Questions and speculation on cohomology theories in arithmetic geometry. Version 1.

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December 20, 2007

Abstract

I have tried to formulate a draft of some of the things I have been thinking about during the first year of my PhD, in order to be able to get some feedback. Any comments or corrections are more then welcome. In particular, I would be very grateful for concrete (possibly partial) answers to some of my questions, for opinions on whether questions are interesting/non-interesting, and for advice on things to read which I am unaware of. I apologise in advance for being too brief and not defining things in many places. I expect to revise this document as I get feedback and learn more, so if you read this after February 2008, please check my website (www.andreasholmstrom.org) for a more recent version.

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1 Introduction

The reason for thinking about cohomology theories in the first place, is the experience of being a student trying to learn number theory. The feeling I have is that the main obstacle to obtaining an overview of number theory is the large number of different cohomology theories. These theories are at the heart of many significant advances in the last fifty years, the most famous being perhaps Deligne's proof of the Weil conjectures using etale cohomology, and Voevodsky's application of motivic cohomology to the Milnor and Bloch-Kato conjectures. Also, many of the hardest and most interesting unsolved problems in the field are directly related to various cohomology theories. However, for each theory, it takes some considerable effort to understand it, and since there are so many, it would seem that one would need a lifetime to understand them all.

It should be possible to find organising principles and underlying mechanisms, so that the maze of cohomology theories becomes accessible and understandable. This hope is the subject of this brief essay. So far, there isn't much understanding, but rather a list of things I would like to understand in the future.

1.1 Acknowledgements

I've had many interesting discussions with various people about the material in this draft. In particular, I would like to thank Tony Scholl, Gereon Quick, Markus Severitt and Alex Shannon for many interesting discussions about some of these things.

1.2 Examples of cohomology theories

To illustrate the problem, we list some notions of cohomology encountered in arithmetic geometry: Etale cohomology, various versions of algebraic K-theory, the concept of "arithmetic vs. geometric" cohomology theories, absolute Hodge cohomology, Hodge cohomology, Amitsur cohomology, archimedean cohomology, Andre-Quillen cohomology, Betti cohomology, Borel-Moore homology, cdh cohomology, Cech cohomology, Chow groups, arithmetic Chow groups, Arakelov Chow groups, group cohomology and continuous group cohomology, crystalline cohomology, crystalline Deligne cohomology, de Rham cohomology, Deligne cohomology, Deligne-Beilinson cohomology, smooth Deligne cohomology, Eichler cohomology, elliptic Bloch groups, equivariant Deligne cohomology, etale K-theory, etale motivic cohomology, flat cohomology, Fontaine-Messing cohomology, Friedlander-Suslin cohomology, Galois cohomology, Hyodo-Kato cohomology, Lawson homology, cohomology of Lie algebras, "log" versions of Betti, de Rham, crystalline and etale cohomology, Milnor K-theory, Kato homology, Monsky-Washnitzer cohomology, morphic cohomology, motivic cohomology, non-abelian cohomology, Nisnevich cohomology, p-adic etale cohomology, parabolic cohomology, rigid cohomology, syntomic cohomology, rigid syntomic cohomology, relative log convergent cohomology, Rost's cycle modules, singular cohomology of arithmetic schemes, Suslin homology, Tate cohomology, unramified cohomology, Weil-etale cohomology, Zariski cohomology, and various theories with compact support. Also, various notions of motives and of mixed motives, and various other kinds of algebraic cycle groups. In addition, many of the theories come with a choice of coefficients. One could also extend the list to theories occurring in other areas of mathematics, there would then be at least a few hundreds of them.

Almost all of the theories in the list are defined on some suitable subcategory of the category of schemes, and we shall restrict our attention to these (the others are much easier to understand).

These theories are of course related in a number of ways. Some are defined in terms of another one, or arise as a special case of another one. Some coincide in suitable circumstances. There are lots of interesting maps from one to another, and lots of spectral sequences from one theory to another.

1.3 Examples of questions related to cohomology theories

We mention five (inter-related) important groups of questions.

- Many questions about motivic L-functions. These are defined in terms of the Galois module structure on ℓ-adic cohomology, and the Hodge structure on the Betti cohomology. Questions asked include questions about special values (regulators, Tamagawa number conjecture, BSD conjecture, ...), and questions about analytic continuation/automorphicity (modularity, Artin conjecture, ...).

- Questions about algebraic cycles (Grothendieck's standard conjectures, Hodge conjecture, Tate conjectures, ...).

- Unsolved problems related to motivic cohomology (finite generation, Quillen-Lichtenbaum conjecture, Beilinson-Soule conjecture, construction of the motivic spectral sequence, . . .)
- Questions about the constructions of various categories of motives.
- Questions about Galois representations "coming from geometry", (Fontaine-Mazur conjecture, . . .)

1.4 Reasons for studying cohomology theories

Trying to understand cohomology theories in general seems like a useful thing to do for the following reasons:

- There should be underlying structures and mechanisms, which would bring some order to the current chaos of cohomology theories. Understanding these would of course be very interesting, and my feeling is that one should study all the theories together.
- Many of the problems mentioned above seem to be beyond reach today. A good fundamental understanding of general mechanisms behind cohomology theories could hopefully make these conjectures more accessible some day.
- Even without making any progress on a "general framework", or on any of the hard questions mentioned, having studied these things for a while should be very helpful when trying to work on other things in arithmetic/algebraic geometry in the future.
- In any case, trying to understand something of all this, and writing it down, should be valuable to future students trying to learn number theory. There is currently no textbook doing this, as far as I am aware.

2 Very brief review of algebraic topology

2.1 Axioms for cohomology

In algebraic topology, we consider the category **Top** of topological spaces (or rather compactly generated spaces, in order to avoid pathologies). A cohomology theory on this category is a functor to graded abelian groups, satisfying the Eilenberg-Steenrod axioms[1]: Functoriality, naturality of the boundary homomorphism, long exact sequence, homotopy invariance, and excision. If one also includes the dimension axiom, the axioms characterise

[1]A cohomology theory should actually be defined on the category of pairs of spaces, and one should also discuss the additivity axiom. We ignore these things for simplicity.

singular cohomology. Without the dimension axiom, one gets lots of cohomology theories; some of the most well-known are K-theory, complex cobordism and versions of elliptic cohomology, but there are lots of other useful theories as well.

