_A(M)$.

Chap. I. $K, k, p, \overline{K}, G_K, \overline{k}, G_k, I_K, \mathbb{Q}_p^{nr}, K_0, \mathbb{Q}_p', e, \sigma, q, f_k, \ell$.

1.1: W_K, $\mathbf{Rep}_E(W_K)$, ρ, Ta, Ta^{-1}, $\Delta\{-1\}$, $'W_K$, N, $\mathbf{Rep}_{E,Ta}('W_K)$, Ta_0, $\mathbf{Rep}_E('W_K)$, $\nu \in \{e, f, h, g\}$, $\mathbf{Rep}_{E,\nu}('W_K)$, $\mathbf{Rep}_{E,p\nu}('W_K)$.
1.2: ρ^{ss}, Δ^{ss}, $P_K(\Delta, u)$, $a(\Delta)$, $\varepsilon(\Delta, \psi, \mu)$.
1.3: ρ_{sl}, $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'W_K)$, $\mathbf{Rep}_{\mathbb{Q}_\ell,\nu}(D'W_K)$, $\mathbf{Rep}_{\mathbb{Q}_\ell}(D'G_K)$, $\mathbf{Rep}_{\mathbb{Q}_\ell,\nu}(D'G_K)$.
1.4: $H^i_\nu(K, \Delta)$, Δ_{st}, $\Delta_{st}\{-1\}$, $C_\nu(K, \Delta)$.
2.1: $\mathbb{C}_p$, $V_p(U)$, R, $W(R)$, $W_{K_0}(R)$, A^0_{cris}, A_{cris}, B^+_{cris}, B^+_{dR}, θ, $\mathscr{M}_p$, U_p, t, B_{dR}, B_{cris}, $V^0_\ell(\Gamma)$, $B_{st,\ell}$, N.
2.2: $D_{\mathrm{HT}}(V)$, $D_{\mathrm{dR}}(V)$, t_V, $\mathrm{Fil}^i D_{\mathrm{dR}}(V)$, $\widehat{D}_{pst}(V)$, $D_{pst}(V)$, $D_{st}(V)$, $D_{\mathrm{cris}}(V)$, $\mathbf{Rep}_{\mathbb{Q}_\ell,\nu}(G_K)$.
3.1: S_ν.
3.2: $T^\frown$.
3.3: $C_\nu(K, V)$, λ_V, $D_\nu(V)$, $H^i_\nu(K, V)$, $H^i_{/f}(K, V)$.
4.1: $L_f(K, V)$, $H^1_f(K, T)$, $\mathrm{Tam}^0_{K,\omega}(T)$, $\mathrm{Tam}^0_K(T)$.

Chap. II.

1.1: F, $\overline{F}$, $F_\mathfrak{p}$, $\overline{F}_\mathfrak{p}$, G_F, $G_\mathfrak{p}$, $S(F)$, $S_\infty(F)$, $S_f(F)$, $S_p(F)$, Σ_f, Σ_p, S, S_∞, F_S, $G_{F,S}$, $\mathbf{Rep}_{\mathbb{Q}_\ell,S}(G_F)$, $H^i(U_S, M)$, $H^i(F, V)$.
1.2: $H^i_0(U_S, V)$.
1.3: $H^1_g(F, V)$, $H^1_{f,\Sigma}(F, V)$, $H^1_f(F, V)$, $\widetilde{H}^i_f(F, V)$.
2.1: t_V.
3.1: $\mathbf{Rep}_{\mathbb{Q}_\ell,pg}(G_F)$, $\mathbf{Rep}_{\mathbb{Q}_\ell,pg,S}(G_F)$, $H^i_g(U_S, V)$, $H^i_g(F, V)$.
3.2: $D_{pst,\mathfrak{p}}(V)$.
3.3: $H^1_{\nu'/\nu}(F_\mathfrak{p}, V)$, $H^1_{/\nu}(F_\mathfrak{p}, V)$.
3.4: $L_\mathfrak{p}(V, s)$, $L_\Sigma(V, s)$, $L(V, s)$.
4.1: $\Delta_S(V)$, $\Delta_{S,\mathbb{Z}_\ell}(V)$, $|\cdot|_{\mathrm{can}}$.
4.2: $\Delta_f(V)$, $|\cdot|_{\mathrm{EP},V}$.
5.3: $T^\frown$, $\Delta'_f(T)$, $\mathrm{Tam}^0_\omega(T)$, $\text{Ш}(T)$, $\widetilde{\Delta}'_f(T)$.

Chap. III.

1.1: $D_{\mathrm{dR}}(V)$, V_{dR}, $V_{\mathbb{C}}$, W_nV, $\mathrm{Fil}^iV_{\mathrm{dR}}$, $\mathbf{SH}_{\mathbb{C}}(\mathbb{R})$, $\mathbf{1}_{\mathbb{C}}(1)$, γ^iV, δ^iV, $H^i_g(\mathbb{C}, V)$, $H^i(\mathbb{C}, V)$, $H^1_e(\mathbb{C}, V)$, $H^1_f(\mathbb{C}, V)$, α_V, $L(V, s)$.

1.2: A^+, V_{dR}, $D_{\mathrm{dR}}(V)$, α_V^+, $\mathbf{SH}_{\mathbb{R}}(\mathbb{R})$, $\mathbf{1}_{\mathbb{R}}(1)$.

2.1: $\mathbf{PSPM}_F(\mathbb{Q})$, M_{dR}, $M_{B,\mathfrak{p}}$, M_ℓ, $M_{(\mathfrak{p})}$, W_nM, $\mathbf{SPM}_F(\mathbb{Q})$, $\mathbf{1}_F$, $\mathrm{Ext}_{F/F'}$, $\mathrm{Res}_{F'/F}$.

2.2: Σ_M, $L_{\mathfrak{p}}(M, s)$, $L_\Sigma(M, s)$, $L(M, s)$, $L_\varnothing(M, s)$.

3.1: $H^i_{\mathbb{M}}(F, M)$, $\mathbb{M}_S$, $H^i_{\mathbb{M}}(U_S, M)$, $H^i_{\mathbb{M},f,\Sigma}(F, M)$, $M^+_{(\mathfrak{p})}$, $\alpha_{M,\mathfrak{p}}$, $H^i_{\mathbb{M},f,\Sigma}(F, M)_{\mathbb{R}}$, u_M, v_M, v'_M, u'_M.

3.2: $C_\ell(\mathbb{M})$, $\Lambda(M, N)$, δ_M, $s_f(M)$.

3.3: $C_{\ell,S}(\mathbb{M})$.

4.1: $\mathbf{SM}_{F,\mathrm{pl}}(\mathbb{Q})$, $\mathbf{SM}_F(\mathbb{Q})$, $\mathbf{SM}_{F,S}(\mathbb{Q})$, $H^i_g(F, M)$, $H^i_g(U_S, M)$, $H^i_{f,\Sigma}(F, M)$, $H^i_{f,\Sigma}(F, M)_{\mathbb{R}}$, $H^i_f(F, M)$.

4.3: $\Delta_f(M)$, ι_M, $L_f(M)$.

RÉFÉRENCES

[Be85] A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet. Math. **30** (1985), 2036–2070.

[Be87] ——, *Height pairing between algebraic cycles*, Contemp. Math. **67** (1987), 1–24.

[Bl80] S. Bloch, *A note on height pairing, Tamagawa numbers and the Birch and Swinnerton-Dyer conjecture*, Invent. Math. **58** (1980), 65–76.

[Bl84] ——, *Height pairing for algebraic cycles*, J. Pure Appl. Algebra **34** (1984), 119–145.

[BK90] S. Bloch et K. Kato, *L-functions and Tamagawa numbers of motives*, in The Grothendieck Festschrift, vol. 1, Prog. in Math. **86**, Birkhäuser, Boston, 1990, pp. 333–400.

[Bu] *Séminaire sur les périodes p-adiques*, (tenu à Bures en 1988), actes en préparation.

[Ca64] J. W. S. Cassels, *Arithmetic on curves of genus* 1, IV Proof of the Hauptvermutung, J. reine u. angew. Math. **211** (1962), 95–112.

[De71] P. Deligne, *Théorie de Hodge*. II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57.

[De73] ——, *Les constantes de l'équation fonctionnelle des fonctions L*, dans *Modular functions of one variable*. II, Lecture Notes in Math. 349, Springer-Verlag, Berlin, 1973, pp. 501–595.

[De74] ——, *Théorie de Hodge*. III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.

[De79] ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979, pp. 313–346.

[De85] ——, Lettre à C. Soulé (1985).

[DM82] P. Deligne and J. Milne, *Tannakian categories*, Hodge Cycles, Motives and Shimura Varieties, Lecture Notes in Math. **900**, Springer-Verlag, Berlin, 1982, pp. 101–128.

[Den91] C. Deninger, *On the Γ-factors attached to motives*, Invent. Math. **104** (1991), 245–265.

[Fl90] M. Flach, *A generalisation of the Cassels-Tate pairing*, J. reine u. angew. Math. **412** (1990), 113–127.

[Fo83] J.-M. Fontaine, *Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques*, dans Algebraic Geometry Tokyo-Kyoto, Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 86–108.

[Fo92] ——, *Valeurs spéciales des fonctions L des motifs*, Séminaire Bourbaki, exposé 751, février 1992.

[FI] J.-M. Fontaine et L. Illusie, *p-adic periods: A survey*, prépublication d'Orsay, 1990.

[FL82] J.-M. Fontaine et G. Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. **15** (1982), 547–608.

[FP91] J.-M. Fontaine et B. Perrin-Riou, *Autour des conjectures de Bloch et Kato*: I; *Cohomologie galoisienne.* II; *Structures motiviques f-closes.* III—*Le cas général*, C. R. Acad. Sci., Paris, 313, série I, 1991, pp. 189–196, 349–356, et 421–428.

[H] O. Hyodo, $H^1_g(K, V) = H^1_{st}(K, V)$, manuscript non publié (1989).

[Il90] L. Illusie, *Cohomologie de de Rham et cohomologie étale p-adique*, Séminaire Bourbaki, exposé 726, juin 1990.

[Ja89] U. Jannsen, *On the ℓ-adic cohomology of varieties over number fields and its Galois cohomology*, Galois groups over $\mathbb{Q}$, MSRI Publications, vol. 16, Springer, Berlin, 1989, pp. 315–360.

[Ja90] ——, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer, Berlin, 1990.

[Ka] K. Kato, *Iwasawa theory and p-adic Hodge theory*, prépublication, 1991.

[Li72] S. Lichtenbaum, *On the values of zeta and L-functions*. I, Ann. of Math. **96** (1972), 338–360.

[Li73] ——, *Values of zeta functions, étale cohomology and algebraic K-theory*, Algebraic K-theory II, Lecture Notes in Math., vol. 342, Springer, Berlin, 1973.

[Mi86] J. S. Milne, *Arithmetic duality theorems*, Perspectives in Mathematics, vol. 1, Academic Press, Boston, 1986.

[N] J. Nekovář, *On p-adic heights pairings*, Séminaire de théorie des nombres de Paris, 1991.

[Ni] W. Niziol, *On a cohomological functor associated to crystalline representations*, prépublication, 1990.

[Ne65] A. Néron, *Quasi-fonctions et hauteurs sur les variétés abéliennes*, Ann. of Math. **82** (1965), 249–331.

[Po67] G. Poitou, *Cohomologie galoisienne des modules finis*, Dunod, Paris, 1967.

[Sc79] P. Schneider, *Über gewisse Galoiscohomologiegruppen*, Math. Z. **168** (1979), 181–205.

[Sc91] A. J. Scholl, *Remarks on special values of L-functions*, L-functions and Arithmetic, Proc. of the Durham Symposium, London Math. Soc. Lecture Notes Series **153**, Cambridge University Press, 1991, pp. 373–392.

[Se67] J.-P. Serre, *Sur les groupes de Galois attachés aux groupes p-divisibles*, in Proc. of a conference on Local fields, Nuffic Summer School at Driebergen, Springer-Verlag, Berlin, 1967, pp. 118–131 (= Oe.72).

[Se68] ——, *Abelian ℓ-adic representations and elliptic curves*, Benjamin, Inc., New York, 1968; 2ème édition, Addison-Wesley, Reading (1989).

[Se70] ——, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Séminaire Delange-Pisot-Poitou, 1969/70, exp. 19 (= Oe.87).

[ST68] J.-P. Serre et J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517 (= J.-P. Serre, Oe.79).

[So79] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombre et cohomologie étale*, Invent. Math. **55** (1979), 251–295.

[Ta62] J. Tate, *Duality theorems in Galois cohomology over number fields*, Proc. ICM Stockholm (1962), 288–295.

[Ta67] ——, *p-divisible groups*, in Proc. of a Conf. on Local Fields, Nuffic Summer School at Driebergen, Springer-Verlag, Berlin, 1967, pp. 153–183.

[Ta76] ——, *Relations between K_2 and Galois cohomology*, Invent. Math. **36** (1976), 257–274.

[Ta79] ——, *Number theory background*, Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math. **33** (2), Amer. Math. Soc., Providence, RI, 1979, pp. 3–26.

[RSS88] *Beilinson's conjectures on special values of L-functions*, (M. Rapoport, N. Schappacher, P. Schneider, eds.), Perspectives in Mathematics, vol. 4, Academic Press, Boston, 1988.

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Motivic L-Functions and Regularized Determinants

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0. Introduction

In the papers [**De1, De2**] we gave an interpretation of local L-factors of pure motives as regularized characteristic power series on infinite-dimensional cohomologies. This led to speculation on an "arithmetic site" whose global cohomologies would be deeply connected with the global L-series of motives. These arguments suggested, in particular, a formula for the Riemann ξ-function as a regularized characteristic power series, which was proved in [**De2**, §4].

In §§1–6 of this article we extend the above interpretation to the local L-factors of mixed motives. For the finite primes we give an improved construction of the infinite-dimensional cohomologies using an elementary case of the Riemann-Hilbert correspondence. This does away with the semisimplicity assumption we had to make in [**De2**]. This new point of view was noted independently by S. Bloch. We also understand better than in [**De1**] the relation between Archimedean and Deligne cohomology.

Apart from this, our main objective is to discuss in some detail the following aspects of the still speculative "arithmetic cohomology": What form should a Lefschetz fixed point formula take? We mention the relation with explicit formulas in analytic number theory. We give a short "proof" in the spirit of [**Se**] of the Riemann hypotheses assuming that a Hodge $*$-operator with standard properties exists on the prospected cohomologies. Following a classical pattern we relate the functional equation for motivic L-series to Poincaré duality. We "explain" the well-known conjectures on the vanishing and pole order of L-functions at integers by certain cohomological conjectures. We point out relations between a Künneth formula and Kurokawa's multiple zeta functions. In §§1–6 everything is proved and we think of mixed motives in terms of realizations [**D5, J2**]. In the speculative §7 we are not

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precise about the meaning of the word motive in the formal discussions.

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1. Regularized determinants and dimensions

We recall the definition of regularized determinants in the following algebraic setting [**De1**, §1]: Let Θ be an endomorphism of a complex vector space V of countable dimension. We say that $\det_\infty \Theta$ (or $\dim_\infty \Theta$) is defined if the following conditions hold.

(1) V is the direct sum of finite-dimensional Θ-invariant subspaces. For any α in $\mathbb{C}$ there are at most finitely many of these subspaces on which α occurs as an eigenvalue.

This is equivalent to

(1′) $V = \bigoplus_{\alpha\in\mathbb{C}} V_\alpha$, where the V_α are Θ-invariant finite-dimensional subspaces such that α is the only eigenvalue of $\Theta|V_\alpha$.

If (1′) holds then V_α is uniquely determined as $V_\alpha = \operatorname{Ker}(\Theta-\alpha)^n$ for n large enough and we call $m(\alpha) := \dim V_\alpha$ the (algebraic) multiplicity of α. We also write $V^{\Theta\sim\alpha}$ for V_α.

(2) Under condition (1) let $\operatorname{Sp}(\Theta)$ be the set of eigenvalues of Θ with their (algebraic) multiplicities. We assume that the Dirichlet series

$$\sum_{\substack{\alpha\in\operatorname{Sp}\Theta\\ \alpha\neq 0}} \frac{1}{\alpha^s} \quad \text{with } \alpha^{-s} = |\alpha|^{-s} e^{-is(\operatorname{Arg}\alpha)},\ -\pi < \operatorname{Arg}\alpha \le \pi,$$

converges absolutely for $\operatorname{Re} s \gg 0$ and has an analytic continuation denoted $\zeta_\Theta(s)$ to the half-plane $\operatorname{Re} s > -\varepsilon$ for some $\varepsilon > 0$ which is holomorphic at $s = 0$.

Under these conditions we set

$$\dim_\infty(\Theta|V) = \dim V_0 + \zeta_{\tilde{\Theta}}(0),$$

where $\tilde{\Theta}$ is the induced endomorphism of V/V_0, and

$$\det_\infty(\Theta|V) = \begin{cases} \exp(-\zeta'_\Theta(0)) & \text{if } 0 \notin \operatorname{Sp}(\Theta), \\ 0 & \text{if } 0 \in \operatorname{Sp}(\Theta). \end{cases} \tag{1.1}$$

Remark. The choice of the principal branch Arg of arg is compatible with the convention in [**De1**, §1] but different from the one in [**De2**, (2.1)]. It leads to a more uniform expression for local L-factors in terms of regularized characteristic power series than the one in [**De2**].

(1.2) Lemma. *Consider a commutative diagram with exact lines*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Theta'} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta''} & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \end{array}$$

in which $\det_\infty \Theta'$ *and* $\det_\infty \Theta''$ *are defined. Then* $\det_\infty \Theta$ *is defined as well and*

$$\det_\infty \Theta = \det_\infty \Theta' \cdot \det_\infty \Theta'',$$
$$\dim_\infty \Theta = \dim_\infty \Theta' + \dim_\infty \Theta''.$$

Proof. By assumption

$$V' = \bigoplus_\alpha V'_\alpha \quad \text{and} \quad V'' = \bigoplus_\alpha V''_\alpha,$$

where the finite-dimensional subspaces V'_α, V''_α are given by

$$V'_\alpha = \operatorname{Ker}(\Theta' - \alpha)^{n'_\alpha} \quad \text{and} \quad V''_\alpha = \operatorname{Ker}(\Theta'' - \alpha)^{n''_\alpha}$$

for n'_α and n''_α large enough. We claim that for $n_\alpha = n'_\alpha + n''_\alpha$ the natural sequence

$$0 \to V'_\alpha \to \operatorname{Ker}(\Theta - \alpha)^{n_\alpha} \to V''_\alpha \to 0$$

is exact. For $v'' \in V''_\alpha$ choose a preimage v in V. Then $(\Theta - \alpha)^{n''_\alpha} v \in V'$. Hence

$$(\Theta - \alpha)^{n_\alpha} v = \sum_{\mu \neq \alpha} v'_\mu \quad \text{for certain } v'_\mu \in V'_\mu.$$

Since $\Theta' - \alpha$ restricted to V'_μ is an isomorphism for $\mu \neq \alpha$, we find $w'_\mu \in V'_\mu$ with $(\Theta - \alpha)^{n_\alpha} w'_\mu = v'_\mu$. Hence, $v - \sum_{\mu\neq\alpha} w'_\mu$ is a preimage of v'' in $\operatorname{Ker}(\Theta - \alpha)^{n_\alpha}$.