2.2 Stable homotopy theory

Because of the homotopy axiom, cohomology theories can be viewed as functors on the homotopy category, i.e. the category obtained from **Top** by declaring homotopy equivalences to be isomorphisms. The Brown representability theorem says that *any* cohomology theory functor $H^n(-)$ is representable - in other words, it is isomorphic to $Hom(-, Y)$ for some suitable space Y (Here Hom is taken in the homotopy category). If we take H^n to be singular cohomology with coefficients in a group G, the representing space is known as the n-th Eilenberg-MacLane space $K(G, n)$.

The Eilenberg-Steenrod axioms (in particular the long exact sequence) translate into the statement that the sequence of spaces representing the cohomology groups H^0, H^1, $\ldots$, form a *spectrum*. A spectrum is defined to be a sequence of spaces $E = \{E_0, E_1, \ldots\}$ together with maps $\sigma_n : S^1 \wedge E^k \to E^{k+1}$ (the first term is smash product with the circle). The spectra form a category Spt, with the obvious notion of morphism. One defines homotopy groups of a spectrum, and these specialise to homology and cohomology groups. By inverting homotopy equivalences in the category if spectra, one obtains the so called *stable homotopy category*, which in some sense is "the category of cohomology theories" (however, beware of phantom maps). There is no doubt that having such a category has led to many interesting developments in topology.

It seems to be the case that all interesting invariants of topological spaces occur as cohomology/homology/homotopy groups, and hence Brown representability can be interpreted as:

> *"All interesting algebraic invariants of topological spaces are of the same form"* (namely homotopy groups of a space or a spectrum).

2.3 Structure on cohomology

A cohomology theory can carry various kinds of "extra structure". For example, there could be a product, making the graded abelian group into a graded ring. The spectrum corresponding to such a theory is called a ring spectrum. Other examples: Maps to/from other theories, group action on the cohomology, ...

In general, one would like to know the relation between structure on cohomology and structure on the corresponding spectrum.

2.4 Cohomology operations

Given two cohomology theories E and F, one is interested in natural transformations from E to F. (Make this more precise, distinguish between stable and unstable operations). The Hom set of all stable operations from E to F is usually denoted F^*E, the star indicating the fact that this is a graded abelian group. In particular, one studies E^*E.

Examples: The Steenrod algebra, the Novikov algebra MU^*MU of operations in complex cobordism.

2.5 Oriented theories

An important class of cohomology theories is the class of oriented theories - these are essentially the ones which have a theory of Chern classes. To every such cohomology theory one can associate a *formal group law*, describing how Chern classes of line bundles behave under multiplication. Conversely, one can prove that every formal group law satisfying a certain condition (Landweber exactness) has a corresponding cohomology theory. Example: The additive group law corresponds to singular cohomology, the multiplicative group law corresponds to K-theory, and the universal formal group law (related to the Lazard ring) corresponds to complex cobordism (therefore complex cobordism is "universal" among oriented theories). One can show that the formal group law associated to an elliptic curve satisfies Landweber exactness, and hence there is a cohomology theory with this formal group law, which one calls elliptic cohomology. There is of course one such thing for each elliptic curve, and unifying these leads to the notion of topological modular forms (see the survey of Lurie), with many exciting recent developments.

2.6 Summary

To summarise, we have seen four kinds of "organising principles" for cohomology theories in algebraic topology:

1. A list of axioms
2. Representability and the stable homotopy category
3. The correspondence with formal groups law
4. A "universal" cohomology theory

The first two apply to all cohomology theories, while the last two apply to the class of orientable theories.

References for this section: Adams: Stable homotopy and generalised homology, Kono and Tamaki: Generalized cohomology, Strickland: Course notes on formal groups (online), Lurie: A survey on elliptic cohomology (online), Hopkins ICM talk (online).

3 Organising principles in algebraic geometry

Given that the collection of all cohomology theories in algebraic geometry is big and difficult to understand, it is interesting to look for ways to organise and explain them. We shall briefly look at the four "organising principles" in the previous section, and discuss their applicability to to algebraic geometry. We shall also consider two others, which we call "Common methods of construction", and "Category-theoretic approaches". Let us write **Geom** for the category of geometric objects on which a cohomology theory is defined - in this section this will be some subcategory of the category of schemes.

3.1 Axioms

There is no list of axioms that apply to all cohomology theories in algebraic geometry. The following lists of axioms are found in the literature:

3.1.1 Weil cohomology

Here the category **Geom** is the category of irreducible smooth projective varieties over an algebraically closed field k. The target category is vector spaces over a "coefficient field" K of characteristic zero (or better perhaps: anticommutative K-algebras). We write X for an object of **Geom**.

The axioms are: Finite generation, vanishing outside the range $[0, \dim X]$, Poincare duality, Kunneth product formula, a cycle class map, and the weak and strong Lefschetz axioms. We will not explain them here, except saying that the cycle class map is a map from the group $C^i(X)$ of algebraic cycles of codimension i, to the cohomology group $H^{2i}(X)$.

The notion of Weil cohomology date back to the Grothendieck school. Examples of Weil cohomologies include ℓ-adic cohomology, crystalline cohomology, Betti cohomology, and (I think) de Rham cohomology. These cohomology theories are extra interesting because they come with "extra structure" (Galois action, filtration, some kind of crystal structure, Hodge structure). For a while I have been trying to think about these various "extra structures", why they appear and what they have in common. A very interesting thing is found in a note by Toen (Homotopy types of algebraic varieties) where he explains that one way of thinking about this is to say that "geometric cohomology theories take values in categories of linear representations of a pro-algebraic group". This apparently "explains" all of the above "extra structure".

3.1.2 Bloch-Ogus cohomology

The notion of Bloch-Ogus cohomology was defined in the 70s, presumably by Bloch and Ogus. It has also been considered by Gillet. Briefly the axioms are: Product, homotopy invariance, purity, cycle class map, and coefficients.

For the details, we refer to Levine's expository article "Mixed motives" in the K-theory handbook, also available on his web page.

Examples include: singular cohomology, ℓ-adic cohomology, de Rham cohomology, Deligne cohomology, and motivic cohomology.

3.1.3 Oriented cohomology

The notion of oriented cohomology in algebraic geometry is more recent, probably from the 90s. See various notes on algebraic cobordism on Levine's web page, or the book "Algebraic cobordism" by Levine and Morel. The category **Geom** is the category of smooth quasi-projective varieties over a field, and the target category is graded rings. Briefly, the axioms are: push-forwards, homotopy property, and a projective bundle formula. This implies a theory of Chern classes.

Examples include: Singular cohomology, topological K-theory, complex cobordism, the Chow ring, algebraic K_0, algebraic cobordism.

3.1.4 Pretheory

Voevodsky sometimes uses the notion of a pretheory. For the definition, see for example Zainoulline: On rational injectivity for pretheories. This should include a number of cohomology theories, probably including etale cohomology and motivic cohomology.

3.2 Formal group laws

Except for the construction of algebraic cobordism, I have not seen the correspondence between formal groups laws and cohomology theories being much exploited in algebraic geometry. However, Morel said in a conversation that he could see no obstruction to why it wouldn't work in the same way as in topology. In particular, it should be possible to define elliptic cohomology for schemes.

3.3 Universal theories

For any list of "axioms", one can look for a universal theory satisfying the axioms. By this we roughly mean a theory which has a map to every other theory satisfying the axioms, such that each theory, considered as a functor on **Geom**, factors through this map. The idea of Grothendieck's (pure) motives is that should be the universal Weil cohomology. Mixed motives are somehow (related to) the universal Bloch-Ogus cohomology. Algebraic cobordism is the universal oriented cohomology.

3.4 Common methods of construction

The most well-known method of constructing cohomology theories in algebraic geometry is by means of a Grothendieck topology. Briefly, one defines a Grothendieck topology on a category $\mathcal{C}$ to be a set of families $\{U_i \to U\}_i$ of morphisms in $\mathcal{C}$. These families are called covering families and are required to satisfy certain natural conditions ("fiber products, compositions, and isomorphisms"). A *site* is a category together with such a set of covering families. A presheaf on $\mathcal{C}$ is a contravariant functor on $\mathcal{C}$ (to abelian groups, say), and one obtains the category of sheaves $Shv(\mathcal{C})$ as the presheaves behaving like a sheaf with respect to the covering families. The category $\mathcal{C}$ is assumed to have fiber products and a final object. The *global sections functor*, $\Gamma : Shv(\mathcal{C}) \to Ab$, is defined by sending a sheaf to its value at the final object. One shows that Γ is additive and left exact, and defines sheaf cohomology of X as the right derived functors of Γ, i.e. $H^i(X, F) = R^i\Gamma(F)$ for a sheaf F.

Examples of cohomology theories obtained through Grothendieck topologies include: etale cohomology with finite coefficients, Zariski cohomology, flat cohomology (I think). Also, ℓ-adic cohomology is obtained from etale cohomology by taking a limit (over sheaf cohomology with coefficients $\mathbb{Z}/\ell^n$) and then tensoring with $\mathbb{Q}_\ell$. Crystalline cohomology is similarly obtained by taking a limit (over the length of a ring of Witt vectors) of sheaf cohomology groups. However, lots of important cohomology theories are not defined through a Grothendieck topology.

Another important way of constructing cohomology theories is by using algebraic cycles, with various equivalence relations. Examples: Chow groups, classical motives. Even for theories not constructed directly in terms of algebraic cycles, cycles still play an important role whenever there is a cycle class map.

3.5 Category-theoretic approaches

It is conceivable that there are category-theoretic gadgets that would be of some help in algebraic geometry. There are various ways of defining cohomology/homology in quite abstract categorical settings; however, I don't know much about this. Examples of such cohomological notions include monad/comonad cohomology, Baues-Wirsching cohomology, Pirashvili-Waldhausen homology, Cohomology of operads (Rezk???), Gorenstein cohomology, Haefliger cohomology, Hochschild-Mitchell cohomology, MacLane cohomology, Generalized Quillen cohomology, and Greenlees' notion of Tate cohomology in axiomatic stable homotopy theory. Possibly, there is also something in the book by Reiten and Beligiannis.

3.6 Representability and stable homotopy theory

Following Morel and Voevodsky's pioneering work in the 90s, one can define a category of spectra starting from varieties/schemes instead of topological spaces or simplicial sets. This is the motivic stable homotopy category. If every cohomology was representable by an object in this category, the slogan would again be: "Every cohomology theory is of the same form". This (it seems to me) would be a great thing, at least psychologically, but probably also in terms of actual consequences. However, there are some serious problems with this. Representable theories include motivic cohomology, algebraic K-theory, algebraic cobordism, Witt groups, and hermitian K-theory. Modifying the motivic stable homotopy category, one can obtain another stable homotopy category in which "etale theories" are representable (in particular, etale cohomology, etale K-theory, and etale cobordism). It is not at all clear what the situation is for all the other theories. We shall discuss these things in some more detail in later sections.

4 Features of cohomology theories in algebraic geometry

Here is a random list of things that should be taken into account in any explanation/organisation of cohomology theories in algebraic geometry.

4.1 Domain category

The "friendliest possibly" category one could have is probably smooth projective varieties over a really nice field, for example algebraically closed, or of characteristic zero, or admitting resolution of singularities. More or less all theories are defined on such varieties, at least. Some theories extend to more general things (relaxing some of the requirements smooth/projective/friendly base scheme), and it can sometimes be an important problem to find the right definition for such an extension.

A fundamental divide seem to be between the cohomology theories defined for things in characteristic p (crystalline, syntomic, $\ldots$), and those defined for things in characteristic zero (most other theories).

For a number theorist, it would be very interesting to systematically study the cohomology theories (whatever that means) on the category of S-schemes, where the base scheme S is not a field, but a number ring or some other arithmetic thing.

4.2 Torsion

Some theories are by definition without any torsion, for example any Weil cohomology theory is defined in this way. Other theories allow torsion, for example algebraic K-theory, and etale cohomology with finite coefficients. Torsion makes things more complicated, but also more interesting, and the size of torsion subgroups sometimes have useful arithmetic interpretations.

4.3 Spectral sequences

A key issue is the subject of spectral sequences. There are lots of spectral sequences, going from one cohomology to another, but it seems difficult to understand when, in general, one should expect a spectral sequence. Of course, there are some general principles, for example:

- Grothendieck's spectral sequence for the composition of two functors between abelian categories
- Filtrations on a spectrum
- Exact couples
- In topology, given a spectrum F which is a module over a ring-spectrum E, one can (under some hypotheses) compute the F-cohomology of a space X by a spectral sequence starting at some Ext groups $Ext(E_*(X), \pi_*(F))$ (here the Ext groups are taken in the category of π_*E-modules, and I haven't really worked out how the indices behave).

However, it is still far from clear how to understand spectral sequences in general. The situation is of course worsened by my own poor understanding of spectral sequences in algebraic topology.

Many spectral sequences in algebraic geometry are special cases of Grothendieck's, but I believe there are also other examples. Unfortunately, the only serious book on spectral sequences (McCleary) focuses exclusively on topology.

Question 1. Are there concrete examples in algebraic geometry of ring spectra and module spectra?