Thus $V = \bigoplus V_\alpha$ with the finite-dimensional Θ-invariant subspaces $V_\alpha = \operatorname{Ker}(\Theta - \alpha)^{n_\alpha}$ is the decomposition as in condition $(1')$. Moreover, it follows that the (algebraic) multiplicity of the eigenvalue α on V is the sum of its multiplicities on V' and V''. The remaining assertions are now obvious.

Note that for a positive real number $\delta > 0$ we have

$$\det_\infty(\delta\Theta | V) = \delta^{\dim_\infty(\Theta|V)} \det_\infty(\Theta|V).$$

In connection with functional equations we will also need the case $\delta = -1$. This will involve regularized superdimensions $\operatorname{sdim}_\infty \theta$ defined as follows:

(1.3) Assume that for a pair (V, Θ) satisfying condition (1) we are given a decomposition $V = V^+ \oplus V^-$ into Θ-invariant subspaces. Then $(V^\pm, \Theta^\pm)$

with $\Theta^{\pm} = \Theta | V^{\pm}$ also satisfy condition (1). We say that $\operatorname{sdim}_{\infty} \Theta$ exists with respect to this decomposition, if the Dirichlet series attached to $\Theta^{\pm}$

$$\sum_{\substack{\alpha \in \operatorname{Sp} \Theta^{\pm} \\ \alpha \neq 0}} \frac{1}{\alpha^s} \quad \text{with } \alpha^{-s} = |\alpha|^{-s} e^{-is(\operatorname{Arg} \alpha)}$$

converge absolutely for $\operatorname{Re} s \gg 0$ and have analytic continuations $\zeta_{\Theta^{\pm}}(s)$ to $\operatorname{Re} s > -\varepsilon$ for some $\varepsilon > 0$ with at most first-order poles at $s = 0$. Writing:

$$\zeta_{\Theta^{\pm}}(s) = \frac{\lambda^{\pm}}{s} + H^{\pm}(s), \qquad \lambda^{\pm} \text{ in } \mathbb{C}, \ H^{\pm} \text{ holomorphic at } s = 0,$$

we set

$$\operatorname{sdim}_{\infty} \Theta = (\dim V_0^+ + H^+(0)) - (\dim V_0^- + H^-(0)).$$

The reason why we have to allow for poles at $s = 0$ of first order will become clear from the discussion below and from considerations on Hecke L-series (7.19), (7.20).

REMARK. If $\dim_{\infty} \Theta^{\pm}$ exists, then $\operatorname{sdim}_{\infty} \Theta$ exists as well and we have

$$\operatorname{sdim}_{\infty} \Theta = \dim_{\infty} \Theta^+ - \dim_{\infty} \Theta^-.$$

In particular, if $\dim V < \infty$ we get

$$\operatorname{sdim}_{\infty} \Theta = \dim V^+ - \dim V^-.$$

We call a decomposition $V = W^+ \oplus W^-$ into Θ-invariant eigenspaces commensurable with (V^+, V^-) if $V^+ \cap W^+$ (resp. $V^- \cap W^-$) is of finite codimension in V^+ and W^+ (resp. in V^- and W^-). Since V decomposes into the generalized eigenspaces V_{α}, we see that commensurability is equivalent to the existence of Θ-invariant decompositions:

$$W^+ = U^+ \oplus F^+, \qquad V^+ = U^+ \oplus E^+,$$
$$W^- = U^- \oplus F^-, \qquad V^- = U^- \oplus E^-,$$

such that E^+, E^-, F^+, F^- are finite dimensional and $F^+ \oplus F^- \cong E^+ \oplus E^-$. Hence we get:

(1.4) LEMMA. *The regularized superdimension of Θ with respect to $V = V^+ \oplus V^-$ exists if and only if it exists with respect to $V = W^+ \oplus W^-$. In case the superdimensions exist their difference is an even integer.*

(1.5) For any pair (V, Θ) as above we define a Θ-invariant decomposition by

$$V^+ = \bigoplus_{\alpha} V_{\alpha}, \quad \text{where } \alpha = 0 \text{ or } \alpha \neq 0 \text{ and } -\pi < \operatorname{Arg} \alpha \leq 0,$$
$$V^- = \bigoplus_{\alpha} V_{\alpha}, \quad \text{where } \alpha \neq 0 \text{ and } 0 < \operatorname{Arg} \alpha \leq \pi.$$

A decomposition of V is called standard, if it is commensurable with this one. We say that the regularized superdimension of Θ exists if it exists with respect to one (and hence any) standard decomposition of V. Note the following simple result.

(1.6) Lemma. *Given* (V, Θ), *assume that* $\det_\infty \Theta$ *and the regularized superdimension of* Θ *exist. Then* $\det_\infty(-\Theta)$ *exists as well and we have*

$$\det_\infty(-\Theta) = e^{i\pi(\operatorname{sdim}_\infty \Theta)} \det_\infty \Theta,$$

where $\operatorname{sdim}_\infty \Theta$ *is made up with respect to any standard decomposition of* V.

Remark. If $\dim_\infty \Theta^{\pm}$ exist, then $\operatorname{sdim}_\infty \Theta$ can be viewed as the η-invariant of Θ in the sense of Atiyah-Patodi-Singer. In this case the lemma is well known (see, e.g., **[Wi2]**).

Proof. According to (1.5) we may assume that $\operatorname{sdim}_\infty \Theta$ is formed with respect to the decomposition in (1.6).

Since

$$\operatorname{Arg}(-\alpha) = \begin{cases} \operatorname{Arg}\alpha + \pi & \text{if } -\pi < \operatorname{Arg}\alpha \le 0, \\ \operatorname{Arg}\alpha - \pi & \text{if } 0 < \operatorname{Arg}\alpha \le \pi, \end{cases}$$

we have

$$\sum_{\substack{\beta \in \operatorname{Sp}(-\Theta) \\ \beta \neq 0}} \frac{1}{\beta^s} = e^{-i\pi s} \sum_{\substack{\alpha \in \operatorname{Sp}\Theta^+ \\ \alpha \neq 0}} \frac{1}{\alpha^s} + e^{i\pi s} \sum_{\substack{\alpha \in \operatorname{Sp}\Theta^- \\ \alpha \neq 0}} \frac{1}{\alpha^s},$$

which converges absolutely for $\operatorname{Re} s$ large. Hence the Dirichlet series on the left is analytically continued to $\operatorname{Re} s > -\varepsilon$, $\varepsilon > 0$ by the function

$$\zeta_{-\Theta}(s) = e^{-i\pi s}\zeta_{\Theta^+}(s) + e^{i\pi s}\zeta_{\Theta^-}(s).$$

On the other hand, we have

$$\zeta_\Theta(s) = \zeta_{\Theta^+}(s) + \zeta_{\Theta^-}(s).$$

Writing

$$\zeta_{\Theta^\pm}(s) = \frac{\lambda^\pm}{s} + H^\pm(s)$$

as in (1.4), we find $\lambda^+ + \lambda^- = 0$ since $\det_\infty \Theta$ is defined. Thus

$$\zeta_{-\Theta}(s) = \frac{1}{s}(\lambda^+ e^{-i\pi s} + \lambda^- e^{i\pi s}) + e^{-i\pi s} H^+(s) + e^{i\pi s} H^-(s)$$

is holomorphic at $s = 0$ and hence $\det_\infty(-\Theta)$ is defined as well. The formula for $\det_\infty(-\Theta)$ follows from an immediate computation.

2. Regularized determinants and Riemann-Hilbert correspondence on $\mathbb{G}_m$

We recall the Riemann-Hilbert correspondence **[D1, H]** in the elementary case where the underlying variety is $\mathbb{G}_m/\mathbb{C}$. Consider a regular singular algebraic differential equation (M, ∇) on $\mathbb{G}_m/\mathbb{C}$. Its sheaf of germs of horizontal sections in the analytic topology defines a local system on $\mathbb{C}^*$ and hence a finite-dimensional complex representation of $\pi_1(\mathbb{C}^*, 1)$. The resulting tensor functor between regular singular differential equations and representations of $\pi_1(\mathbb{C}^*, 1)$ is an equivalence of tensor categories.

Let us fix a choice of $i = \sqrt{-1}$ and hence an orientation of $\mathbb{C}$. We identify $\mathbb{Z}$ with $\pi_1(\mathbb{C}^*, 1)$ by mapping 1 to the loop $e^{2\pi i t}$, $0 \le t \le 1$. Set

$\mathbb{L} = \Gamma(\mathbb{G}_m, \mathscr{O}) = \mathbb{C}[z, z^{-1}]$, $\Theta = z\,d/dz$, and $\Delta = \mathbb{L}[\Theta]$. By D.E.R.S. $(\mathbb{G}_m)$ we denote the category of left $\Delta = \mathbb{L}[\Theta]$-modules D regular singular at $0, \infty$ which are free of finite rank over $\mathbb{L}$. Since $\mathbb{L}$ is principal, we obtain an equivalence $\mathbb{H}$ between D.E.R.S. $(\mathbb{G}_m)$ and the category of finite-dimensional complex representations of $\mathbb{Z}$.

By construction we have

$$\text{rk}_{\mathbb{L}} D = \dim_{\mathbb{C}} \mathbb{H}(D). \tag{2.1}$$

Explicitly the functor $\mathbb{H}$ is given as follows: Let $e: \mathbb{C} \to \mathbb{C}^*$, $e(\tau) = \exp(2\pi i\tau)$ be the universal covering of $\mathbb{C}^*$. We will view the composition

$$\mathbb{L} \subset \mathscr{O}(\mathbb{C}^*) \overset{e^*}{\hookrightarrow} \mathscr{O}(\mathbb{C})$$

as an inclusion of $\mathbb{C}$-algebras. If we let Θ act on $\mathscr{O}(\mathbb{C})$ by the derivation $(2\pi i)^{-1} d/d\tau$ then $\mathscr{O}(\mathbb{C})$ becomes a left Δ-module. The group $\mathbb{Z}$ acts Δ-linearly on $\mathscr{O}(\mathbb{C})$ by translations $(\nu^*\varphi)(\tau) = \varphi(\tau + \nu)$ for ν in $\mathbb{Z}$.

Then we have

$$\mathbb{H}(D) = (D \otimes_{\mathbb{L}} \mathscr{O}(\mathbb{C}))^{\Theta=0}, \tag{2.2}$$

the kernel of $\Theta \hat{=} \Theta \otimes \text{id} + \text{id} \otimes \Theta$ on $D \otimes_{\mathbb{L}} \mathscr{O}(\mathbb{C})$ with the induced $\mathbb{Z}$-operation.

Quite generally the hypercohomology of a regular singular algebraic differential equation equals the cohomology of the corresponding local system **[D1,** Chapter II, 6.2 and 6.3]. In our case we deduce the canonical and elementary isomorphisms

$$H^w(\mathfrak{t}, D) \xrightarrow{\sim} H^w(\mathbb{Z}, \mathbb{H}(D)) \tag{2.3}$$

where the one-dimensional real Lie algebra $\mathfrak{t} = \mathbb{R}$ acts on D by mapping t to $t\Theta$. In other words

$$D^{\Theta=0} \xrightarrow{\sim} \mathbb{H}(D)^{\mathbb{Z}} \quad \text{and} \quad D/\Theta D \xrightarrow{\sim} \mathbb{H}(D)_{\mathbb{Z}}. \tag{2.4}$$

For α in $\mathbb{C}$ let $\mathbb{L}(\alpha)$ denote the Δ-module which as an $\mathbb{L}$-module is $\mathbb{L}$ itself and on which Θ acts by $\Theta_{\mathbb{L}(\alpha)} = \Theta_{\mathbb{L}} - \alpha\,\text{id}$. For λ in $\mathbb{C}^*$ let $\mathbb{C}(\lambda)$ be the $\mathbb{Z}$-module whose underlying vector space is $\mathbb{C}$ and on which $\nu \in \mathbb{Z}$ acts by multiplication with λ^ν. Then there is a natural isomorphism $\mathbb{H}(\mathbb{L}(\alpha)) \cong \mathbb{C}(e(\alpha))$. The following remark is now trivial.

(2.5) Remark. Every nonzero object of D.E.R.S. $(\mathbb{G}_m)$ is a successive extension of objects $\mathbb{L}(\alpha)$.

An object D of D.E.R.S. $(\mathbb{G}_m)$ is in particular a $\mathbb{C}$-vector space with an action by Θ. Let $\text{Sp}(\Theta)$ denote the set of eigenvalues of Θ on D. Write F for the (inverse of the monodromy) automorphism on $\mathbb{H}(D)$ given by the action of $-1 \in \mathbb{Z}$. With this notation we have:

(2.6) Corollary. *As a $\mathbb{C}$-vector space D decomposes into a countable direct sum of finite-dimensional Θ-invariant subspaces. Assigning (algebraic)* *multiplicities to eigenvalues, we have* $\mathrm{Sp}(\Theta) = e^{-1}\,\mathrm{Sp}(F)$ *as sets with multiplicities.*

Proof. On $\mathbb{L}(\alpha)$ viewed as a $\mathbb{C}$-vector space Θ has eigenvalues $\nu - \alpha$ for $\nu \in \mathbb{Z}$ with multiplicity one. Since the only eigenvalue of F on $\mathbb{C}(e(\alpha))$ is $e(-\alpha)$, we obtain the assertion for $\mathbb{L}(\alpha)$. The general case follows by induction using (2.5) and the proof of (1.3).

Remark. For objects D of D.E.R.S.$(\mathbb{G}_m)$ and representations H of $\mathbb{Z}$ we introduce twists by

$$D(\alpha) = D \otimes_{\mathbb{L}} \mathbb{L}(\alpha) \quad \text{and} \quad H(\lambda) = H \otimes_{\mathbb{C}} \mathbb{C}(\lambda) \qquad \text{for } \alpha \in \mathbb{C},\, \lambda \in \mathbb{C}^* .$$

Clearly,

$$D(\alpha)^{\Theta=0} = D^{\Theta=\alpha} \quad \text{and} \quad H(\lambda)^{F=\mathrm{id}} = H^{F=\lambda\,\mathrm{id}} .$$

Since $\mathbb{H}$ is a tensor functor, we have natural isomorphisms:

$$\mathbb{H}(D(\alpha)) \cong \mathbb{H}(D)(e(\alpha));$$

hence, applying (2.4) to $D(\alpha)$ we get $D^{\Theta=\alpha} \xrightarrow{\sim} \mathbb{H}(D)^{F=e(\alpha)}$. It follows again that α is an eigenvalue of Θ if and only if $e(\alpha)$ is an eigenvalue of F. Moreover, their geometric multiplicities are equal.

(2.7) Lemma. *For* $\gamma \in \mathbb{C}^*$, $z \in \mathbb{C}$ *consider the regularized product* **[Del, §1]**

$$\prod_{\nu\in\mathbb{Z}} \gamma(z+\nu) := \begin{cases} \exp(-\zeta'_{\gamma,z}(0)) & \text{if } z \notin \mathbb{Z}, \\ 0 & \text{if } z \in \mathbb{Z}, \end{cases}$$

where $\zeta_{\gamma,z}(s) = \sum_{\nu\in\mathbb{Z}}[\gamma(z+\nu)]^{-s}$ *is defined for* $\mathrm{Re}\, s > 1$ *by taking* $-\pi < \mathrm{Arg}(\gamma(z+\nu)) \leq \pi$. *Then we have*

$$\prod_{\nu\in\mathbb{Z}} \gamma(z+\nu) = \begin{cases} 1 - e^{-2\pi i z} & \text{if } \mathrm{Im}\,\gamma > 0 \text{ or } \gamma > 0,\ \mathrm{Im}\, z < 0, \\ & \text{or if } \gamma < 0,\ \mathrm{Im}\, z \leq 0; \\ 1 - e^{2\pi i z} & \text{if } \mathrm{Im}\,\gamma < 0 \text{ or } \gamma > 0,\ \mathrm{Im}\, z \geq 0, \\ & \text{or if } \gamma < 0,\ \mathrm{Im}\, z > 0. \end{cases}$$

Proof. The assertion is trivial if z is an integer. We use the Hurwitz zeta function $\zeta(s, z)$ which is defined for $\mathrm{Re}\, s > 1$, $z \neq 0, -1, -2, \ldots$ by the series

$$\zeta(s, z) = \sum_{\nu=0}^{\infty} \frac{1}{(z+\nu)^s}, \qquad -\pi < \mathrm{Arg}(z+\nu) \leq \pi,$$

with analytic continuation to s in $\mathbb{C}\backslash\{1\}$. It is known that

$$\zeta(0, z) = \tfrac{1}{2} - z \quad \text{and} \quad \partial_s\zeta(0, z) = \log\Gamma(z) - \tfrac{1}{2}\log 2\pi$$

for a suitable branch of $\log\Gamma(z)$.

For a complex number $\gamma \neq 0$ we introduce the functions

$$\zeta_\gamma(s, z) = \sum_{\nu=0}^{\infty} \frac{1}{(\gamma(z+\nu))^s}, \qquad -\pi < \arg\gamma(z+\nu) \leq \pi,$$

and

$$\zeta_\gamma^-(s, z) = \sum_{\nu=0}^{\infty} \frac{1}{(\gamma(z-\nu))^s}, \qquad -\pi < \arg\gamma(z-\nu) \le \pi$$
$$= \zeta_{-\gamma}(s, -z).$$

If $\gamma \neq 0$ is not a negative real number, then we have

$$\operatorname{Arg}(\gamma(z+\nu)) = \operatorname{Arg}\gamma + \operatorname{Arg}(z+\nu) \quad \text{for almost all } \nu \ge 0$$

since $\lim_{\nu\to\infty} \operatorname{Arg}(z+\nu) = 0$ and $-\pi < \operatorname{Arg}\gamma < \pi$. Hence we have

$$\zeta_\gamma(s, z) = \gamma^{-s}\tilde{\zeta}(s, z),$$

where $\tilde{\zeta}(s, z)$ differs from the Hurwitz zeta function only by taking non-principal arguments in the definition of $(z+\nu)^{-s}$ for at most finitely many ν. Therefore, we still have

$$\tilde{\zeta}(0, z) = \frac{1}{2} - z \quad \text{and} \quad \exp(-\partial_s\tilde{\zeta}(0, z)) = \left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1}$$

and hence

$$\text{(2.7.1)}\quad \zeta_\gamma(0, z) = \frac{1}{2} - z \quad \text{and} \quad \exp(-\partial_s\zeta_\gamma(0, z)) = \gamma^{1/2-z}\left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1}.$$

If $\gamma < 0$, we have for almost all $\nu \ge 0$

$$\operatorname{Arg}(\gamma(z+\nu)) = \begin{cases} \operatorname{Arg}(z+\nu) + \pi & \text{if } \operatorname{Im} z \le 0, \\ \operatorname{Arg}(z+\nu) - \pi & \text{if } \operatorname{Im} z > 0; \end{cases}$$

hence, by the same argument as before

(2.7.2)

$$\zeta_\gamma(0, z) = \tfrac{1}{2} - z \quad \text{and}$$

$$\exp(-\partial_s\zeta_\gamma(0, z)) = \begin{cases} |\gamma|^{\frac{1}{2}-z} e^{i\pi(\frac{1}{2}-z)}(\frac{1}{\sqrt{2\pi}}\Gamma(z))^{-1} & \text{if } \operatorname{Im} z \le 0, \\ |\gamma|^{\frac{1}{2}-z} e^{-i\pi(\frac{1}{2}-z)}(\frac{1}{\sqrt{2\pi}}\Gamma(z))^{-1} & \text{if } \operatorname{Im} z > 0. \end{cases}$$

We have

$$\zeta_{\gamma, z}(s) = \zeta_\gamma(s, z) + \zeta_\gamma^-(s, z) - (z\gamma)^{-s}$$

and hence $\zeta_{\gamma, z}(0) = 0$. The claim now follows from the equation

$$\prod_{\nu\in\mathbb{Z}} \gamma(z+\nu) = \exp(-\partial_s\zeta_\gamma(0, z))\exp(-\partial_s\zeta_{-\gamma}(0, -z))(z\gamma)^{-1}$$

using the formula

$$\frac{1}{z}\left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1}\left(\frac{1}{\sqrt{2\pi}}\Gamma(-z)\right)^{-1} = i(e^{i\pi z} - e^{-i\pi z}).$$

REMARK. The case $\gamma = 1$ of the lemma was pointed out to me by N. Kurokawa in a response to **[De2,** (2.3)].