4.4 Cohomology operations

Another central notion is cohomology operations. As soon as one tries to think about some kind of "category of cohomology theories", it becomes very natural to consider endomorphims and automorphisms of theories. We have already seen briefly how these are interpreted in algebraic topology, as endomorphism rings in the category of spectra. Let us take some examples of operations in algebraic geometry.

- Voevodsky's operations on motivic cohomology with finite coefficients. These are similar to the Steenrod operations in topology, and played an important role in the proofs of the Milnor and Bloch-Kato conjectures.

- Since the absolute Galois group of the base field acts on the etale cohomology of a variety, each element of the Galois group can be seen as a cohomology operation
- In his thesis, Joel Riou has given a very nice treatment of operations in algebraic K-theory, from the new viewpoint of motivic homotopy theory, with some interesting consequences.
- One can think of Weil cohomology theories as taking values in some categories of representations of pro-algebraic groups. These groups could then be thought of as part of the cohomology operations for that cohomology theory. See Toen: Homotopy types of algebraic varieties.
- There are several suggestions for the construction of the category of mixed motives. One of the most promising is the one given by Nori. The idea is roughly to consider the ring of operations on singular cohomology of schemes, modify this ring to obtain a certain Hopf algebra, and then define a mixed motive to be a comodule over this Hopf algebra. See Levine's chapter in the K-theory handbook.
- Morel has done some interesting work on relating the endomorphism ring of the sphere spectrum to the Grothendieck-Witt ring over a field.

I have never seen a unified point of view on these things, but they are all very interesting, and it seems to me that they should certainly be studied together.

4.5 Grading and twists

Another point of confusion (at least for a student) is the fact that cohomology theories in algebraic geometry often come naturally as bigraded things. The notation is usually $H^p(X, A(q))$, where A is some kind of coefficients, and p and q are the two grading indices. The q variable is usually called the twist. It would be very interesting to understand why one needs bigraded theories in algebraic geometry, but not in topology. In motivic homotopy theory, this is related to the fact that there are several different circles in the category of motivic spaces, while in the category of topological spaces, there is only one (more on this later). However, it is still not clear what the underlying mechanisms behind this phenomenon are.

4.6 Coefficients

In algebraic geometry, coefficents are always a sheaf, but in topology, one doesn't really consider sheaves, but only constant coefficients. Why?

5 Motives

This document should really contain more about motives. However, for now we only refer to the relevant survey articles in the Motives volumes (Kleiman, Deligne, Scholl, Jannsen, Lichtenbaum, Nekovar, Ramakrishnan) and to articles in the K-theory handbook (Friedlander, Grayson, Kahn, Levine). Again, motives are a bit difficult to understand, partially because there are so many different notions. Things that should be understood/explained: Classical (pure) motives for various different kinds of admissible equivalence relations, various conjectural constructions of mixed motives, various categories constructed by Voevodsky, etale motives, various approaches by Deligne (1-motives, absolute Hodge cycles, realizations, ...). Hopefully there will be more about this in a future version of this document.

6 Very brief review of category theory

6.1 Abelian categories

Add later.

6.2 Triangulated categories

Add later.

6.3 Model categories

Add later. An excellent first introduction is the survey article by Dwyer and Spalinski, available online. For a more substantial introduction, see the books by Hovey and Hirschhorn.

6.4 Abstract Brown representability

There are various purely category-theoretical representability theorems for cohomology-like functors. We will give one example; there are several other similar theorems using slightly different hypotheses.

Definition 1. A functor F from a triangulated category to an abelian category is *cohomological* if it is additive and, for any distinguished triangle $X \to Y \to Z \to TX$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact.

(Should define t-generators)

Theorem 1. Let D be a triangulated category admitting small direct sums and a system $\mathcal{F}$ of t-generators. Let H be a contravariant cohomological functor from D to abelian groups, which commutes with small products. Then H is representable. ("Commutes with small products" means that for any small family $\{X_i\}$ of objects in D, we have an isomorphism $H(\oplus X_i) \cong \prod H(X_i)$.)

These theorems can be used in a proof of Brown representability in topology, but it does not seem to be straight-forward to verify the hypotheses in situations arising from algebraic geometry. It is possible that Neeman and Voevodsky know a lot about why this is difficult.

7 Very brief review of motivic homotopy theory

7.1 Spaces

Fix a base scheme S and consider the category Sm/S of smooth S-schemes. We want to construct a model category ("motivic spaces") containing Sm/S. There are several ways of doing this.

One could for example simply take a motivic space to be a contravariant functor from Sm/S to simplicial sets - this is the same thing a simplicial presheaf on Sm/S, i.e. a simplicial object in the category of presheaves (of sets) on Sm/S. One could also replace simplicial presheaves by simplicial sheaves for the Nisnevich topology. The two categories obtained in this way are Quillen equivalent (the Quillen pair is induced from the associated sheaf/inclusion pair on sheaves). In any case, any simplicial set can be viewed as motivic space (take the constant functor). Also, any object of Sm/S (actually any S-scheme) represents a functor to Sets, and hence a motivic space, because any set can be viewed as a discrete simplicial set.

A pointed motivic space is a motivic space X together with a morphism $S \to X$.

The category of pointed motivic spaces carry a symmetric monoidal structure: define the smash product through the usual smash product of simplicial sets.

To fix ideas, we from now on take a motivic space to be a simplicial Nisnevich sheaf, and we take S to be a field k.

The category of motivic spaces has limits and colimits, and is a model category, and also a topos. An important example of a colimit is the smash product of motivic spaces, which is by definition a pushout, which does not exist in general in the category of varieties.

There are three sensible notions of weak equivalence on the category of simplicial presheaves on a site (or Quillen equivalently simplicial sheaves, I think). These are: Objectwise WE, local WE (i.e. stalkwise, on a site with enough points) and $\mathbb{A}^1$-local (the minimal thing generated by local WEs and things of the form $X \times \mathbb{A}^1 \to X$ for $X \in Sm/k$). We want to work with the $\mathbb{A}^1$-local one.

7.2 Spectra

In topology, the category of spectra is defined in terms of smashing with the circle S^1. In some sense, one inverts the circle. In the category of motivic spaces, there are several sensible choices of a "circle". We make the following definitions: The *Tate circle* S^1_t is the motivic space given by the variety $\mathbb{A}^1 - \{0\}$ (sometimes denoted by $\mathbb{G}_m$). The *simplicial circle* S^1_s is the motivic space given by the usual circle in the category of simplicial sets. A *mixed sphere* is a motivic space formed as an iterated smash product of the above two circles. We will also consider the projective line $\mathbb{P}^1$, which is isomorphic, in the homotopy category of pointed motivic spaces, to $S^1_s \wedge S^1_t$.

Given any "circle" S, we define S-spectra as follows:

Definition 2. An S-spectrum is a sequence $\{E_n\}_{n\geq 0}$ of pointed spaces together with structure maps $S \wedge E_n \to E_{n+1}$. A morphism of spectra consist of maps $f_n : E_n \to E'_n$ commuting with the structure maps.

The category of spectra is a model category (should define stable weak equivalences of spectra) and its homotopy category is by definition the motivic stable homotopy category. We will explain in next section how cohomology theories in algebraic geometry can be represented by spectra in this category.

We omit for now any discussion of symmetric spectra, S-modules, and smash products of spectra. See Jardine: Motivic symmetric spectra, and Hu: S-modules in the category of schemes.

8 Representability of cohomology theories in algebraic geometry

Reading the classical references on motivic stable homotopy theory, by Morel and Voevodsky in the 90s, only three cohomology theories are considered: motivic cohomology, algebraic K-theory, and algebraic cobordism. Later (around 2002?), Hornbostel proved representability of Hermitian K-theory and Witt groups. Around 2006, Gereon Quick constructed an alternative motivic stable homotopy category, in which etale theories are representable (at least etale cohomology, etale K-theory, and etale cobordism).

However, it is not clear how stable homotopy theory relates to all the other theories which are classically considered in algebraic and arithmetic geometry. It seems natural to look for Brown representability theorems for cohomology theories in algebraic geometry. In particular, the following questions seem natural:

- Given a cohomology theory defined by a Grothendieck topology, what are the conditions on the topology and on the sheaf for the cohomology theory to be representable?

- Given a group of algebraic cycles mod some (admissible) equivalence relation, what are the conditions on the equivalance relation for the cycle group to be representable? (for example, I guess it is true that Chow groups are representable in some sense, since they form part of motivic cohomology)
- What are the conditions on a Weil cohomology theory for it to be representable?
- Assuming that a Weil cohomology theory (l-adic, de Rham, crystalline, Betti) is representable, can one view its "extra structure" (Galois action, filtration, some kind of crystal structure, Hodge structure) as coming from the level of the representing spectrum? For example, Quick is now working on the Galois action for etale cohomology, and thinks it should work.
- The abstract Brown representability theorems I have seen are for cohomological functors on a triangulated category. Assuming the triangulated category is the homotopy category of a stable model category, can one formulate abstract Brown representability in terms of some conditions on the level of the model category?
- Quick constructed a new stable homotopy category. Are there other useful variants of the stable homotopy category? I think there is a remark by Jardine somewhere, saying that "every Grothendieck topology gives rise to its own stable homotopy category". What does this mean, and does it mean "the same" as Quick's construction for the etale topology? What are the relations in general between Grothendieck topologies and stable homotopy categories? There are people who probably have a very good understanding of this, for example Jardine and Toen. What is a model topos?

For the two last questions, one should look at Jardine's recent preprint: Representability for model categories, and work out to what extent it answers the questions. Note that Jardine does not seem to be completely satisfied with the answer he gets, because of some problem with the etale topology.

8.1 Brown representability theorems in the literature

There are already some versions of Brown representability in the literature, but the picture is far from complete. Note: In the following subsections, all varieties are smooth.

8.1.1 Morel

Morel writes in a recent (2006), unfinished preprint that there exists conditions on functors from simplicial varieties to bigraded abelian groups, which determine exactly when such a functor is representable. However, he does not write out what the conditions actually are. From a brief conversation with Morel in August 2007, my understanding is that there are only two conditions, namely

1. Homotopy invariance
2. The Nisnevich-Mayer-Vietoris property

The second condition is sometimes referred to as Nisnevich excision, or the Brown-Gersten property, or, I think, Nisnevich descent. It is yet not clear to me what the condition actually means in this setting. Hopefully, Morel will write out some details in a future version of his preprint. In Hornbostel's setting described below, the Nisnevich-Mayer-Vietoris property is defined for functors taking values in simplicial sets, and then the condition says that the functor must take fundamental Nisnevich squares to homotopy cartesian squares.

8.1.2 Hornbostel

Since the details of the above theorem are not yet in the literature, we quickly review the theorem used by Hornbostel in his article.

Theorem 2 (Morel? Voevodsky? Hornbostel?)**.** Let k be a (nice?) field. Let P be a presheaf of simplicial sets on the smooth Nisnevich site of k. Assume that P satisfies homotopy invariance and Nisnevich-Mayer-Vietoris. Then for any regular scheme X, we have a natural isomorphism:

$$\pi_n(P(X)) \cong Hom(S^n \wedge X_+, aP_f)$$

Here the Hom is taken in the $\mathbb{A}^1$-homotopy category of k (presumably the same as the above homotopy category of pointed motivic spaces). Also, X_+ is X with an added disjoint basepoint, and I believe that aP_f is a fibrant replacement of the sheafification of P.

This theorem allows Hornbostel to show unstable representability of Balmer's Witt groups, and of hermitian K-theory. However, one would really like a theorem which applies to arbitrary functors taking values in (bi-)graded abelian groups, not only functors which are a priori given as homotopy groups of a simplicial presheaf.

Question 2. There is a distinction between unstable representability and stable representability. I have not really understood what this means.

8.2 Proving that a cohomology theory is representable

Assuming that the above formulation of Brown representability, as interpreted from the communications of Morel, is correct, we can, in principle, determine whether a given cohomology theory is representable or not, *as long as the cohomology theory is defined for all simplicial varieties.* The problem here is that all theories considered classically are defined only on the category of varieties, and not on the larger category of simplicial varieties. From the conversation with Morel, and from Voevodsky's ICM talk, this seems to be the key reason that there are no (affirmative or negative) results about representability of classical cohomology theories (except for the few ones considered by Morel, Voevodsky and Hornbostel). Both Morel and Voevodsky seem to stop asking about Brown representability theorems at this point, but they do not say why.

Suppose that we are given a cohomology theory, and we would like to prove that it is representable. From the above, it seems there are three steps to consider.

1. Extend the definition of the cohomology from varieties to simplicial varieties
2. Check homotopy invariance
3. Check Nisnevich-Mayer-Vietoris

The second step should be easy, and my guess is that for most cohomology theories, homotopy invariance would hold. Also, if it's not, it might be possible to modify it so that it becomes homotopy invariant (possibly Weibel did something like this for algebraic K-theory in the 80s).

Since I have not understood what the third step really means, I cannot say much about it. In particular, I have no idea about whether it is easy to check, and also no idea about when one would expect it to be true. What is the moral of the Nisnevich descent condition? And for what kinds of cohomology theories would one expect Nisnevich descent to hold?