From (2.6), (2.7), and (1.7) we draw the following:

(2.8) Corollary. *Fix real numbers $\delta > 0$ and $q > 1$. For any D in D.E.R.S.$(\mathbb{G}_m)$ set $\Theta_q = (2\pi i/\log q)\Theta$ acting on D. Then we have for complex s:*

$$\det_\infty(\delta(s \pm \Theta_q)|D) = \det(1 - q^{-s}F^{\mp 1}|\mathbb{H}(D)). \tag{2.9}$$

A standard decomposition $D = D^+ \oplus D^-$ of D with respect to the operator $-\Theta_q$ is also standard for any operator $\delta(s - \Theta_q)$ with s in $\mathbb{C}$, and we have

$$\det_\infty(-\delta(s - \Theta_q)|D) = \varepsilon(s)\det_\infty(\delta(s - \Theta_q)|D), \tag{2.10}$$

where

$$\begin{aligned}\varepsilon(s) &= \exp i\pi(\dim_\infty(\delta(s - \Theta_q)|D^+) - \dim_\infty(\delta(s - \Theta_q)|D^-)) \\ &= (-q^s)^{\dim \mathbb{H}(D)} \det(F|\mathbb{H}(D))^{-1}.\end{aligned}$$

Proof. For any $\lambda \neq 0$ let τ_λ be a complex number with $e(\tau_\lambda) = \lambda$. According to (2.6) the eigenvalues of $\delta(s \pm \Theta_q)$ are the numbers

$$\frac{2\pi i\delta}{\log q}\left(\left(\frac{s\log q}{2\pi i} \pm \tau_\lambda\right) + \nu\right) \quad \text{for } \lambda \in \mathrm{Sp}(F),\ \nu \in \mathbb{Z},$$

with the appropriate multiplicities. Now the first formula follows from (2.7).

It is clear that a decomposition $D = D^+ \oplus D^-$ is standard for $-\Theta_q$ if and only if it is standard with respect to $\delta(s - \Theta_q)$ for any value of s. Thus (1.7) implies formula (2.10) for

$$\varepsilon(s) = \exp i\pi(\dim_\infty(\delta(s - \Theta_q)|D^+) - \dim_\infty(\delta(s - \Theta_q)|D^-))$$

taking into account that the regularized dimensions exist by the following argument. Let D^+ (resp. D^-) be the direct sum of the generalized eigenspaces of the operator $-\Theta_q$ for the eigenvalues $2\pi i(-\tau_\lambda + \nu)/\log q$ for $\nu \leq 0$ (resp. $\nu > 0$) and $\lambda \in \mathrm{Sp}(F)$. Then the decomposition $D = D^+ \oplus D^-$ is standard for all operators $\delta(s - \Theta_q)$, and using (2.7.1) and (2.7.2) we have for *all* complex s

$$\dim_\infty(\delta(s - \Theta_q)|D^+) = \sum_{\lambda \in \mathrm{Sp}(F)} \left(\frac{1}{2} + \frac{s\log q}{2\pi i} - \tau_\lambda\right)$$

and

$$\dim_\infty(\delta(s - \Theta_q)|D^-) = \sum_{\lambda \in \mathrm{Sp}(F)} \left(-\frac{1}{2} - \frac{s\log q}{2\pi i} + \tau_\lambda\right).$$

Hence,

$$\begin{aligned}&\exp \pi i(\dim_\infty(\delta(s - \Theta_q)|D^+) - \dim_\infty(\delta(s - \Theta_q)|D^-)) \\ &= \prod_{\lambda \in \mathrm{Sp}(F)} \exp \pi i(1 + \tfrac{s\log q}{\pi i} - 2\tau_\lambda) \\ &= (-q^s)^{\dim \mathbb{H}(D)} \det(F|\mathbb{H}(D))^{-1}.\end{aligned}$$

REMARK. The following particular case of formula (2.9):

$$\det_\infty(i\Theta|D) = \det(1 - F^{-1}|\mathbb{H}(D))$$

is closely related with Lemma 2 in **[A]**. Note that F^{-1} is the usual monodromy operator.

(2.11) For $q > 0$ let $\mathscr{D}_q$ be the category D.E.R.S. $(\mathbb{G}_m)$ but with the following notion of twist: Any object D of $\mathscr{D}_q$ is viewed as a representation of the one-dimensional real Lie algebra $\mathfrak{t} = \mathbb{R}$ by mapping 1 to $\Theta_q = (2\pi i/\log q)\Theta$. For α in $\mathbb{C}$ the twist $D(\alpha)$ of D in $\mathscr{D}_q$ is defined to be D itself as an $\mathbb{L}$-module but with $\mathfrak{t}$-action given by

$$\Theta_{D(\alpha),q} = \Theta_{D,q} - \alpha\,\mathrm{id}; \quad \text{i.e.,}\ \Theta_{D(\alpha)} = \Theta_D - \frac{\alpha\log q}{2\pi i}\,\mathrm{id}\,.$$

We have natural isomorphisms

$$\mathbb{H}(D(\alpha)) \cong \mathbb{H}(D)(q^\alpha)\,.$$

(2.12) Since $\mathbb{H}$ is an equivalence of categories, we can choose a quasi-inverse functor $\mathbb{D}$. For the sequel it will not be important which quasi-inverse we take. Nonetheless we mention the following canonical choice to explain the relation with the construction in **[De2]**.

Let $B = \mathbb{C}[\mathbb{C}]$ be the group algebra over $\mathbb{C}$ with coefficients in $\mathbb{C}$. The typical element will be written in the form $\sum r_\alpha e^\alpha$ with α, r_α in $\mathbb{C}$ and a *symbol* e^α obeying the rule $e^{\alpha+\alpha'} = e^\alpha e^{\alpha'}$. We can view B as a subalgebra of $\mathscr{O}(\mathbb{C})$ by mapping e^α to the function $\tau \mapsto \exp(\alpha\tau)$. Then B inherits a $\mathbb{Z}$- and a Δ-action from $\mathscr{O}(\mathbb{C})$:

$$\nu^*\left(\sum r_\alpha e^\alpha\right) = \sum r_\alpha \exp(\alpha\nu)e^\alpha \quad \text{for } \nu \text{ in } \mathbb{Z},$$

$$\mathbb{L} = B^{\mathbb{Z}}, \qquad \Theta_B\left(\sum r_\alpha e^\alpha\right) = \sum (2\pi i)^{-1}\alpha r_\alpha e^\alpha\,.$$

For a finite-dimensional complex representation H of $\mathbb{Z}$ let F be the automorphism corresponding to the action of -1. Decompose F into its semisimple and unipotent parts $F = F_{\mathrm{s}}F_{\mathrm{u}}$. Let H_{s} be the representation of $\mathbb{Z}$ on H where -1 acts by F_{s}. We set

$$\mathbb{D}(H) = (H_{\mathrm{s}} \otimes_{\mathbb{C}} B)^{\mathbb{Z}}\,.$$

This is naturally an $\mathbb{L}$-module, and it becomes a Δ-module by letting Θ act by

$$\frac{1}{2\pi i}\log F_{\mathrm{u}} \otimes \mathrm{id}_B + \mathrm{id}_{H_{\mathrm{s}}} \otimes \Theta_B\,.$$

Decomposing H into eigenspaces of F_{s} and noting that the λ-eigenspace of $(-1)^*$ on B is isomorphic to $\mathbb{L}(\tau_\lambda)$ for any τ_λ such that $e(\tau_\lambda) = \lambda$, we see that $\mathbb{D}(H)$ is an object of D.E.R.S. $(\mathbb{G}_m)$.

To see that $\mathbb{D}$ is a quasi-inverse to $\mathbb{H}$, we proceed as follows. The map

$$H \to (H_s \otimes \mathscr{O}(\mathbb{C}))^{\Theta=0},$$
$$h \mapsto F_{\mathrm{u}}^{-\tau}(h) = \exp(-\tau \log F_{\mathrm{u}})(h) = \sum_{\nu=0}^{\infty}(-\log F_{\mathrm{u}})^{\nu}(h) \otimes \frac{\tau^{\nu}}{\nu!}$$

is an isomorphism of $\mathbb{C}$-vector spaces by the theory of ordinary differential equations. It is also $\mathbb{Z}$-equivariant:

$$(-1)^* F_{\mathrm{u}}^{-\tau}(h) = F_{\mathrm{u}}^{-(\tau-1)}(F_{\mathrm{s}}h) = F_{\mathrm{u}}^{-\tau}(F_{\mathrm{u}}F_{\mathrm{s}}h)$$
$$= F_{\mathrm{u}}^{-\tau}((-1)^* h) \quad \text{for } h \text{ in } H.$$

Hence, we obtain a natural transformation $\mathbb{HD} \to \mathrm{id}$ defined by the commutative diagram:

$$\begin{array}{ccc} \mathbb{HD}(H) = ((H_{\mathrm{s}} \otimes_{\mathbb{C}} B)^{\mathbb{Z}} \otimes_{\mathbb{L}} \mathscr{O}(\mathbb{C}))^{\Theta=0} & \longrightarrow & (H_{\mathrm{s}} \otimes_{\mathbb{C}} B \otimes_{\mathbb{L}} \mathscr{O}(\mathbb{C}))^{\Theta=0} \\ \downarrow & & \downarrow \text{id} \otimes \text{multipl.} \\ H & \xrightarrow{\sim} & (H_{\mathrm{s}} \otimes_{\mathbb{C}} \mathscr{O}(\mathbb{C}))^{\Theta=0} \end{array}$$

One checks that it induces isomorphisms $\mathbb{HD}(\mathbb{C}(\lambda)) \xrightarrow{\sim} \mathbb{C}(\lambda)$ for all λ in $\mathbb{C}^*$. Since $\mathbb{H}$ and $\mathbb{D}$ are exact and since every $\mathbb{Z}$-representation is a successive extension of $\mathbb{C}(\lambda)$'s, we find that $\mathbb{HD} \to \mathrm{id}$ is an isomorphism of functors. Hence $\mathbb{H}$ is a quasi-inverse of $\mathbb{D}$.

REMARK. In **[De2]** we took the derivation $\mathrm{id}_{H_s} \otimes \Theta_B$ on $\mathbb{D}(H)$ which is the right one only if F is semisimple. Thus we had to assume certain (conjectured) semisimplicity properties of the Frobenius action on l-adic cohomology in §2 of loc. cit. These assumptions can now be discarded.

3. The non-Archimedean local L-factors

Using the results of §2 we rewrite the local non-Archimedean L-factors of a motive in terms of regularized characteristic power series.

Let K be a local non-Archimedean field with prime ideal $\mathfrak{p}$ in $\mathscr{O}_K$, inertia group I, and geometric Frobenius automorphism F in $\mathrm{Gal}(\overline{\kappa}/\kappa)$, where $\kappa = \mathscr{O}_K/\mathfrak{p}$. Fix a prime number l different from the residue characteristic of K and an embedding $\iota: \mathbb{Q}_l \hookrightarrow \mathbb{C}$. Now assume that $\mathrm{char}\, K = 0$. For a finite extension $E/\mathbb{Q}$ let $\mathscr{MM}_K(E)$ be the category of mixed motives over K in the sense of **[D5]** or **[J2]** with multiplication by E. If $M_l = H^\bullet(M \otimes_K \overline{K}, \mathbb{Q}_l)$ denotes the l-adic realization of M, we obtain a functor

$$M \mapsto M_l^I \otimes_{\mathbb{Q}_{l,\iota}} \mathbb{C} = M_{l,\iota}^I \tag{3.1}$$

from $\mathscr{MM}_K(E)$ into the category of $(E \otimes \mathbb{C})[F]$-modules of finite rank over $E \otimes \mathbb{C}$. It is expected that these functors for different l and ι are isomorphic **[T, (4.2.4)]**. Since $E \otimes \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$, we may view $M_{l,\iota}^I$ as an array of complex vector spaces

$$M_{l,\iota}^I = (M_{l,\iota,\sigma}^I)_{\sigma \in \mathrm{Hom}(E,\mathbb{C})}, \quad \text{where } M_{l,\iota,\sigma}^I = M_{l,\iota}^I \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}.$$

The $(E \otimes \mathbb{C})$-valued local L-factor of M is defined by

$$L_K(M, s) = (\det_{\mathbb{C}}(1 - FN\mathfrak{p}^{-s}|M^I_{l,\iota,\sigma})^{-1})_{\sigma\in \mathrm{Hom}(E,\mathbb{C})}.$$

A priori it depends on the choice of l and ι although by the above remark it is expected to be independent of this choice. In certain cases this independence is known [T, (4.3.1)].

For any $\mathbb{Q}$-linear category $\mathscr{A}$ let $\mathscr{A}(E)$ denote the E-linear category of objects A in $\mathscr{A}$ with multiplication by E:

$$E \to \mathrm{End}\, A, \qquad 1 \mapsto \mathrm{id}_A$$

and with the evident morphisms.

Composing the functor (3.1) with a quasi-inverse $\mathbb{D}$ to the Riemann-Hilbert correspondence $\mathbb{H}$ on $\mathbb{G}_m/\mathbb{C}$ we obtain an E-linear functor

(3.1.1) $$\mathscr{F} = \mathscr{F}_{l,\iota}: \mathscr{M}\mathscr{M}_K(E) \to \text{D.E.R.S. } (\mathbb{G}_m)(E).$$

Note that via $\mathscr{F}^\bullet(M) := \mathscr{F}(H^\bullet(M))$ the functor $\mathscr{F}$ is naturally $\mathbb{Z}$-graded. We equip $\mathscr{F}(M)$ with a Lie-algebra ind-representation of $\mathfrak{t}$ by sending t to $t\Theta_{N\mathfrak{p}}$. We will view $\mathscr{F}$ as a functor to $\mathscr{D}_{N\mathfrak{p}}(E)$. Up to natural isomorphisms it commutes with twists by integers.

Let $\mathscr{M}\mathscr{M}_K^{\text{good}}(E)$ denote the full subcategory of motives M with good reduction in the sense that $M_l^I = M_l$. The restriction $H^\bullet(\_/\mathbb{L})$ of $\mathscr{F}^\bullet$ to $\mathscr{M}\mathscr{M}_K^{\text{good}}(E)$ is a tensor functor between Tannakian categories. In particular, $H^\bullet(\_/\mathbb{L})$ can be viewed as a cohomology theory on motives in the sense of Grothendieck.

Let us write $H^w(X/\mathbb{L}) = H(H^w(X)/\mathbb{L})$ for smooth projective varieties X over K such that $H^w(X)$ has good reduction $(E = \mathbb{Q})$. For this cohomology theory $H^w(\ /\mathbb{L})$ we have Poincaré duality, a Künneth formula, Chern classes, etc. The definition of $H^w(\_/\mathbb{L})$ via l-adic cohomology is of course not satisfactory. It just shows that such a theory exists. An independent construction would be of great interest. It would also be important to know what the groups $H^w(\_/\mathbb{L})$ should be in the bad reduction case. There we have only constructed the analogue of $H^w(X_{\overline{K}}, \mathbb{Q}_l)^I$.

Up to now we have kept $\mathbb{L}$ fixed and chosen $\Theta_{N\mathfrak{p}}$ which depends on the field K as our preferred derivation. However, we can also keep the derivation fixed and vary the spaces as follows: For $q > 1$ consider the subring

$$\mathbb{L}_q = \mathbb{C}\left[\exp\left(\frac{2\pi i}{\log q}\xi\right), \exp\left(-\frac{2\pi i}{\log q}\xi\right)\right] \subset \mathscr{O}(\mathbb{C})$$

equipped with the derivation $\Theta = d/d\xi$, and set $\mathbb{L}_{\mathfrak{p}} = \mathbb{L}_{N\mathfrak{p}}$. By the change of variable $z = \exp(\frac{2\pi i}{\log N\mathfrak{p}}\xi)$ we can identify the pair $(\mathbb{L}, \Theta_{N\mathfrak{p}})$ with $(\mathbb{L}_{\mathfrak{p}}, \Theta)$. As $(\mathbb{L}_{\mathfrak{p}}, \Theta)$-module we write $(\mathscr{F}_{\mathfrak{p}}, \Theta)$ for the pair $(\mathscr{F}, \Theta_{N\mathfrak{p}})$.

Let us write $\mathscr{D}_{\mathfrak{p}}$ for the category with twists $\mathscr{D}_{N\mathfrak{p}}$ of (2.11) if we make the identification $(\mathbb{L}, \Theta_{N\mathfrak{p}}) = (\mathbb{L}_{\mathfrak{p}}, \Theta)$ in its construction. Thus we view $\mathscr{F}$

and $H^\bullet(\_/\mathbb{L})$ as functors:

$$\mathscr{F}_{\mathfrak{p}} = \mathscr{F}_{\mathfrak{p},l,\iota}\colon \mathscr{MM}_K(E) \to \mathscr{D}_{\mathfrak{p}}(E) \tag{3.1.2}$$

and

$$H^\bullet(\_/\mathbb{L}_{\mathfrak{p}})\colon \mathscr{MM}_K^{\text{good}}(E) \to \mathscr{D}_{\mathfrak{p}}(E).$$

Note that there is an isomorphism of categories

$$\mathscr{D}_{\mathfrak{p}}(E) \cong \prod_{\sigma\in\operatorname{Hom}(E,\mathbb{C})} \mathscr{D}_{\mathfrak{p}}, \qquad D \mapsto (D_\sigma),$$

where $D_\sigma = D \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}$.

(3.2) Proposition. *For M in $\mathscr{MM}_K(E)$, s in $\mathbb{C}$, $\delta > 0$ we have*

$$L_K(M, s) = \det{}_\infty(\delta(s-\Theta)|\mathscr{F}_{\mathfrak{p}}(M)_\sigma)^{-1}_{\sigma\in\operatorname{Hom}(E,\mathbb{C})}$$

and, in particular, for a smooth projective variety X/K with $H^w(X)$ of good reduction

$$L_K(H^w(X), s) = \det{}_\infty(\delta(s-\Theta)|H^w(X/\mathbb{L}_{\mathfrak{p}})_\sigma)^{-1}_{\sigma\in\operatorname{Hom}(E,\mathbb{C})}.$$

Here it is understood that the same pair l, ι is used for the definition of the local factor and for the definition of $\mathscr{F}_{\mathfrak{p}} = \mathscr{F}_{\mathfrak{p},l,\iota}$. The proposition is then an immediate consequence of the definitions and (2.8).

(3.3) If K is a local field of characteristic $p > 0$, we do not yet have a useful category of motives available. However, for any local field we can still define a functor $\mathscr{F}^\bullet$ from the category of smooth projective varieties X/K to D.E.R.S. $(\mathbb{G}_m)$ by setting

$$\mathscr{F}^\bullet(X) = \mathbb{D}(H^\bullet(X_{\overline{K}}, \mathbb{Q}_l)^I \otimes_{\mathbb{Q}_{l,\iota}} \mathbb{C}).$$

We equip $\mathscr{F}^\bullet(X)$ with a Lie-algebra action of $\mathfrak{t}$ by sending t to $t\Theta_{N\mathfrak{p}}$ and view $\mathscr{F}^\bullet(X)$ as a graded object of $\mathscr{D}_{N\mathfrak{p}}$. If $\operatorname{char}(K) = 0$, we have $\mathscr{F}^\bullet(X) = \mathscr{F}(H^\bullet(X))$ in our earlier notation.