We will discuss the first step in the next subsection.

8.3 Extending a cohomology theory to simplicial schemes

Has anyone thought systematically about this? Consider first the case when the cohomology theory is defined in terms of a Grothendieck topology. Then, it seems to me that the methods used by Friedlander in his book "Etale homotopy of simplicial schemes" should go through, i.e. one can apply the whole machinery of sheaf cohomology to simplicial schemes, and get a good definition of cohomology, which restricts to the usual thing for schemes, and also some nice spectral sequences which allows the cohomology to be computed. However, I haven't had time to start checking the details of this process. Perhaps it only works for the etale topology for some reason - however, I don't see why this should be the case. See Jardine, K-theory archive preprint 0706, for the site of a simplicial scheme. Having worked out the above process, one also have to check how the limit process works, for ℓ-adic and crystalline cohomology.

Consider now cohomology theories defined in terms of algebraic cycles. Is there any obstruction to developing a theory of algebraic cycles on simplicial schemes? If one can do that, the cohomology theories should also be defineable.

I haven't thought about other theories at all - perhaps they can be extended, perhaps not. In particular, one could ask about oriented theories, or about Bloch-Ogus theories.

8.4 Arithmetic schemes

Something which appears to be completely untouched, is the question of representability for cohomology theories on arithmetic schemes. The motivic stable homotopy category is there for any base scheme, so in particular for schemes like $Spec(\mathbb{Z})$. Is there any relation between this category and the cohomology theories classically defined in arithmetic geometry for arithmetic schemes?

9 Homotopy types of varieties

What is meant in general by "homotopy type"? (E.g. Toen's solution to Grothendieck's schematization problem.) How does the notion of homotopy type relate to number-theoretic questions? There is a notion of arithmetic cohomology theories vs geometric theories (see e.g. Nekovar in the Motives volumes). Toen seems to describe one notion of homotopy type for each geometric (Weil) cohomology. Would it make any sense to talk about "arithmetic homotopy type"?

10 The notion of space in algebraic geometry

In some sense, an (non-arithmetic) algebraic geometer is primarily interested in varieties over a field. Extending the perspective to the whole category of schemes is useful for well-known reasons, even if one only wants to study varieties, and it also brings arithmetic objects into the picture. A fundamental question when thinking about cohomology theories is: What is the right geometric category to work in? In fact, finding the right such category seems to be absolutely necessary if one hopes to get a really good understanding of cohomology theories in general. There are many things indicating that it could sometimes be useful to look at categories which are bigger than the usual categories of schemes. Let us first look at some examples of such categories, and then on these indications.

10.1 Examples of categories of geometric objects

10.1.1 Stacks

Various notions of stacks. Some arise naturally as quotients of schemes by group actions. There is also an interesting notion of n-stack (Toen and others).

10.1.2 Algebraic spaces

See Artin or Knutson. Algebraic spaces arise naturally in deformation theory (I think).

10.1.3 Toposes

Toposes was one of the great inventions of Grothendieck, and in some sense a topos is a very general kind of space. For any site, the category of its sheaves form a topos, so in particular, whenever we choose a scheme and a topology on it, we get a topos.

10.1.4 Simplicial varieties/schemes

Let Δ be the category whose objects are sets of the form $\{0, 1, \dots, n\}$ and whose morphisms are the weakly order-preserving maps, i.e. maps f such that $x \leq y$ implies $f(x) \leq f(y)$. For any category $\mathcal{C}$, a simplicial $\mathcal{C}$-object is defined to be a contravariant functor from Δ to $\mathcal{C}$. The notation for the category of such functors is $\Delta^{op}\mathcal{C}$, and there is an embedding of $\mathcal{C}$ into $\Delta^{op}\mathcal{C}$. In particular, if we take $\mathcal{C}$ to be the category of schemes, we obtain the definition of simplicial scheme.

10.1.5 Motivic spaces

We have already seen these; there are several different constructions, presumably all Quillen equivalent. We defined them here as simplicial Nisnevich sheaves on the smooth site of a base scheme.

10.1.6 Sheaves

In some sense, a sheaf can often be thought of as a geometric object in itself. For example, motivic spaces are sheaves by definition, and every scheme represents a sheaf on any (subcanonical) "geometric" site. A geometric cat embeds into its category of sheaves through Yoneda, and somehow it is sometimes right to consider all the sheaves, or at least more sheaves than just the representables.

10.1.7 Noncommutative spaces

I know too little to say anything about these, but they are certainly interesting.

10.1.8 Other ideas of Grothendieck

Grothendieck were thinking about some new ideas (tame topology?) in the 80s - I don't know if this lead to anything or if anyone has understood it after him.

10.1.9 Various kinds of categories

Given a scheme X, there are several different categories naturally associated to it (in addition to its topos) and one could ask whether one can define cohomology for a certain class of categories, and then recover the definition for schemes as a special case.

10.2 Things indicating that the category of schemes could be too small

We have already seen the need for considering simplicial schemes when discussing Brown representability theorems. Also, it seems like simplicial schemes pop up here and there in the literature, often in articles by very clever authors, such as Friedlander or Annette Huber. There seems to be some deep intrinsic reason for studying simplicial things in general - but I have not grasped what lies behind this.

If one believes (very optimistically) that "all cohomology theories are representable", and also believes that "all interesting invariants of schemes are of a cohomological nature", then it follows that every interesting invariant of schemes is defined on the much bigger category of motivic spaces. Even if this is not true in full generality, one still wonders whether life would be simpler in the category of motivic spaces than in the category of schemes.

In many places in the literature, people do algebraic geometry, and in particular cohomological invariants, for some of the geometric objects above. Some more examples.

- Joshua has defined lots of cohomology theories (absolute cohomology, K-theory, higher Chow groups, ...) for stacks, and proved results like Riemann-Roch, and existence of Adams operations in K-theory, which reduce to the usual things for schemes.

- There is a notion of intersection theory for stacks, which should lead to various kinds of cycle class groups.
- K-theory has been considered for toposes, model categories (Sagave), Segal categories, exact (and hence abelian categories) by Quillen and for triangulated categories (although it is perhaps not clear how this should work, see Neeman in K-theory handbook).
- Balmer's Witt groups of schemes are defined for triangulated categories with duality. Applying this to certain derived categories, one obtains definitions of Witt groups for schemes.
- Quite a lot of things have been written about K-theory and other cohomology theories for non-commutative C^*-algebras, and I believe some people have thought about developing stable homotopy theory for noncommutative spaces.
- There are certainly other examples which I have not heard about.