4. Interlude: Varieties over finite fields

In this short section we look at the Weil conjectures from the point of view of a D.E.R.S. $(\mathbb{G}_m)$-valued cohomology theory.

For a variety X over $\mathbb{F}_q$ we set

$$H^\bullet(X/\mathbb{L}) := \mathbb{D}(H^\bullet(\overline{X}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_{l,\iota}} \mathbb{C}), \qquad \overline{X} = X \otimes \overline{\mathbb{F}}_q,$$

where l-adic cohomology is equipped with the geometric Frobenius $F = \mathrm{Fr}_q^* = (\mathrm{id} \times (\ )^q)^{*-1}$ and $H^\bullet(X/\mathbb{L})$ is viewed as an object of $\mathscr{D}_q$. Since $\mathbb{D}$ is an exact tensor functor, the theory $H^\bullet(\_/\mathbb{L})$ inherits all the usual properties of a cohomology theory. Let

$$H^w(X/\mathbb{L}) = H^w(X/\mathbb{L})^+ \oplus H^w(X/\mathbb{L})^-$$

be a standard decomposition of $H^w(X/\mathbb{L})$ with respect to the operator $-\Theta_q$. Corollary (2.8) implies:

(4.1) Proposition. *For any variety $X/\mathbb{F}_q$, $w \geq 0$, and s in $\mathbb{C}$ we have*

$$\imath \det_{\mathbb{Q}_l}(1 - q^{-s}\operatorname{Fr}_q^* | H^w(\overline{X}, \mathbb{Q}_l)) = \det_\infty(s - \Theta_q | H^w(X/\mathbb{L}))$$

and

$$\det_\infty(-s + \Theta_q | H^w(X/\mathbb{L})) = \varepsilon_w(s)\det_\infty(s - \Theta_q | H^w(X/\mathbb{L})),$$

where

$$\begin{aligned}\varepsilon_w(s) &= \exp i\pi(\dim_\infty((s - \Theta_q)|H^w(X/\mathbb{L})^+) - \dim_\infty((s - \Theta_q)|H^w(X/\mathbb{L})^-)) \\ &= (-q^s)^{b_w}\imath \det(\operatorname{Fr}_q^* | H^w(\overline{X}, \mathbb{Q}_l))^{-1}, \qquad b_w = \dim_{\mathbb{Q}_l} H^w(\overline{X}, \mathbb{Q}_l).\end{aligned}$$

If the variety $X/\mathbb{F}_q$ is smooth and proper, Poincaré duality gives a perfect pairing of objects in $\mathscr{D}_q$:

$$H^w(X/\mathbb{L}) \times H^{2d-w}(X/\mathbb{L}) \to H^{2d}(X/\mathbb{L}) \overset{\operatorname{Tr}}{\xrightarrow{\sim}} \mathbb{L}(-d) \qquad \text{[note (2.11)]},$$

where $d = \dim X$. In particular, Θ_q on $H^w(X/\mathbb{L})$ has the same eigenvalues with the same (algebraic) multiplicities as $d \cdot \operatorname{id} - \Theta_q$ on $H^{2d-w}(X/\mathbb{L})$. Hence,

$$\begin{aligned}\det_\infty(s - \Theta_q | H^w(X/\mathbb{L})) &= \varepsilon_w(s)^{-1}\det_\infty(-s + \Theta_q | H^w(X/\mathbb{L})) \\ &= \varepsilon_w(s)^{-1}\det_\infty((d - s) - \Theta_q | H^{2d-w}(X/\mathbb{L})).\end{aligned}$$

This implies the functional equation for $\zeta_X(s)$ in the form

$$\zeta_X(s) = e^{i\pi(\chi^+(s) - \chi^-(s))}\zeta_X(d - s),$$

where $\chi^\pm(s) = \sum_{w=0}^{2d}(-1)^w \dim_\infty(s - \Theta_q | H^w(X/\mathbb{L})^\pm)$. Note that by Deligne's theorem the eigenvalues of Θ_q on $H^w(X/\mathbb{L})$ have weight w, i.e., real part $= w/2$.

5. Logarithmic connections and filtered vector spaces

In this section we relate certain algebraic vector bundles on $\mathbb{G}_{a,\mathbb{C}}$ together with a logarithmic connection to filtered vector spaces. This will be used in §6 to construct the analogs for Archimedean $\mathfrak{p}$ of the functors $\mathscr{F}_\mathfrak{p}$ introduced in §3.

Let D.E.L.S.$(\mathbb{G}_{a,\mathbb{C}})$ be the category of algebraic vector bundles $\mathscr{V}$ on $\mathbb{G}_{a,\mathbb{C}}$ together with a connection

$$\nabla : \mathscr{V} \to \mathscr{V} \otimes \Omega^1_{\mathbb{G}_{a,\mathbb{C}}/\mathbb{G}_{m,\mathbb{C}}}\langle 0\rangle$$

having at most a logarithmic pole at zero.

Set $\mathbb{L}^+ = \Gamma(\mathbb{G}_{a,\mathbb{C}}, \mathscr{O}) = \mathbb{C}[z]$ and $\Delta^+ = \mathbb{L}^+[\Theta]$ where $\Theta = z\,d/dz$. The category D.E.L.S.$(\mathbb{G}_{a,\mathbb{C}})$ is canonically equivalent to the category of Δ^+-modules D^+ which are free of finite rank over $\mathbb{L}^+$.

For $K = \mathbb{C}$ or $\mathbb{R}$ set $e_K = [\mathbb{C} : K]$ and $\Theta_K = -e_K\Theta$. Let $\mathscr{D}_K$ be the category D.E.L.S.$(\mathbb{G}_{a,\mathbb{C}})$ equipped with the following notion of twist: Any object of $\mathscr{D}_K$ is viewed as a representation of $\mathfrak{t}$ by mapping 1 to Θ_K. For α in $\mathbb{C}$ the twist $D^+(\alpha)$ of D^+ in $\mathscr{D}_K$ is defined to be D^+ itself as an $\mathbb{L}^+$-module but with $\mathfrak{t}$-action given by

$$\Theta_{D^+(\alpha),K} = \Theta_{D^+,K} - \alpha\,\mathrm{id}; \quad \text{i.e., } \Theta_{D^+(\alpha)} = \Theta_{D^+} + \frac{\alpha}{e_K}\,\mathrm{id}.$$

We need the following categories $\mathscr{Fil}_K$:

$\mathscr{Fil}_{\mathbb{C}}$ is the additive category of finite-dimensional complex vector spaces V with a decreasing filtration $\mathrm{Fil}^r V$ such that $\mathrm{Fil}^{r_1} V = 0$, $\mathrm{Fil}^{r_2} V = V$ for some r_1, r_2. Morphisms are supposed to respect the filtrations.

$\mathscr{Fil}_{\mathbb{R}}$ is the additive category of finite-dimensional complex vector spaces with a filtration as above and with an involution F_∞ which respects the filtration and induces multiplication by $(-1)^\bullet$ on the associated graded vector space $\mathrm{Gr}^\bullet V$. Morphisms are supposed to respect the filtration and to commute with F_∞.

The categories $\mathscr{Fil}_K$ are obviously pseudoabelian, i.e., additive such that kernels and images of projectors exist. In $\mathscr{Fil}_K$ the twist $V(n)$ of an object V by an integer n is defined to be V itself as a vector space but with filtration

$$\mathrm{Fil}^r V(n) = \mathrm{Fil}^{r+n} V$$

and in case $K = \mathbb{R}$ with $F_\infty | V(n) = (-1)^n F_\infty | V$. Write $\mathbb{C}(0)$ for the object in $\mathscr{Fil}_K$ whose underlying vector space is $\mathbb{C}$ with filtration given by

$$\mathrm{Fil}^r\mathbb{C}(0) = \mathbb{C}(0) \quad \text{for } r \le 0 \qquad \text{and} \qquad \mathrm{Fil}^r\mathbb{C}(0) = 0 \quad \text{for } r > 0.$$

In case $K = \mathbb{R}$ we set $F_\infty = \mathrm{id}$ on $\mathbb{C}(0)$. Note that

$$\mathrm{Hom}_{\mathscr{Fil}_K}(\mathbb{C}(n), \mathbb{C}(m)) = \begin{cases} \mathbb{C} & \text{for } m \le n,\ m \equiv n \bmod e_K, \\ 0 & \text{otherwise}. \end{cases}$$

On $\mathbb{L} = \Gamma(\mathbb{G}_{m,\mathbb{C}}, \mathscr{O})$ consider the filtration and involution given by

$$\mathrm{Fil}^r\mathbb{L} = z^r\mathbb{L}^+, \qquad F_\infty(z) = -z.$$

For V in $\mathscr{Fil}_{\mathbb{C}}$ set

$$\mathbb{D}^+(V) = \mathrm{Fil}^0(V \otimes_{\mathbb{C}} \mathbb{L}),$$

an $\mathbb{L}^+ = \mathrm{Fil}^0\mathbb{L}$-module with action by $\Theta_{\mathbb{C}} \hat{=} \mathrm{id} \otimes \Theta_{\mathbb{C}}$. For V in $\mathscr{Fil}_{\mathbb{R}}$ set

$$\mathbb{D}^+(V) = \mathrm{Fil}^0(V \otimes_{\mathbb{C}} \mathbb{L})^{F_\infty = \mathrm{id}}.$$

Let sq be the injection $\mathbb{L} \to \mathbb{L}$ induced by $sq(z) = z^2$. Note that it corresponds to the squaring map on $\mathbb{G}_{m,\mathbb{C}}$. We view $\mathbb{D}^+(V)$ as an $\mathbb{L}^+$-module via sq and let $\Theta_{\mathbb{R}}$ operate via $\mathrm{id} \otimes \Theta_{\mathbb{C}}$.

For $K = \mathbb{R}, \mathbb{C}$ there are natural isomorphisms $\mathbb{D}^+(\mathbb{C}(n)) \cong \mathbb{L}^+(n)$ in $\mathscr{D}_K$. Similarly as in **[De1, §6]** we have the following easy proposition.

(5.1) Proposition. *In $\mathscr{Fil}_K$ any object is isomorphic to a finite direct sum of objects $\mathbb{C}(n)$ for n in $\mathbb{Z}$. The $\mathbb{D}^+$ as constructed above induces an additive functor*

$$\mathbb{D}^+ : \mathscr{Fil}_K \to \mathscr{D}_K$$

which commutes with $\otimes$-products and internal Homs. *For any integer there are natural isomorphisms*

$$\mathbb{D}^+(V(n)) \cong \mathbb{D}^+(V)(n).$$

For all V we have

$$\mathrm{rk}_{\mathbb{L}^+} \mathbb{D}^+(V) = \dim V.$$

$\mathbb{D}^+$ induces an equivalence of categories between $\mathscr{Fil}_K$ and $\mathscr{D}_K^{\mathrm{ad}}$, the full subcategory of $\mathscr{D}_K$ generated by objects which are isomorphic to finite direct sums of $\mathbb{L}^+(n)$ for n in $\mathbb{Z}$.

Remarks. (1) The subcategory $\mathscr{D}_K^{\mathrm{ad}}$ of $\mathscr{D}_K$ is closed under twists by integers.

(2) Using the real structure $\mathbb{R}[z, z^{-1}] \subset \mathbb{L}$ on $\mathbb{L}$ the functor $\mathbb{D}^+$ maps real structures on objects of $\mathscr{Fil}_K$ to real structures on objects of $\mathscr{D}_K$.

As in **[De1]** we set $\Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$.

(5.2) Proposition. *For V in $\mathscr{Fil}_K$ set $d_\nu = \dim \mathrm{Gr}^\nu V$. Then we have for all complex s*

$$\dim_\infty\left(\frac{1}{2\pi}(s - \Theta_K)|\mathbb{D}^+(V)\right) = \left(\frac{1}{2} - \frac{s}{e_K}\right)\dim V + \frac{1}{e_K}\sum_\nu \nu d_\nu$$

and

$$\det_\infty\left(\frac{1}{2\pi}(s - \Theta_K)|\mathbb{D}^+(V)\right) = \prod_{\nu\in\mathbb{Z}} \Gamma_K(s-\nu)^{-d_\nu}.$$

Proof. Since any object in $\mathscr{Fil}_K$ is isomorphic to a direct sum of $\mathbb{C}(n)$'s and since both sides of the equations are additive (respectively multiplicative), we are reduced to $\mathbb{C}(n)$. Since an n-twist changes s to $s+n$ in all expressions, we are reduced to $\mathbb{C}(0)$. Since $\mathbb{D}^+(\mathbb{C}(0)) = \mathbb{L}^+$ and since Θ_K acts on $\mathbb{L}^+$ with eigenvalues $-e_K\nu$ for $\nu = 0, 1, \ldots$ of multiplicity one, we have to consider the Dirichlet series

$$\sum_{\nu=0}^{\infty} \frac{1}{[\frac{1}{2\pi}(s + e_K\nu)]^u} = \zeta_{e_K/2\pi}\left(u, \frac{s}{e_K}\right), \qquad \mathrm{Re}\, u > 1,$$

in the notation of the proof of (2.7). Now the assertion follows from (2.7.1).

Remark. One may wonder about the factor $(2\pi)^{-1}$ in the above formulas. For the first it is irrelevant, but for the second it seems to be the best choice. Namely, let δ be in $\mathbb{C}^*$ and set

$$\Gamma_{\mathbb{C},\delta}(s) = \delta^{s-\frac{1}{2}}\sqrt{2\pi}^{-1}\Gamma(s) \quad \text{and} \quad \Gamma_{\mathbb{R},\delta}(s) = (2\delta)^{\frac{s-1}{2}}\sqrt{2\pi}^{-1}\Gamma\left(\frac{s}{2}\right).$$

Then we have for all complex s

$$\dim_\infty(\delta(s-\Theta_K)|\mathbb{D}^+(V)) = \left(\frac{1}{2}-\frac{s}{e_K}\right)\dim V + \frac{1}{e_K}\sum_\nu \nu d_\nu$$

and in case δ is not a negative real number

$$\det_\infty(\delta(s-\Theta_K)|\mathbb{D}^+(V)) = \prod_{\nu\in\mathbf{Z}} \Gamma_{K,\delta}(s-\nu)^{-d_\nu}.$$

This follows as above from (2.7.1) and (2.7.2). For $\delta < 0$ the formula for $\det_\infty$ depends on $\operatorname{Im} s$. Using the factor $\Gamma_{\mathbf{R},\delta}(s)$ to complete the Riemann zeta function at infinity introduces the ε-factor $(2\pi\delta)^{s-\frac{1}{2}}$ in its functional equation.

6. The Archimedean local factors

In the first part of this section we express the Archimedean local L-factors of a motive as regularized characteristic power series.

Let $\mathscr{MH}_\mathbb{C}$ (resp. $\mathscr{MH}_\mathbb{R}$) denote the category of mixed Hodge structures with coefficients in $\mathbb{R}$ (equipped with the action of an infinite Frobenius F_∞ which respects the weight filtration and maps F^i to $\overline{F}^i$) **[D2; Be,** §7]. Fix a finite extension $E/\mathbb{Q}$ as a field of multiplication. We refer to **[F-PR, De3]** or the proof of (6.3) for the definition of the $(E\otimes\mathbb{C})$-valued L-factor $L(H, s)$ of a mixed Hodge structure H in $\mathscr{MH}_K(E)$.

Let us define an additive functor

$$\mathscr{V}: \mathscr{MH}_K \to \mathscr{Fil}_K \tag{6.1}$$

by

$$\mathscr{V}((H, W_\bullet H, F^\bullet H_\mathbb{C})) = (H_\mathbb{C}, \operatorname{Fil}^i H_\mathbb{C} = F^i H_\mathbb{C} \cap \overline{F^i H_\mathbb{C}})$$

in case $K=\mathbb{C}$ and by

$$\mathscr{V}((H, W_\bullet H, F^\bullet H_\mathbb{C}, F_\infty))$$
$$= \left(H_\mathbb{C}, \operatorname{Fil}^i H_\mathbb{C} = (F^i H_\mathbb{C}\cap\overline{F^i H_\mathbb{C}})^{(-1)^i} \oplus (F^{i+1}H_\mathbb{C}\cap\overline{F^{i+1}H_\mathbb{C}})^{(-1)^{i+1}}, F_\infty\right)$$

in case $K=\mathbb{R}$. Here the exponent ± 1 denotes the ± 1-eigenspace of F_∞.

Note that $\mathscr{V}(H)$ carries the real structure

$$\mathscr{V}^\mathbf{R}(H) = (H, \operatorname{Fil}^i H = F^i H_\mathbb{C}\cap H) \quad \text{in case } K=\mathbb{C}$$

and

$$\mathscr{V}^\mathbf{R}(H) = \left(H, \operatorname{Fil}^i H = (F^i H_\mathbb{C}\cap H)^{(-1)^i} \oplus (F^{i+1}H_\mathbb{C}\cap H)^{(-1)^{i+1}}, F_\infty\right)$$

in case $K=\mathbb{R}$.

The functor $\mathscr{V}$ commutes with twists and sends $\mathbb{R}(n)$ in $\mathscr{MH}_K$ to $\mathbb{C}(n)$ in $\mathscr{Fil}_K$. For $K=\mathbb{C}$ it commutes with $\otimes$-products but not for $K=\mathbb{R}$. In both cases $\mathscr{V}$ does not commute with duals.

Composing $\mathscr{V}$ with the functor $\mathbb{D}^+ : \mathscr{F}il_K \to \mathscr{D}_K^{\mathrm{ad}}$, we obtain a functor

$$\mathbb{D}^+_{\mathscr{H}} : \mathscr{M}\mathscr{H}_K(E) \to \mathscr{D}_K^{\mathrm{ad}}(E). \tag{6.2}$$

Using the second remark after (5.1) we see that any object $\mathbb{D}^+_{\mathscr{H}}(H)$ in the image of $\mathbb{D}^+_{\mathscr{H}}$ carries a canonical $E\otimes\mathbb{R}$ structure, which we denote by $\mathbb{D}^{+,\mathbb{R}}_{\mathscr{H}}(H)$.

Note that there is a natural isomorphism of categories

$$\mathscr{D}_K(E) \cong \prod_{\sigma\in\mathrm{Hom}(E,\mathbb{C})} \mathscr{D}_K, \qquad D \mapsto (D_\sigma),$$

where $D_\sigma = D \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}$.

REMARK. The construction of the functor $\mathbb{D}$ in **[Del, §3]** uses only the Hodge filtration and not the weight filtration of a Hodge structure. Hence it makes sense for mixed Hodge structures as well, and it turns out to be equal to $\mathbb{D}^{+,\mathbb{R}}_{\mathscr{H}}$ above (after the substitution $z = T^{-e_K}$).

(6.3) PROPOSITION. *For H in $\mathscr{M}\mathscr{H}_K(E)$ and all complex s we have*

$$L(H,s) = \det_\infty\left(\frac{1}{2\pi}(s-\Theta_K)|\mathbb{D}^+_{\mathscr{H}}(H)_\sigma\right)^{-1}_{\sigma\in\mathrm{Hom}(E,\mathbb{C})}.$$

PROOF. Let e_σ be the idempotent in $E\otimes\mathbb{C} = \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$ corresponding to the embedding $\sigma : E \hookrightarrow \mathbb{C}$. Then

$$\mathbb{D}^+_{\mathscr{H}}(H)_\sigma = e_\sigma \mathbb{D}^+_{\mathscr{H}}(H) = e_\sigma \mathbb{D}^+(\mathscr{V}(H)) = \mathbb{D}^+(e_\sigma\mathscr{V}(H)).$$

Note that $\mathbb{D}^+$ is $E\otimes\mathbb{C}$ linear and that $\mathscr{F}il_K(E)$ is pseudo-abelian. We view $e_\sigma\mathscr{V}(H)$ as an object of $\mathscr{F}il_K$. Proposition (5.2) implies

$$\det_\infty\left(\frac{1}{2\pi}(s-\Theta_K)|\mathbb{D}^+_{\mathscr{H}}(H)_\sigma\right)^{-1} = \prod_{\nu\in\mathbb{Z}} \Gamma_K(s-\nu)^{d_{\nu,\sigma}},$$

where $d_{\nu,\sigma} = \dim_{\mathbb{C}} \mathrm{Gr}^\nu(e_\sigma\mathscr{V}(H)) = \dim_{\mathbb{C}}(e_\sigma \mathrm{Gr}^\nu \mathscr{V}(H))$.

Consider the filtration γ on $H_{\mathbb{C}}$:

$$\gamma^\nu H_{\mathbb{C}} = F^\nu H_{\mathbb{C}} \cap \overline{F}^\nu H_{\mathbb{C}},$$

where F^ν is the Hodge filtration. For $K = \mathbb{C}$ we have $\mathrm{Gr}^\nu \mathscr{V}(H) = \mathrm{Gr}^\nu_\gamma H_{\mathbb{C}}$, and since by definition

$$L(H,s)_\sigma = \prod_\nu \Gamma_{\mathbb{C}}(s-\nu)^{n_{\nu,\sigma}},$$

where $n_{\nu,\sigma} = \dim_{\mathbb{C}}(e_\sigma \mathrm{Gr}^\nu_\gamma H_{\mathbb{C}})$, the claim follows. For $K = \mathbb{R}$ set

$$n^\pm_{\nu,\sigma} = \dim_{\mathbb{C}} e_\sigma(\mathrm{Gr}^\nu_\gamma H_{\mathbb{C}})^\pm.$$

Then by definition

$$\begin{aligned} L(H,s)_\sigma &= \prod_\nu \Gamma_{\mathbb{R}}(s+\varepsilon_\nu-\nu)^{n^+_{\nu,\sigma}}\Gamma_{\mathbb{R}}(s+1-\varepsilon_\nu-\nu)^{n^-_{\nu,\sigma}} \\ &= \prod_\nu \Gamma_{\mathbb{R}}(s-\nu)^{d'_{\nu,\sigma}}, \end{aligned}$$

where $\varepsilon_\nu \in \{0, 1\}$, $\varepsilon_\nu \equiv \nu \bmod 2$, and

$$d'_{\nu,\sigma} = n^{(-1)^\nu}_{\nu+1,\sigma} + n^{(-1)^\nu}_{\nu,\sigma}.$$

Clearly

$$\mathrm{Gr}^\nu \mathscr{V}(H) \cong \left(\frac{\gamma^\nu H_{\mathbb{C}}}{\gamma^{\nu+2} H_{\mathbb{C}}}\right)^{(-1)^\nu},$$

and thus there is an $(E \otimes \mathbb{C})$-equivariant exact sequence

$$0 \to (\mathrm{Gr}^{\nu+1}_\gamma H_{\mathbb{C}})^{(-1)^\nu} \to \mathrm{Gr}^\nu \mathscr{V}(H) \to (\mathrm{Gr}^\nu_\gamma H_{\mathbb{C}})^{(-1)^\nu} \to 0.$$

This implies that

$$d'_{\nu,\sigma} = \dim e_\sigma \mathrm{Gr}^\nu \mathscr{V}(H) = d_{\nu,\sigma}$$

and hence the assertion.

Up to now we have kept $\mathbb{L}^+$ fixed and chosen Θ_K depending on the ground field K. As in §3 and in **[De1]** we can also keep the derivation fixed and vary the spaces: Let $\tau: K \hookrightarrow \mathbb{C}$ be any embedding, and denote by $\mathfrak{p} = \{\tau, \overline{\tau}\}$ the unique place of K. Consider the subring

$$\mathbb{L}^+_{\mathfrak{p}} = \mathbb{C}[\exp(-e_K\xi)] \subset \mathscr{O}(\mathbb{C})$$

equipped with the derivation $\Theta = d/d\xi$. The change of variables $z = \exp(-e_K\xi)$ identifies the pair $(\mathbb{L}^+, \Theta_K)$ with $(\mathbb{L}^+_{\mathfrak{p}}, \Theta)$. We write $\mathscr{D}_{\mathfrak{p}}$ for the category with twists $\mathscr{D}_K$ of §5 if we make the identification $(\mathbb{L}^+, \Theta_K) = (\mathbb{L}^+_{\mathfrak{p}}, \Theta)$ in its construction. We can view $\mathbb{D}^+_{\mathscr{H}}$ as a functor

$$\mathbb{D}^+_{\mathscr{H}}: \mathscr{MH}_K(E) \to \mathscr{D}^{\mathrm{ad}}_{\mathfrak{p}}(E).$$

For $K = \mathbb{C}$ or $\mathbb{R}$ the real Betti realization

$$M_B = H^\bullet_{\mathrm{sing}}(M \otimes_K \overline{K}, \mathbb{R})$$

of a motive M over K induces a functor

$$\mathscr{MM}_K(E) \to \mathscr{MH}_K(E), \qquad M \mapsto M_B.$$

Its composition with $\mathbb{D}^+_{\mathscr{H}}$ is denoted by

$$\mathscr{F}_{\mathfrak{p}}: \mathscr{MM}_K(E) \to \mathscr{D}^{\mathrm{ad}}_{\mathfrak{p}}(E).$$

It will turn out to be the Archimedean analogue of (3.1.1). Note that for any M in $\mathscr{MM}_K(E)$ there is a canonical $E\otimes\mathbb{R}$ structure $\mathscr{F}^{\mathbb{R}}_{\mathfrak{p}}(M)$ on $\mathscr{F}_{\mathfrak{p}}(M)$. For $E = \mathbb{Q}$ the Archimedean cohomology $H^\bullet_{\mathrm{Ar}}(M)$ of the motive M introduced in **[De1]** is just $\mathscr{F}^{\mathbb{R}}_{\mathfrak{p}}(M)$. The functor $\mathscr{F}_{\mathfrak{p}}$ is naturally $\mathbb{Z}$-graded via

$$\mathscr{F}^\bullet_{\mathfrak{p}}(M) = \mathscr{F}_{\mathfrak{p}}(H^\bullet(M)),$$

and up to natural isomorphisms it commutes with twists. We view $\mathscr{F}_{\mathfrak{p}}(M)$ as a Lie algebra representation of $\mathfrak{t}$ by sending 1 to Θ.

(6.4) Since $\mathcal{F}_{\mathrm{p}}$ does not commute with duals and for $K = \mathbb{R}$ not with $\otimes$-products as well, we cannot view $\mathcal{F}_{\mathrm{p}}$ as a *geometric* cohomology theory on motives. This is in accordance with the philosophy of Arakelov theory **[Ma]** that the "reduction" of the infinite fibres of an arithmetic variety should be viewed as totally degenerate. So $\mathcal{F}_{\mathrm{p}}(M)$ would be something like the "fixed module under inertia" of the "true cohomology" of M over K. This point of view is compatible with the fact that $\mathcal{F}_{\mathrm{p}}$ is left exact but not exact and with a later argument about weights (7.14).

As a trivial consequence of (6.3) we get

(6.5) COROLLARY. *For M in $\mathcal{MM}_K(E)$ and all complex s we have*

$$L_K(M, s) = \det{}_\infty\left(\frac{1}{2\pi}(s - \Theta)|\mathcal{F}_{\mathrm{p}}(M)_\sigma\right)^{-1}_{\sigma\in\mathrm{Hom}(E,\mathbb{C})}.$$

We now discuss a canonical pairing between the kernel of Θ_K on $\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}$ and certain Ext-groups in the category of real mixed Hodge structures over K. For H in $\mathcal{MH}_K$ we have if $K = \mathbb{C}$

$$\begin{aligned}\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}(H)^{\Theta=0} &= \mathcal{F}il^0(\mathcal{V}^{\mathbb{R}}(H)\otimes_{\mathbb{R}}\mathbb{R}[z, z^{-1}])^{\Theta=0}\\ &= \left(\sum_{\nu+\mu\geq 0}\mathcal{F}il^{\nu}\mathcal{V}^{\mathbb{R}}(H)\otimes_{\mathbb{R}} z^{\mu}\mathbb{R}[z]\right)^{\Theta=0}\\ &= \mathcal{F}il^0\mathcal{V}^{\mathbb{R}}(H) = F^0H_{\mathbb{C}}\cap H\end{aligned}$$

and if $K = \mathbb{R}$

$$\begin{aligned}\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}(H)^{\Theta=0} &= (\mathcal{F}il^0\mathcal{V}^{\mathbb{R}}(H))^{F_\infty=\mathrm{id}} = (F^0H_{\mathbb{C}}\cap H)^{F_\infty=\mathrm{id}}\\ &= (F^0H_{\mathbb{C}}\cap H)^{\overline{F}_\infty=\mathrm{id}}.\end{aligned}$$

For any Hodge structure H define $\pi_n\colon H_{\mathbb{C}}\to H_{\mathbb{C}}$ by $\pi_n(h) = \frac{1}{2}(h+(-1)^n\overline{h})$, where $- = \mathrm{id}_H\otimes c$. Thus $\pi_n(H_{\mathbb{C}}) = (2\pi i)^n H$. The canonical pairing

$$H_{\mathbb{C}}\times H^*(1)_{\mathbb{C}} \xrightarrow{\langle\,,\,\rangle} \mathbb{R}(1)_{\mathbb{C}} \xrightarrow{\pi_1} \mathbb{R}$$

induces an $\mathbb{R}$-linear pairing

$$(F^0H_{\mathbb{C}}\cap H)\times\frac{H^*(1)_{\mathbb{C}}}{H^*(1)+F^0H^*(1)_{\mathbb{C}}}\to\mathbb{R} \tag{6.6}$$

since $\langle H, H^*(1)\rangle\subset\mathbb{R}(1)\subset\operatorname{Ker}\pi_1$ and $\langle F^0H_{\mathbb{C}}, F^0H^*(1)_{\mathbb{C}}\rangle\subset F^0\mathbb{R}(1)_{\mathbb{C}} = 0$. There are canonical isomorphisms

$$\begin{aligned}\frac{H^*(1)_{\mathbb{C}}}{H^*(1)+F^0H^*(1)_{\mathbb{C}}} &= \frac{H^*(1)_{\mathbb{C}}/H^*(1)}{F^0H^*(1)_{\mathbb{C}}/(F^0H^*(1)_{\mathbb{C}}\cap H^*(1))} \overset{\pi_1}{\cong} \frac{H^*}{\pi_1(F^0H^*(1)_{\mathbb{C}})}\\ &= \frac{H^*}{\pi_0(F^1H^*_{\mathbb{C}})},\end{aligned}$$

and (6.6) translates into the pairing

$$(6.7) \qquad (F^0 H_{\mathbb{C}} \cap H) \times \frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})} \to \mathbb{R}$$

induced by the canonical pairing $H \times H^* \to \mathbb{R}$.

We claim that (6.7) and hence (6.6) is nondegenerate:

$$\left(\frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})}\right)^* = \operatorname{Ker}(H \to (\pi_0(F^1 H^*_{\mathbb{C}}))^*)$$

consists of all h in H such that $\operatorname{Re}(\psi(h)) = 0$ for all $\mathbb{C}$-linear maps $\psi\colon H_{\mathbb{C}} \to \mathbb{C}$ with $\psi(F^0(H_{\mathbb{C}})) = 0$. Replacing ψ by $i\psi$ we find that in fact $\psi(h) = 0$ for all such ψ and hence that h is in $F^0 H_{\mathbb{C}}$. Thus we have

$$\left(\frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})}\right)^* = F^0 H_{\mathbb{C}} \cap H$$

as claimed.

In case $K = \mathbb{R}$ the de Rham conjugation $\overline{F}_\infty$ is selfadjoint with respect to the pairing in (6.6). Hence the pairing of fixed modules

$$(F^0 H_{\mathbb{C}} \cap H)^{\overline{F}_\infty} \times \left(\frac{H^*(1)_{\mathbb{C}}}{H^*(1) + F^0 H^*(1)_{\mathbb{C}}}\right)^{\overline{F}_\infty} \to \mathbb{R}$$

is nondegenerate as well.

Now recall **[Be,** (1.4) and (1.6)] that there are canonical isomorphisms

$$\operatorname{Ext}^1_{\mathcal{MH}_{\mathbb{C}}}(\mathbb{R}(0), H) \cong \frac{W_0 H_{\mathbb{C}}}{W_0 H + F^0 W_0 H_{\mathbb{C}}},$$

$$\operatorname{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), H) \cong \left(\frac{W_0 H_{\mathbb{C}}}{W_0 H + F^0 W_0 H_{\mathbb{C}}}\right)^{\overline{F}_\infty}.$$

Thus we have proved:

(6.8) Proposition. *For any H in $\mathcal{MH}_K$ and M in $\mathcal{MM}_K$ there are canonical pairings of finite-dimensional $\mathbb{R}$-vector spaces*

$$\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}(H)^{\Theta=0} \times \operatorname{Ext}^1_{\mathcal{MH}_K}(\mathbb{R}(0), H^*(1)) \to \mathbb{R},$$

$$\mathcal{F}^{\mathbb{R}}_{\mathrm{p}}(M)^{\Theta=0} \times \operatorname{Ext}^1_{\mathcal{MH}_K}(\mathbb{R}(0), M^*_B(1)) \to \mathbb{R},$$

which are nondegenerate if the weights of H and M respectively are ≥ -2.

Now let us use the proposition to clarify the relation between Archimedean and Deligne cohomology. Let X/K be a smooth projective variety of dimension d. Then we have canonical isomorphisms

$$H^{w+1}_{\mathcal{D}}(X, \mathbb{R}(n)) \cong \operatorname{Ext}^1_{\mathcal{MH}_K}(\mathbb{R}(0), H^w_B(X)(n)) \quad \text{if } w+1 < 2n$$
$$\cong (\mathcal{F}^{\mathbb{R}}_{\mathrm{p}}(H^w(X)^*(1-n))^{\Theta=0})^*.$$

The induced isomorphism for $w+1<2n$

$$(6.9) \qquad \alpha\colon H_{\mathscr{D}}^{w+1}(X, \mathbb{R}(n)) \to (\mathscr{F}_{\mathfrak{p}}^{\mathbb{R}}(H^{2d-w}(X)(d+1-n))^{\Theta=0})^*$$

will play a role in §7. If we compose with the inverse of a strong Lefschetz isomorphism $H^w(X) \xrightarrow{\sim} H^{2d-w}(X)(d-w)$, we get the first part of

(6.10) PROPOSITION. *For $m = w+1-n \le w/2$ there is a natural perfect pairing*

$$H_{\mathscr{D}}^{w+1}(X, \mathbb{R}(n)) \times \mathscr{F}_{\mathfrak{p}}^{\mathbb{R}}(H^w(X))^{\Theta=m} \to \mathbb{R}.$$

For $m > w/2$ the groups $\mathscr{F}_{\mathfrak{p}}^{\mathbb{R}}(H^w(X))^{\Theta=m}$ vanish. In particular, the weights of Θ on $\mathscr{F}_{\mathfrak{p}}^{\mathbb{R}}(H^w(X))$ are $\le w$.

By (6.5) it is clear that

$$(6.11) \qquad \operatorname{ord}_{s=m} L_K(H^w(X), s) = -\dim_{\mathbb{R}} \mathscr{F}_{\mathfrak{p}}^{\mathbb{R}}(H^w(X))^{\Theta=m} \quad \text{for } m \text{ in } \mathbb{Z}.$$

Since $L_K(H^w(X), s)$ has no poles for $s > w/2$, the second assertion of the proposition follows. Note that the duality in (6.10) gives a satisfactory algebraic explanation for Beilinson's observation that

$$(6.12) \qquad \operatorname{ord}_{s=m} L_K(H^w(X), s) = -\dim_{\mathbb{R}} H_{\mathscr{D}}^{w+1}(X, \mathbb{R}(n)) \quad \text{if } m \le w/2.$$

The present explanation of (6.12) is to be preferred to that in **[De1,** §5].

7. Arithmetic cohomology?

In this section we extensively discuss aspects of the still speculative site $\mathscr{S}$ of **[De2,** §3] (called $\mathscr{T}$ in loc. cit.). The discussion is meant as an approximation, so on occasion we will deliberately be somewhat vague. The motives attached to algebraic Hecke characters will serve us to test consequences of our considerations. In some cases we are suggested new formulas in analytic number theory which can be proved by classical methods (7.16), (7.18), and (7.20).

$\mathscr{S}$ should be a site ringed by a sheaf of $\mathbb{C}$-algebras $\mathscr{C}$. The objects in the underlying category of $\mathscr{S}$ should be equipped with functorial actions of the group $T = (\mathbb{R}, +)$. Schemes and Arakelov varieties should give rise to objects of $\operatorname{cat}\mathscr{S}$ which we denote by the same symbol. There should be a morphism of ringed sites from varieties over $\mathbb{C}$ with the analytic topology and ringed by $\mathbb{C}$ into $\mathscr{S}$. For any embedding $\sigma\colon \mathbb{Q}_l \hookrightarrow \mathbb{C}$ a $\mathbb{Q}_l$-sheaf on a variety over a field of characteristic zero should give rise to a $\mathscr{C}$-sheaf on the corresponding object of $\operatorname{cat}\mathscr{S}$.

Because of the T-actions, the cohomology groups $H^w(X, \mathscr{C})$ of any object in $\operatorname{cat}\mathscr{S}$ would be $\mathbb{C}$-vector spaces with an action of T. We expect them to decompose into the direct sum of finite-dimensional T-spaces. Thus they would carry an action by the Lie-algebra $\mathfrak{t}$ of T and hence an action by the endomorphism Θ corresponding to $1 \in \mathfrak{t} = \mathbb{R}$.

If $X \xrightarrow{\pi} Y$ is a morphism, cup product would turn $H^w(X, \mathscr{C})$ into an $H^0(Y, \mathscr{C})$-module. For a non-Archimedean local field K of characteristic zero we expect $H^0(\operatorname{Spec} K, \mathscr{C}) \cong \mathbb{L}_\mathfrak{p}$ in $\mathscr{D}_\mathfrak{p}$. If X is a smooth projective variety over K with good reduction, we now expect

$$H^w(X, \mathscr{C}) \cong H^w(X/\mathbb{L}_\mathfrak{p}) \quad \text{in } \mathscr{D}_\mathfrak{p} .$$

There is a similarity to the situation for crystalline cohomology: coefficients vary with the ground field, and the action of T which corresponds to $\mathbb{Z} \cong \langle \mathrm{Fr} \rangle$ is σ-linear with respect to the coefficient module structure on cohomology. In our situation the action of T on $\mathbb{L}_\mathfrak{p}$ is given by $\sigma \colon T \to \operatorname{Aut}_\mathbb{C} \mathbb{L}_\mathfrak{p}$, $(\sigma(t)\ell)(\xi) = \ell(\xi + t)$.