Question 3. It is striking that K-theory has been developed for a huge variety of mathematical objects, while this is not so much the case for other cohomology theories. What is so special about K-theory? What *is* K-theory anyway?

Question 4. For each of the categories of geometric objects in the above list, how much algebraic geometry can you do with them? Are there examples of things you cannot do, but which you can do with schemes? For which of them can you do homotopy theory? Intersection theory? Birational geometry? Cohomology?

Question 5. Could it be that some of the hard questions in number theory would be easier to attack if one started to think in terms of a bigger category than the usual categories of schemes? A naive analogue: If one had tried for a while to develop elementary calculus, working over the field of rational numbers, it would be a great relief to discover that one could actually work over $\mathbb{R}$ instead. This is in spite of the fact that $\mathbb{R}$ is an extremely complicated and mysterious object (a bit like for example the category of motivic spaces), while $\mathbb{Q}$ is very concrete and approachable (a bit like the category of schemes).

11 Some speculation and ideas

I strongly believe that there are interesting connections between number theory and homotopy theory. Here is a brief list of optimistic ideas, which would be longer if only I knew more about both of these subjects.

11.1 Algebraic geometry over Spec $\mathbb{Z}$

I guess a lot of number theorists have worked on understanding the category of schemes over *Spec* $\mathbb{Z}$. I know almost nothing about this, but it would be very interesting to understand the connections between the following things:

- Arakelov geometry
- Existing cohomology theories for arithmetic schemes (Kato homology, Schmidt's singular homology, ...)
- Motivic (stable) homotopy theory over $\mathbb{Z}$ and other arithmetic base schemes

One could also ask if there is any chance that Deninger's conjectural cohomology theory, giving a cohomological approach to motivic L-functions, could be represented by a spectrum in some kind of arithmetic stable homotopy category.

Note also that the above representability theorems used by Morel and Hornbostel, seem to be formulated only for the case where the base scheme is a field. It would be very interesting to see what one can say about Brown representability for other base schemes.

11.2 Local Langlands and classical homotopy theory

Speaking about relations between number theory and homotopy theory, one should also mention that there seems to be many interesting things going on in parts of classical stable homotopy theory. For example, the integers appearing as orders of summands of the stable homotopy groups of spheres, also pop up in number theory. Most well-known is the appearance of the Bernoulli numbers (see for example Hopkins ICM talk), but there are also deeper connections, see for example the article by Behrens and Lawson on topological automorphic forms, and also the last paragraph of Morava: Complex cobordism and algebraic topology, where he expresses a hope of some deep connections to the local Langlands program. The last two articles are both very recent (2007) and available on the arXiv. Thanks to Matthias Strauch and Mark Behrens for interesting discussions about this.

11.3 Regulator maps

Regulator maps are used for studying special values of motivic L-functions - they typically go from one cohomology groups to another. Is it meaningful to ask whether these maps can be viewed as coming from a map of spectra, and would this yield any kind of interesting information?

11.4 Torsion

A vague idea: The (orders of the) torsion parts of various K-theory groups have arithmetic significance. Could it be the case that other (perhaps yet unknown) cohomology theories have torsion parts which could explain special values of L-functions or other interesting number-theoretic things?

11.5 New Galois representations?

Could etale cohomology groups of more general things than varieties (e.g. simplicial varities or motivic spaces) give rise to new and interesting Galois representations? Could this be of some help in answering classical questions? Would such Galois representations satisfy the same properties as those which come from ordinary varieties? For example, would they form a compatible system?

11.6 Formal group laws

Formal group laws pop up here and there in number theory. Possibly it could be interesting to see what implications it has that a cohomology theory in algebraic geometry is (expected to be) associated to every (Landweber exact) formal group law.

A particular, speculative connection is the following. Consider first the multiplicative formal group law $\mathbb{G}_m$. It gives rise to two very interesting pieces of mathematics:

1. By adjoining torsion points of $\mathbb{G}_m$ one obtains cyclotomic extensions and towers of number fields, related to classical Iwasawa theory.

2. By taking the cohomology theory associated to $\mathbb{G}_m$, one obtains K-theory.

Now the interesting thing is that these two are intimately related. One such connection is the following. Let $\Lambda = \mathbb{Z}_\ell[[T]]$ be the Iwasawa algebra and let G be a finite group of order $\ell - 1$. Then the algebra $\Lambda[G]$ arises as the ring of degree zero cohomology operations $[\hat{K}, \hat{K}]$, where $\hat{K}$ is the complex K-theory spectrum, Bousfield completed at ℓ. Also, let F be a number field, let $F_\infty = F(\mu_{\ell^\infty})$ and let M_∞ be the Galois group of the maximal abelian ℓ-extension of F_∞ unramified away from ℓ. Now M_∞ can be expressed in terms of $\hat{K}$ and the algebraic K-theory spectrum of $\mathcal{O}_F[\frac{1}{\ell}]$. As a consequence, things like Leopoldt's conjecture and all other statement about Iwasawa-theoretic invariants can be expressed in terms of the above K-theory spectra. For example, the Leopoldt conjecture is equivalent to the finiteness of a certain homotopy group of a localisation of the above algebraic K-theory spectrum.

There might also be another relation between Iwasawa theory and K-theory in terms of vanishing orders of some L-functions (is this true?).

Now it might of course be a coincidence that the multiplicative formal group law gives rise to two things that are closely related. But it might also be the case that there is something else underlying this connection. Consider a formal group law $\hat{E}$ from an elliptic curve. From it, we obtain:

1. Interesting extensions and towers of number fields, by adjoining torsion points.
2. A cohomology theory, namely elliptic cohomology. Presumably, elliptic cohomology can also be defined for objects in algebraic geometry, if the formal group law correspondence works.

What is the relation between the two things now? Could this be of any help in noncommutative Iwasawa theory? Or could Iwasawa theory give some information on elliptic cohomology?

For algebraic cobordism, the formal group law is the universal one. I have no idea about whether one can adjoin torsion points of this to a number field, and get something interesting. If one can, then this connection could perhaps lead to something interesting already now, since algebraic cobordism is already well established and more managable than elliptic cohomology. However, this might not be possible, as it seems likely that the concept of torsion points only make sense for formal groups laws coming from algebraic groups, and it seems unlikely that this is the case for the universal formal group law. Would there be a Galois action on the torsion points in this (cobordism) case, or for other formal group laws?

12 Some more questions

Question 6. (Very vague) Fix some geometric category **Geom** and some algebraic category **Alg**, for example abelian groups. Is it possible to formulate conditions on axiom systems for functors **Geom** $\to$ **Alg** that guarantees the existence of a universal theory?