A very optimistic suggestion to explain the T-action would be the following: There should be a "ground point" P in $\mathscr{S}$ (which is not a scheme) and an extension $\widehat{P} \to P$ with $\operatorname{Aut}_P(\widehat{P}) \cong T$. What was written $H^w(X, \mathscr{C})$ above should in fact be $H^w(X \times_P \widehat{P}, \mathscr{C})$ with the T-action coming from transport of structure. This guess may be too naive: for reasons that become clear later when we discuss zeta-functions we would have $H^0(\widehat{P}, \mathscr{C}) \cong \mathbb{C}$ with trivial T-action. If $\operatorname{Spec} K \to P$ were geometrically connected (with respect to $- \times_P \widehat{P}$), I would expect

$$H^0(\operatorname{Spec} K \times_P \widehat{P}, \mathscr{C}) \cong H^0(\widehat{P}, \mathscr{C}) \cong \mathbb{C},$$

a contradiction. Hence $\operatorname{Spec} K \times_P \widehat{P}$ should not be connected. But then $H^0(\operatorname{Spec} K \times_P \widehat{P}, \mathscr{C}) \cong \mathbb{L}_\mathfrak{p}$ would not be an integral domain, which is also a contradiction. So the crystalline picture may be more appropriate than this analogy with the étale site over a finite field.

For any local field K, we expect to have a model $\mathscr{Y}_K$ of $\operatorname{Spec} K$ in $\mathscr{S}$ with closed point $\mathfrak{p}$. For non-Archimedean K, $\mathscr{Y}_K$ would be $\operatorname{Spec} \mathscr{O}_K$. For any motive M in $\mathscr{M}\mathscr{M}_K(E)$ we expect to have a canonical T-sheaf $\mathscr{F}(M)$ with E-action on the $\mathscr{S}$-site of $\mathscr{Y}_K$ such that its stalk at $\mathfrak{p}$ is given by

$$\mathscr{F}(M)_\mathfrak{p} = H^0(\mathfrak{p}, \mathscr{F}(M)) = \mathscr{F}_\mathfrak{p}(M)$$

in $\mathscr{D}_\mathfrak{p}(E)$ with the objects $\mathscr{F}_\mathfrak{p}(M)$ of §§3 and 6.

If k is a number field, let $\mathscr{Y}$ be the object in $\mathscr{S}$ corresponding to the Arakelov compactification $\operatorname{Spec} \mathscr{O}_k \cup \{\mathfrak{p}|\infty\}$ of $\operatorname{Spec} \mathscr{O}_k$. As in the local case we expect for any M in $\mathscr{M}\mathscr{M}_k(E)$ a canonical T-sheaf $\mathscr{F}(M)$ with endomorphisms in E on the $\mathscr{S}$-site of $\mathscr{Y}$ such that

$$\mathscr{F}(M)_\mathfrak{p} \cong \mathscr{F}_\mathfrak{p}(M) := \mathscr{F}_\mathfrak{p}(M \otimes_k k_\mathfrak{p}) \text{ in } \mathscr{D}_\mathfrak{p}(E) \tag{7.1}$$

for all places $\mathfrak{p}$ of k. The formation of $\mathscr{F}$ should commute with twists and duals.

If for example $M = H^w(X)$, where $\pi \colon X \to \operatorname{Spec} k$ is a smooth and projective variety over k, we expect

$$\mathscr{F}(M) = j_* R^w \pi_* \mathscr{C}_X , \tag{7.1.1}$$

where $j\colon \operatorname{Spec} k \hookrightarrow \mathcal{Y}$ is the inclusion. For the motive $\mathbb{Q}(0) = H^0(\operatorname{Spec} k)$ in $\mathcal{M}_K$ we expect

$$\mathcal{F}(\mathbb{Q}(0)) = j_* \mathcal{C}_{\operatorname{Spec} k} = \mathcal{C}_{\mathcal{Y}}. \tag{7.1.2}$$

For suitable analytic functions Φ a functional calculus in the algebra of correspondences should allow one to construct $\Phi(\Theta)$ as a correspondence on $\mathcal{Y}$. Let S be a finite set of places of k, and set $\mathcal{Y}_S = \mathcal{Y} \backslash S$. If Φ is such that $\Phi(\Theta)$ is of trace class on all stalks $\mathcal{F}(M)_{\mathfrak{p}}$ and on the cohomologies $H^i_c(\mathcal{Y}_S, \mathcal{F}(M))$ of $\mathcal{Y}_S$ with compact support, we are led to expect the following Lefschetz trace formula:

$$\sum_{\mathfrak{p} \in |\mathcal{Y}_S|} \operatorname{Tr}_E(\Phi(\Theta) | \mathcal{F}(M)_{\mathfrak{p}}) = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}_E(\Phi(\Theta) | H^i_c(\mathcal{Y}_S, \mathcal{F}(M))). \tag{7.2}$$

Here $|\mathcal{Y}_S|$ are the finite and infinite places of k that are not contained in S. For an endomorphism φ of an $E \otimes \mathbb{C}$-module V we set

$$\operatorname{Tr}_E(\varphi | V) = (\operatorname{Tr}(\varphi | V_\sigma))_{\sigma \in \operatorname{Hom}(E, \mathbb{C})} \in E \otimes \mathbb{C}$$

if the traces of φ on all $V_\sigma = V \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C}$ exist.

Now let us fix z, s with real parts sufficiently large and consider $\Phi(\tau) = (s - \tau)^{-z}$. Let $\zeta_{\mathfrak{p},\sigma}(z)$ resp. $\zeta_{i,\sigma}(z)$ denote the zeta functions of the operator $\frac{1}{2\pi}(s - \Theta)$ on $\mathcal{F}_{\mathfrak{p}}(M)_\sigma$ resp. on $H^i_c(\mathcal{Y}_S, \mathcal{F}(M))_\sigma$. Then (7.2) combined with (7.1) implies that

$$\sum_{\mathfrak{p} \in |\mathcal{Y}_S|} \zeta_{\mathfrak{p},\sigma}(z) = \sum_{i=0}^{2} (-1)^i \zeta_{i,\sigma}(z). \tag{7.3}$$

If the convergence on the left is locally uniform in $\operatorname{Re} z > -\varepsilon$, $\varepsilon > 0$, we obtain

$$\sum_{\mathfrak{p} \in |\mathcal{Y}_S|} \zeta'_{\mathfrak{p},\sigma}(0) = \sum_{i=0}^{2} (-1)^i \zeta'_{i,\sigma}(0)$$

and hence by definition of regularized determinants

$$\prod_{\mathfrak{p} \in |\mathcal{Y}_S|} \det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta) | \mathcal{F}_{\mathfrak{p}}(M)_\sigma \right)^{-1}$$
$$= \prod_{i=0}^{2} \det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i_c(\mathcal{Y}_S, \mathcal{F}(M))_\sigma \right)^{(-1)^{i+1}}.$$

According to (3.2) and (6.5) the left-hand side equals

$$\hat{L}_S(M, s)_\sigma = \prod_{\mathfrak{p} \in |\mathcal{Y}_S|} L_{\mathfrak{p}}(M, s)_\sigma,$$

where $L_{\mathfrak{p}}(M, s) = L_{k_{\mathfrak{p}}}(M \otimes_k k_{\mathfrak{p}}, s)$ in earlier notation and where we have put the hat on L to indicate the possible presence of Archimedean factors. Thus we finally get

$$(7.4)\qquad \hat{L}_S(M, s) = \prod_{i=0}^{2} \det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i_c(\mathcal{Y}_S, \mathcal{F}(M))\right)^{(-1)^{i+1}}$$

for $\operatorname{Re} s$ large enough.

I expect that the expressions

$$\det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i_c(\mathcal{Y}_S, \mathcal{F}(M))\right)$$

extend to entire functions. If $\det_\infty$ were defined via Weierstraß products, this would be clear. For regularized determinants I do not know whether this property is automatic without further regularity conditions as in **[CV]**.

The zeros of $\hat{L}_S(M, s)$ would be eigenvalues of Θ on $H^1_c(\mathcal{Y}_S, \mathcal{F}(M))$, and

$$\det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^1_c(\mathcal{Y}_S, \mathcal{F}(M))\right)$$

should have order one as an entire function.

(7.5) If M is a pure motive of weight w in $\mathcal{M}_k(E)$ then by analogy with **[D4,** Theorem (3.2.3)] and taking (7.1.1) into account we expect that $H^i(\mathcal{Y}, \mathcal{F}(M))$ is pure of weight $w + i$, i.e., that the eigenvalues of Θ have real part $(w+i)/2$. From the expression

$$(7.6)\qquad \hat{L}(M, s) = \prod_{i=0}^{2} \det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i(\mathcal{Y}, \mathcal{F}(M))\right)^{(-1)^{i+1}}$$

we would then get: The poles of $\hat{L}(M, s)$ are exactly the eigenvalues of Θ on the finite-dimensional spaces $H^i(\mathcal{Y}, \mathcal{F}(M))$ for $i = 0$ and 2; they have real part $w/2$ or $w/2 + 1$. The zeros of $\hat{L}(M, s)$ are exactly the eigenvalues of Θ on the infinite-dimensional space $H^1(\mathcal{Y}, \mathcal{F}(M))$ and hence should have real part $(w+1)/2$. That the zeros of $\hat{L}(M, s)$ should lie on the line of symmetry $\operatorname{Re} s = (w+1)/2$ for the (expected) functional equation of $\hat{L}(M, s)$ is of course a generalization of Riemann's conjecture to pure motives. It is compatible with the Riemann conjecture for Dirichlet L-series which is commonly considered in analytic number theory and with the investigations about nontrivial zeros of automorphic L-functions in **[Mo]**.

(7.7) Example. Let χ be an algebraic Hecke character over the number field k with values in the number field, E; see, e.g., **[Sch]**. For any embedding σ of E into $\mathbb{C}$ let

$$L(\chi^\sigma, s) = \prod_{\mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1}, \quad \text{where } \mathfrak{f}_\chi \text{ is the conductor of } \chi,$$

be the L-series of $\chi^\sigma = \sigma\chi$. We set

$$\hat{L}(\chi^\sigma, s) = L(\chi^\sigma, s) \prod_{\mathfrak{p}|\infty} L_\mathfrak{p}(\chi^\sigma, s),$$

where the local factors at infinity $L_\mathfrak{p}(\chi^\sigma, s)$ are defined as follows: Let χ^σ have infinity type $\sum_\tau n_\tau(\sigma) \cdot \tau \in \mathbb{Z}[\mathrm{Hom}(k, \mathbb{C})]$; i.e.,

$$\chi^\sigma((\alpha)) = \prod_\tau (\alpha^\tau)^{n_\tau(\sigma)} \quad \text{for totally positive } \alpha \in k^*, \ \alpha \equiv 1 \bmod \mathfrak{f}_\chi.$$

Let $w = n_\tau(\sigma) + n_{\bar{\tau}}(\sigma)$ denote the weight of χ. If $\mathfrak{p}$ is a real place of k (whose existence implies that all $n_\tau(\sigma) = w/2$), put

$$L_\mathfrak{p}(\chi^\sigma, s) = \Gamma_\mathbb{R}(s + \varepsilon_\mathfrak{p}^\sigma - w/2),$$

where $\varepsilon_\mathfrak{p}^\sigma \in \{0, 1\}$ is such that the $\mathfrak{p}$-component $\chi_\mathfrak{p}^\sigma : k_\mathfrak{p}^* \to \mathbb{C}^*$ of the idèle class character attached to χ^σ satisfies $\chi_\mathfrak{p}^\sigma(-1) = (-1)^{\varepsilon_\mathfrak{p}^\sigma + w/2}$. If $\mathfrak{p}$ is a complex place corresponding to the pair $\tau, \bar{\tau} : k \hookrightarrow \mathbb{C}$, we put

$$L_\mathfrak{p}(\chi^\sigma, s) = \Gamma_\mathbb{C}(s - \min(n_\tau(\sigma), n_{\bar{\tau}}(\sigma))).$$

Let $M(\chi)$ in $\mathcal{M}_k(E)$ denote the motive of χ in the sense of **[D3,** §8; **Sch,** Chapter I, Theorem 4.1]. See also **[DeMu,** (3.4)]. Then we have

$$L(M(\chi), s) = (L(\chi^\sigma, s))_{\sigma \in \mathrm{Hom}(E, \mathbb{C})},$$
$$\hat{L}(M(\chi), s) = (\hat{L}(\chi^\sigma, s))_{\sigma \in \mathrm{Hom}(E, \mathbb{C})}.$$

If $\chi \neq N_{k/\mathbb{Q}}^n$ for all $n \in \mathbb{Z}$ then each L-series $\hat{L}(\chi^\sigma, s)$ is an entire function, and according to the discussion in (7.5) we are led to expect that

$$H^i(\mathcal{Y}, \mathcal{F}(M(\chi))) = 0 \quad \text{for } i = 0, 2, \tag{7.8}$$

and hence that

$$\hat{L}(M(\chi), s) = \det_\infty\left(\frac{1}{2\pi}(s - \Theta) | H^1(\mathcal{Y}, \mathcal{F}(M))\right). \tag{7.8.1}$$

As regards the trivial character we have

$$L(\mathbb{Q}(0), s) = \zeta_k(s) \quad \text{and} \quad \hat{L}(\mathbb{Q}(0), s) = \zeta_k(s)\Gamma_\mathbb{R}(s)^{r_1}\Gamma_\mathbb{C}(s)^{r_2}.$$

Thus $\hat{L}(\mathbb{Q}(0), s)$ has first-order poles at $s = 0, 1$ and is holomorphic in $\mathbb{C}\backslash\{0, 1\}$. Because of (7.1.2) and (7.5) we therefore expect

$$\begin{aligned} H^0(\mathcal{Y}, \mathcal{C}) &= H^0(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0))) = \mathbb{C}(0), \\ H^2(\mathcal{Y}, \mathcal{C}) &= H^2(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0))) \overset{\mathrm{Tr}}{\cong} \mathbb{C}(-1) \end{aligned} \tag{7.9}$$

as $\mathbb{C}[\Theta]$-modules, where $\mathbb{C}(\alpha)$ is $\mathbb{C}$ with Θ acting by $-\alpha\,\mathrm{id}$, i.e. $\Theta_{\mathbb{C}(\alpha)} = \Theta_{\mathbb{C}(0)} - \alpha\,\mathrm{id}$. For the ξ-function of k

$$\xi_k(s) = \frac{s}{2\pi}\frac{s-1}{2\pi}\zeta_k(s)\Gamma_\mathbb{R}(s)^{r_1}\Gamma_\mathbb{C}(s)^{r_2},$$

formula (7.6) now implies

$$\xi_k(s) = \det_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^1(\mathcal{Y},\mathcal{C})\right). \qquad (7.10)$$

The case where $\chi = N^n_{k/\mathbb{Q}}$ follows by twisting with $-n$.

(7.11) Assuming the existence of suitable Hodge $*$-operators, the conjectures on weights in (7.5) would ensue by adapting Serre's argument in **[Se]** as follows: Assume $E=\mathbb{Q}$ for simplicity. For a pure motive M in $\mathcal{M}_k$ of weight w let M^* denote the dual motive. Cup product and the trace map (7.9) would give a T-equivariant pairing

$$H^i(\mathcal{Y},\mathcal{F}(M))\times H^{2-i}(\mathcal{Y},\mathcal{F}(M)^*(1-i-w))) \xrightarrow{\cup} H^2(\mathcal{Y},\mathcal{F}(\mathbb{Q}(1-i-w)) \xrightarrow[\sim]{\mathrm{Tr}} \mathbb{C}(-i-w).$$

Hence Θ would behave as a derivation with respect to $\cup$. For f_1 in $H^i(\mathcal{Y},\mathcal{F}(M))$ and f_2 in $H^{2-i}(\mathcal{Y},\mathcal{F}(M)^*(1-i-w)))$ we would get

$$(w+i)f_1\cup f_2 = \Theta(f_1\cup f_2) = \Theta f_1\cup f_2 + f_1\cup\Theta f_2. \qquad (7.12)$$

Now assume the existence of a $\mathbb{C}$-antilinear isomorphism

$$*\colon H^i(\mathcal{Y},\mathcal{F}(M)) \xrightarrow{\sim} H^{2-i}(\mathcal{Y},\mathcal{F}(M)^*(1-i-w))$$

which is T- and hence also $\mathfrak{t}$-equivariant such that the Hermitian bilinear form on $H^i(\mathcal{Y},\mathcal{F}(M))$ defined by $\langle f,f'\rangle = \mathrm{Tr}(f\cup *f')$ is positive definite; cf. **[Weil, V.2]**.

Because of $\Theta\circ * = *\circ\Theta$, relation (7.12) implies

$$(w+i)\langle f,f'\rangle = \langle\Theta f,f'\rangle + \langle f,\Theta f'\rangle \qquad (7.13)$$

for all f, f' in $H^i(\mathcal{Y},\mathcal{F}(M))$. If $\Theta f=\rho f$ for some $f\neq 0$, setting $f'=f$ implies

$$(w+i)\|f\|^2 = \rho\|f\|^2 + \overline{\rho}\|f\|^2;\quad \text{i.e., } \operatorname{Re}\rho = (w+i)/2.$$

Note that (7.13) is equivalent to the relation $\Theta = (w+i)/2 + iS$ where S is symmetric. The completion of $H^i(\mathcal{Y},\mathcal{F}(M))$ with respect to $\langle\ ,\ \rangle$ would be a Hilbert space with a T-action and an unbounded operator Θ satisfying (7.13) on its domain of definition.

(7.14) We leave the discussion of weights with one more remark: The weight of Θ on $H^0(\operatorname{Spec}\mathbb{Q}_p,\mathcal{C}) = \mathbb{L}_p$ is zero for $p<\infty$. For $p=\infty$ the weights of Θ on

$$\mathbb{L}_\infty \cong \mathcal{F}_\infty(H^0(\operatorname{Spec}(\mathbb{R})) \overset{?}{=} H^0(\infty,\mathcal{C}) \overset{?}{=} \text{“}H^0(\operatorname{Spec}(\mathbb{R}),\mathcal{C})^{I_\infty}\text{”}$$

are $0,-4,-8,\ldots$. Thus $\operatorname{Spec}\mathbb{Q}_p$ should be “smooth, proper” in the geometry of cat$\mathcal{S}$ for $p<\infty$ whereas $\operatorname{Spec}\mathbb{R}$ would have totally degenerate reduction at ∞. This argument extends to arbitrary motives and it is compatible with Manin's point of view in **[Ma]**.

(7.15) Let us turn to some evidence in favour of the above formalism for the motives $M(\chi)$ attached to algebraic Hecke characters as in (7.7). We first discuss the Lefschetz trace formula (7.2): According to (3.2) and (6.5) the eigenvalues of Θ on $\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma}$ with their algebraic multiplicities equal the poles with their order of

$$L_{\mathfrak{p}}(M(\chi), s)_{\sigma} = L_{\mathfrak{p}}(\chi^{\sigma}, s),$$

i.e., the numbers

- $(\log N\mathfrak{p})^{-1}(\log \chi^{\sigma}(\mathfrak{p}) + 2\pi i\nu)$ for $\nu \in \mathbb{Z}$ if $\mathfrak{p} \nmid \mathfrak{f}_{\chi}$ is finite,
- $w/2 - \varepsilon_{\mathfrak{p}}^{\sigma} - 2\nu$ for $\nu \geq 0$ if $\mathfrak{p}$ is real,
- $\min(n_{\tau}(\sigma), n_{\overline{\tau}}(\sigma)) - \nu$ for $\nu \geq 0$ if $\mathfrak{p} = \{\tau, \overline{\tau}\}$ is complex,

in the notation of (7.7). Hence $\mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma})$ is equal to

- $\sum_{\nu\in\mathbb{Z}} \Phi((\log N\mathfrak{p})^{-1}(\log \chi^{\sigma}(\mathfrak{p}) + 2\pi i\nu))$ for finite $\mathfrak{p} \nmid \mathfrak{f}_{\chi}$,
- zero for $\mathfrak{p}|\mathfrak{f}_{\chi}$,
- $\sum_{\nu=0}^{\infty} \Phi(w/2 - \varepsilon_{\mathfrak{p}}^{\sigma} - 2\nu)$ for real $\mathfrak{p}$,
- $\sum_{\nu=0}^{\infty} \Phi(\min(n_{\tau}(\sigma), n_{\overline{\tau}}(\sigma)) - \nu)$ for complex $\mathfrak{p} = \{\tau, \overline{\tau}\}$

if the sums converge.

According to (7.2), (7.8), (7.8.1) and (7.9), (7.10) we expect that

$$\sum_{i=0}^{2} (-1)^i \,\mathrm{Tr}(\Phi(\Theta)|H^i(\mathcal{Y}, \mathcal{F}(M(\chi)))_{\sigma})$$

equals

- $-\sum_{\rho} \Phi(\rho)$ if $\chi \neq N_{k/\mathbb{Q}}^{n}$ for all $n \in \mathbb{Z}$,
- $\Phi(0) - \sum_{\rho} \Phi(\rho) + \Phi(1)$ if $\chi = 1$

if the sums over the nontrivial zeros ρ of $L(\chi^{\sigma}, s)$ converge. The following theorem which is proved in **[De4,** Corollary (1.8)] is therefore compatible with (7.2).

(7.16) THEOREM. *Fix some $w/2 + 1 < a$, and consider a holomorphic function $\Phi(z)$ in an open subset of $\mathbb{C}$ containing $\mathrm{Re}\, z \leq a$. Assume that there exist constants $c_1 > 0$, $c_2 > 0$, $\alpha > 1$ such that*

$$|\Phi(x + iy)| \leq c_1(|y| + c_2)^{-\alpha} \quad \text{for } x \leq a \text{ and } y \in \mathbb{R}.$$

Then the above sums for the local traces $\mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma})$ are absolutely convergent and we have an identity of absolutely convergent series

$$\sum_{\mathfrak{p}} \mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma}) = \delta(\Phi(0) + \Phi(1)) - \sum_{\rho} \Phi(\rho),$$

where ρ runs over the nontrivial zeros of $L(\chi^{\sigma}, s)$ and $\delta = 0$ if $\chi \neq N_{k/\mathbb{Q}}^{n}$ for all $n \in \mathbb{Z}$ and $\delta = 1$ if $\chi = 1$.

A more general result is given in **[De4,** Theorem (1.7)].

Note that the function $\Phi(\tau) = (s-\tau)^{-z}$ is included for $\mathrm{Re}\, z > 1$ and $\mathrm{Re}\, s > w/2+1$. For Φ as in the theorem we also have explicit formulas à la Weil **[B, W]** expressing $\delta(\Phi(0)+\Phi(1)) - \sum_\rho \Phi(\rho)$ as a sum over local contributions $W_\mathfrak{p}(F;\chi^\sigma)$ defined using the inverse Mellin transform F of Φ. For functions Φ as in (7.16) we have in fact

$$\mathrm{Tr}(\Phi(\Theta)|\mathscr{F}_\mathfrak{p}(M(\chi))_\sigma) = W_\mathfrak{p}(F;\chi^\sigma).$$

Weil's way to write explicit formulas has the advantage to apply to more general Φ than those covered by (7.16). Our way to write them makes the interpretation as a Lefschetz trace formula plausible. Perhaps the $W_\mathfrak{p}(F,\chi^\sigma)$ could be interpreted as regularized traces if $\Phi(\Theta)$ is not of trace class on $\mathscr{F}_\mathfrak{p}(M(\chi))_\sigma$.

As mentioned above we expect that $\det_\infty(\frac{1}{2\pi}(s-\Theta)|H^1(\mathscr{Y},\mathscr{F}(M(\chi)))$ defines an entire function. Thus we are led to consider the series

$$\xi(z,s) = \sum_\rho \left(\frac{1}{2\pi}(s-\rho)\right)^{-z}, \tag{7.17}$$

where ρ runs over the nontrivial zeros of $L(\chi^\sigma, s)$. The following theorem which generalizes the result of **[De2,** §4**]** was proved by Soulé **[So2]** for the Riemann zeta function using the work of Cramér **[Cr]**. The extension to Hecke characters is due to Illies **[I]**:

(7.18) Theorem **[So2, I]**. *Let Ω be the set of complex numbers which are not of the form $\rho-\lambda$, $\lambda \geq 0$.*

(i) *For any s in Ω the series for the function $\xi(z,s)$ converges absolutely if* $\mathrm{Re}\, z > 1$. *It extends to a meromorphic function of s in Ω and z in $\mathbb{C}$ which is regular for $z=0$.*

(ii) *For s in Ω we have*

$$\prod_\rho \left(\frac{1}{2\pi}(s-\rho)\right) = \left(\frac{s}{2\pi}\cdot\frac{s-1}{2\pi}\right)^\delta \hat{L}(\chi^\sigma, s)$$

with δ as in (7.16).

Remark. The method of proof in **[De2,** §4**]** extends easily to Hecke characters but gives (7.18) only for $\mathrm{Re}\, s > w/2+1$.

(7.19) We now wish to discuss the functional equation. Assume that for a motive M in $\mathscr{MM}_k(E)$ its $\hat{L}_S$-series is given by (7.4) and that the regularized determinants make sense for all complex s in a set Ω as above. Poincaré duality should provide a T-equivariant isomorphism between the (smooth) dual:

$H_c^\nu(\mathscr{Y}_S,\mathscr{F}(M))_\sigma^* :=$ direct sum of the duals of the (finite-dimensional) irreducible T-subspaces in a T-invariant decomposition of $H_c^\nu(\mathscr{Y}_S,\mathscr{F}(M))_\sigma$

and $H^{2-\nu}(\mathscr{Y}_S, \mathscr{F}(M)^*(1))_\sigma$. We get

$$\begin{aligned} \hat{L}_S(M, s) &= \prod_{\nu=0}^{2} \det_\infty \left(\frac{1}{2\pi}(s - \Theta)|H^\nu_c(\mathscr{Y}_S, \mathscr{F}(M))\right)^{(-1)^{\nu+1}} \\ &= \prod_{\nu=0}^{2} \det_\infty \left(\frac{1}{2\pi}(s + \Theta)|H^\nu_c(\mathscr{Y}_S, \mathscr{F}(M))^*\right)^{(-1)^{\nu+1}} \\ &= \prod_{\nu=0}^{2} \det_\infty \left(\frac{1}{2\pi}(s + \Theta)|H^{2-\nu}(\mathscr{Y}_S, \mathscr{F}(M)^*(1))\right)^{(-1)^{\nu+1}} \\ &= \prod_{\nu=0}^{2} \det_\infty \left(-\frac{1}{2\pi}((1 - s) - \Theta)|H^\nu(\mathscr{Y}_S, \mathscr{F}(M^*)\right)^{(-1)^{\nu+1}} \end{aligned} \tag{7.19.1}$$

Now let S be empty, $\mathscr{Y}_S = \mathscr{Y}$. A standard decomposition (1.6)

$$H^\nu(\mathscr{Y}, \mathscr{F}(M)) = H^\nu(\mathscr{Y}, \mathscr{F}(M))^+ \oplus H^\nu(\mathscr{Y}, \mathscr{F}(M))^-$$

with respect to $-\Theta$ would also be standard with respect to $1 - s - \Theta$ for any s. We expect that the regularized superdimension of $1 - s - \Theta$

$$D_\nu(s) := \operatorname{sdim}_\infty(1 - s - \Theta|H^\nu(\mathscr{Y}, \mathscr{F}(M^*))$$

exists with respect to such a decomposition; cf. (1.4), (1.6). Using (1.7) we would get

$$\hat{L}(M, s) = \varepsilon(M, s)\hat{L}(M^*, 1 - s),$$

where

$$\varepsilon(M, s) = \exp\left(i\pi \sum_{\nu=0}^{2}(-1)^{\nu+1} D_\nu(s)\right).$$

If one could show, for example, from (7.6) that $\hat{L}(M, s)$ and $\hat{L}(M^*, 1 - s)$ had genus one, it would follow from the definition of the genus that

$$\varepsilon(M, s) = ae^{bs} \quad \text{for some } a \in (E \otimes \mathbb{C})^*, \ b \in E \otimes \mathbb{C}.$$

For algebraic Hecke characters there is the following result which was suggested by our considerations of regularized superdimensions:

(7.20) Theorem [I]. *For s in Ω set*

$$\xi^\pm(z, s) = \sum_\rho{}^{\pm} \left(\frac{1}{2\pi}(s - \rho)\right)^{-z},$$

where in $\sum_\rho^\pm$ the sum is over the nontrivial zeros of $L(\chi^\sigma, s)$ with $\operatorname{Im}\rho \gtreqless 0$. Then the sum for $\xi^\pm(z, s)$ converges absolutely if $\operatorname{Re} z > 1$. It extends to a meromorphic function of s in Ω and z in $\mathbb{C}$. For fixed s in Ω it has at most a first-order pole at $z = 0$.

That (7.20) follows for $\chi = 1$ from results of Cramér **[Cr]** is evident from **[So2]**.

(7.21) We now turn to a discussion of the order of L-functions at integral values of s. For M in $\mathcal{MM}_k(E)$ and S any finite set of places of k formula (7.19.1) gives

$$\hat{L}_S(M, s) = \prod_{\nu=0}^{2} \det{}_\infty \left(\frac{1}{2\pi}(s+\Theta)|H^\nu(\mathcal{Y}_S, \mathcal{F}(M^*(1))\right)^{(-1)^{\nu+1}}$$

and hence

$$\operatorname{ord}_{s=0} \hat{L}_S(M, s) = \sum_{\nu=0}^{2}(-1)^{\nu+1} \dim_{\mathbb{C}} H^\nu(\mathcal{Y}_S, \mathcal{F}(M^*(1)))^{\Theta\sim 0}$$

in the notation of (1.1).

Now assume for the moment that $k = \mathbf{Q}$, $S = \{\infty\}$, and $E = \mathbb{Q}$ such that in particular $\hat{L}_S(M, s) = L(M, s)$. According to conjecture B of **[Scho]** for motives M over $\mathcal{Y}_S = \operatorname{Spec}\mathbb{Z}$ we should have

$$\operatorname{ord}_{s=0} L(M, s) = \sum_{\nu=0}^{1}(-1)^{\nu+1} \dim_{\mathbb{Q}} \operatorname{Ext}^\nu_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M^*(1)).$$

This identity would be explained by canonical isomorphisms

$$\operatorname{Ext}^\nu_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^\nu(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(M))^{\Theta\sim 0} \tag{7.22}$$

for M in $\mathcal{MM}_{\mathbb{Z}}$. Replacing M by $M(n)$ gives

$$\operatorname{Ext}^\nu_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M(n)) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^\nu(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(M))^{\Theta\sim n}. \tag{7.23}$$

The expected vanishing of $\operatorname{Ext}^2_{\mathcal{MM}_{\mathbb{Z}}}$ fits in with the idea that $H^2(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(M))$ should vanish since $\operatorname{Spec}\mathbb{Z}$ is "noncompact" and 1-dimensional.

The following consideration is compatible with (7.22). Let δ be the connecting morphism in the relative exact sequence for the pair $(\overline{\operatorname{Spec}\mathbb{Z}}, \operatorname{Spec}\mathbb{Z})$, where $\overline{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$:

$$H^1(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(M)) \xrightarrow{\delta} H^2_\infty(\overline{\operatorname{Spec}\mathbb{Z}}, \mathcal{F}(M)).$$

By Poincaré duality we expect an isomorphism

$$H^2_\infty(\overline{\operatorname{Spec}\mathbb{Z}}, \mathcal{F}(M)) \cong H^0(\infty, \mathcal{F}(M^*(1)))^* \cong \mathcal{F}_\infty(M^*(1))^*. \tag{7.24}$$

In (6.8) we have shown that there is a natural map

$$\operatorname{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), M_B) \otimes_{\mathbb{R}} \mathbb{C} \to (\mathcal{F}_\infty(M^*(1))^{\Theta=0})^* \tag{7.25}$$

which is an isomorphism if the weights of M are ≤ 0. It should fit into a commutative diagram

$$\begin{array}{ccc} H^1(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(M))^{\Theta\sim 0} & \xrightarrow{\ \delta\ } & H^2_\infty(\overline{\operatorname{Spec}\mathbb{Z}}, \mathcal{F}(M))^{\Theta\sim 0} = (\mathcal{F}_\infty(M^*(1))^{\Theta=0})^* \\ \uparrow & & \uparrow \\ \operatorname{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M) \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\text{canonical}} & \operatorname{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), M_B) \otimes_{\mathbb{R}} \mathbb{C}. \end{array}$$

It was this argument by the way which led to the map (7.25) in (6.8) and thus helped to understand the relation between $\mathscr{F}_\infty^{\Theta=m}$ and Deligne cohomology in (6.10).

The fact that the regulator map should be viewed as a boundary map using Arakelov compactification is due to Beilinson (e.g., [**Be**, 0.3]. However, the relation between his exact sequence of topological groups with volume forms and the above exact sequence is not at all obvious to me.

Note that we cannot have a Gysin isomorphism

$$H^2_\infty(\overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{F}(M)) \cong H^0(\infty, \mathscr{F}(M)(-1)) = \mathscr{F}_\infty(M)(-1)$$

because in connection with (7.24) it would imply that $\mathscr{F}_\infty$ commutes with duals which it does not: for example, Θ has weights $\dots -8, -4, 0$ on $\mathscr{F}_\infty(\mathbb{Q}(0)) = \mathbb{L}_\infty$ whereas the weights on $\mathscr{F}_\infty(\mathbb{Q}(0))^*$ are $0, 4, 8, \dots$. Again this supports the philosophy that ∞ is a singular point of $\overline{\operatorname{Spec}\mathbb{Z}}$ in the geometry of cat$\mathscr{S}$. However, we expect $\overline{\operatorname{Spec}\mathbb{Z}}$ to be "smooth compact" since its cohomologies should be pure and satisfy Poincaré duality. Classically regular curves cannot have singular points of course. One could also imagine that $\overline{\operatorname{Spec}\mathbb{Z}}$ is singular such that the $H^i(\overline{\operatorname{Spec}\mathbb{Z}}, j_*\mathscr{F}(M))$ would be intersection cohomology groups. In this case we would not expect that $j_*\mathscr{F}(\mathbb{Q}(0)) = \mathscr{C}_{\overline{\operatorname{Spec}\mathbb{Z}}}$.

Let X be smooth and proper over $\mathbb{Q}$, and consider the motive $M = H^i(X)(n)$ in $\mathscr{M}\mathscr{M}_\mathbb{Z}$. Conjecturally the Ext-groups $\operatorname{Ext}^\nu_{\mathscr{M}\mathscr{M}_\mathbb{Z}}(\mathbb{Q}(0), H^i(X)(n))$ can then be expressed in terms of Chow- and algebraic K-groups [**Scho**, III]. In conjunction with (7.22) we are thus led to expect the following isomorphisms:

(7.26)

$$H^0(\operatorname{Spec}\mathbb{Z}, \mathscr{F}(H^i(X)))^{\Theta\sim n} = \begin{cases} (CH^n(X)/CH^n(X)^0)\otimes\mathbb{C} & \text{if } i = 2n, \\ 0 & \text{if } i \neq 2n; \end{cases}$$

$$H^1(\operatorname{Spec}\mathbb{Z}, \mathscr{F}(H^i(X)))^{\Theta\sim n} = \begin{cases} CH^n(X)^0\otimes\mathbb{C} & \text{if } i+1 = 2n, \\ H^{i+1}_\mathscr{M}(X, \mathbb{Q}(n))_\mathbb{Z}\otimes\mathbb{C} & \text{if } i+1 \neq 2n. \end{cases}$$

Here $CH^n(X)^0$ is the subgroup of classes of cycles in $CH^n(X)$ which are homologically equivalent to zero and $H^k_\mathscr{M}(X, \mathbb{Q}(n))_\mathbb{Z}$ is the image of the K-theory of a regular model proper and flat over $\mathbb{Z}$ for X in $H^k_\mathscr{M}(X, \mathbb{Q}(n)) := \operatorname{Gr}^n_\gamma K_{2n-k}(X)_\mathbb{Q} \cong K^{(n)}_{2n-k}(X)_\mathbb{Q}$.

Extrapolating (7.26) we are led to think of the groups

$$H^1(\operatorname{Spec}\mathbb{Z}, \mathscr{F}(H^i(X)))^{\Theta\sim\alpha} \text{ for arbitrary } \alpha \text{ in } \mathbb{C}$$

as something like $H^{i+1}_\mathscr{M}(X, \mathbb{C}(\alpha))_\mathbb{Z}$, i.e., "$K$-theory indexed by complex numbers"—an idea proposed by N. Kurokawa.

Let us return to the general case and assume that M is a mixed motive over $\mathscr{Y}_S$ with coefficients in E. For $S \supset \{\mathfrak{p}|\infty\}$ we expect isomorphisms of

$E \otimes \mathbb{C}$ modules

$$\text{(7.27)} \qquad \operatorname{Ext}^{\nu}_{\mathcal{MM}_{\mathscr{Y}_S}(E)}(E(0), M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{\nu}(\mathscr{Y}_S, \mathscr{F}(M))^{\Theta \sim 0}$$

and, in particular, taking the limit over all finite $S \supset \{\mathfrak{p}|\infty\}$

$$\operatorname{Ext}^{\nu}_{\mathcal{MM}_k(E)}(E(0), M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{\nu}(\operatorname{Spec} k, \mathscr{F}(M))^{\Theta\sim 0}.$$

For $k = E = \mathbb{Q}$ the left-hand side is again expressed in terms of Chow- and K-groups in **[Scho, III]**: just replace $H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$ by $H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))$ in (7.26).