Question 7. Historically, important invariants of algebro-geometric objects have turned out (often long after their original definition) to be of a cohomological nature. For example the genus of a curve. This, I guess, is one of the main reasons that cohomology theories are important. Are there interesting invariants which are NOT of a cohomological/homotopical nature? Of course, since we don't exactly what "cohomological" and "homotopical" means, this is a vague question, but perhaps one can suggest some examples. Galois groups are in some sense of a homotopical nature, since they are like the fundamental group. What about the Mordell-Weil group of an abelian variety - can it be regarded as a cohomology group in some sense?

Question 8. Is there a way of systematically studying/classifying admissible equivalence relations on algebraic cycles? In the articles by Kleiman, for example, I do not feel that there is anything in this direction. For example, how do people find new interesting equivalence relations (e.g. Voevoedsky's smash equivalence)?

Question 9. In topology, there is a condition (Landweber exactness) on a FGL for it to come from a cohomology theory. Is this condition the same in algebraic geometry? There is a map from the classical stable homotopy category to the motivic one, I think (or is it the other way around?). Does this map commute with "taking the FGL of a cohomology theory"?

Question 10. What is meant in general by "deformation"?

Question 11. What is meant in general by "descent"?

Question 12. Can one use operads to understand loop spaces in the category of motivic spaces?

Question 13. What is the relation between morphisms of spectra and maps on cohomology? (Full/faithful???)

Question 14. For each cohomology theory in algebraic geometry, one wonders: is there a monad/comonad somewhere???

Question 15. For some cohomology theories, it is a nontrivial task to define it for non-smooth varieties. If one can show that the theory is representable, then one automatically gets a definition for any motivic space, and hence in particular for any S-scheme. How does this compare to classical definitions for e.g. non-smooth varieties?

Question 16. Many cohomology theories come with extra structure, for example Weil cohomology theories. Do they come from the level of the spectrum??? In particular, what can we say about the Galois action? I do have some vague ideas about the latter, which I plan to include in the next version of this document.

Question 17. What are the moral/conceptual reason for that the Nisnevich topology appears everywhere, and not any of all the other topologies?

13 Category-theoretic questions

13.1 Constructions from m-categories to n-categories

The following types of "maps" are often considered and systematically studied:

- Maps from sets to sets preserving some kind of structure. These maps are the morphisms in a suitable category.
- Maps from a category to a category (possibly preserving some kind of structure, e.g. limits or tensor products). These are functors.
- Maps from a functor to a functor. These are natural transformations.
- It is also not uncommon to see maps from a 2-category to a 2-category, e.g. the construction assigning to a model category its homotopy category.

All these "maps" have in common that their target and domain are of the same nature. However, there are "maps" (in algebraic geometry and probably elsewhere) that are of great importance but whose target and domain seem to be of different nature. For example, take the construction of a motivic L-function of a motive (defined over Q, say). It is not a functor, but it respects product and shifting, and duality. This is not a map between sets (i.e. a homomorphism of some kind), nor is it a map between categories. Instead, it is a map from a category to a set (e.g. the set of power series convergent in some halfplane), and because it respects products, one could think of it as some kind of "homomorphism", but from a 1-category to a 0-category. Similarly, the nerve construction goes from the 2-category of categories to the 1-category of simplicial sets (although simplicial sets can probably be thought of as an infinity-category or something like that, but that's not the point here). Another example would be the map sending a scheme to its associated topos (etale, say). This would be a map from the 1-category of schemes to the 2-category of toposes (do toposes form a 2-category?). Also, the construction sending a scheme to its derived category of sheaves. Other potential sources of examples: The representation ring, formal group laws.

How should one think of maps of this kind? What are they called, and have they been systematically studied? What is the right notion of a (structure-preserving) map from an m-category to an n-category? When is information lost by applying such a map? One could imagine maps from, say, the category of motives to another set, respecting products and shifting, but being easier to analyze than the motivic L-function construction. Perhaps one can show that the motivic L-function construction is unique with these properties, and obtain useful information through uniqueness. One could also imagine other similar constructions being very useful, and that a systematic approach could be interesting. Is this nonsense???

13.2 Representable functors

One often talks about a "category A object in the category B"? Examples:

- A group object in Top is a topological group
- A spectrum object in the category of spectra is a bispectrum
- A group object in the category of schemes is a group scheme
- A formal group law can be defined as a group object in a certain category.
- An example of K-vector space objects in Yoshida's course on non-abelian Lubin-Tate theory.

What are the conditions on the categories A and B for the above expression to make sense? For example, why not talk about a topological space object in the category of groups?

Is the following statement true: A category A object in the category B is an object X such that the functor represented by X (a priori a functor from B to Set) lifts to a functor from B to A. (Here one could take either the covariant or the contravariant functor represented by X - usually I think one takes the contravariant one.)

In general, suppose that a cohomology theory (in topology or in algebraic geometry) is represented by a spectrum in some kind of stable homotopy category. Given some extra structure on the cohomology theory, when is it immediate that this structure can be seen as coming from the level of the stable homotopy category? For example, if the cohomology groups come with a group action, can one automatically say that the spectrum carries an action by the same group, and that the action on cohomology groups is induced by this action?

I have other related questions which I cannot yet formulate precisely, which I guess have to do with enriched categories, and what they really are, morally. What does one mean by "enriched"? For example, is an additive category enriched over abelian groups? (The abelian group structure on the Hom sets is there already because of "non-enriched" properties of the category.) What I really want to understand is how one should understand representability of functors in general, when the target category is not Sets.

13.3 Geometric and algebraic categories

Can a category theorist give a precise definition of the terms "algebraic category" (e.g. Rings, Groups etc) and "geometric category" (Top, Schemes, Stacks, etc)?

14 Omissions

We have not really discussed the following things:

- Symmetric spectra/S-modules for schemes (see Jardine: Motivic symmetric spectra, and a book by Hu.) Also the motivic functors of Dundas, Röndigs and Østvær.
- Operads
- Automorphic representations and automorphic L-functions
- Homotopy theory for rigid analytic spaces
- Lurie's heavy machinery: can it be related to some of the above questions?
- Equivariant motivic homotopy theory
- Lots of other things by Toen and his coauthors - much of this looks really interesting, but I know too little about it

15 General references

The Motives volumes, eds Jannsen et al. The K-theory handbook, 2 volumes. Lots of relevant things by Grothendieck, Deligne, Quillen, Thomason, Voevodsky, Morel, Beilinson, Levine, Jardine, Toen, Hovey, Rezk, Hopkins, Lurie, Hornbostel, Bloch and others. Should include a proper bibliography in next version.