(7.28) Having discussed L-series of motives in some detail let us quickly consider Hasse Weil zeta functions of quasi-projective schemes $X/\mathbb{Z}$

$$\zeta_X(s) = \prod_{x \in |X|} (1 - Nx^{-s})^{-1}.$$

Similarly as in (7.2)–(7.4) a Lefschetz trace formula of the form

$$\text{(7.29)} \qquad \sum_{x\in|X|} \operatorname{Tr}(\Phi(\Theta)|\mathscr{C}_x) = \sum_i (-1)^i \operatorname{Tr}(\Phi(\Theta)|H^i_c(X, \mathscr{C}))$$

and the relation $\mathscr{C}_x = \mathbb{L}_{Nx}$ imply at least formally that

$$\zeta_X(s) = \prod_i \operatorname{det}_{\infty}\left(\frac{1}{2\pi}(s - \Theta)|H^i_c(X, \mathscr{C})\right)^{(-1)^{i+1}}.$$

Now assume that X is regular connected of dimension d. Then Poincaré duality should give

$$H^i_c(X, \mathscr{C}(n))^* \cong H^{2d-i}(X, \mathscr{C}(d-n)),$$

and we would get

$$\zeta_X(s) = \prod_{i=0}^{2d} \operatorname{det}_{\infty}\left(\frac{1}{2\pi}(s - d + \Theta)|H^i(X, \mathscr{C})\right)^{(-1)^{i+1}}.$$

In particular, we would have

$$\operatorname{ord}_{s=d-n} \zeta_X(s) = \sum_{i=0}^{2d} (-1)^{i+1} \dim_{\mathbb{C}} H^i(X, \mathscr{C}(n))^{\Theta\sim 0}.$$

We expect formal analogues of Tate's conjecture:

$$H^i_{\mathcal{M}}(X, \mathbb{C}(n)) := \operatorname{Gr}^n_{\gamma} K_{2n-i}(X) \otimes \mathbb{C} \xrightarrow{\sim} H^i(X, \mathscr{C}(n))^{\Theta\sim 0}$$

and, in particular, that

$$H^i(X, \mathscr{C}(n))^{\Theta\sim 0} = 0 \quad \text{for } i > 2n.$$

This would give

$$\operatorname{ord}_{s=d-n} \zeta_X(s) = \sum_{i=0}^{2n} (-1)^{i+1} \dim_{\mathbb{Q}} \operatorname{Gr}^n_{\gamma}(K_{2n-i}(X) \otimes \mathbb{Q}),$$

a conjecture due to Soulé [**Sol**]. Assume that there exists a compactification $\overline{X}$ of X over $\overline{\operatorname{Spec}\mathbb{Z}}$ with smooth projective infinite fibre $X_\infty = \overline{X}\setminus X$ and satisfying Poincaré duality

$$H^{i+1}_{X_\infty}(\overline{X}, \mathscr{C}(n)) = H^{2d-i-1}(X_\infty, \mathscr{C}(d-n))^* = \mathscr{F}_\infty(H^{2d-i-1}(X_\infty)(d-n))^* .$$

The natural map α of (6.9) which is an isomorphism for $i < 2n$ should fit into a commutative diagram

$$\begin{array}{ccc} H^i(X, \mathscr{C}(n))^{\Theta\sim 0} \xrightarrow{\delta} H^{i+1}_{X_\infty}(\overline{X}, \mathscr{C}(n))^{\Theta\sim 0} & = (\mathscr{F}_\infty(H^{2d-i-1}(X_\infty)(d-n))^{\Theta=0})^* \\ \uparrow\wr & & \alpha\uparrow \\ H^i_{\mathscr{M}}(X, \mathbb{Q}(n)) \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{r_{\mathscr{D}}} & H^i_{\mathscr{D}}(X_\infty, \mathbb{R}(n)) \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

where $r_{\mathscr{D}}$ is the regulator map.

(7.30) It is not clear to me how the Lefschetz trace formula should look if we replace X by $\overline{X}$ in (7.29). What are the "fixed points under Θ" at infinity? The connection between the values of L-functions at integral points and volumes conjectured by Beilinson, Bloch-Kato et al. still remains mysterious even at our formal level. Note, however, that in theoretical physics integrals over path spaces are often defined formally by regularized determinants of certain operators [**Wil,** §3]. Hence there should be a relation between measure theory on infinite-dimensional spaces and such determinants. I expect such a theory to be necessary for a proof of the conjectures on special values of L-series.

(7.31) Let $CH^1(\overline{\operatorname{Spec}\mathbb{Z}})$ denote the first Arakelov Chow group of $\overline{\operatorname{Spec}\mathbb{Z}}$. I expect a cycle class map such that composition with the trace map:

$$CH^1(\overline{\operatorname{Spec}\mathbb{Z}}) \to H^2(\overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{C}(1))^{\Theta=0} \overset{\operatorname{Tr}}{\underset{\sim}{\to}} \mathbb{C}$$

equals the Arakelov degree. Since the latter surjects onto $\mathbb{R}$, we are somewhat confirmed in our basic assumption that arithmetic cohomology should have coefficients in a field C at least containing $\mathbb{R}$. The local constructions in §3 render $C = \mathbb{R}$ improbable, $C = \mathbb{C}$ seems to be the minimal choice.

Of course, higher-dimensional Arakelov Chow groups should have cycle maps into arithmetic cohomology, but we will not discuss this here. No doubt the ideas of Gillet and Soulé in [**GiSo**] will fit into the picture.

(7.32) We close this section with some remarks about a Künneth formula for arithmetic cohomology. The category cat$\mathscr{S}$ should have a product $\underline{\times}$ such that if X and Y are schemes, $X \underline{\times} Y \neq X \times_{\operatorname{Spec}\mathbb{Z}} Y$ is no longer a scheme in general. Since Θ comes from a Lie algebra representation, a Künneth formula

$$H^\bullet_c(X \underline{\times} Y, \mathscr{C}) \cong H^\bullet_c(X, \mathscr{C}) \otimes_{\mathbb{C}} H^\bullet_c(Y, \mathscr{C})$$

would imply that Θ on the left corresponds to $\Theta \otimes \mathrm{id} + \mathrm{id} \otimes \Theta$ on the right. In particular, we would have

$$\det_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^2(\overline{\operatorname{Spec}\mathbb{Z}} \underline{\times} \overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{C}) \right) = \left(\frac{s-1}{2\pi} \right)^2 \prod_{\rho, \rho'} \frac{1}{2\pi}(s - (\rho + \rho')),$$

where ρ and ρ' run over the nontrivial zeros of $\zeta(s)$. Of course, the regularized product on the right does not make sense. However, similar products but with the restriction $\operatorname{Im}\rho, \operatorname{Im}\rho' \geq 0$ or $\operatorname{Im}\rho, \operatorname{Im}\rho' < 0$ have been proposed by Kurokawa [K] as zeta-functions of $\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$. His functions are of order two, and there is hope that they have similar arithmetic properties as classical L-functions. Let us illustrate this by an example where no convergence problems arise: Let K be $\mathbb{R}$ or $\mathbb{C}$ with unique place $\mathfrak{p}$, set $e_\mathfrak{p} = [\mathbb{C} : K]$, and let X be an algebraic scheme over $\mathbb{Z}$. Then the zeta-function of $X \underline{\times} \mathfrak{p}$ should be given by the infinite product

$$\zeta_{X \underline{\times} \mathfrak{p}}(s) := \prod_{\nu=0}^{\infty} \zeta_X(s + \nu e_\mathfrak{p}) \tag{7.33}$$

which converges absolutely to an analytic function in the region where ζ_X is analytic: note that the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ for $\zeta_X(s)$ has $a_1 = 1$ as its first coefficient and hence that $\zeta_X(s) \to 1$ strongly for $\operatorname{Re} s \to \infty$. The following formal computation justifies the definition (7.33):

$$\begin{aligned}
\zeta_{X \underline{\times} \mathfrak{p}}(s) &= \prod_i \det_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i_c(X \underline{\times} \mathfrak{p}, \mathscr{C}) \right)^{(-1)^{i+1}} \\
&= \prod_i \det_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i_c(X, \mathscr{C}) \otimes_\mathbb{C} \mathbb{L}_\mathfrak{p} \right)^{(-1)^{i+1}} \\
&= \prod_i \prod_{\nu=0}^{\infty} \det_\infty \left(\frac{1}{2\pi}(s - \Theta) | H^i_c(X, \mathscr{C})(\nu e_\mathfrak{p}) \right)^{(-1)^{i+1}} \\
&= \prod_i \prod_{\nu=0}^{\infty} \det_\infty \left(\frac{1}{2\pi}(s + \nu e_\mathfrak{p} - \Theta) | H^i_c(X, \mathscr{C}) \right)^{(-1)^{i+1}} \\
&= \prod_{\nu=0}^{\infty} \zeta_X(s + \nu e_\mathfrak{p}).
\end{aligned}$$

Clearly $\zeta_{X \underline{\times} \mathfrak{p}}(s)$ has the Euler product $\prod_p \zeta_{X_p \underline{\times} \mathfrak{p}}(s)$, where

$$\zeta_{X_p \underline{\times} \mathfrak{p}}(s) = \prod_{\nu=0}^{\infty} \zeta_{X_p}(s + \nu e_\mathfrak{p}), \qquad X_p = X \otimes_\mathbb{Z} \mathbb{F}_p,$$

and it can be written as a Dirichlet series. The case where $X = \operatorname{Spec}\mathscr{O}_k$ with k a finite extension of $\mathbb{Q}$ is considered in detail by Cohen and Lenstra [CL] in their heuristic study of class groups of number fields as was kindly pointed out to me by Manin. They prove a functional equation for $\zeta_{X \underline{\times} \mathrm{p}}(s)$ in loc. cit. Theorem 7.1 which would also be explained by Poincaré duality on X. If $K = \mathbb{C}$ and $X = \operatorname{Spec}\mathbb{Z}$ their corollary (3.7) specializes to the formula

$$\zeta_{\operatorname{Spec}\mathbb{Z}\underline{\times}\mathrm{p}}(s) = \prod_{\nu=0}^{\infty} \zeta(s+\nu) = \sum_{\mathscr{A}} \frac{|\mathscr{A}|}{|\operatorname{Aut}\mathscr{A}|} \frac{1}{|\mathscr{A}|^s}, \qquad \operatorname{Re} s > 1.$$

Here $\mathscr{A}$ runs over all finite abelian groups (i.e., coherent torsion sheaves on $\operatorname{Spec}\mathbb{Z}$) up to isomorphism.

(7.34) In order to carry out the program of §7 the most important step is of course to find the objects and morphisms of the category $\operatorname{cat}\mathscr{S}$ and to study the new geometry that they give rise to. Foundations again!

References

[A] M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, Colloque en l'honneur de Laurent Schwartz, Astérisque **131** (1985), 43–59.

[B] K. Barner, *On A. Weil's explicit formula*, J. Reine Angew. Math. **323** (1981), 139–152.

[Be] A. A. Beilinson, *Notes on absolute Hodge cohomology*, Contemp. Math. **55** (1986), 35–68.

[CL] H. Cohen and H. W. Lenstra, Jr., *Heuristics on class groups of number fields*, Lecture Notes in Math., vol. 1068, Springer-Verlag, New York, 1984, pp. 33–62.

[CV] P. Cartier and A. Voros, *Une nouvelle interprétation de la formule des traces de Selberg*, Grothendieck Festschrift II, Birkhäuser, Basel, 1991.

[Cr] H. Cramér, *Studien über die Nullstellen der Riemannschen Zetafunktion*, Math. Z. **4** (1919), 104–130.

[D1] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Math., vol. 163, Springer-Verlag, New York, 1970.

[D2] ——, *Théorie de Hodge*. II, Inst. Hautes Études Sci. Publ. Math. **40** (1972), 5–57.

[D3] ——, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math. **33** (1979), 313–342.

[D4] ——, *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 313–428.

[D5] ——, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over $\mathbb{Q}$, Math. Sci. Res. Inst. Publ., Springer-Verlag, New York, 1989, pp. 79–297.

[De1] C. Deninger, *On the Γ-factors attached to motives*, Invent. Math. **104** (1991), 245–261.

[De2] ——, *Local L-factors of motives and regularized determinants*, Invent. Math. **107** (1992), 135–150.

[De3] ——, *L-functions of mixed motives*, these Proceedings, vol. 1, pp. 517–525.

[De4] ——, *Lefschetz trace formulas and explicit formulas in analytic number theory*, J. Reine Angew. Math. **441** (1993), 1–15.

[DeMu] C. Deninger and J. Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.

[GISo] H. Gillet and Ch. Soulé, *Arithmetic analogs of the standard conjectures*, these Proceedings, vol. 1, pp. 129–140.

[F-PR] J. M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: Cohomologie Galoisienne et valeurs de fonctions L*, these Proceedings, vol. 1, pp. 599–706.

[H] A. Haefliger, *Local theory of meromorphic connections in dimension one (Fuchs theory)*, Algebraic *D*-Modules, Perspect. Math., Academic Press, Boston, MA, 1987, pp. 129–149.

[I] G. Illies, *Diplomarbeit*, Münster, Germany, 1992.

[J1] U. Jannsen, *Deligne homology, Hodge-𝒟-conjecture, and motives, Beilinson's conjectures on special values of L-functions*, Perspect. Math., vol. 4, Academic Press, New York, 1988, pp. 305–372.

[J2] ——, *Mixed motives and algebraic K-theory*, Lecture Notes in Math., vol. 1400, Springer-Verlag, New York, 1990.

[K] N. Kurokawa, *Multiple zeta functions: An example*, Zeta Functions in Geometry, Adv. Stud. Pure Math., 1991.

[Ma] Yu. I. Manin, *Three-dimensional hyperbolic geometry as ∞-adic Arakelov geometry*, Invent. Math. **104** (1991), 223–244.

[Mo] C. J. Moreno, *Explicit formulas in the theory of automorphic forms*, Number Theory Day, Lecture Notes in Math., vol. 626, Springer-Verlag, New York, 1977, pp. 73–216.

[Sch] N. Schappacher, *Periods of Hecke characters*, Lecture Notes in Math., vol. 1301, Springer-Verlag, New York, 1988.

[Scho] A. J. Scholl, *Remarks on special values of L-functions*, L-functions and Arithmetic, London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge and New York, 1991, pp. 373–392.

[Se] J.-P. Serre, *Analogues kählériens de certaines conjectures de Weil*, Ann. of Math. (2) **71** (1960), 392–394.

[So1] C. Soulé, *K-théorie et zéros aux points entiers de fonctions zêta*, Proc. ICM, Warsaw, 1983, pp. 437–445.

[So2] ——, Letter to the author, February 13, 1991.

[T] J. Tate, *Number theoretical background*, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 3–26.

[W] A. Weil, *Sur les formules explicites de la théorie des nombres*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 3–18.

[Well] R. O. Wells, *Differential analysis on complex manifolds*, Graduate Texts in Math., vol. 65, Springer-Verlag, New York, 1980.

[Wi1] E. Witten, *Physics and geometry*, Proc. Internat. Congr. Math., Berkeley, CA, 1986.

[Wi2] ——, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

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On a Result of Deninger Concerning Riemann's Zeta Function

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Let R be the set of nontrivial zeroes of the Riemann zeta function $\zeta(z)$ and Ω the set of complex numbers which are not of the form $\rho - \lambda$, with $\rho \in R$ and λ a nonnegative real number. Given $z \in \Omega$ and $\rho \in R$ we write $z - \rho = |z - \rho| e^{i \operatorname{Arg}(z-\rho)}$ where the argument $\operatorname{Arg}(z-\rho)$ is chosen to lie between $-\pi$ and π, and for every complex number s we define $(z-\rho)^s = |z-\rho|^s e^{is \operatorname{Arg}(z-\rho)}$. Now consider the infinite series

$$\xi(s, z) = \sum_{\rho \in R} \frac{1}{((z-\rho)/2\pi)^s}.$$

THEOREM. (i) *When* $\operatorname{Re}(s) > 1$ *the series* $\xi(s, z)$ *is absolutely convergent for any* $z \in \Omega$. *It extends to a meromorphic function of* $s \in \mathbb{C}$ *and* $z \in \Omega$. *For every fixed value of* z, $\xi(s, z)$ *has a simple pole when* $s = 1$ *and is regular when* $s = 0$.

(ii) *Let* $\partial_s \xi(0, z)$ *be the derivative of* $\xi(s, z)$ *at* $s = 0$. *For every* $z \in \Omega$ *the following formula holds*:

$$2^{-1/2}(2\pi)^{-2}\pi^{-z/2}\Gamma(z/2)\zeta(z)z(z-1) = \exp(-\partial_s \xi(0, z)).$$

PROOF. When $\operatorname{Re}(z) > 1$ this result is due to Deninger [**D**, Theorem 3.3]. Therefore, by the unicity of analytic continuation, we just need to prove (i).

Fix a positive real number $T > 0$, and consider the series

$$V(t) = \sum_{\operatorname{Im}(\rho) > T} \exp(it\rho).$$

According to Cramér [**C**], the series $V(t)$ is absolutely convergent for any choice of a positive real number t, and

$$V(t) - \frac{\log t}{2\pi i(1 - \exp(-it))} + \frac{C + \log(2\pi)}{2\pi t}$$

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extends to a smooth function around $t = 0$, where C is the Euler constant. Consequently

$$V(t) = -\frac{\log t}{2\pi t} - \frac{C + \log(2\pi)}{2\pi t} + \frac{\log t}{4\pi i} + W_1(t) + W_2(t) t \log t, \tag{1}$$

where $W_1(t)$ and $W_2(t)$ are smooth around $t = 0$.

For any $z \in \mathbb{C}$ such that $|\mathrm{Im}(z)| < T$ consider

$$\theta^+(t, z) = V(t/2\pi) e^{-izt/2\pi},$$
$$\theta^-(t, z) = V(t/2\pi) e^{-(1-z)it/2\pi},$$
$$\xi^+(s, z) = \frac{e^{i\pi s/2}}{\Gamma(s)} \int_0^\infty \theta^+(t, z) t^{s-1}\, dt,$$
$$\xi^-(s, z) = \frac{e^{-i\pi s/2}}{\Gamma(s)} \int_0^\infty \theta^-(t, z) t^{s-1}\, dt.$$

When $\mathrm{Re}(s) > 1$, since $V(t)$ is convergent, we may integrate term-by-term and one gets

$$\xi^+(s, z) = \sum_{\mathrm{Im}(\rho) > T} \frac{1}{((z - \rho)/2\pi)^s}$$

and

$$\xi^-(s, z) = \sum_{\mathrm{Im}(\rho) < -T} \frac{1}{((z - \rho)/2\pi)^s}.$$

From (1) and the identities

$$\int_0^1 t^n t^{s-1}\, dt = \frac{1}{s + n}$$

and

$$\int_0^1 t^n (\log t) t^{s-1}\, dt = -\frac{1}{(s + n)^2}$$

($n \in \mathbb{Z}$), we deduce that

$$\begin{aligned} \xi^+(s, z) &= \frac{e^{i\pi s/2}}{\Gamma(s)} \left(\frac{2z - 1}{4\pi i s^2} + \frac{1}{(s - 1)^2} - \frac{C}{s - 1} + \cdots \right) \\ &= \frac{2z - 1}{4\pi i s} + \frac{i}{(s - 1)^2} - \frac{\pi}{2(s - 1)} + \eta^+(s, z), \end{aligned}$$

and similarly

$$\xi^-(s, z) = \frac{1 - 2z}{4\pi i s} - \frac{i}{(s - 1)^2} - \frac{\pi}{2(s - 1)} + \eta^-(s, z),$$

where $\eta^+(s, z)$ and $\eta^-(s, z)$ are meromorphic functions of (s, z), $|\mathrm{Im}(z)| < T$, which are regular around $s = 1$ and $s = 0$. Adding up these two equalities, since $\xi(s, z)$ differs from $\xi^+(s, z) + \xi^-(s, z)$ by the sum of finitely many terms, we get

$$\xi(s, z) = -\frac{\pi}{s - 1} + \eta(s, z),$$

where $\eta(s, z)$ is meromorphic in (s, z), and regular around $s = 1$ and $s = 0$. The theorem follows.

This note is an updated version of a letter of the second author to C. Deninger (Feb. 13, 1991). The result has in the meantime been generalized to the case of Hecke L-series by Illies **[I]**.

References

[C] H. Cramér, *Studien über die Nullstellen der Riemannschen Zetafunktion*, Math. Z. **4** (1919), 104–130.

[D] C. Deninger, *Local L-factors of motives and regularized determinants*, Invent. Math. **107** (1992), 135–150.

[I] G. Illies, *Regularisierte Produkte und Heckesche L-Reihen*, Diplomarbeit, Münster, 1992.

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